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Research article

pth moment exponential stability and convergence analysis of semilinear stochastic evolution equations driven by Riemann-Liouville fractional Brownian motion

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Abstract: Many works have been done on Brownian motion or fractional Brownian motion, but few of them have considered the simpler type, Riemann-Liouville fractional Brownian motion. In this paper, we investigate the semilinear stochastic evolution equations driven by Riemann-Liouville fractional Brownian motion with Hurst parameter $H < 1/2$. First, we prove the pth moment exponential stability of mild solution. Then, based on the maximal inequality from Lemma 10 in [1], the uniform boundedness of pth moment of both exact and numerical solutions are studied, and the strong convergence of the exponential Euler method is established as well as the convergence rate. Finally, two multi-dimensional examples are carried out to demonstrate the consistency with theoretical results.

Keywords: semilinear stochastic evolution equation; Riemann-Liouville fractional Brownian motion; pth moment exponential stability; exponential Euler method; strong convergence
Mathematics Subject Classification: 60H15, 60G15

1. Introduction

Fractional Brownian motion (fBm) was first introduced by Kolmogorov [2] in 1940, where it was defined in a Hilbert space and called "Wiener Helix". Mandelbrot and Van Ness [3] established a stochastic integral representation in terms of a standard Brownian motion. As an extension of the classic Brownian motion, fBm has attracted the increasing attention due to its wide applications in mathematical finance [4–7]; in biology [8, 9]; in telecommunication networks [10–12]; in population dynamic systems [13, 14] and in chemistry [15, 16] etc.

A fBm of Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $\{B^H(t), t \geq 0\}$ with covariance function

$$R_H(t, s) = E(B^H(t)B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$$

for every $s, t \in \mathbb{R}^+$. It is a self-similar process, that is, $B^H(\alpha t)$ has the same distribution as $\alpha^H B^H(t)$. The constant H determines the sign of the covariance of the future and past increments. The covariance function is positive when $H > \frac{1}{2}$ and negative when $H < \frac{1}{2}$. According to Beran's definition [17], when $H > \frac{1}{2}$ then fBm is called a long memory process, and when $H < \frac{1}{2}$ then fBm is called a short memory process, and when $H = \frac{1}{2}$ then fBm becomes the standard Brownian motion without memory.

It is known that the increments of fBm are dependent for $H \neq \frac{1}{2}$, so that fBm is neither a Markov process, nor a semimartingale, the classical Itô calculus cannot be used in stochastic differential equations (SDEs) driven by fBm with $H \neq \frac{1}{2}$, and different approaches have been proposed in order to define stochastic integrals respect to fBm in the past three decades.

Lin [18] and Dai and Heyde [19] have defined the stochastic integral of Stratonovich type for fBm in the case of $0 < H < \frac{1}{2}$. However, this integral does not satisfy the property $\mathbb{E}\left(\int_0^t \phi_s dB^H(s)\right) = 0$, which is important in modeling problems of SDEs driven by fractional Gaussian noise. For this reason, Duncan et al. [20] have introduced a new stochastic integral of Itô type for fBm with zero mean which is the limit of Riemann sums defined by means of the Wick product rather than ordinary products. Zähle [21] has introduced a pathwise stochastic integral with respect to the fBm B with parameter $H \in (0, 1)$ by using the notions of fractional integral and derivative. If the integrator has λ-Hölder continuous paths with $\lambda > 1 - H$, then this integral can be interpreted as a Riemann-Stieltjes integral. Many interesting works on defining stochastic integrals respect to fBm with Hurst parameter $H \neq \frac{1}{2}$ we refer the literatures [1, 22–24].

Mandelbrot and Van Ness [3] introduced the following integral representation:

$$B^H(t) = B^H(0) + \frac{1}{\Gamma(1-\alpha)} \left\{ Z(t) + \int_0^t (t-s)^{-\alpha} dW(s) \right\}, \quad t > 0,$$

where

$$Z(t) = \int_{-\infty}^0 ((t-s)^{-\alpha} - (-s)^{-\alpha}) dW(s),$$

$\{W(s), s \in \mathbb{R}\}$ is a standard Brownian motion, Γ represents the gamma function, $\alpha = \frac{1}{2} - H \in \left(0, \frac{1}{2}\right)$. Considering that the process $Z(t)$ has absolutely continuous trajectories, it suffices to consider the term $\int_0^t (t-s)^{-\alpha} dW(s)$, which is called Riemann-Liouville fractional integral and we denote it by $W^H(t)$ along the paper, that is,

$$W^H(t) = \int_0^t (t-s)^{-\alpha} dW(s). \tag{1.1}$$

For more details about the relation between Riemann-Liouville fractional Brownian motion (RLfBm) and fBm we refer [25–27].

Recently, the exponential methods for semilinear stochastic evolution equations have attracted the interest of many researchers. Based on the variation-of-constants formula, the exponential integrator methods such as exponential Runge-Kutta methods [28, 29], exponential Taylor methods [30], explicit exponential general linear methods [31], exponential multistep methods [32] have been constructed to solve the deterministic semilinear problems. These numerical methods have excellent linear stability and can be performed explicitly. For their stability analysis, Shi et al. [33] have demonstrated that the exponential Euler method is convergent with the strong order 1/2 for semilinear SDEs, and it can also reproduce the mean-square exponential stability for any nonzero stepsize. Komori and Burrage [34] have derived an explicit exponential Euler scheme for multi-dimensional, non-commutative stochastic differential equations with a semilinear drift term to simulate stiff biochemical reaction systems. Li and Zhang [35] have proved the mean-square exponential stability of the exponential Euler method for semilinear stochastic delay differential equations under Poisson white noise excitations. Hu and Huang [36] have derived the necessary and sufficient condition of the numerical delay dependent stability of the exponential Euler method for a class of stochastic delay differential equations, and showed that the exponential method can fully preserve the asymptotic mean-square stability. In [37], an exponential Euler method has been proposed to approximate the exact solutions of stochastic functional differential equations driven by weighted fBm $B^{a,b}$, and the convergence rate and strong convergence of the method of the true solutions have obtained.

Up to now, to our best knowledge, lots of work have focused on Brownian motion or fBm, but few of them have considered the simpler type, RLfBm. In this paper, we investigate the semilinear stochastic evolution equations driven by RLfBm with Hurst parameter $H < 1/2$. Considering that exponential integrator methods have great properties and can be performed explicitly, we study the uniform boundedness of exponential Euler approximation solution and exact solution as well as strong convergence.

The rest of the paper is arranged as follows. In Section 2, some necessary concepts about fractional RLfBm are introduced. A technical lemma which is crucial in our stability analysis is proved. In Section 3, we prove the pth moment exponential stability of the semilinear stochastic evolution equations driven by RLfBm. In Section 4, we investigate the uniform boundedness of pth moment of both exact and numerical solutions of Eq (2.1), and show the convergence of the exponential Euler method for semilinear stochastic evolution equations driven by RLfBm as well as the convergence rate of the method. Finally, some numerical experiments are carried out to support the theoretical results in Section 5.

