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*Research article*

# On the modified of the one-dimensional Cahn-Hilliard equation with a source term

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**Abstract:** We consider the modified Cahn-Hilliard equation that govern the relative concentration $\phi$ of one component of a binary system. This equation is characterized by the presence of the additional inertial term $\tau_D \frac{d^2\phi}{dt^2}$ which stands for the relaxation of the diffusion flux. This equation is associated with Dirichlet boundary conditions. We study the existence, uniqueness and regularity of solutions in one space dimension. We also prove the existence of the global attractor and exponential attractors.

**Keywords:** Cahn-Hilliard equation; well posedness; strongly continous semigroups; globall attractors; robust exponential attractors
**Mathematics Subject Classification:** 35G20, 35L60, 35Q92, 35B41

## 1. Introduction

The Cahn-Hilliard equation,

$$\frac{d\phi}{dt} + \Delta(\Delta\phi - f(\phi)) = 0 \tag{1.1}$$

is very important in science of materials. This equation is a simple model for the phase separation processes of a binary at a fixed temperature. We refer the reader to [11, 12] for more details. The function $f : \mathbb{R} \to \mathbb{R}$ is the derivative of a double-well potential $F$ whose correspond to the phases of material. A typical nonlinear model is given by

$$F(s) = \tfrac{1}{4}(s^2 - 1)^2$$

i.e.

$$f(s) = s^3 - s$$

The function $\phi(x,t)$ represents the concentration of one of the metallic components of the alloy. It also worth to note that the sole given by Eq (1.1) is not sufficient for the accurate description of the whole variety of physical phenomena arising in this theory, so a number of various modifications this equation was introduced (see [2, 13–16, 18, 19]). On the interesting from both mathematical and physical point of view modifications of the Cahn-Hilliard equation is the following hyperbolic relaxation of the Cahn-Hilliard equation or the Hyperbolic Cahn-Hilliard equation:

$$\tau_D \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + \Delta(\Delta\phi - f(\phi)) = 0 \tag{1.2}$$

which was introduced by P. Galenko and Coauthors (see [3–10]) in order to treat in a more accurate way the non-equilibrium effects in spinodal decomposition. In a fact, the inertial term $\tau_D \frac{d^2\phi}{dt^2}$ changes the type of the equation (from parabolic to hyperbolic) and the analytical properties of its solutions. Equation (1.2) endowed with Dirichlet boundary conditions has been studied in [1] by S. Gatti, V. Pata and M. Grasselli, A. Miranville who proved the existence of Global attractors and exponential attractors for the usual cubic nonlinear term in dimension one of space. In this paper, we consider the problem in [1] by adding a source term that is,

$$\tau_D \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + \Delta(\Delta\phi - f(\phi)) = g(\phi) \tag{1.3}$$

where $g$ is defined by

$$g(\phi) = \frac{k\phi}{k' + |\phi|}, \qquad k, k' \in \mathbb{R}_+^*$$

and function $g : \mathbb{R} \to \mathbb{R}$ is of class $C^1$, Lipschitzian and bounded.

This article is organized as follows. In section 2, we present the problem and fix some notations of operators and spaces. In section 3, we derive a priori estimates which allow us to prove, in section 4, the existence, uniqueness and regularity of solutions. In section 5, setting of dissipativity and in section 6, dissipativity in higher-order spaces. Finally, in section 7, the existence of exponential attractors.

## 2. Setting of the problem

Let us consider the following initial and boundary value problem in a bounded open interval of $\mathbb{R}$, with boundary $\Gamma$ :

$$\tau_D \frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial t} + \Delta^2\phi - \Delta f(\phi) - g(\phi) = 0 \tag{2.1}$$

$$\phi(0) = \phi_0 \qquad and \qquad \frac{\partial\phi(0)}{\partial t} = \phi_1 \tag{2.2}$$

$$\phi = \Delta\phi = 0 \quad on \quad \Gamma \tag{2.3}$$

where $\phi$ is the order parameter, $\tau_D$ is the relaxation time of the diffusion flow, $f$ is the nonlinear regular potentials and $g$ the source term.

Assume that :

$$f \ \ is \ \ of \ \ class \ \ C^2 \tag{2.4}$$

$$f(0) = 0 \tag{2.5}$$

$$f(s)s \geq F(s) - c_1 \geq c_2, \quad c_1, c_2 \geq 0, \quad where \ \ F(s) = \int_0^s f(t)dt \tag{2.6}$$

$$F(s) \geq c_3 s^4 - c_4, \quad c_3 > 0, c_4 \geq 0 \tag{2.7}$$

$$f'(s) \geq -c_5, \quad c_5 \geq 0 \tag{2.8}$$

The source term $g$ satisfies the following properties:

$$g \ \ is \ \ of \ \ class \ \ C^1 \tag{2.9}$$

$$g \ \ is \ \ bounded \tag{2.10}$$

$$g \ \ is \ \ lipschitz \tag{2.11}$$

**Notations :** We denote by $((.,.))$ the usual scalar product on $L^2(0,l)$ associated to the norm $\|.\|$. Also we set $\|.\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}\|$, where $-\Delta$ stands for the minus Laplace operator associated with (homogeneous) Dirichlet boundary conditions (it is a strictly positive, self-adjoint and unbounded linear operator with compact inverse $(-\Delta)^{-1}$) ). Note that $\|.\|_{-1}$ is equivalent to the usual norm on $H^{-1}(0,l)$, where $H^{-1}(0,l) = H_0^1(0,l)'$. More generally, $\|.\|_X$ denotes the norm on the Banach space X. We pose :

$$A = -\Delta = -\frac{\partial^2}{\partial x^2} \quad \text{and} \quad \Omega = ]0,l[ \subset \mathbb{R}$$

## 3. A priori estimates

In this section, we will establish a number of important inegalities that will be used later in the proof of existence, uniqueness, regularity of solution and the existence of finite-dimensional attractors. In

what follows, the poincaré, Holder and young inegalities are extensively used, without further referring to them. We rewrite (2.1) in the equivalent form:

$$\tau_D \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial t} + A^2\phi + Af(\phi) - g(\phi) = 0 \tag{3.1}$$

We multiply (3.1) by $A^{-1}\phi$, integrate over $\Omega$ and have

$$\frac{d}{dt}(2\tau_D(\frac{d\phi}{dt}, A^{-1}\phi) + \|\phi\|_{-1}^2) - 2\tau_D\|\frac{d\phi}{dt}\|_{-1}^2 + 2\|\nabla\phi\|^2 + 2\int_\Omega f(\phi)\phi dx - 2(g(\phi), A^{-1}\phi) = 0 \tag{3.2}$$

We then multiply (3.1) by $A^{-1}\frac{d\phi}{dt}$ and have

$$\frac{d}{dt}(\tau_D\|\frac{d\phi}{dt}\|_{-1}^2+\|\nabla\phi\|^2+2\int_\Omega F(\phi)dx)+2\|\frac{d\phi}{dt}\|_{-1}^2-2(g(\phi),A^{-1}\frac{d\phi}{dt})=0 \tag{3.3}$$

The sum of (3.3) and $\alpha$ times (3.2), (where $\alpha>0$ is small enough) we find

$$\frac{d}{dt}H+(2-2\alpha\tau_D)\|\frac{d\phi}{dt}\|_{-1}^2+2\alpha\|\nabla\phi\|^2+2\alpha\int_\Omega f(\phi)\phi dx-2\alpha(g(\phi),A^{-1}\phi)-2(g(\phi),A^{-1}\frac{d\phi}{dt})=0 \tag{3.4}$$

where

$$H=2\alpha\tau_D(\frac{d\phi}{dt},A^{-1}\phi)+\alpha\|\phi\|_{-1}^2+\tau_D\|\frac{d\phi}{dt}\|_{-1}^2+\|\nabla\phi\|^2+2\int_\Omega F(\phi)dx \tag{3.5}$$

Let $\alpha>0$ such that $\alpha\in(0,\frac{1}{2\tau_D})$ we have

$$|2\alpha\tau_D(\frac{d\phi}{dt},A^{-1}\phi)|\leqslant\frac{\tau_D}{2}\|\frac{d\phi}{dt}\|_{-1}^2+\alpha\|\phi\|_{-1}^2$$

Owing to (2.7) we have

$$\int_\Omega F(\phi)dx\geq c_3\|\phi\|_{L^4(\Omega)}^4-c_4(\Omega)$$

The Eq (3.5) satisfies

$$c_6(\tau_D\|\frac{d\phi}{dt}\|_{-1}^2\|+\|\phi\|_{-1}^2+\|\nabla\phi\|^2+\|\phi\|_{L^4(\Omega)}^4)-c_4(\Omega)\leqslant H\leqslant c_7(\tau_D\|\frac{d\phi}{dt}\|_{-1}^2\| \tag{3.6}$$

$$+\|\phi\|_{-1}^2+\|\nabla\phi\|^2+\|\phi\|_{L^4(\Omega)}^4)+c_8(\Omega),\ c_6,c_7>0 \text{ and } c_8,c_4\geq 0$$

Owing (2.10) and thanks to Young and Poincare inegalities we have

$$|(g(\phi),A^{-1}\frac{d\phi}{dt})|\leqslant\frac{1}{4}\|\frac{d\phi}{dt}\|_{-1}^2+c_9$$

and

$$\alpha|(g(\phi),A^{-1}\phi)|\leqslant\frac{\alpha}{4}\|\phi\|_{-1}^2+c_{10}$$

For $\frac{3}{4}-\alpha\tau_D>0$, the Eq (3.4) satisfies

$$\frac{d}{dt}H+c_{11}H\leqslant c_{12} \tag{3.7}$$

Apply the Gronwall's lamma to (3.7), we obtain

$$\tau_D\|\frac{d\phi(t)}{dt}\|_{-1}^2+\|\phi(t)\|_{-1}^2+\|\nabla\phi(t)\|^2+\|\phi(t)\|_{L^4(\Omega)}^4\leqslant e^{-c_{11}t}(H(0))+c_{13} \tag{3.8}$$

