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Research article

Some coupled fixed point theorems on multiplicative metric spaces with an application

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Abstract: This article aims to prove some coupled fixed point (FP) theorems for nonlinear contractive type mapping in multiplicative metric space (MM-space). Our presented work consists of the maximum type and some other expressions in the framework of MM-space. We also provide illustrative examples and an application in support of our generalized results. Our offered results expand and develop a variety of the latest outcomes in the existing literature. Moreover, we present an application of the two Urysohn integral equations to support our work.

Keywords: coupled fixed point; multiplicative metric space; contraction conditions, Urysohn integral equations
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1. Introduction

Fixed point (FP) theory is one of the largest and primary areas of research. Many mathematicians have made contributions to FP theory by using different contraction type mappings and different types of spaces. In 1922, Banach [1] proved a “Banach contraction principle” which is stated as: “a self-mapping on a complete metric space verifying the contraction condition has a unique FP”. Later on, many authors generalized this principle in many directions and proved different contractive type FP results in the context of metric spaces (M-space) for single and multi-valued mappings. Some of their results can be found in (e.g, see: [2–5] the references therein).

In 1972 Grossman and Katz [6] introduced an innovative kind of calculus called multiplicative calculus by interchanging the roles of subtraction and addition with the role of division and multiplication, respectively. By using the terminology of Grossman and Katz [6], Bashirov *et al.* [7] demonstrate its usefulness and also provide some application of this calculus and defined the notion of multiplicative metric space (MM-space). In [8], Rom and Sarwar discussed the characterization of MM-space completeness. Zada and Riaz [9] proved some FP theorems on multiplicative metric like spaces. Shukla [10] presented some critical remarks on MM-spaces and FP theorems. Ali [11], Ozavsar and Cevikel [12], Kumar *et al.* [13], He *et al.* [14], Kang *et al.* [15, 16], Gu *et al.* [17], and Mongkolkeha *et al.* [18] proved different contractive type FP and coupled FP theorems in MM-spaces.

The concept of coupled FP was firstly initiated by Lakshmikantham and Circ [19]. He provides the important mechanism of coupled FP and find coupled quasi-solution of initial value problems for ordinary differential equation and give some existence theorem of coupled FP for any continuous and discontinuous mappings. Gordji *et al.* [20] proved some coupled FP theorems in partially ordered M-space with an application. While Cho *et al.* [21] established some nonlinear coupled FP theorems in ordered generalized M-spaces with integral type application. In [22], Huang *et al.* proved some FP and coupled FP theorems in ordered M-spaces by using multi-valued operators. Sintunavarat [23] established some coupled coincidence point theorems in intuitionistic fuzzy normed spaces without commutativity conditions. While Li *et al.* [24] proved some strong coupled FP results in fuzzy M-spaces with an application. Sintunavart and Kumam [25] proved some coupled coincidence and coupled FP theorems in partially ordered M-spaces. Sabetghadam and Masiha [26] proved some coupled FP theorems in cone M-space. Rehman *et al.* [27] proved some strong coupled FP theorems by using cyclic type mappings in cone M-spaces with an application.

In [28], Jiang and Gu proved some common coupled FP theorems in MM-spaces with applications. Shanjit *et al.* [29] proved some coupled FP theorems in partially ordered MM-space and its applications. Recently, Rugumisa and Kumar [30] studied the idea of Grossman and Katz [6] and proved some results in MM-spaces.

In this paper, we prove some coupled FP theorems for nonlinear function in MM-spaces under generalized contraction conditions. Our presented work consists of the maximum type and some other expressions in the framework of MM-space. We also provide some illustrative examples and an application in support of our generalized results in MM-spaces. Our offered results expand and develop a variety of the latest outcomes in the existing literature. Moreover, we present an application of the two Urysohn integral equations to support our work. By using this concept, one can prove more different types of contractive results for nonlinear functions in MM-spaces with different types of integral equations applications.

2. Preliminaries

Definition 2.1. [7] Let $B \neq \emptyset$ set, then a mapping $\dot{d} : B \times B \to \mathbb{R}^+$ is called a multiplicative metric if it satisfies the following conditions:

(m1) $\dot{d}(\xi, \eta) \geq 1$ and $\dot{d}(\xi, \eta) = 1 \Leftrightarrow \xi = \eta$, for all $\xi, \eta \in B$.

(m2) $\dot{d}(\xi, \eta) = \dot{d}(\eta, \xi)$ for all $\xi, \eta \in B$.