2. Preliminaries

We here use the notation of [38–40], which gives a complete survey of fractional integrals and derivatives.

Let f be a deterministic real-valued function that belongs to $L^1([a,b])$, where (a,b) is a finite interval of $\mathbb{R}$. The left-sided fractional Riemann-Liouville integrals of order $\alpha > 0$ are determined at almost every $x \in (a,b)$ and defined as

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)dy.$$

These integrals extend the usual nth-order iterated integrals of f for $\alpha = n \in \mathbb{N}$. It is easy to verify the composition formula

$$I^{\alpha}_{a+}(I^{\beta}_{a+}f) = I^{\alpha+\beta}_{a+}f.$$

Fractional differentiation may be introduced as an inverse operation. Let us assume in the sequel that $0 < \alpha < 1$ and $p > 1$. We will denote by $I^{\alpha}_{a+}(L^p)$ the class of functions f in $L^p([a,b])$ which may be represented as an I^{β}_{a+}-integral of some function $\phi \in L^p([a,b])$. If $f \in I^{\alpha}_{a+}(L^p)$, the function ϕ such that $f = I^{\alpha}_{a+}\phi$ is unique in L^p and it agrees with the left-sided Riemann-Liouville derivative of f of order α defined by

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}dt.$$

Moreover, this derivative has the following Weyl representation

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha-1}}dy\right),$$

where the convergence of the integrals at the sigularity $x = t$ holds in the L^p-sense.

Next, we consider an RLfBm with values in a Hilbert space.

Let $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ and $(K, \|\cdot\|_K, \langle\cdot,\cdot\rangle_K)$ be two real, separable Hilbert spaces and let $\mathcal{L}(K, U)$ be the space of bounded linear operator from K to U. Let $Q \in \mathcal{L}(K, K)$ be a non-negative self-adjoint operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty}\lambda_n < \infty$, where $\lambda_n \geq 0$ $(n = 1, 2, ...)$ are non-negative real numbers and $\{e_n\}_{n\geq 1}$ is a complete orthonormal basis in K. We define the infinite-dimensional RLfBm on K with covariance Q as follows:

$$W^H_Q(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta^H_n(t), \quad t \geq 0,$$

where $\{\beta^H_n(t)\}_{n\in\mathbb{N}}$ be a sequence of one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. $W^H(t)$ is a K-valued Gaussian process, starts from 0, and has zero mean and covariance:

$$\mathbb{E}\left\langle W^H(t), x\right\rangle\left\langle W^H(s), y\right\rangle = R(s,t)\langle Q(x), y\rangle \quad for\ all\ x, y \in K\ and\ t, s \in [0, T].$$

Denote by $\mathcal{L}^0_2(K, U)$ the space of all $\xi \in \mathcal{L}(K, U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{\mathcal{L}^0_2(K,U)} = \sum_{n=1}^{\infty}\|\sqrt{\lambda_n}\xi(s)e_n\|^2 = tr(\xi Q\xi^*) < \infty.$$

Then ξ is called a Q-Hilbert-Schmidt operator from K to U.

In this paper, we shall consider the following semilinear stochastic evolution equation driven by RLfBm:

$$\begin{cases} dX(t) = (AX(t) + f(t, X(t)))dt + g(t)dW^H_Q(t), & t \in [0, T], \\ X(0) = X_0, \end{cases} \tag{2.1}$$

where $W^H_Q(t)$ is a Riemann-Liouville fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$, $f : [0, T] \times (\Omega, U) \to U$ is a continuous function, $g : [0, T] \to \mathcal{L}^0_2(K, U)$ is a deterministic time-dependent function, A is the generator of a strongly continuous analytic semigroup $S = \{S(t)\}_{t\geq 0}$ on a Hilbert space.

Throughout this paper, we impose the following assumptions:

(H1) Let $A : D(A) \subset U \to U$ be the infinitesimal generator of an analytic semigroup $S(\cdot)$ on U, there exist two constants $M \geq 1$ and $\lambda \geq 0$ for all $t \geq 0$ such that

$$|S(t)| \leq Me^{-\lambda t}. \tag{2.2}$$

(H2) Assume that f in Eq (2.1) satisfies the globally Lipschitz condition, that is, there exists a constant L_1 such that

$$|f(t, X(t)) - f(t, Y(t))| \leq L_1|X(t) - Y(t)|, \tag{2.3}$$

for all $X(t), Y(t) \in L^2(\Omega, U)$.

(H3) Assume that g in Eq (2.1) satisfies the following Hölder condition, that is, there exists a constant L_2 such that

$$|g(t) - g(s)| \leq L_2 |t - s|^{\gamma}, \tag{2.4}$$

where $1/p \leq \gamma < 1$ for all $X(t), Y(t) \in L^2(\Omega, U)$.

(H4) There exists two constants L_3, L_4 such that

$$\begin{aligned} |f(t, X) - f(s, X)| &\leq L_3 |t - s|^{\beta} |X|, \\ |g(t)| &\leq L_4, \end{aligned} \tag{2.5}$$

where β is a fixed positive constant.

(H5) There exists a non-negative real numbers $Q_1 \geq 0$ and continuous function $\xi(t) : [0, \infty) \to \mathbb{R}^+$ such that

$$\mathbb{E}|f(t, X(t))|^p \leq Q_1 \mathbb{E}|X(t)|^p + \xi(t), \quad t \geq 0, \tag{2.6}$$

and there exists a non-negative real number $\xi_1 \geq 0$ such that $|\xi(t)| \leq \xi_1 e^{-p\lambda t}$.

(H6) For any $t \geq 0$, let

$$Q_2 := \int_0^{+\infty} e^{p\lambda t} |g(t)|^p dt. \tag{2.7}$$

Definition 2.1. *A U-valued stochastic process $X(t)$ is called a mild solution of Eq (2.1) if for all $t \in [0, T]$, $X(t)$ satisfies*

$$X(t) = S(t)X_0 + \int_0^t S(t - s) f(s, X(s)) ds + \int_0^t S(t - s) g(s) dW_Q^H(s), \quad \mathbb{P} - a.s. \tag{2.8}$$

Theorem 2.2. *Let **(H1)–(H3)** hold. Then the Eq* (2.1) *have a unique mild solution $X(t)$ on $t \in [0, T]$.*

Proof. The proof of existence and uniqueness is almost similar to Theorem 1 of [41], so it is omitted here. □

Lemma 2.3. *For any $g : [0, T] \to \mathcal{L}_2^0(K, U)$ such that **(H1)** and **(H6)** hold. Then for any $0 < H < \frac{1}{2}$, $0 < \alpha < \frac{1}{2}$ and $p > 2/(1 - 2\alpha)$,*

$$\mathbb{E}\left| \int_0^t S(t - s) g(s) dW_Q^H(s) \right|^p \leq C(p, M, \alpha, \lambda, Q_2) e^{-\lambda t}, \tag{2.9}$$

where $C(p, M, \alpha, \lambda, Q_2)$ is a constant depends on p, M, α, λ, Q_2 only.