We multiply (3.1) by $\phi$, integrate over $\Omega$ and have

$$\frac{d}{dt}[2\tau_D(\frac{d\phi}{dt},\phi)+\|\phi\|^2]-2\tau_D\|\frac{d\phi}{dt}\|^2+2\|\Delta\phi\|^2+2(f'(\phi)\nabla\phi,\nabla\phi)-2(g(\phi),\phi)=0 \tag{3.9}$$

We then multiply (3.1) by $\frac{d\phi}{dt}$ and have

$$\frac{d}{dt}[\tau_D\|\frac{d\phi}{dt}\|^2 + \|\Delta\phi\|^2] + 2(f'(\phi)\nabla\phi, \nabla\frac{d\phi}{dt}) + 2\|\frac{d\phi}{dt}\|^2 - 2(g(\phi), \frac{d\phi}{dt}) = 0 \tag{3.10}$$

The sum of (3.10) and $\lambda$ times (3.9), (where $\lambda > 0$ is small enough ) we find

$$\frac{d}{dt}Y(t) + 2(f'(\phi)\nabla\phi, \nabla\frac{d\phi}{dt}) + 2\|\frac{d\phi}{dt}\|^2 - 2(g(\phi), \frac{d\phi}{dt}) - 2\lambda\tau_D\|\frac{d\phi}{dt}\|^2 + 2\lambda\|\Delta\phi\|^2 \tag{3.11}$$
$$+2\lambda(f'(\phi)\nabla\phi, \nabla\phi) - 2\lambda(g(\phi), \phi) = 0$$

where

$$Y = \tau_D\|\frac{d\phi}{dt}\|^2 + \|\Delta\phi\|^2 + 2\lambda\tau_D(\frac{d\phi}{dt}, \phi) + \lambda\|\phi\|^2 \tag{3.12}$$

Let $\lambda > 0$ such that $\lambda \in (0, \frac{1}{2\tau_D})$ we have

$$|2\alpha\tau_D(\frac{d\phi}{dt}, \phi)| \leqslant \frac{\tau_D}{2}\|\frac{d\phi}{dt}\|^2 + \lambda\|\phi\|^2$$

The Eq (3.12) satisfies

$$c_{14}(\tau_D\|\frac{d\phi}{dt}\|^2 + \|\phi\|^2 + \|\Delta\phi\|^2) \leqslant Y \leqslant c_{15}(\tau_D\|\frac{d\phi}{dt}\|^2\| + \|\phi\|^2 + \|\Delta\phi\|^2) \tag{3.13}$$

Owing (2.10) and thanks to Young and Poincare inegalities we have

$$|(g(\phi), \frac{d\phi}{dt})| \leqslant \frac{1}{4}\|\frac{d\phi}{dt}\|^2 + c_{16}$$

and

$$\lambda|(g(\phi), \phi)| \leqslant \frac{\lambda}{4}\|\phi\|^2 + c_{17}$$

Owing (2.8), we have

$$\lambda(f'(\phi)\nabla\phi, \nabla\phi) \geq -\lambda c5\|\nabla\phi\|^2 \tag{3.14}$$

Owing (2.2) and thanks to Young inegalitie and $H^1(\Omega) \subset L^\infty(\Omega)$, we have

$$\begin{aligned} |(f'(\phi)\nabla\phi, \nabla\frac{d\phi}{dt})| &\leqslant c_{18}(1 + \|\phi\|^2_{L^\infty(\Omega)})\|\nabla\phi\|\|\nabla\frac{d\phi}{dt}\| \\ &\leqslant c_{19}(1 + \|\nabla\phi\|^2)\|\nabla\phi\|\|\nabla\frac{d\phi}{dt}\| \\ &\leqslant c_{19}\|\nabla\phi\|^2 + c_{19}\|\nabla\phi\|^6_{L^6(\Omega)} + \frac{1}{4}\|\nabla\frac{d\phi}{dt}\|^2 \end{aligned}$$

For $\frac{3}{2} - 2\lambda\tau_D > 0$ and $(c_{19} - \lambda c_5) > 0$, the Eq (3.4) satisfies

$$\frac{d}{dt}Y(t) + c_{20}Y(t) + c_{21}(\|\nabla\phi\|^2 + \|\nabla\phi\|^6_{L^6(\Omega)} + \|\nabla\frac{d\phi}{dt}\|^2) \leqslant c_{22} \tag{3.15}$$

Apply the Gronwall's lamma to (3.15), we obtain

$$\tau_D\|\frac{d\phi}{dt}\|^2 + \|\phi\|^2 + \|\Delta\phi\|^2 \leqslant e^{-c_{20}t}(\tau_D\|\frac{d\phi(0)}{dt}\|^2\| + \|\phi(0)\|^2 + \|\Delta\phi(0)\|^2) + c_{23} \tag{3.16}$$

and

$$\int_0^t(\|\nabla\phi\|^2 + \|\nabla\phi\|^6_{L^6(\Omega)} + \|\nabla\frac{d\phi}{dt}\|^2)dx \leqslant e^{-c_{20}t}(\tau_D\|\frac{d\phi(0)}{dt}\|^2 + \|\phi(0)\|^2 + \|\Delta\phi(0)\|^2) + c_{23} \tag{3.17}$$

## 4. Existence, uniqueness and regularity of solutions

L'existence of solution of the system is based on the Galerkin method.

**Theorem 4.1.** *Suppose that the hypotheses (2.4), (2.6), (2.7), (2.8) and (2.11) are verified and for* $(\phi_0, \phi_1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ *then the system (2.1)-(2.3) has a unique solution* $(\phi, \frac{d\phi}{dt})$ *such that:*

$$(\phi, \tfrac{d\phi}{dt}) \in L^\infty(\mathbb{R}^+, H_0^1(\Omega) \times H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega) \times L^2(\Omega)), \qquad \textit{for all} \quad T > 0$$

Proof.
a) Existence
Consider a spectral basis $(\omega_i)_{i\geq 1}$ of $-\Delta$ associated with eigenvalues $0 < \lambda_1 < \lambda_2 < .... < ...$ which forms an orthonornal basis in $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$. We pose $V_n = span\{\omega_1, ........, \omega_n\}$ this spectral basis and

$$\phi_n = \sum_{i=1}^{n} \phi_{i,n}\omega_i$$

Then we consider the following approximating problem, written in the functional form

$$\tau_D \frac{d^2\phi_n}{dt^2} + \frac{d\phi_n}{dt} + \Delta^2\phi_n - \Delta f(\phi_n) - g(\phi_n) = 0 \tag{4.1}$$

$$\phi_n(0) = \phi_{0,n} \qquad and \qquad \frac{d\phi_n(0)}{dt} = \phi_{1,n} \tag{4.2}$$

Replacing $\phi$ by $\phi_n$ in Eqs (3.17), (3.16) and 3.8), we find:

1. $\phi_n$ is bounded in $L^\infty(\mathbb{R}^+, H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$
2. $\frac{d\phi_n}{dt}$ is bounded in $L^\infty(\mathbb{R}^+, H^{-1}(\Omega)) \cap L^2(0, T; L^2(\Omega))$

Finally, the passage to the limit is based on classical (Aubin-Lions type) compactness results, we find the result of first part of Theorem (4.1).
b) Uniqueness.
Let us consider $(\phi_1, \frac{d\phi_1}{dt})$ and $(\phi_2, \frac{d\phi_2}{dt})$ two solutions of system (2.1)-(2.3) with respective initial conditions $(\phi_{0,1}, \frac{d\phi_{0,1}}{dt})$ and $(\phi_{0,2}, \frac{d\phi_{0,2}}{dt})$ in $H_0^1(\Omega) \times H^{-1}(\Omega)$. we pose

$$(\phi, \tfrac{d\phi}{dt}) = (\phi_1 - \phi_2, \tfrac{d\phi_1}{dt} - \tfrac{d\phi_2}{dt})$$

The system (2.1)-(2.3) becomes

$$\tau_D \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + A^2\phi + A(f(\phi_1) - f(\phi_2)) - (g(\phi_1) - g(\phi_2)) = 0 \tag{4.3}$$

$$\phi(0) = \phi_{0,1} - \phi_{0,2} \qquad and \qquad \frac{d\phi(0)}{dt} = \phi_{1,1} - \phi_{1,2} \tag{4.4}$$

We multiply (4.3) by $A^{-1}\phi$, integrate over $\Omega$ and have

$$\frac{d}{dt}[2\tau_D(\frac{d\phi}{dt}, A^{-1}\phi) + \|\phi\|_{-1}^2] - 2\tau_D\|\frac{d\phi}{dt}\|_{-1}^2 + 2\|\nabla\phi\|^2 + 2(f(\phi_1) - f(\phi_2), \phi) \tag{4.5}$$

$$- 2(g(\phi_1) - g(\phi_2), A^{-1}\phi) = 0$$

We then multiply (4.3) by $A^{-1}\frac{d\phi}{dt}$ and have

$$\frac{d}{dt}[\tau_D\|\frac{d\phi}{dt}\|_{-1}^2 + \|\nabla\phi\|^2] + 2\|\frac{d\phi}{dt}\|_{-1}^2 + 2(f(\phi_1) - f(\phi_2), \frac{d\phi}{dt}) - 2(g(\phi_1) - g(\phi_2), A^{-1}\frac{d\phi}{dt}) = 0 \tag{4.6}$$

The sum of (4.6) and $\alpha$ times (4.5), (where $\alpha > 0$ is small enough) we find

$$\frac{d}{dt}X(t) - 2\alpha\tau_D\|\frac{d\phi}{dt}\|_{-1}^2 + 2\alpha\|A^{\frac{1}{2}}\phi\|^2 + 2\|\frac{d\phi}{dt}\|_{-1}^2 + 2\alpha(f(\phi_1) - f(\phi_2), \phi) + 2(f(\phi_1) - f(\phi_2), \frac{d\phi}{dt}) \tag{4.7}$$

$$-2\alpha(g(\phi_1) - g(\phi_2), A^{-1}\phi) - 2(g(\phi_1) - g(\phi_2), A^{-1}\tfrac{d\phi}{dt}) = 0$$

where

$$X = 2\alpha\tau_D(\frac{d\phi}{dt}, A^{-1}\phi) + \alpha\|\phi\|_{-1}^2 + \tau_D\|\frac{d\phi}{dt}\|_{-1}^2 + \|A^{\frac{1}{2}}\phi\|^2) \tag{4.8}$$