Proof. Choosing $p > 2/(1-2\alpha)$, applying Hölder's inequality, we obtain

$$
\begin{aligned}
\mathbb{E}\left|\int_0^t S(t-s)g(s)dW_Q^H(s)\right|^p =&\mathbb{E}\left|\int_0^t S(t-s)g(s)(t-s)^{-\alpha}dW(s)\right|^p \\
\le& C_p\left(\mathbb{E}\left|\int_0^t S(t-s)g(s)(t-s)^{-\alpha}dW(s)\right|^2\right)^{\frac{p}{2}} \\
\le& C_p\left(\mathbb{E}\left|\int_0^t |S(t-s)|^2 g^2(s)(t-s)^{-2\alpha}ds\right|\right)^{\frac{p}{2}} \\
\le& C_pM^p\left(\mathbb{E}\left|\int_0^t e^{-2\lambda(t-s)}g^2(s)(t-s)^{-2\alpha}ds\right|\right)^{\frac{p}{2}} \\
\le& C_pM^p\left(\left(\int_0^t \left|(t-s)^{-2\alpha}\right|^{\frac{p}{p-2}}ds\right)^{\frac{p-2}{p}}\mathbb{E}\left(\int_0^t \left|e^{-2\lambda(t-s)}g^2(s)\right|^{\frac{p}{2}}ds\right)^{\frac{2}{p}}\right)^{\frac{p}{2}} \\
\le& C_pM^p\left(\left(\int_0^t (t-s)^{-\frac{2p\alpha}{p-2}}ds\right)^{\frac{p-2}{p}}\mathbb{E}\left(\int_0^t e^{-p\lambda(t-s)}|g(s)|^pds\right)^{\frac{2}{p}}\right)^{\frac{p}{2}} \\
\le& C_pM^p\left(\frac{p-2}{p-2-2p\alpha}\right)^{\frac{p-2}{2}}t^{\frac{(p-2)(p-2-2p\alpha)}{2}}e^{-p\lambda t}\mathbb{E}\left(\int_0^t e^{p\lambda s}|g(s)|^pds\right) \\
\le& e^{-\lambda t}C_pM^p\left(\frac{p-2}{p-2-2p\alpha}\right)^{\frac{p-2}{2}}t^{\frac{(p-2)(p-2-2p\alpha)}{2}}e^{-(p-1)\lambda t}\mathbb{E}\left(\int_0^t e^{p\lambda s}|g(s)|^pds\right) \\
\le& e^{-\lambda t}C_pM^p\left(\frac{p-2}{p-2-2p\alpha}\right)^{\frac{p-2}{2}}t^{\frac{(p-2)(p-2-2p\alpha)}{2}}e^{-(p-1)\lambda t}Q_2 \\
\le& C(p,M,\alpha,\lambda,Q_2)e^{-\lambda t}.
\end{aligned}
\tag{2.10}
$$

□

Lemma 2.4. *([1]) Suppose that g is an adapted stochastic process satisfying the following condition:*

$$\int_0^T \mathbb{E}|g(s)|^pds < \infty \tag{2.11}$$

for some $p > 2/(1-2\alpha)$, where $0 < \alpha < 1/2$. Then the process $\int_0^t g(s)dW^H(s)$ has a version which is τ-Hölder continuous for any $\tau < \frac{1}{2} - \alpha - \frac{1}{p}$, and we have the maximal inequality

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t g(s)dW^H(s)\right|^p\right) \le C(p)\int_0^T \mathbb{E}|g(t)|^pdt. \tag{2.12}$$

3. Exponential stability of solution

In this section, we are interested in investigating the pth moment exponential stability of the semilinear stochastic evolution equation (2.1).

Theorem 3.1. *Under **(H1)**, **(H5)** and **(H6)**, there exists a constant* $0 < \mu < \lambda - M_2$ *such that*

$$\limsup_{t\to\infty}\left(\frac{1}{t}\right)\log \mathbb{E}|X(t)|^p \le -\mu.$$

In other words, the mild solution exponentially decays to zero in L^p*-sense.*

Proof. Due to the fact $(a+b+c)^p \le 3^{p-1}a^p + 3^{p-1}b^p + 3^{p-1}c^p$, using (2.8), we then have

$$\begin{aligned}\mathbb{E}|X(t)|^p \le& 3^{p-1}\mathbb{E}|S(t)X_0|^p + 3^{p-1}\mathbb{E}\left|\int_0^t S(t-s)f(s,X(s))ds\right|^p\\ &+ 3^{p-1}\mathbb{E}\left|\int_0^t S(t-s)g(s)dW_Q^H(s)\right|^p\\ =:& \sum_{i=1}^3 J_i.\end{aligned}$$

It is obvious that $J_1 \le 3^{p-1}M^p e^{-p\lambda t}\mathbb{E}|X_0|^p \le 3^{p-1}M^p e^{-\lambda t}\mathbb{E}|X_0|^p$. Combine with **(H1)** and **(H5)**, by using Hölder's inequality, yields

$$\begin{aligned}J_2 \le& 3^{p-1}M^p\mathbb{E}\left(\int_0^t e^{-\frac{\lambda}{q}(t-u)}e^{-\frac{\lambda}{p}(t-u)}|f(u,X(u))|du\right)^p\\ \le& 3^{p-1}M^p\left(\int_0^t e^{-\lambda(t-u)}du\right)^{p/q}\int_0^t e^{-\lambda(t-u)}\mathbb{E}|f(u,X(u))|^p du\\ \le& 3^{p-1}M^p\left(\frac{1-e^{-\lambda t}}{\lambda}\right)^{p/q}e^{-\lambda t}\int_0^t e^{\lambda u}(Q_1\mathbb{E}|X(u)|^p + \xi(u))du\\ \le& 3^{p-1}M^p\left(\frac{1}{\lambda}\right)^{p/q}Q_1 e^{-\lambda t}\int_0^t e^{\lambda u}\mathbb{E}|X(u)|^p du + 3^{p-1}M^p\left(\frac{1}{\lambda}\right)^{p/q}\xi_1 e^{-\lambda t},\end{aligned} \tag{3.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>2$.

Also, Lemma 2.3 implies

$$J_3 \le 3^{p-1}C(p,M,\alpha,\lambda,Q_2)e^{-\lambda t}. \tag{3.2}$$

Let

$$\begin{aligned}M_1 =& 3^{p-1}M^p\mathbb{E}|X_0|^p,\\ M_2 =& 3^{p-1}M^p\left(\frac{1}{\lambda}\right)^{p/q}Q_1,\\ M_3 =& 3^{p-1}M^p\left(\frac{1}{\lambda}\right)^{p/q}\xi_1,\\ M_4 =& 3^{p-1}C(p,M,\alpha,\lambda,Q_2).\end{aligned}$$

Consequently,

$$\mathbb{E}|X(t)|^p \le (M_1+M_3+M_4)e^{-\lambda t} + M_2 e^{-\lambda t}\int_0^t e^{\lambda u}\mathbb{E}|X(u)|^p du. \tag{3.3}$$

Therefore, for arbitrary $\mu \in \mathbb{R}^+$ with $0 < \mu < \lambda - M_2$, we have

$$\int_0^t e^{\mu s}\mathbb{E}|X(s)|^p ds \le (M_1 + M_3 + M_4)\int_0^t e^{-(\lambda-\mu)s}ds + M_2\int_0^t e^{-(\lambda-\mu)s}\left(\int_0^s e^{\lambda u}\mathbb{E}|X(u)|^p du\right)ds. \quad (3.4)$$

Due to the continuity of $X(t)$, we have

$$\begin{aligned} & M_2\int_0^t e^{-(\lambda-\mu)s}\left(\int_0^s e^{\lambda u}\mathbb{E}|X(u)|^p du\right)ds \\ &= M_2\int_0^t e^{\lambda u}\mathbb{E}|X(u)|^p du\int_u^t e^{-(\lambda-\mu)s}ds \\ &= M_2\int_0^t e^{\lambda u}\mathbb{E}|X(u)|^p\left(\frac{e^{-(\lambda-\mu)u} - e^{-(\lambda-\mu)t}}{\lambda-\mu}\right)du \\ &\le \frac{M_2}{\lambda-\mu}\int_0^t e^{\lambda u}e^{-(\lambda-\mu)u}\mathbb{E}|X(u)|^p du \\ &= \frac{M_2}{\lambda-\mu}\int_0^t e^{\mu u}\mathbb{E}|X(u)|^p du. \end{aligned} \quad (3.5)$$

Then, together with (3.4) and (3.5), yields

$$\int_0^t e^{\mu s}\mathbb{E}|X(s)|^p ds \le (M_1 + M_3 + M_4)\int_0^t e^{-(\lambda-\mu)s}ds + \frac{M_2}{\lambda-\mu}\int_0^t e^{\mu u}\mathbb{E}|X(u)|^p du.$$

Hence, for $t \in [0, T]$ we obtain

$$\begin{aligned} \int_0^t e^{\mu s}\mathbb{E}|X(s)|^p ds \le & \frac{1}{1-\frac{M_2}{\lambda-\mu}}(M_1 + M_3 + M_4)\int_0^t e^{-(\lambda-\mu)s}ds \\ = & \frac{1}{1-\frac{M_2}{\lambda-\mu}}(M_1 + M_3 + M_4)\frac{1}{\lambda-\mu}(1 - e^{-(\lambda-\mu)t}) \\ \le & C''(p, M, \lambda, \xi_1, Q_2). \end{aligned} \quad (3.6)$$

By virtue of (3.3), (3.4) and (3.6), we can show that

$$\begin{aligned} \mathbb{E}|X(t)|^p \le & (M_1 + M_3 + M_4)e^{-\mu t} + M_2 C''(p, M, \lambda, \xi_1, Q_2)e^{-\mu t} \\ = & C'(p, M, \lambda, \xi_1, Q_1, Q_2)e^{-\mu t}. \end{aligned} \quad (3.7)$$

The proof is complete. □

4. Strong convergence

In our analysis, it is more natural to work with the equivalent expression

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}f(s, X(s))ds + \int_0^t e^{A(t-s)}g(s)dW_Q^H(s). \quad (4.1)$$

Now, we introduce the exponential Euler method for Eq (2.1). Choose integer $m \ge 1$ such that $N = T\frac{m}{\tau}$ is an integer number. Let the step size $h \in (0, 1)$ be a fraction of T ($T > 0$) and τ, i.e.,

$h = \frac{T}{N} = \frac{\tau}{M}$ for some integer $N > T$ and $M > \tau$, then the discrete exponential Euler approximate solution is defined as follows

$$y_{k+1} = e^{Ah} y_k + e^{Ah} f(t_k, y_k)h + e^{Ah} g(t_k)\Delta W_k^H, \tag{4.2}$$

where y_k is an approximation to $X(t_k)$ with $t_k = kh$ for $k = 0, 1, 2, ..., N-1$, $y_0 = X_0$ and $\Delta W_k^H = W_Q^H(t_{k+1}) - W_Q^H(t_k)$. Then it is clear to define the continuous exponential Euler approximate solution $y(t)$ for $0 \le t \le T$ in the following form

$$y(t) = e^{At} y_0 + \int_0^t e^{A(t-\bar{s})} f(\bar{s}, Z(s))ds + \int_0^t e^{A(t-\bar{s})} g(\bar{s})dW_Q^H(s), \tag{4.3}$$

where $\bar{s} = [s/h]h$ and $[x]$ denote the largest integer which is smaller than x, and $Z(t)$ is the step function defined by

$$Z(t) = \sum_{k=0}^{\infty} \chi_{[t_k, t_{k+1})}(t) y_k, \tag{4.4}$$

where $\chi_{[E]}$ is the indicator function of set E. For any integer $k \ge 0$, it is obvious that $y(t_k) = Z(t_k) = y_k$, that is the continuous exponential Euler solution $y(t)$ and the step function $Z(t)$ coincide with the discrete solution at the grid point.

Lemma 4.1. *Under **(H4)**, **(H5)** and Lemma 2.4, there exists a constant C_1 independent of h such that*

$$\mathbb{E}\left(\sup_{0\le t\le T} |y(t)|^p\right) \le C_1, \qquad \mathbb{E}\left(\sup_{0\le t\le T} |X(t)|^p\right) \le C_1. \tag{4.5}$$

Proof. Due to the inequality $(a+b+c)^p \le 3^{p-1}a^p + 3^{p-1}b^p + 3^{p-1}c^p$, using condition (4.3) we have

$$|y(t)|^p \le 3^{p-1}\left(\left|e^{At}y_0\right|^p + \left|\int_0^t e^{A(t-s)} f(\bar{s}, Z(s))ds\right|^p + \left|\int_0^t e^{A(t-\bar{s})} g(\bar{s})dW_Q^H(s)\right|^p\right)$$

Taking the expectation of both sides, it follows from **(H4)**, **(H5)**, Lemma 2.4 and Hölder's inequality that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T} |y(t)|^p\right) \le & 3^{p-1}e^{p|A|T}\mathbb{E}|y_0|^p + 3^{p-1}T^{p-1}e^{p|A|T}\int_0^t (Q_1\mathbb{E}|Z(s)|^p + \xi(s))ds \\
& + 3^{p-1}e^{p|A|T}\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t g(\bar{s})dW_Q^H(s)\right|^p\right) \\
\le & 3^{p-1}e^{p|A|T}\mathbb{E}|y_0|^p + 3^{p-1}T^{p-1}e^{p|A|T}Q_1\int_0^T \mathbb{E}|Z(s)|^p ds \\
& + 3^{p-1}T^p e^{p|A|T}\xi_1 + 3^{p-1}e^{p|A|T}\int_0^T \mathbb{E}|g(t)|^p dt \\
\le & 3^{p-1}e^{p|A|T}\mathbb{E}|y_0|^p + 3^{p-1}T^{p-1}e^{p|A|T}Q_1\int_0^T \mathbb{E}(\sup_{0\le r\le s}|y(r)|^p)ds \\
& + 3^{p-1}T^p e^{p|A|T}\xi_1 + 3^{p-1}e^{p|A|T}L_4^p T.
\end{aligned}$$