Let $\alpha > 0$ such that $\alpha \in (0, \frac{1}{2\tau_D})$ we have

$$|2\alpha\tau_D(\frac{d\phi}{dt}, A^{-1}\phi)| \leqslant \frac{\tau_D}{2}\|\frac{d\phi}{dt}\|_{-1}^2 + \alpha\|\phi\|_{-1}^2$$

The Eq (4.8) satisfies

$$c_{24}(\tau_D\|\frac{d\phi}{dt}\|_{-1}^2 + \alpha\|\phi\|_{-1}^2 + \|A^{\frac{1}{2}}\phi\|^2) \leqslant X \leqslant c_{25}(\tau_D\|\frac{d\phi}{dt}\|_{-1}^2 + \alpha\|\phi\|_{-1}^2 + \|A^{\frac{1}{2}}\phi\|^2) \tag{4.9}$$

Thanks to (2.8) and inequality of Poincare, we have

$$\begin{aligned}\alpha(f(\phi_1) - f(\phi_2), \phi) &= \alpha(\phi\int_0^1 f'(s\phi_1 + (1-s)\phi_2)ds\phi, \phi)\\ &\geq -\alpha c_5\|A^{\frac{1}{2}}\phi\|^2\end{aligned}$$

and

$$\begin{aligned}|(f(\phi_1) - f(\phi_2), \frac{d\phi}{dt})| &= |A^{\frac{1}{2}}(\phi\int_0^1 f'(s\phi_1 + (1-s)\phi_2)ds), A^{-\frac{1}{2}}\frac{d\phi}{dt})\\ &\leqslant \|A^{\frac{1}{2}}(\phi\int_0^1 f'(s\phi_1 + (1-s)\phi_2)ds)\|\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|\end{aligned}$$

Thanks to the inegality of Holder and $H^1(\Omega) \cap L^\infty(\Omega)$ (where n=1), we have

$$\begin{aligned}\|A^{\frac{1}{2}}(\phi\int_0^1 f'(s\phi_1 + (1-s)\phi_2)ds)\| &\leqslant \|A^{\frac{1}{2}}\phi\int_0^1 f'(s\phi_1 + (1-s)\phi_2)ds\|\\ &\quad + \|\phi\int_0^1 f''(s\phi_1 + (1-s)\phi_2)(sA^{\frac{1}{2}}\phi_1 + (1-s)A^{\frac{1}{2}}\phi_2)ds\|\\ &\leqslant c_{25}(1 + \|A^{\frac{1}{2}}\phi_1\|^2 + \|A^{\frac{1}{2}}\phi_2\|^2)\|A^{\frac{1}{2}}\phi\|\end{aligned}$$

Thanks to Young's inequality, we have

$$|(f(\phi_1) - f(\phi_2), \frac{d\phi}{dt})| \leqslant C(\|\nabla\phi_1\|, \|\nabla\phi_1\|)\|\nabla\phi\|^2 + \frac{1}{4}\|\frac{d\phi}{dt}\|_{-1}^2 \tag{4.10}$$

Thanks to (2.21) and the inegalities of Young and Poincare, we have

$$\begin{aligned}\alpha|((g(\phi_1) - g(\phi_2), A^{-1}\phi))| &\leqslant \alpha\|g(\phi_1) - g(\phi_2)\|\|A^{-1}\phi\| \\ &\leqslant \alpha c_{26}\|\phi_1 - \phi_2\|\|A^{-\frac{1}{2}}\phi\| \\ &\leqslant \alpha c_{26}\|\nabla\phi\|^2 + \frac{\alpha}{4}\|\phi\|_{-1}^2\end{aligned}$$

and

$$\begin{aligned}|((g(\phi_1) - g(\phi_2), A^{-1}\frac{d\phi}{dt}))| &\leqslant \|g(\phi_1) - g(\phi_2)\|\|A^{-1}\frac{d\phi}{dt}\| \\ &\leqslant c_{27}\|\phi_1 - \phi_2\|\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\| \\ &\leqslant c_{27}\|A^{\frac{1}{2}}\phi\|^2 + \frac{1}{4}\|\frac{d\phi}{dt}\|_{-1}^2\end{aligned}$$

For $\frac{3}{2} - 2\alpha\tau_D > 0$ and $C(\alpha, \|\nabla\phi_{0,1}\|, \|\nabla\phi_{0,2}\|) = Q \in L^1(0, T)$( because $\phi_{0,1}, \phi_{0,2} \in H_0^1(\Omega)$), the Eq (4.7) satisfies

$$\frac{d}{dt}X(t) + QX(t) \leqslant 0 \tag{4.11}$$

Applying Gronwall's lemma to (4.11), we obtain

$$X(t) \leqslant e^{-Qt}X(0) \tag{4.12}$$

Hence the uniqueness, as well as the continuous dependence with respect to the initial data. □

**Theorem 4.2.** *Suppose that the hypotheses of Theorem (4.1) are verified and for* $(\phi_0, \phi_1)$ $\in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ *then the system (2.1)-(2.3) has a unique solution* $(\phi, \frac{d\phi}{dt})$ *such that:*

$$(\phi, \tfrac{d\phi}{dt}) \in L^\infty(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega) \times H_0^1(\Omega)), \quad \textit{for all} \quad T > 0$$

Proof. Thanks to the estimates (3.17), (3.16) and 3.8), we find the result of the Theorem (4.2). □

## 5. Dissipativity

We set $\Phi_0^{\tau_D} = H_0^1(\Omega) \times H^{-1}(\Omega)$.
It follows from Theorem (4.1) that we can define the continuous family of operators

$$\begin{aligned}S_{\tau_D}(t) :&\Phi_0^{\tau_D} \to \Phi_0^{\tau_D} \\ &(\phi_0, \phi_1) \mapsto (\phi(t), \frac{d\phi(t)}{dt})\end{aligned}$$

where $(\phi(t), \frac{d\phi(t)}{dt})$ is the unique solution to our system.

**Theorem 5.1.** *The semigroup $\{S_{\tau_D}(t), t \geq 0\}$ associated with (2.1)-(2.3) possesses a bounded absorbing set $\beta_0$ in $\Phi_0^{\tau_D}$ such that, for every bounded set $B \subset \Phi_0^{\tau_D}$, there exist $t_0 = t_0(B) \geq 0$ such that $t \geq t_0$ implies $S_{\tau_D}(t) \subset \beta_0$. Therefore, $S_{\tau_D}(t)$ is dissipative in $\Phi_0^{\tau_D}$.*

Proof.
The result of theorem follows directly from (3.8). □

**Remark 5.2.** *We can assume, without loss of generality that $\beta_0$ is positively invariant by $S_{\tau_D}(t)$, i.e., $S_{\tau_D}(t)\beta_0 \subset \beta_0, \forall t \geq 0$.*

**Lemma 5.3.** *For any $R \geq 0$, there exist $K = K(R) \geq 0$ such that, for any two initial data $\phi_0, \phi_1 \in \Phi_0^{\tau_D}$ with $||\phi_i||_{\Phi_0^{\tau_D}} \leqslant R$, there holds*

$$||S_{\tau_D}(t)\phi_0 - S_{\tau_D}(t)\phi_1||_{\Phi_0^{\tau_D}} \leqslant e^{Kt}||\phi_0 - \phi_1||_{\Phi_0^{\tau_D}} \tag{5.1}$$

*where $\phi_0 = (\phi_0^1, \phi_0^2)$ and $\phi_1 = (\phi_1^1, \phi_1^2)$*

Proof.
Given two solutions $\phi^1$ and $\phi^2$ corresponding to different initial data $\phi_0, \phi_1 \in \Phi_0^{\tau_D}$, the difference $\phi = \phi^1 - \phi^2$ fulfills

$$\tau_D \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + A^2\phi + A(f(\phi^1) - f(\phi^2)) - (g(\phi^1) - g(\phi^2)) = 0 \tag{5.2}$$

Multiplying (5.2) by $A^{-1}\frac{d\phi}{dt}$ , we get

$$\frac{d}{dt}(\tau_D||\frac{d\phi}{dt}||_{-1}^2 + ||\nabla\phi||^2) + 2||\frac{d\phi}{dt}||_{-1}^2 = -2(f(\phi^1) - f(\phi^2), \frac{d\phi}{dt}) + 2(g(\phi^1) - g(\phi^2), A^{-1}\frac{d\phi}{dt}) \tag{5.3}$$

Thanks to Theorem(5.1), we have

$$||(A^{\frac{1}{2}}(f(\phi^1) - f(\phi^2)|| \leqslant c_{28}||A^{\frac{1}{2}}\phi||$$

Thanks to (2.21) and the inegalities of Young and Poincare, we have

$$\begin{aligned}
|(g(\phi^1) - g(\phi^2), A^{-1}\frac{d\phi}{dt})| &\leqslant ||g(\phi^1) - g(\phi^2)||||A^{-1}\frac{d\phi}{dt}|| \\
&\leqslant c_{29}||\phi^1 - \phi^2||||A^{-\frac{1}{2}}\frac{d\phi}{dt}|| \\
&\leqslant c_{29}||A^{\frac{1}{2}}\phi||^2 + \frac{1}{4}||\frac{d\phi}{dt}||_{-1}^2
\end{aligned}$$

Thus

$$\frac{d}{dt}||(\phi(t), \frac{d\phi(t)}{dt})||_{\Phi_0^{\tau_D}} \leqslant C||(\phi(t), \frac{d\phi(t)}{dt})||_{\Phi_0^{\tau_D}} \tag{5.4}$$

Applying Gronwall's lemma to (5.4), we obtain the resultant of the Lemma. □

**Theorem 5.4.** *Under the hypotheses of Theorems (5.1) and (4.1), the Semi-group $\{S_{\tau_D}(t), t \geq 0\}$ defined on the phase $\Phi_0^{\tau_D}$ itself has a global attractor $A_{\tau_D}$ in $\Phi_1^{\tau_D}$ .*