By the Gronwall's inequality we hence obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|y(t)|^p\right)\leq C_{11},$$

where

$$C_{11}=3^{p-1}e^{p|A|T}(\mathbb{E}|y_0|^p+T^p\xi_1+L_4^pT)e^{3^{p-1}T^pe^{p|A|T}Q_1}.$$

Also, the proof process of the uniform boundedness of pth moment of exact solution $X(t)$ is similar to $y(t)$, for a constant C_{12} produced. Then the result will be obtained by considering $C_1=\max\{C_{11},C_{12}\}$. □

Lemma 4.2. *Under **(H4)**, **(H5)** and Lemma 2.4, there exists a constant C_2 such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|y(t)-Z(t)|^p\right)\leq C_2h. \tag{4.6}$$

Proof. Through the definition of $y(t)$ and $Z(t)$, for $t\in[t_k,t_{k+1})$ the following equality yields

$$y(t)-Z(t)=e^{A(t-t_k)}y_k+\int_{t_k}^t e^{A(s-t_k)}f(t_k,y_k)ds+\int_{t_k}^t e^{A(s-t_k)}g(t_k)dW_Q^H(s)-y_k.$$

By the Hölder's inequality, we can show

$$\begin{aligned}|y(t)-Z(t)|^p\leq 3^{p-1}\Bigg(&|e^{A(t-t_k)}-I|^p|y_k|^p+h^{p-1}\int_{t_k}^t|e^{A(s-t_k)}|^p|f(t_k,y_k)|^pds\\&+\left|\int_{t_k}^t e^{A(s-t_k)}g\left(t_k\right)dW_Q^H(s)\right|^p\Bigg),\end{aligned}$$

where I is an identity matrix. For every $t_k\leq t<t_{k+1}$, we have $|e^{A(t-t_k)}-I|\leq e^{|A|h}-1\leq|A|he^{|A|h}$. Appling Lemma 2.4, **(H3)** and Lemma 4.1, we derive

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq k\leq n}\sup_{t_k\leq t<t_{k+1}}|y(t)-Z(t)|^p\right)\leq&3^{p-1}|A|^ph^pe^{p|A|T}\mathbb{E}\left(\sup_{0\leq k\leq n}|y_k|^p\right)\\&+3^{p-1}h^{p-1}e^{p|A|T}Q_1\left(\int_{t_{k_1}}^{t_{k_1+1}}\mathbb{E}|y_k|^pds\right)\\&+3^{p-1}h^pe^{p|A|T}\xi_1\\&+3^{p-1}e^{p|A|T}\left(\int_{t_{k_2}}^{t_{k_2+1}}\mathbb{E}|g(t_k)|^pdt\right)\\\leq&3^{p-1}h^pe^{p|A|T}(|A|^pC_1+Q_1C_1+\xi_1)+3^{p-1}e^{p|A|T}L_4^ph\\\leq&3^{p-1}e^{p|A|T}L_4^ph+O(h^p),\end{aligned}$$

where in the first inequality two last suprema are taken for some $0\leq k_1,k_2\leq n$. Therefore, the result would be

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|y(t)-Z(t)|^p\right)\leq C_2h,$$

where $C_2=3^{p-1}e^{p|A|T}L_4^p$ is a constant independent of h. □

Theorem 4.3. *Under **(H2)**–**(H5)** and Lemma 2.4, for* $0 < H < 1/2$*, the numerical approximation solution produced by the exponential Euler method converges to the exact solution of Eq* (2.1)*, that is*

$$\lim_{h\to 0}\mathbb{E}\left(\sup_{0\le t\le T}|X(t)-y(t)|^p\right)=0. \tag{4.7}$$

In fact, we prove the more efficient inequality, that is, there exists a positive constant C such that

$$\mathbb{E}\left(\sup_{0\le t\le T}|X(t)-y(t)|^p\right)\le Ch. \tag{4.8}$$

Proof. It follows from (4.1) and (4.3) that

$$\begin{aligned}
X(t)-y(t)\le&\int_0^t\left(e^{A(t-s)}f(s,X(s)-e^{A(t-\bar{s})}f(\bar{s}),Z(s))\right)ds+\int_0^t\left(e^{A(t-s)}g(s)-e^{A(t-\bar{s})}g(\bar{s})\right)dW_Q^H(s)\\
=&\int_0^t(e^{A(t-s)}-e^{A(t-\bar{s})})f(s,X(s))ds+\int_0^t e^{A(t-\bar{s})}(f(s,X(s))-f(\bar{s},X(s)))ds\\
&+\int_0^t e^{A(t-\bar{s})}(f(\bar{s},X(s))-f(\bar{s},Z(s)))ds+\int_0^t(e^{A(t-s)}-e^{A(t-\bar{s})})g(s)dW_Q^H(s)\\
&+\int_0^t(g(s)-g(\bar{s}))e^{A(t-\bar{s})}dW_Q^H(s)\\
=:&\sum_{i=1}^5 I_i(t)
\end{aligned} \tag{4.9}$$

In sequence, we will consider the upper bounds to $\mathbb{E}(\sup_{0\le t\le T}|I_i(t)|^p)$ for every $1\le i\le 5$ with repect to $\mathbb{E}(\sup_{0\le t\le T}|X(t)-y(t)|^p)$ and h. For the first term, by the Hölder's inequality, **(H5)** and Lemma 4.1 we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|I_1(t)|^p\right)=&\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t e^{A(t-\bar{s})}(e^{A(\bar{s}-s)}-I)f(s,X(s))ds\right|^p\right)\\
\le&T^{p-1}e^{2p|A|T}|A|^p h^p\mathbb{E}\left(\sup_{0\le t\le T}\int_0^t|f(s,X(s))|^p ds\right)\\
\le&T^p e^{2p|A|T}|A|^p h^p\left(Q_1\mathbb{E}\left(\sup_{0\le t\le T}|X(t)|^p\right)+\xi_1\right)\\
\le&T^p e^{2p|A|T}|A|^p h^p(Q_1C_1+\xi_1).
\end{aligned} \tag{4.10}$$

For the second term, it follows from Hölder's inequality, **(H4)** and Lemma 4.1 that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|I_2(t)|^p\right)\le&T^{p-1}e^{p|A|T}\mathbb{E}\left(\sup_{0\le t\le T}\int_0^t|(f(s,X(s))-f(\bar{s},X(s)))|^p ds\right)\\
\le&T^p e^{p|A|T}L_3^p|s-\bar{s}|^{\beta p}\mathbb{E}\left(\sup_{0\le t\le T}|X(t)|^p\right)\\
\le&T^p e^{p|A|T}L_3^p h^{\beta p}C_1.
\end{aligned} \tag{4.11}$$

For the third term, it follows from **(H2)** and Lemma 4.2 that

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|I_3(t)|^p\right) \le& T^{p-1}e^{p|A|T}L_1^p\mathbb{E}\left(\sup_{0\le t\le T}\int_0^t|X(s)-Z(s)|^p ds\right)\\
\le& 2^pT^{p-1}e^{p|A|T}L_1^p\int_0^T\mathbb{E}|X(s)-y(s)|^p ds\\
&+2^pT^{p-1}e^{p|A|T}L_1^p\mathbb{E}\left(\sup_{0\le t\le T}\int_0^t|y(s)-Z(s)|^p ds\right)\\
\le& 2^pT^{p-1}e^{p|A|T}L_1^p\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|X(u)-y(u)|^p\right)ds+2^pT^pe^{p|A|T}L_1^pC_2h.
\end{aligned}
\tag{4.12}
$$

For the fourth term, together with **(H4)** and Lemma 2.4, yields

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|I_4(t)|^p\right) =& \mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t e^{A(t-\bar{s})}(e^{A(\bar{s}-s)}-I)g(s)dW_Q^H(s)\right|^p\right)\\
\le& e^{2p|A|T}|A|^ph^p\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t g(s)dW_Q^H(s)\right|^p\right)\\
\le& e^{2p|A|T}|A|^ph^p\int_0^T\mathbb{E}|g(t)|^p dt\\
\le& e^{2p|A|T}|A|^ph^pL_4^pT.
\end{aligned}
\tag{4.13}
$$

For the last term, combine with **(H3)** and Lemma 2.4, we can show easily that

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|I_5(t)|^p\right) \le& e^{p|A|T}\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t(g(s)-g(\bar{s}))dW_Q^H(s)\right|^p\right)\\
\le& e^{p|A|T}\int_0^T\mathbb{E}|g(s)-g(\bar{s})|^p ds\\
\le& e^{p|A|T}L_2^ph^{p\gamma}T\\
\le& e^{p|A|T}L_2^phT.
\end{aligned}
\tag{4.14}
$$

Substituting (4.10)–(4.14) into (4.9), we then have

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|X(t)-y(t)|^p\right) \le& 5^{p-1}\sum_{i=1}^{5}\mathbb{E}\left(\sup_{0\le t\le T}|J_i(t)|^p\right)\\
=& 5^{p-1}T^pe^{2p|A|T}|A|^ph^p(Q_1C_1+\xi_1)+5^{p-1}T^pe^{p|A|T}L_3^ph^{\beta p}C_1\\
&+5^{p-1}2^pT^{p-1}e^{p|A|T}L_1^p\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|X(u)-y(u)|^p\right)ds\\
&+5^{p-1}2^pT^pe^{p|A|T}L_1^pC_2h+5^{p-1}e^{2p|A|T}|A|^ph^pL_4^pT+5^{p-1}e^{p|A|T}L_2^phT\\
=& C_3h^p+C_4\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|X(u)-y(u)|^p\right)ds+C_5h+C_6h^{\beta p},
\end{aligned}
$$

where

$$\begin{aligned}C_3 =&5^{p-1}\left(T^p e^{2p|A|T}|A|^p(Q_1C_1+\xi_1)+e^{2p|A|T}|F|^pL_4^pT\right),\\ C_4 =&5^{p-1}2^pT^{p-1}e^{p|A|T}L_1^p,\\ C_5 =&5^{p-1}2^pT^pe^{p|A|T}L_1^pC_2+5^{p-1}e^{p|A|T}L_2^pT,\\ C_6 =&5^{p-1}T^pe^{p|A|T}L_3^pC_1.\end{aligned}$$

are all nonnegative constants. Hence, we obtain the following results by Gronwall inequality:

$$\mathbb{E}\left(\sup_{0\le t\le T}|X(t)-y(t)|^p\right)\le Ch^{\min\{1,\beta p\}}. \tag{4.15}$$

where C is a new constant. Finally, we have

$$\lim_{h\to 0}\mathbb{E}\left(\sup_{0\le t\le T}|X(t)-y(t)|^p\right)=0. \tag{4.16}$$

□

5. Numerical experiments

Let step size $h>0$ and define the discrete version of RLfBm in the following form

$$W_n^H=\sum_{k=0}^{n-1}(t_n-t_k)^{-\alpha}\Delta W_k, \tag{5.1}$$

where W_n^H is an approximation to $W^H(t)$ with $t_k=kh$ for $k=1,2,...,n-1$, $\Delta W_k=W(t_{k+1})-W(t_k)$ is the discretized Wiener increment. Figures 1 and 2 are Riemann-Liouville fractional Brownian motion with Hurst parameter of 0.1 and 0.4 respectively.

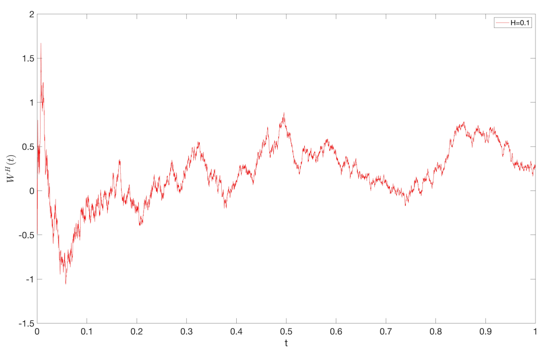

Figure 1. Riemann-Liouville fractional Brownian motion with $H=0.1$.

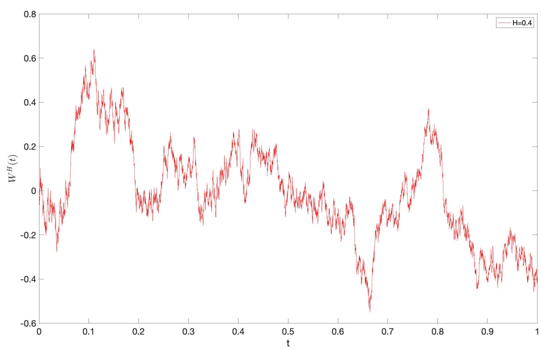

Figure 2. Riemann-Liouville fractional Brownian motion with $H=0.4$.

Next, we provide the following examples to verify the pth moment exponential stability and convergence result of the exponential Euler method for semilinear stochastic evolution equation (2.1).