Proof.
We decompose the semi-group $\{S_{\tau_D}(t), t \geq 0\}$ into a sum of two semi-groups so that the first tends to 0 when t tends to infinity and the second is asymptotically compact in $\Phi_1^{\tau_D}$

$$S_{\tau_D}(t) = S^1_{\tau_D}(t) + S^2_{\tau_D}(t), \quad t \geq 0$$

Consider the following decomposition:

$$(\phi(t), \frac{d\phi(t)}{dt}) = (w(t), \frac{dw(t)}{dt}) + (v(t), \frac{dv(t)}{dt}) \tag{5.5}$$

where $(v(t), \frac{dv(t)}{dt})$ is the solution of system

$$\tau_D \frac{d^2 v}{dt^2} + \frac{dv}{dt} + A^2 v + A(f(\phi) - f(w)) + c_5 v = 0 \tag{5.6}$$

$$v(0) = \phi_0 \qquad \frac{dv(0)}{dt} = \phi_1 \tag{5.7}$$

associated with the semi-group $\{S^1_{\tau_D}(t), t \geq 0\}$ and $(w(t), \frac{dw(t)}{dt})$ is the solution of system

$$\tau_D \frac{d^2 w}{dt^2} + \frac{dw}{dt} + A^2 w + Af(w) + c_5 w - c_5 \phi = g(\phi) \tag{5.8}$$

$$w(0) = 0 \qquad \frac{dw(0)}{dt} = 0 \tag{5.9}$$

associated with the semi-group $\{S^2_{\tau_D}(t), t \geq 0\}$. To continue the proof, we need the following lemma.

**Lemma 5.5.** *Under the hypotheses of Theorem (5.1), the semi-group $\{S^2_{\tau_D}(t), t \geq 0\}$ associated with system (5.8)-(5.9) is dissipative in $\Phi_0^{\tau_D}$.*

Proof of the Lemma (5.5).
Multiplying (5.8) by $A^{-1}w$ and $A^{-1}\frac{dw}{dt}$, we have

$$\frac{d}{dt}(2\tau_D \|\frac{dw}{dt}\|^2_{-1} + \|w\|^2_{-1}) - 2\tau_D \|\frac{dw}{dt}\|^2_{-1} + 2\int_0^l f(w)w dx + 2c_5\|w\|^2_{-1} + 2\|\nabla w\|^2 \tag{5.10}$$
$$-2c_5(\phi, A^{-1}w) = 2(g(\phi), A^{-1}w)$$

and

$$\frac{d}{dt}(\tau_D \|\frac{dw}{dt}\|^2_{-1} + \|\nabla w\|^2 + 2\int_0^l F(w)dx + c_5\|w\|^2_{-1}) + 2\|\frac{dw}{dt}\|^2_{-1} - 2c_5(\phi, A^{-1}\frac{dw}{dt}) = 2(g(\phi), A^{-1}\frac{dw}{dt}) \tag{5.11}$$

The sum of (5.11) and $\alpha$ times (5.10), (where $\alpha > 0$ is small enough) we find

$$\frac{d}{dt}H(t) + 2\|\frac{dw}{dt}\|^2_{-1} - 2c_5(\phi, A^{-1}\frac{dw}{dt}) - 2\alpha\tau_D\|\frac{dw}{dt}\|^2_{-1} + 2\alpha\int_0^l f(w)w dx + 2\alpha c_5\|w\|^2_{-1} \tag{5.12}$$
$$+2\alpha\|\nabla w\|^2 - 2\alpha c_5(\phi, A^{-1}w) = 2\alpha(g(\phi), A^{-1}w) + 2(g(\phi), A^{-1}\frac{dw}{dt})$$

where

$$H = \tau_D\|\frac{dw}{dt}\|^2_{-1} + \|\nabla w\|^2 + 2\int_0^l F(w)dx + c_5\|w\|^2_{-1} + 2\alpha\tau_D(\frac{dw}{dt}, A^{-1}w) + \alpha\|w\|^2_{-1} \tag{5.13}$$

Thanks to (2.6), we get

$$\alpha(f(w), w) \geq \alpha \int_0^l F(w)dx - c_1|\Omega|$$

Owing the Theorem (5.1) and the inegality of Poincare, we have

$$-\alpha c_5(\phi, A^{-1}w) - c_5(\phi, A^{-1}\frac{dw}{dt}) \geq -\frac{\alpha c_5}{2}||w||_{-1}^2 - \frac{c_5}{2}||\frac{dw}{dt}||_{-1}^2 - c_{30}$$

The Eq (5.12), satisfies ( for $\alpha > 0$ such that $(\frac{3}{4} - 2\alpha\tau_D) > 0$)

$$\frac{d}{dt}H(t) + c_{31}H(t) \leqslant c_{32} \tag{5.14}$$

Applying Gronwall's lemma to (5.14), we obtain the result of the Lemma. □
To continue the proof of the theorem, we will multiply Eq (5.6) by $A^{-1}v$ and $A^{-1}\frac{dv}{dt}$, we get

$$\frac{d}{dt}(2\tau_D(\frac{dv}{dt}, A^{-1}) + ||v||_{-1}^2) + 2||\nabla v||^2 + 2c_5||v||_{-1}^2 - 2\tau_D||\frac{dv}{dt}||_{-1}^2 + 2(f(\phi) - f(w), v) = 0 \tag{5.15}$$

and

$$\frac{d}{dt}(\tau_D||\frac{dv}{dt}||_{-1}^2 + ||\nabla v||^2 + c_5||v||_{-1}^2) + 2||\frac{dv}{dt}||_{-1}^2 + 2(f(\phi) - f(w), \frac{dv}{dt}) = 0 \tag{5.16}$$

The sum of (5.16) and $\lambda$ times (5.15), (where $\lambda > 0$ is small enough) we find

$$\frac{d}{dt}Y(t) + 2||\frac{dv}{dt}||_{-1}^2 + 2\alpha||\nabla v||^2 + 2\alpha c_5||v||_{-1}^2 - 2\alpha\tau_D||\frac{dv}{dt}||_{-1}^2 + 2\alpha(f(\phi) - f(w), v) \tag{5.17}$$

$$-2(f(\phi) - f(w) - vf'(w), \tfrac{dw}{dt}) = 0$$

where

$$Y = \tau_D||\frac{dv}{dt}||_{-1}^2 + ||\nabla v||^2 + c_5||v||_{-1}^2 + 2\alpha\tau_D(\frac{dv}{dt}, A^{-1}) + \alpha||v||_{-1}^2 + 2\int_0^l (F(\phi) - F(w) - vf(w))dx \tag{5.18}$$

Thanks to (2.8), we get

$$2\int_0^l (F(\phi) - F(w) - vf(w))dx \geq -\frac{1}{2}||\nabla v||^2 - \frac{c_5}{2}||v||_{-1}^2 \tag{5.19}$$

$$(f(\phi) - f(w), v) \geq -\frac{1}{2}||\nabla v||^2 - \frac{c_5}{2}||v||_{-1}^2 \tag{5.20}$$

The Eq (3.18), satisfies

$$c_{33}||(v, \frac{dv}{dt})||_{\Phi_0^{\tau_D}}^2 \leqslant Y \leqslant c_{34}||(v, \frac{dv}{dt})||_{\Phi_0^{\tau_D}}^2 \tag{5.21}$$

Thanks to the Lemma (5.5) and $H_0^1(0, l)$ is an algebra, we have

$$2(f(\phi) - f(w) - vf'(w), \frac{dw}{dt}) \leqslant c_{35}||\frac{dw}{dt}||_{-1}||A^{\frac{1}{2}}(f(\phi) - f(w) - vf'(w))||$$

$$\leqslant c_{35}\|\frac{dw}{dt}\|_{-1}\|\nabla v\|^2$$
$$\leqslant c_{36}\|\nabla v\|^2$$

The Eq (5.17), satisfies (for $\alpha > 0$ such that $(2 - 2\alpha\tau_D) > 0$)

$$\frac{d}{dt}Y(t) + c_{37}Y(t) \leqslant 0 \tag{5.22}$$

Applying Gronwall's lemma to (5.22), we obtain the result of the first part of the theorem.
Multiplying (5.8) by $w$ and $\frac{dw}{dt}$, we have

$$\frac{d}{dt}(\tau_D\|\frac{dw}{dt}\|^2 + \|Aw\|^2 + c_5\|w\|^2 + \int_0^l f'(w)|\nabla w|^2 dx) + 2\|\frac{dw}{dt}\|^2 - 2c_5(\phi, \frac{dw}{dt}) \tag{5.23}$$
$$- \int_0^l f''(w)|\nabla w|^2 \tfrac{dw}{dt} dx = 2(g(\phi), \tfrac{dw}{dt})$$

and

$$\frac{d}{dt}(2\tau_D(\frac{dw}{dt}, w) + \|w\|^2) - 2\tau_D\|\frac{dw}{dt}\|^2 + \|Aw\|^2 + \int_0^l f(w)Awdx + c_5\|w\|^2 - 2c_5(\phi, w) = 2(g(\phi), w) \tag{5.24}$$

The sum of (5.23) and $\alpha$ times (5.24), (where $\alpha > 0$ is small enough) we find

$$\frac{d}{dt}X(t) + 2\|\frac{dw}{dt}\|^2 - 2c_5(\phi, \frac{dw}{dt}) - 2\alpha\tau_D\|\frac{dw}{dt}\|^2 + \alpha\|Aw\|^2 - \int_0^l f''(w)|\nabla w|^2\frac{dw}{dt}dx \tag{5.25}$$
$$+\alpha \int_0^l f(w)Awdx + \alpha c_5\|w\|^2 - 2\alpha c_5(\phi, w) = 2\alpha(g(\phi), w) + 2(g(\phi), \tfrac{dw}{dt})$$

where

$$X = \tau_D\|\frac{dw}{dt}\|^2 + \|Aw\|^2 + c_5\|w\|^2 + \int_0^l f'(w)|\nabla w|^2 dx + 2\tau_D(\frac{dw}{dt}, w) + \|w\|^2$$

for $\alpha > 0$ such that $\alpha \in (0, \frac{1}{2\tau_D})$, we have

$$c_{38}(\tau_D\|\frac{dw}{dt}\|^2 + \|Aw\|^2 + \|w\|^2 + \int_0^l f'(w)|\nabla w|^2 dx) \leqslant X(t) \leqslant c_{39}(\tau_D\|\frac{dw}{dt}\|^2 + \|Aw\|^2$$
$$+\|w\|^2 + \int_0^l f'(w)|\nabla w|^2 dx)$$

Thanks to Theorem (5.1) and the inegalities of Young and Poincare, we have

$$-2c_5(\phi, \frac{dw}{dt}) - 2\alpha c_5(\phi, w) \geq -c_5\|\phi\|\|\frac{dw}{dt}\| - \alpha c_5\|\phi\|\|w\|$$
$$\geq -c_5\|A^{\frac{1}{2}}\phi\|\|\frac{dw}{dt}\| - \alpha c_5\|A^{\frac{1}{2}}\phi\|\|w\|$$
$$\geq -\frac{1}{2}\|\frac{dw}{dt}\|^2 - \frac{\alpha c_5}{2}\|w\|^2 - \alpha c_{40}$$

Thanks to Lemma (5.5 ) and the inegalities of Young and Poincare, we get

$$\begin{aligned}\int_0^l f''(w)|\nabla|^2\frac{dw}{dt}dx &\leqslant c_{41}\|w\|_{L^\infty(\Omega)}\|\nabla w\|^2\|\frac{dw}{dt}\|\\ &\leqslant c_{41}\|Aw\|^{\frac{1}{2}}\|\frac{dw}{dt}\|\\ &\leqslant \frac{1}{4}\|Aw\|^2+\frac{1}{4}\|\frac{dw}{dt}\|^2+c_{42}\end{aligned}$$

The Eq (5.25), satisfies (for $\alpha > 0$ such that $(\frac{3}{4} - 2\alpha\tau_D) > 0$)

$$\frac{d}{dt}X(t) + c_{43}X(t) \leqslant c_{44}, \quad c_{43} > 0, \;\; c_{44} \geq 0 \tag{5.26}$$

Applying Gronwall's lemma to (5.26), we obtain the result of the second part of the theorem. □

**Corollary 5.6.** *The global attractor $A_{\tau_D}$ is bounded in $\Phi_1^{\tau_D}$, with a bound independent of $\tau_D$.*

## 6. Dissipativity in higher-order spaces

Our goal in this part is to show that for $f$ and $g$ satisfying compatibility conditions, the semi-group $\{S_{\tau_D}(t), \quad t \geq 0\}$ defined on the space $\Phi_n^{\tau_D}$( for $1 \leqslant n \leqslant 4$) has an absorbing and closed set $B^n_{\tau_D}$ in $\Phi_n^{\tau_D}$. We assume that:

$$f \in C^{n+1}(\mathbb{R}) \;\; and \;\; f''(0) = 0 \;\; for \;\; n = 3, 4 \tag{6.1}$$

and

$$g^{(n-1)}(\phi) \;\; is \;\; bounded \tag{6.2}$$

If $n = 1$, then (6.1)-(6.2) does not add anything to our previous assumptions on $f$ and $g$. Notice that the derivative of the classical Cahn-Hilliard potential satisfies (6.1) for every $n \in \mathbb{N}$.

**Theorem 6.1.** *Let (6.1)-(6.2) hold for some $1 \leqslant n \leqslant 4$.Then there exists $R_n > 0$ such that the closed ball $B^n_{\tau_D}$ of $\Phi_n^{\tau_D}$ centered at zero of radius $R_n$ is a (bounded) absorbing set for $\{S_{\tau_D}(t), \quad t \geq 0\}$ in $\Phi_n^{\tau_D}$.That is, for every bounded set $B \subset \Phi_n^{\tau_D}$, there exists $t_n = t_n(B)$ such that*

$$S_{\tau_D}(t)B \subset B^n_{\tau_D}, \quad t \geq t_n$$

Theorem (6.1) is a straightforward consequence of Lemma (6.2). The proof follows directly from the proof of the lemma.

**Lemma 6.2.** *Let the hypotheses of Theorem (6.1) hold. Given $\rho_{n-1}, \rho_n \geq 0$, there are $K_{n-1} = K_{n-1}(\rho_{n-1}) \geq 0$ , $K_n = K_n(\rho_n) \geq 0$ and $\nu_n > 0$ such that, if*

$$\|(\phi_0, \phi_1)\|_{\Phi_{n-1}^{\tau_D}} \leqslant \rho_{n-1} \quad and \quad \|(\phi_0, \phi_1)\|_{\Phi_n^{\tau_D}} \leqslant \rho_n$$

*the following inegality holds:*

$$\|S_{\tau_D}(t)(\phi_0, \phi_1)\|_{\Phi_n^{\tau_D}} \leqslant K_n e^{-\nu_n t} + K_{n-1}$$

**Proof.** Throughout this proof, $c_j \geq 0$ for $j \in \mathbb{N}$ may be depend on $\rho_{n-1}$. We exploit an inductive argument on $n \geq 1$. So, we assume that

$$\sup_{\|(\phi_0,\phi_1)\|_{\Phi^{\tau_D}_{n-1}} \leqslant \rho_{n-1}} (\|A^{\frac{n}{2}}\phi\|^2 + \tau_D\|A^{\frac{n-2}{2}}\frac{d\phi}{dt}\|^2) \leqslant c, \quad \forall t \geq 0 \tag{6.3}$$

Notice that, for $n = 1$, (6.3) holds thanks to estimate (3.8). In particular, the assumptions on $f$ yield that

$$\sup_{t\geq 0}\sum_{k=0}^{n+1}\|f^{(k)}(\phi)\|_{L^\infty} \leqslant c \quad and \quad f(\phi) \in D(A^{\frac{n}{2}}) \tag{6.4}$$

Thanks to formula the derivative of a composition of functions of the Faa di Bruno, there is

$$D^n f(\phi) = f'(\phi)D^n\phi + \Gamma \tag{6.5}$$

with $\Gamma = 0$ if $n = 1$, and

$$\Gamma = \sum_{k=2}^{n+1}\Gamma_k f^{(k)}(\phi)$$

where $\Gamma_k$ is a linear combination of terms of the for

$$(D^1\phi)^{i_1}(D^2\phi)^{i_2}......(D^{n-1}\phi)^{i_{n-1}}$$

for some nonnegative integers $i_1, ......., i_{n-1}$ satisfying

$$i_1 + ....... + i_{n-1} = k \quad and \quad 2i_1 + ....... + (n-1)i_{n-1} = n$$

We multiply (3.1) by $A^{n-1}\phi$ and $A^{n-1}\frac{d\phi}{dt}$ and have

$$\frac{d}{dt}(2\tau_D(\frac{d\phi}{dt}, A^{n-1}\phi) + \|A^{\frac{n-1}{2}}\phi\|^2) + 2\|A^{\frac{n+1}{2}}\phi\|^2 - 2\tau_D\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2 \tag{6.6}$$

$$= -2(A^{\frac{n}{2}}f(\phi), A^{\frac{n}{2}}\phi) + 2(g(\phi), A^{n-1}\phi)$$

$$\frac{d}{dt}(\tau_D\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2 + \|A^{\frac{n+1}{2}}\phi\|^2) + 2\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2 = -2(A^{\frac{n}{2}}f(\phi), A^{\frac{n}{2}}\frac{d\phi}{dt}) + 2(g(\phi), A^{n-1}\frac{d\phi}{dt}) \tag{6.7}$$

The sum of (6.7) and $\alpha$ times (6.6), (where $\alpha > 0$ is small enough) we find

$$\frac{d}{dt}H(t) + 2\alpha\|A^{\frac{n+1}{2}}\phi\|^2 - 2\alpha\tau_D\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2 + 2\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2 = -2\alpha(A^{\frac{n}{2}}f(\phi), A^{\frac{n}{2}}\phi) \tag{6.8}$$

$$+2\alpha(g(\phi), A^{n-1}\phi) - 2(A^{\frac{n}{2}}f(\phi), A^{\frac{n}{2}}\frac{d\phi}{dt}) + 2(g(\phi), A^{n-1}\frac{d\phi}{dt})$$

Thanks to (6.3), (6.4) and (6.5), we get

$$\|A^{\frac{1}{2}}\Gamma\| \leqslant c_{45}$$

and

$$-2\alpha(A^{\frac{n}{2}}f(\phi),A^{\frac{n}{2}}\phi)\leqslant \alpha c_{46}$$

Owing (6.5), we have

$$\begin{aligned}-2(A^{\frac{n}{2}}f(\phi),A^{\frac{n}{2}}\frac{d\phi}{dt}) &= -2(f'(\phi)A^{\frac{n}{2}}\phi,A^{\frac{n}{2}}\frac{d\phi}{dt})-2(\Gamma,A^{\frac{n}{2}}\frac{d\phi}{dt})\\ &= \int_0^l |A^{\frac{n}{2}}\phi|^2 f''(\phi)\frac{d\phi}{dt}dx-\frac{d}{dt}\int_0^2 |A^{\frac{n}{2}}\phi|^2 f'(\phi)dx-2(\Gamma,A^{\frac{n}{2}}\frac{d\phi}{dt})\end{aligned}$$

Keeping in mind (6.3)-(6.4), we now evaluate the terms of the right-hand side of the above egality. If $n=1$, as in the proof of Theorem (5.4), we have

$$\int_0^l |A^{\frac{1}{2}}\phi|^2 f''(\phi)\frac{d\phi}{dt}dx\leqslant \frac{1}{2}\|A\phi\|^2+\frac{1}{4}\|\frac{d\phi}{dt}\|^2+c_{47}$$

$1\leqslant n\leqslant 4$, then

$$\int_0^l |A^{\frac{n}{2}}\phi|^2 f''(\phi)\frac{d\phi}{dt}dx\leqslant c_{48}\|\frac{d\phi}{dt}\|_{L^\infty}\leqslant \frac{1}{4}\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2+c_{49}$$