Example 1. Consider a two-dimensional semilinear stochastic evolution equation driven by RLfBm with Hurst parameter $H = 0.4$ as follows:

$$\begin{cases} dX(t) = (AX(t) - tX(t))dt + e^{-2t}dW_Q^H(t), \\ X(t) = 1 + t, \end{cases} \tag{5.2}$$

where

$$A = \begin{pmatrix} -2 & 0.5 \\ 0 & 1 \end{pmatrix}.$$

Then from (4.1) we know the exact solution of (5.5) is

$$X(t) = e^{At} - e^{At}\int_0^t e^{-As}s(1+s)ds + e^{At}\int_0^t e^{-(|A|+2)s}dW_Q^H(s). \tag{5.3}$$

Take $t \in [0, 5]$, $H = 0.4$ and the initial value $X(0) = X_0 = (1, 1)^T$. It is easy to verify that the conditions of Theorem 3.1 are satisfied. We use the exponential Euler method to simulate the paths of the numerical solution of Eq (5.2), according to Figure 3 we know that the Eq (5.2) is the pth moment exponential stable.

Consider the convergence result, we assume $H = 0.4$, step size $h = 2^{-7}, 2^{-8}, 2^{-9}$ and $p = 4, 5, 6$. Take $t \in [0, 1]$, Figure 4 is the simulations of both exact and numerical solutions of (5.2). The average maximum of pth moment of error of exponential Euler method with 1000 paths simulation is obtained as

$$e(h) = \frac{1}{1000}\sum_{k=1}^{1000} \max_{1\le k\le \frac{1}{h}} |X(t_k) - y(t_k)|^p. \tag{5.4}$$

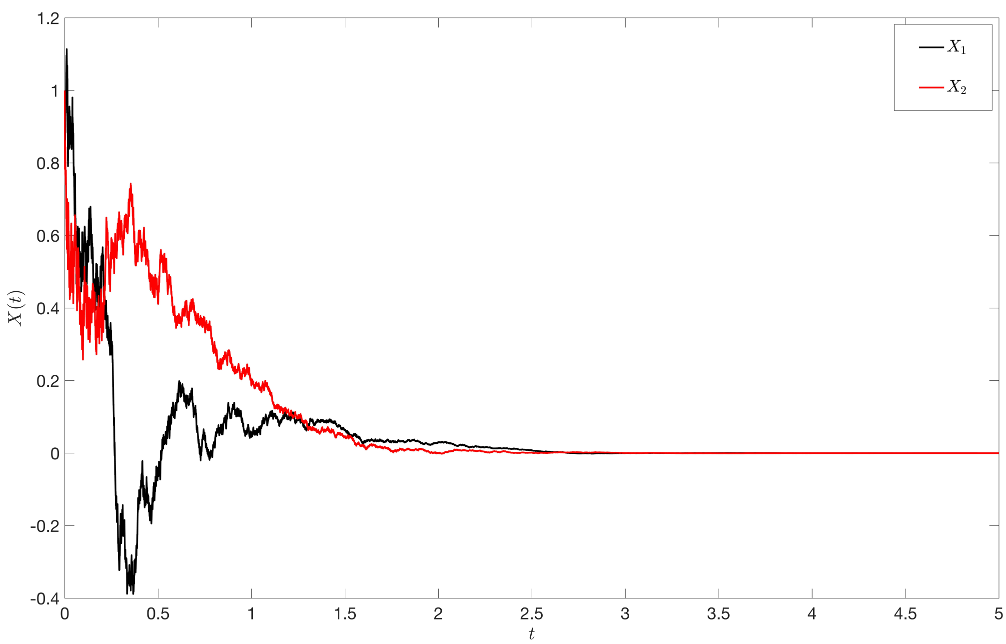

Figure 3. pth moment stability of semilinear stochastic evolution equation driven by RLfBm with $H = 0.4$.

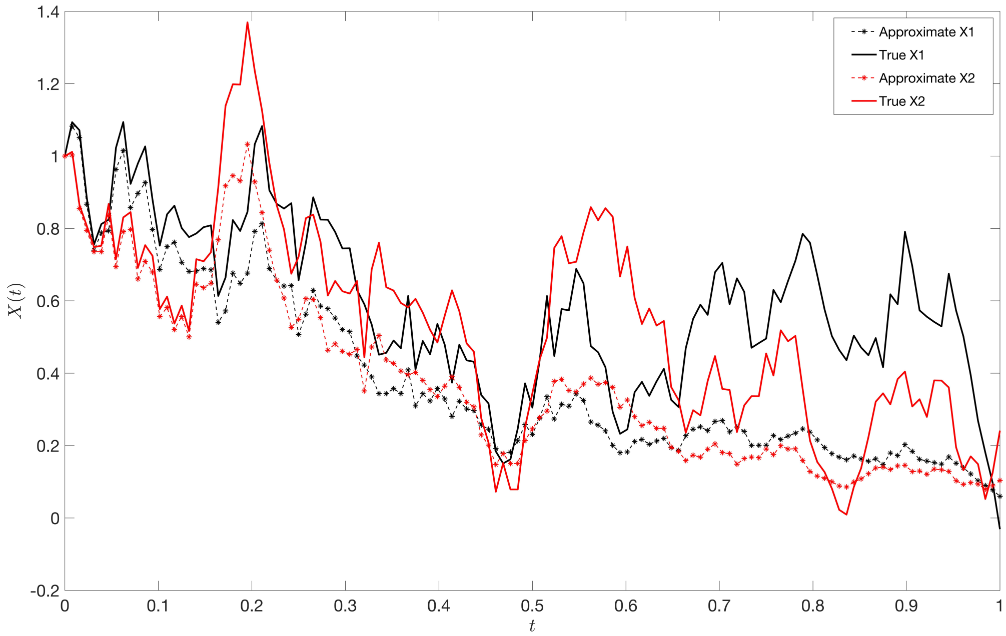

Figure 4. The paths of exact and numerical of Eq (5.2) with $H = 0.4$.

By calculating, we derive the constant C in (4.15) as shown in Table 1, and the average maximum of pth moment of error with 1000 paths simulation shows that Theorem 4.3 holds.

Table 1. The average maximum of pth moment and coefficient in (4.15).

$h = 2^{-7}$			$h = 2^{-8}$	$h = 2^{-9}$
p	$e(h)$	C	$e(h)$	$e(h)$
5	25.2985	$5^4e^{10}(2^5 \cdot 3^4e^{10} + 1)e^{5^4 2^5 e^{10}}$	25.5599	26.8570
6	65.4866	$5^5e^{12}(2^6 \cdot 3^5e^{12} + 1)e^{5^5 2^6 e^{12}}$	64.3370	67.7299
7	188.7418	$5^6e^{14}(2^7 \cdot 3^6e^{14} + 1)e^{5^6 2^7 e^{14}}$	211.7740	204.0455

Example 2. Consider a three-dimensional semilinear stochastic evolution equation driven by RLfBm with Hurst parameter $H = 0.4$ as follows:

$$\begin{cases} dX(t) = (AX(t) - X(t))dt + e^{-2t}dW_Q^H(t), \\ X(t) = cos(t), \end{cases} \tag{5.5}$$

where

$$A = \begin{pmatrix} -2 & 3 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Then from (4.1) we know the exact solution of (5.5) is

$$X(t) = e^{At} - e^{At}\int_0^t e^{-As}\cos(s)ds + e^{At}\int_0^t e^{-(|A|+2)s}dW_Q^H(s). \tag{5.6}$$

Take $t \in [0,5]$, $H = 0.4$ and the initial value $X(0) = X_0 = (1,1)^T$. It is easy to verify that the conditions of Theorem 3.1 are satisfied. We use the exponential Euler method to simulate the paths of the numerical solution of Eq (5.5), according to Figure 5 we know that the Eq (5.5) is the pth moment exponential stable.