Concerning the last term, we have

$$-2(\Gamma,A^{\frac{n}{2}}\frac{d\phi}{dt})\leqslant -2(A^{\frac{1}{2}}\Gamma,A^{\frac{n-1}{2}}\frac{d\phi}{dt})\leqslant c_{50}\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|\leqslant \frac{1}{4}\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2+c_{51}$$

for $\alpha>0$ such that $\alpha\in(0,\frac{1}{2\tau_D})$, we have

$$c_{52}\|S_{\tau_D}(t)(\phi_0,\phi_1)\|^2_{\Phi_n^{\tau_D}}\leqslant H\leqslant c_{53}\|S_{\tau_D}(t)(\phi_0,\phi_1)\|^2_{\Phi_n^{\tau_D}} \tag{6.9}$$

Thanks to (6.2)-(6.3), we have

$$\begin{aligned}2\alpha(g(\phi),A^{n-1}\phi) &= 2\alpha(A^{\frac{n-1}{2}}g(\phi),A^{\frac{n-1}{2}}\phi)\\ &\leqslant \alpha c_{54}\end{aligned}$$

and

$$\begin{aligned}2(g(\phi),A^{n-1}\frac{d\phi}{dt}) &= 2(A^{\frac{n-1}{2}}g(\phi),A^{\frac{n-1}{2}}\frac{d\phi}{dt})\\ &\leqslant c_{55}\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|\\ &\leqslant \frac{1}{2}\|A^{\frac{n-1}{2}}\frac{d\phi}{dt}\|^2+c_{56}\end{aligned}$$

Finally, collecting the above inegalities, we end up with

$$\frac{d}{dt}H(t)+c_{57}H(t)\leqslant c_{58},\quad c_{57}>0, c_{58}\geq 0 \tag{6.10}$$

The conclusion follows from the Gronwall lemma. □

Actually, up to (possibly) enlarging the radius $R_n$, the absorbing set $B^n_{\tau_D}$ is exponentially attracting in $\Phi^0_{\tau_D}$ as well.

**Proposition 6.3.** *Let the hypotheses of Theorem (6.1) for* $1 \leqslant n \leqslant 4$ *hold. Then there exist* $\omega_n > 0$ *and a positive increasing function* $J_n$ *such that, for any bounded set* $B \subset \Phi^0_{\tau_D}$,

$$dist_{\Phi^0_{\tau_D}}(S_{\tau_D}(t)B, B^n_{\tau_D}) \leqslant J_n(R)e^{-\omega_n t}, \quad \forall t \geq 0$$

*where* $R = \sup_{(\phi_0,\phi_1)\in B} \|(\phi_0, \phi_1)\|_{\Phi^0_{\tau_D}}$

Proof.
In the proof we will use the following two lemmas.

**Lemma 6.4.** *Let* $S(t)$ *be a strongly continuous semi group on a Banach space Phi. Let* $B_0, B_1, B_3 \subset \Phi$ *be such that*

$$dist_\Phi(S(t)B_0, B_1) \leqslant K_1 e^{-\nu_1 t}, \quad dist_\Phi(S(t)B_1, B_2) \leqslant K_2 e^{-\nu_2 t}$$

*for some* $\nu_1, \nu_2 > 0$ *and* $K_1, K_2 \geq 0$*. Assume also that, for* $z_1, z_2 \in \Phi$*, then hold*

$$\|S(t)z_1 - S(t)z_2\|_\Phi \leqslant e^{\nu_0 t}\|z_1 - z_2\|_\Phi$$

*for some* $\nu_0 \geq 0$*. Then it follows that*

$$dist_\Phi(S(t)B_0, B_2) \leqslant (K_1 + K_2)e^{-\nu t}$$

*where* $\nu = \frac{\nu_1\nu_2}{\nu_0+\nu_1+\nu_2}$*. Here,* $dist_\Phi$ *denotes the usual Hausdorff semidistance in* $\Phi$*.*

The proof of Lemma (6.4) is given in [17].

**Lemma 6.5.** *Let (6.1)-(6.2) hold for some* $1 \leqslant n \leqslant 4$*. Then, up to toking a possibly larger* $R_n$*, we have*

$$dist_{\Phi^0_{\tau_D}}(S_{\tau_D}(t)B^{n-1}_{\tau_D}, B^n_{\tau_D}) \leqslant L_n e^{-\omega_n t}, \quad \forall t \geq 0$$

*for some* $L_n \geq 0$ *and some* $\omega_n > 0$

**Proof.**
We need to show that the solution map is the sum of a term exponentially decreasing (in norm) in the space $\Phi^0_{\tau_D}$, and a term uniformly bounded in $\Phi^n_{\tau_D}$. Adapting the proof of the Theorem (5.4) (in view of the proof of Lemma (6.2), it is immediate to see that the system (5.6)-(5.7) is exponentially stable in $\Phi^0_{\tau_D}$, whereas the solution $w$ to (5.8)-(5.9) satisfies the uniform bound

$$\|(w(t), \frac{dw}{dt})\|_{\Phi^n_{\tau_D}} \leqslant c, \quad \forall t \geq 0$$

for some $c$ depending on the radius $R_{n-1}$ of $B^{n-1}_{\tau_D}$. Redefining $R_n$ to be greater than or equal to the above constant $c$, we reach the desired conclusion. □

**Proof of the Proposition 6.3.** Till the end of the proof, we agree to redefine inductively the radius $R_n$ so that Lemma (6.5) holds. Then, on account of the Lemma (5.3), Theorem (5.1), and Lemma (6.5), applying recursively Lemma (6.5), we get the result. □

**Corollary 6.6.** *Let (6.1)-(6.2) hold for some* $1 \leqslant n \leqslant 4$. *Then the global attractor* $A_{\tau_D}$ *is bounded in* $\Phi_n^{\tau_D}$, *with a bound independent of* $\tau_D$.

**Theorem 6.7.** *Let (6.1)-(6.2) hold for some* $1 \leqslant n \leqslant 4$. *Then*

$$\lim_{\tau_D \to 0}[dist_{\Phi_1^0}(A_{\tau_D}, \widetilde{A}_0)] = 0$$

*where*

$$\widetilde{A}_0 = \{(\phi, \psi) : \phi \in A_0, \psi = -A(A\phi + f(\phi)) + g(\phi)\}$$

We point out that the Hausdorff semidistance in the above theorem is taken in $\Phi_1^0$ (and not just in $\Phi_{\tau_D}^0$). Clearly, this is a stronger stability result.

## 7. Existence of exponential attractors

Provided that (6.1)-(6.2) hold for $n = 4$, there a robust family of exponential attractors $\{M_{\tau_D}\}$ which is uniformly bounded in $\Phi_{\tau_D}^n$. Besides, the basin of attraction of each $M_{\tau_D}$ coincides with the whole phase-space $\Phi_{\tau_D}^0$. In particular, $A_{\tau_D} \subset M_{\tau_D}$.
We define the application $\mathbb{J}$ by

$$\mathbb{J} : \mathbb{B}_0^4 \to D(A^{-\frac{1}{2}})$$
$$\phi \mapsto \mathbb{J}(\phi) = -A(A\phi + f(\phi)) + g(\phi)$$

We then introduce the lifting maps $\mathbb{L}_{\tau_D}$: $\mathbb{B}_0^4 \to \Phi_{\tau_D}^0$ as

$$\mathbb{L}_{\tau_D}\phi = \begin{cases} (\phi, \mathbb{J}\phi), & if \;\; \tau_D > 0 \\ \phi, & if \;\; \tau_D = 0 \end{cases}$$

**Remark 7.1.** *Endowing* $\mathbb{B}_0^4$ *with the metric topology of* $D(A^{-\frac{1}{2}})$, *it is straightforward to check that* $\mathbb{J}$ *is* $\frac{1}{2}$*-Holder continuous from* $\mathbb{B}_0^4$ *into* $D(A^{-\frac{1}{2}})$. *Indeed, such a Holder continuity is essential in order to apply the following Lemma (7.3) this is the reason why we shall work in* $\mathbb{B}_0^4$.

**Theorem 7.2.** *Let (6.1)-(6.2) hold for* $n = 4$. *Then the semigroups* $S_{\tau_D}(t)$ *possess compact positively invariant sets* $M_{\tau_D} \subset \mathbb{B}_0^4$ *(called exponential attractors ) with fulfill the following conditions.*

*(T1) There exist* $\omega > 0$ *and a positive increasing function* $\mathbb{J}$ *such that, for any bounded set* $\mathbb{B} \subset \Phi_{\tau_D}^0$, *there holds*

$$dist_{\Phi_{\tau_D}^0}(S_{\tau_D}(t)\mathbb{B}, M_{\tau_D}) \leqslant J(R)e^{-\omega t}, \quad \forall t \geq 0$$

*where* $R = \sup_{(\phi_0,\phi_1)\in B} \|(\phi_0, \phi_1)\|_{\Phi_{\tau_D}^0}$

*(T2) The fractal dimension of* $M_{\tau_D}$ *is uniformly bounded with respect to* $\tau_D$.