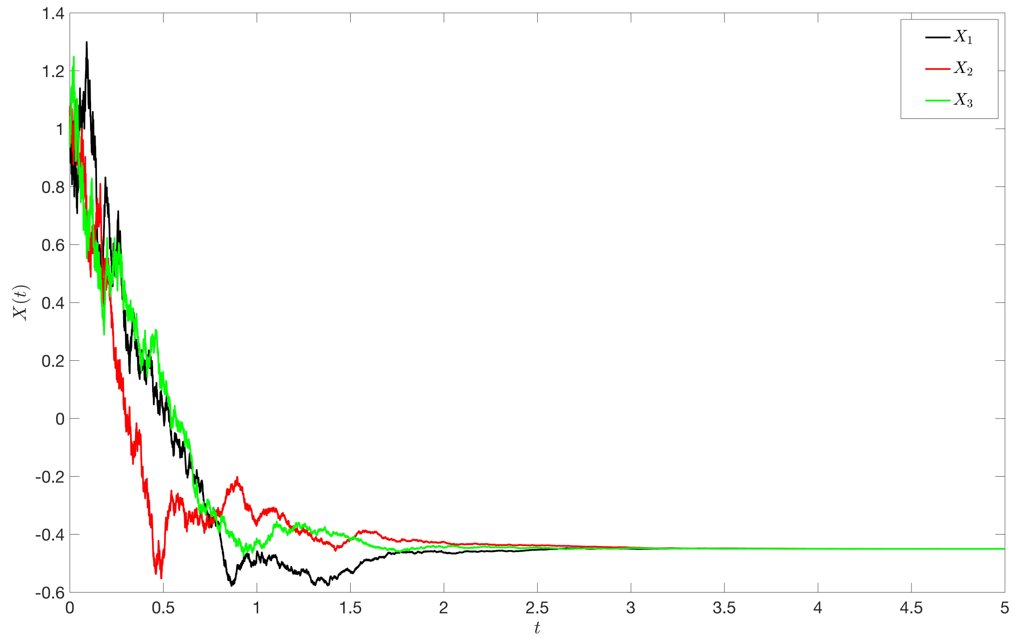

Figure 5. pth moment stability of semilinear stochastic evolution equation driven by RLfBm with $H = 0.4$.

Consider the convergence result, we assume $H = 0.4$, stepsize $h = 2^{-7}, 2^{-8}, 2^{-9}$ and $p = 4,5,6$. Take $t \in [0,1]$, Figure 6 is the simulations of both exact and numerical solutions of (5.5). The average maximum of pth moment of error of exponential Euler method with 1000 paths simulation is obtained as

$$e(h) = \frac{1}{1000} \sum_{k=1}^{1000} \max_{1 \le k \le \frac{1}{h}} |X(t_k) - y(t_k)|^p. \tag{5.7}$$

By calculating, we derive the constant C in (4.15) as shown in Table 2, and the average maximum of pth moment of error with 1000 paths simulation shows that Theorem 4.3 holds.

Table 2. The average maximum of pth moment and coefficient in (4.15).

$h = 2^{-7}$			$h = 2^{-8}$	$h = 2^{-9}$
p	$e(h)$	C	$e(h)$	$e(h)$
5	19.5974	$5^4 e^{10}(2^5 \cdot 3^4 e^{10} + 1)e^{5^4 2^5 e^{10}}$	20.9878	20.1836
6	46.2188	$5^5 e^{12}(2^6 \cdot 3^5 e^{12} + 1)e^{5^5 2^6 e^{12}}$	47.8385	55.1631
7	133.7892	$5^6 e^{14}(2^7 \cdot 3^6 e^{14} + 1)e^{5^6 2^7 e^{14}}$	126.1493	152.6237

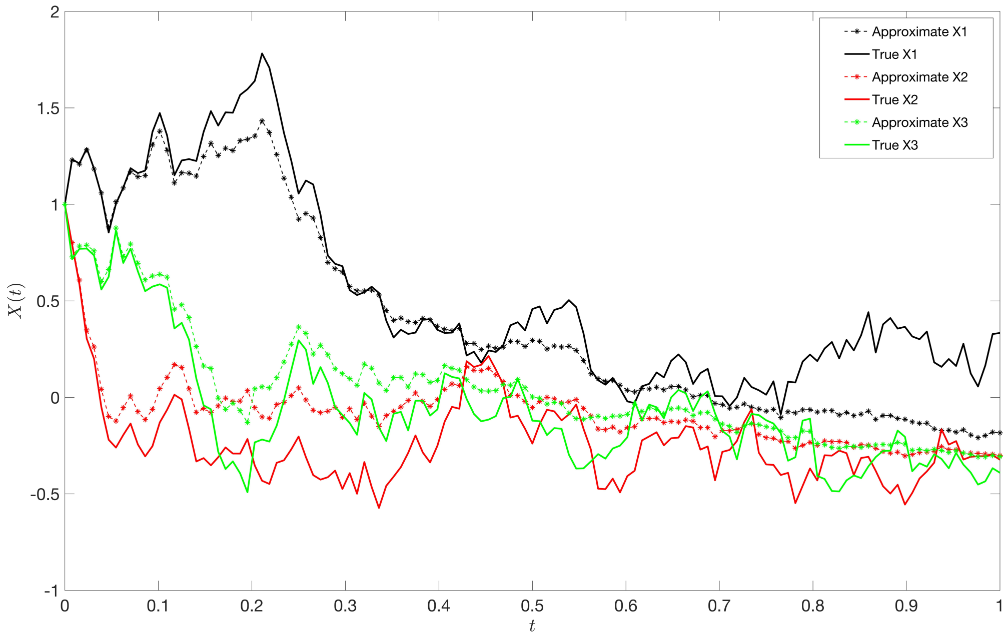

Figure 6. The paths of exact and numerical of Eq (5.5) with $H = 0.4$.

6. Conclusions

In this paper, the pth moment exponential stability of mild solution and convergence result of the numerical solution for the exponential Euler method to semilinear stochastic evolution equation driven by RLfBm with Hurst parameter $H < 1/2$ have been discussed. The proof of pth moment exponential stability of mild solution has been analytically given in a Hilbert space, and we show that the exponential Euler approximation solution converges to the analytical solution to semilinear stochastic evolution equation when the step size h is very small. We only consider the simple exponential Euler method here, in the future, we will study several higher order exponential integrator methods such as Runge-Kutta methods, exponential Taylor methods and etc. Based on Lemma 10 in [1], the connections between the numerical and exact solutions of stochastic differential equations driven by RLfBm are also worthy of investigating.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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