*(T3) There exist* $\epsilon \in (0, \frac{1}{8})$ *and* $C > 0$ *such that*

$$dist^{sym}_{\Phi_{\tau_D}^0}(M_{\tau_D}, \mathbb{L}_{\tau_D}M_0) \leqslant C\tau_D^{\epsilon}$$

*The quantities $\omega$, $J$, $\epsilon$ and $C$ are independent of $\tau_D$. Here, $M_{\tau_D}$ is the symmetric Hausdorff distance in $\Phi^0_{\tau_D}$.*

**Lemma 7.3.** *There exist $\Delta_j \geq 0$ , $k \in (0, \frac{1}{2})$ and $t^* \geq t_n$( all independent of $\tau_D$ such that the following conditions hold.*

*(L1) The map $S_{\tau_D} = S_{\tau_D}(t^*)$ satisfies, every $z_1 = (\phi^1_0, \phi^1_1)$, $z_2 = (\phi^2_0, \phi^2_1)$ in $\mathbb{B}^4_0$,*

$$S_{\tau_D} z_1 - S_{\tau_D} z_2 = \mathbb{L}_{\tau_D}(z_1, z_2) + \mathbb{N}_{\tau_D}(z_1, z_2)$$

*where*

$$\|\mathbb{L}_{\tau_D}(z_1, z_2)\|_{\Phi^0_{\tau_D}} \leqslant k\|z_1 - z_2\|_{\Phi^0_{\tau_D}}$$

$$\|\mathbb{N}_{\tau_D}(z_1, z_2)\|_{\Phi^1_{\tau_D}} \leqslant \Delta_1\|z_1 - z_2\|_{\Phi^0_{\tau_D}}$$

*(L2) The lifting map $\mathbb{L}_{\tau_D}$ fulfills*

$$\|S^k_{\tau_D} z - \mathbb{L}_{\tau_D} S^k_0 \mathbb{P}_{\tau_D} z\|_{\Phi^0_{\tau_D}} \leqslant \Delta^k_2 \sqrt[8]{\tau_D}, \quad \forall z \in \mathbb{B}^4_0, \quad \forall k \in \mathbb{N}$$

*where $\mathbb{P}_{\tau_D}$: $\mathbb{B}^4_{\tau_D} \to \mathbb{B}^4_0$ is the projection onto the first component when $\tau_D > 0$, and the identity map, otherwise.*

*(L3) For any $z \in \mathbb{B}^4_{\tau_D}$, there holds*

$$\|S_{\tau_D} z - \mathbb{L}_{\tau_D} S_0 \mathbb{P}_{\tau_D} z\|_{\Phi^0_{\tau_D}} \leqslant \Delta_3 \sqrt[8]{\tau_D}, \quad \forall t \in [t^*, 2t^*]$$

*(L4) The map*

$$(t, z) :\mapsto S_{\tau_D}(t) z : [t^*, 2t^*] \times \mathbb{B}^4_{\tau_D} \to \mathbb{B}^4_{\tau_D}$$

*is $(\frac{1}{2})$ – Hölder continuous. Besides, the map $z :\mapsto S_{\tau_D}(t) z$ is lipschitz continuous on $\mathbb{B}^4_{\tau_D}$, with a lipschitz constant independent of $\tau_D$ and $t \in [t^*, 2t^*]$*

**Proof of Lemma 7.3**

Throughout the proof, the generic constant $c_i$ may be depend on the (common) radius $R_n$ of the absorbing balls $\mathbb{B}^4_{\tau_D}$.

**Proof of (L1)**

Let $z_1 = (\phi^1_0, \phi^1_1)$, $z_2 = (\phi^2_0, \phi^2_1)$ in $\mathbb{B}^4_0$ . Then we write the difference $S_{\tau_D} z_1 - S_{\tau_D} z_2 = (\widetilde{\phi}, \frac{d\widetilde{\phi}}{dt}) = (\widetilde{w}(t), \frac{d\widetilde{w}}{dt}) + (\widetilde{v}, \frac{d\widetilde{v}}{dt})$, where $\widetilde{w}$ and $\widetilde{v}$ are the solutions to the problems

$$\tau_D \frac{d^2\widetilde{v}}{dt^2} + \frac{d\widetilde{v}}{dt} + A^2\widetilde{v} + A(f(\widetilde{\phi}) - f(\widetilde{w})) + c_5\widetilde{v} = 0 \tag{7.1}$$

$$\widetilde{v}(0) = \widetilde{\phi}_0 \qquad \frac{d\widetilde{v}(0)}{dt} = \widetilde{\phi}_1 \tag{7.2}$$

and

$$\tau_D \frac{d^2\widetilde{w}}{dt^2} + \frac{d\widetilde{w}}{dt} + A^2\widetilde{w} + Af(\widetilde{w}) + c_5\widetilde{w} - c_5\widetilde{\phi} = g(\widetilde{\phi}) \tag{7.3}$$

$$\widetilde{w}(0) = 0 \qquad \frac{d\widetilde{w}(0)}{dt} = 0 \tag{7.4}$$

Multiplying the equation (7.3) by $\frac{d\widetilde{w}}{dt}$, we find

$$\frac{d}{dt}(\|A\widetilde{w}\|^2 + \tau_D\|\frac{d\widetilde{w}}{dt}\|^2 + c_5\|\widetilde{w}\|^2 + \int_0^l f'(\widetilde{w})|\nabla\widetilde{w}|^2dx) + 2\|\frac{d\widetilde{w}}{dt}\|^2 = \int_0^l f''(\widetilde{w})|\nabla\widetilde{w}|^2\frac{d\widetilde{w}}{dt}dx \tag{7.5}$$

$$+2(g(\widetilde{\phi}), \tfrac{d\widetilde{w}}{dt}) + 2c_5(\widetilde{\phi}, \tfrac{d\widetilde{w}}{dt})$$

Thanks to Lemma (5.5), Lemma (5.3) and Lemma of Gronwall, we find

$$\|(\widetilde{w}, \frac{d\widetilde{w}}{dt})\|_{\Phi^1_{\tau_D}} \leqslant c_5e^{Kt}\|(\widetilde{\phi}_0, \widetilde{\phi}_1\|_{\Phi^0_{\tau_D}} \tag{7.6}$$

Multiplying the Eq (7.1) by $A^{-1}\widetilde{v}$ and $A^{-1}\frac{d\widetilde{v}}{dt}$ and thanks to first part of the proof the Theorem (5.4) we have

$$\frac{d}{dt}\|(\widetilde{v}, \frac{d\widetilde{v}}{dt})\|_{\Phi^0_{\tau_D}} + c\|(\widetilde{v}, \frac{d\widetilde{v}}{dt})\|_{\Phi^0_{\tau_D}} \leqslant 0 \tag{7.7}$$

Applying Gronwall's lemma to (7.7), we obtain

$$\|(\widetilde{v}, \frac{d\widetilde{v}}{dt})\|_{\Phi^0_{\tau_D}} \leqslant e^{-ct}\|(\widetilde{\phi}_0, \widetilde{\phi}_1)\|_{\Phi^0_{\tau_D}}$$

Taking $t^* \geq t_n$ large enough, we have

$$\mathbb{N}_{\tau_D}(z_1, z_2) = (\widetilde{w}(t^*), \frac{d\widetilde{w}}{dt}(t^*))$$

and

$$\mathbb{L}_{\tau_D}(z_1, z_2) = (\widetilde{v}(t^*), \frac{d\widetilde{v}}{dt}(t^*))$$

**Proof of (L2) and (L3)**

Both assertions follows by the same form

$$\|S_{\tau_D}z - \mathbb{L}_{\tau_D}S_0\mathbb{P}_{\tau_D}z\|_{\Phi^0_{\tau_D}} \leqslant ce^{ct}\sqrt[8]{\tau_D} + ce^{\frac{-t}{2\tau_D}}, \quad \forall t \geq 0, \quad z \in \mathbb{B}^4_{\tau_D} \tag{7.8}$$

To prove the above estimate, we first need

**Lemma 7.4.** *There holds*

$$\sup_{\phi_0\in\mathbb{B}^4_0}\int_0^\infty \|A^{-\frac{1}{2}}\frac{d^2\phi^0}{dt^2}(y)\|dy < \infty \tag{7.9}$$

*where* $S_0(t)\phi^0_0 = \phi^0$

**Proof of Lemma 7.4**

Since $\phi_0$ is bounded in $D(A^{\frac{5}{2}})$, then $\mathbb{J}(\phi_0)$ is bounded in $D(A^{\frac{1}{2}})$. Consider the problem obtained by differentiating with respect to time (2.1) for $\tau_D = 0$, that is

$$\frac{d\psi}{dt} + A(A\psi + f'(\phi^0)\psi) = \psi g'(\phi^0) \tag{7.10}$$

and

$$\psi(0) = \mathbb{J}(\phi_0) \tag{7.11}$$

where $\psi = \frac{d\phi^0}{dt}$

It is a standard matter to see that there exists a unique solution for the problem (7.10)-(7.11) in $C([0,T], D(A^{\frac{1}{2}})$, for any $T > 0$.
Multiplying the Eq (7.10) by $\alpha A^{-1}\psi + A^{-1}\frac{d\psi}{dt}$, we have

$$\frac{d}{dt}(\|A^{\frac{1}{2}}\psi\|^2 + \alpha\|A^{-\frac{1}{2}}\psi\|^2) + 2\alpha\|A^{\frac{1}{2}}\psi\|^2 + 2\|A^{-\frac{1}{2}}\frac{d\psi}{dt}\|^2 = -2\alpha(A^{\frac{1}{2}}(\psi f'(\phi^0), A^{-\frac{1}{2}}\psi) \tag{7.12}$$

$$-2(A^{\frac{1}{2}}(\psi f'(\phi^0), A^{-\frac{1}{2}}\frac{d\psi}{dt}) + 2\alpha(g'(\phi^0)\psi, A^{-1}\psi) + 2(g'(\phi^0)\psi, A^{-1}\frac{d\psi}{dt})$$

On account of the controls

$$\|A^{\frac{1}{2}}(\psi f'(\phi^0))\| \leqslant c\|A^{\frac{1}{2}}\psi\|,$$

and

$$\|(g'(\phi^0)\psi\| \leqslant c\|\psi\| \leqslant c\|A^{\frac{1}{2}}\psi\|$$

The Eq (7.12), satisfies (for $\alpha > 0$ large enough)

$$\frac{d}{dt}(\|A^{\frac{1}{2}}\psi\|^2 + \|A^{-\frac{1}{2}}\psi\|^2) + \|A^{-\frac{1}{2}}\frac{d\psi}{dt}\|^2 \leqslant c(\|A^{\frac{1}{2}}\psi\|^2 + \|A^{-\frac{1}{2}}\psi\|^2) \tag{7.13}$$

Applying Gronwall's lemma to (7.13), we obtain the estimate. □

Given $z = (\phi_0, \phi_1) \in \mathbb{B}^4_{\tau_D}$, we set

$$(\phi^{\tau_D}, \frac{d\phi^{\tau_D}}{dt}) = S_{\tau_D}(t)z \quad \textit{and} \quad (\phi^0, \frac{d\phi^0}{dt}) = \mathbb{L}_{\tau_D}S_0(t)\mathbb{P}_{\tau_D}z$$

The difference $\phi = \phi^{\tau_D} - \phi^0$ solves the problem

$$\tau_D\frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + A(A\phi + f(\phi^{\tau_D}) - f(\phi^0)) = -\tau_D\frac{d^2\phi^0}{dt^2} + g(\phi^{\tau_D}) - g(\phi^0) \tag{7.14}$$

and

$$\phi(0) = 0 \qquad\qquad \frac{d\phi}{dt}(0) = \phi_1 - \mathbb{J}(\phi_0) \tag{7.15}$$

Multiplying the Eq (7.14) by $A^{-1}\frac{d\phi}{dt}$, we get

$$\frac{d}{dt}(\tau_D\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|^2) + 2\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|^2 = -2(A^{\frac{3}{2}}\phi, A^{-\frac{1}{2}}\frac{d\phi}{dt}) - 2(A^{\frac{1}{2}}f(\phi^{\tau_D}) - f(\phi^0)), A^{-\frac{1}{2}}\frac{d\phi}{dt}) \tag{7.16}$$

$$+2(g(\phi^{\tau_D}) - g(\phi^0), A^{-1}\tfrac{d\phi}{dt}) - 2\tau_D(A^{-\frac{1}{2}}\tfrac{d^2\phi^0}{dt^2}, A^{-\frac{1}{2}}\tfrac{d\phi}{dt})$$

The right-hand side of the above equality seen to be controlled by

$$\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|^2 + c + c\tau_D\|A^{-\frac{1}{2}}\frac{d^2\phi^0}{dt^2}\|^2$$

Hence, by (7.9) and the Gronwall lemma, we conclude that

$$\tau_D\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|^2 \leqslant ce^{-\frac{t}{2\tau_D}} + 2c\tau_D \tag{7.17}$$

On the other hand, we may rewrite (7.16) as

$$\frac{d}{dt}\|(\phi, \frac{d\phi}{dt})\|^2_{\Phi^0_{\tau_D}} + 2\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|^2 = 2(A^{\frac{1}{2}}f(\phi^{\tau_D}) - f(\phi^0)), A^{-\frac{1}{2}}\frac{d\phi}{dt})$$

$$+2(g(\phi^{\tau_D}) - g(\phi^0), A^{-1}\tfrac{d\phi}{dt}) - 2\tau_D(A^{-\frac{1}{2}}\tfrac{d^2\phi^0}{dt^2}, A^{-\frac{1}{2}}\tfrac{d\phi}{dt})$$

This time, we control the right-hand side by

$$2\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|^2 + c\|(\phi, \frac{d\phi}{dt})\|^2_{\Phi^0_{\tau_D}} + c\tau_D\|A^{-\frac{1}{2}}\frac{d^2\phi^0}{dt^2}\|^2$$

So to obtain

$$\frac{d}{dt}\|(\phi, \frac{d\phi}{dt})\|^2_{\Phi^0_{\tau_D}} \leqslant c\|(\phi, \frac{d\phi}{dt})\|^2_{\Phi^0_{\tau_D}} + c\tau_D\|A^{-\frac{1}{2}}\frac{d^2\phi^0}{dt^2}\|^2 \tag{7.18}$$

Applying Gronwall's lemma to (7.18) on $(\sqrt{\tau_D}, t + \sqrt{\tau_D})$,along with (7.17 and (7.9), we obtain

$$\|A^{\frac{1}{2}}\phi(t + \sqrt{\tau_D})\|^2 + \tau_D\|A^{-\frac{1}{2}}\frac{d\phi}{dt}(t + \sqrt{\tau_D})\|^2 \leqslant e^{ct}(\|A^{\frac{1}{2}}\phi(\sqrt{\tau_D})\|^2 + \tau_D\|A^{-\frac{1}{2}}\frac{d\phi}{dt}(\sqrt{\tau_D})\|^2) \tag{7.19}$$

$$\leqslant e^{ct}(\|A^{\frac{1}{2}}\phi(\sqrt{\tau_D})\|^2 + ce^{-\frac{1}{2\sqrt{\tau_D}}} + 2c\tau_D) \tag{7.20}$$

But, from Theorem (5.1), we get at once that

$$\|A^{\frac{1}{2}}\phi(t)\| \leqslant \|A^{\frac{3}{2}}\phi\|\|A^{-\frac{1}{2}}\phi\| \leqslant c\int_0^t \|A^{-\frac{1}{2}}\frac{d\phi}{dt}(y)\|dy \leqslant c\sqrt{t}$$

Hence,

$$\|A^{\frac{1}{2}}\phi(\sqrt{\tau_D})\| \leqslant c\sqrt[4]{\tau_D}$$

The inegality (7.20), satisfies

$$\|(\phi, \frac{d\phi}{dt})\|^2 \leqslant e^{ct}(c\sqrt[4]{\tau_D} + ce^{-\frac{1}{2\sqrt{\tau_D}}} + 2c\tau_D)$$

Hence the proof of $(L2)$ and $(L3)$. □

**Proof of the (L4)**

For any $t', t \in [t^*, 2t^*]$ and for any $z_1, z_2 \in \mathbb{B}^4_{\tau_D}$, in view of Lemma (5.3), there holds

$$\|S_{\tau_D}(t)z_1 - S_{\tau_D}(t')z_2\|_{\Phi^0_{\tau_D}} \leqslant \|S_{\tau_D}(t)z_1 - S_{\tau_D}(t')z_1\|_{\Phi^0_{\tau_D}} + \|S_{\tau_D}(t')z_1 - S_{\tau_D}(t')z_2\|_{\Phi^0_{\tau_D}}$$

$$\leqslant \|S_{\tau_D}(t)z_1 - S_{\tau_D}(t')z_1\|_{\Phi^0_{\tau_D}} + e^{2Kt^*}\|z_1 - z_2\|_{\Phi^0_{\tau_D}}$$

Setting $t = t'$, we get at once the second assertion of ($L4$). To prove the first one, we will show that

$$\sup_{z\in\mathbb{B}^4_{\tau_D}} \int_{t^*}^{2t^*} \|(\frac{d\phi}{dt}(y), \frac{d^2\phi}{dt}(y))\|^2_{\Phi^0_{\tau_D}}\, dy \leqslant c \tag{7.21}$$

where, as usual, $\phi(t)$ denotes the first component of $S_{\tau_D}(t)z$.
Multiplying (3.1) by $A\frac{d\phi}{dt}$, we have

$$\frac{d}{dt}(\tau_D\|A^{\frac{1}{2}}\frac{d\phi}{dt}\|^2 + \|A^{\frac{3}{2}}\phi\|^2) + 2\|A^{\frac{1}{2}}\frac{d\phi}{dt}\|^2 = -2(A^{\frac{3}{2}}f(\phi), A^{\frac{1}{2}}\frac{d\phi}{dt}) + 2(A^{\frac{1}{2}}g(\phi), A^{\frac{1}{2}}\frac{d\phi}{dt}) \tag{7.22}$$

Thanks to the proof of Lemma (6.2), we have

$$\frac{d}{dt}(\tau_D\|A^{\frac{1}{2}}\frac{d\phi}{dt}\|^2 + \|A^{\frac{3}{2}}\phi\|^2) + \|A^{\frac{1}{2}}\frac{d\phi}{dt}\|^2 \leqslant c$$

Hence, integrating on $(t^*, 2t^*)$, we find

$$\int_{t^*}^{2t^*} \|A^{\frac{1}{2}}\frac{d\phi}{dt}(y)\|^2 dy \leqslant c \tag{7.23}$$

Multiplying (3.1) by $A\frac{d^2\phi}{dt^2}$, we get

$$\frac{d}{dt}(\|A^{-\frac{1}{2}}\frac{d\phi}{dt}\|^2 + 2(A^{\frac{3}{2}}\phi, A^{-\frac{1}{2}}\frac{d\phi}{dt}) + 2(A^{\frac{1}{2}}f(\phi), A^{-\frac{1}{2}}\frac{d\phi}{dt})) + 2\tau_D\|A^{-\frac{1}{2}}\frac{d^2\phi}{dt^2}\|^2 = 2\|A^{\frac{1}{2}}\frac{d\phi}{dt}\|^2$$
$$+2(f'(\phi)\tfrac{d\phi}{dt}, \tfrac{d\phi}{dt}) + 2(g(\phi), A^{-1}\tfrac{d^2\phi}{dt})$$

On account of the above estimates, an integrating on $(t^*, 2t^*)$ yields the remaining part of (7.21).

**Remark 7.5.** *Actually, the coefficient in the Hölder continuity does not need to be independent of $\tau_D$ to deduce the uniform(with respect to $\tau_D$) finite fractal dimensionality of the family of robust exponential attractors $M_{\tau_D}$. Recall that the fractal dimension of a(relatively) compact set $\mathbb{K}$ in a metric space $\mathbb{X}$ is defined by*

$$dim_F(\mathbb{K}, \mathbb{X}) = \limsup_{\mu\to 0} \frac{\mathbb{H}_\mu(\mathbb{K}, \mathbb{X})}{\log(\frac{1}{\mu})}$$

*where $\mathbb{H}_\mu(\mathbb{K}, \mathbb{X}) = \log N_\mu(\mathbb{H}_\mu(\mathbb{K}, \mathbb{X})$ denotes the so-called Kolmogorov $\mu$-entropy of $\mathbb{K}$ in $\mathbb{X}$, $N_\mu(\mathbb{H}_\mu(\mathbb{K}, \mathbb{X})$ being the minimal number of the $\mu$-balls of $\mathbb{X}$ to cover $\mathbb{K}$. So, if the constant in the Hölder continuity is independent of $\tau_D$, as it is the case here, we further deduce that the Kolmogorov $\mu$-entropy of the family of robust exponential attractors is, for fixed $\mu > 0$, bounded from above independently of $\tau_D$.*

## Conflict of interest

The author declares that there is no conflict of interests in this paper.

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