(m3) $\dot{d}(\xi, \eta) \leq \dot{d}(\xi, \zeta) \cdot \dot{d}(\zeta, \eta)$ for all $\xi, \eta, \zeta \in B$.

And a pair $(B, \dot{d})$ is said to be an MM-space.

Example 2.2. [12] Consider $B = (\mathbb{R}_+)^\kappa$ contain all κ-tuples of positive real numbers. Let $\dot{d} : (\mathbb{R}_+)^\kappa \times (\mathbb{R}_+)^\kappa \to \mathbb{R}$ be defined as follows:

$$\dot{d}(\xi, \eta) = \left|\frac{\xi_1}{\eta_1}\right| \cdot \left|\frac{\xi_2}{\eta_2}\right| \cdots \left|\frac{\xi_\kappa}{\eta_\kappa}\right|,$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_\kappa),\ \eta = (\eta_1, \eta_2, \ldots, \eta_\kappa) \in (\mathbb{R}_+)^\kappa$ and $|\cdot| : \mathbb{R}_+ \to [1, +\infty)$ is defined as follows:

$$|k| = \left\{k \quad \text{if } k \geq 1, \text{ and } \frac{1}{k} \quad \text{if } k < 1\right\}.$$

Then it is conclusive that all given conditions of an MM-space are satisfied hence $((\mathbb{R}_+)^\kappa, d)$ is an MM-space.

Lemma 2.3. *[12] Let $(B, \dot{d})$ be an MM-space, $\{\xi_\kappa\}$ be a sequence in B, then $\xi_\kappa \to \xi$ as $\kappa \to \infty$ if and only if $d(\xi_\kappa, \xi) \to 1$ as $\kappa \to \infty$.*

Lemma 2.4. *[12] Let $(B, \dot{d})$ be an MM-space, $\{\xi_\kappa\}$ be a sequence in B. If sequence $\{\xi_\kappa\}$ is multiplicative convergent, then the multiplicative limit is unique.*

Definition 2.5. [12] Let $(B, \dot{d})$ be an MM-space and $\{\xi_\kappa\}$ be a sequence in B. Then the sequence is called a multiplicative Cauchy sequence if it holds that for all $\varepsilon > 1$, there exists a natural number $\kappa_0 \in N$ such that $\dot{d}(\xi_\kappa, \xi_j) < \varepsilon$ for all $\kappa, j > \kappa_0$.

Definition 2.6. [12] A space $(B, \dot{d})$ is called multiplicative complete if every multiplicative Cauchy sequence in $(B, \dot{d})$ is multiplicative convergent in B.

Proposition 2.7. *[12] Let $(B, \dot{d})$ be an MM-space, $\{\xi_\kappa\}$ and $\{\eta_\kappa\}$ be two sequences in B such that $\xi_\kappa \to \xi$ and $\eta_\kappa \to \eta$ as $\kappa \to \infty$. Then $\dot{d}(\xi_\kappa, \eta_\kappa) \to \dot{d}(\xi, \eta)$ as $\kappa \to \infty$.*

Example 2.8. Let $(B, \dot{d})$ be an MM-space and $\dot{d} : B \times B \to \mathbb{R}^+$ be defined by

$$\dot{d}(\xi, \eta) = e^{|\xi - \eta|}, \qquad \forall\, \xi, \eta \in B. \tag{2.1}$$

If we take a natural log of (2.1). Then the MM-space became a usual metric space, which satisfies all the conditions of the usual metric space.

Theorem 2.9. *[12] Let $(B, \dot{d})$ be an MM-space and sequence $\{\xi_\kappa\}$ in B multiplicative convergent, then it is a multiplicative Cauchy sequence.*

Definition 2.10 (Banach-contraction)**.** [12] Let $(B, \dot{d})$ be an MM-space. A mapping $\Gamma : B \to B$ is said to be multiplicative contractive if there exists $\lambda \in [0, 1)$, such that $\dot{d}(\Gamma\xi, \Gamma\eta) \leq (d(\xi, \eta))^\lambda$, $\forall\, \xi, \eta \in B$.

Definition 2.11 (Coupled Banach Contraction mapping)**.** [31] Let $(B, \dot{d})$ is an MM-space. A mapping $\Gamma : B \times B \to B$ is called a coupled Banach contraction for $\lambda \in (0, 1)$ such that for all $(\xi, \eta), (\mu, \nu) \in B \times B$, then the following inequality holds,

$$\dot{d}(\Gamma(\xi, \eta), \Gamma(\mu, \nu)) \leq \frac{\lambda}{2}\left(\dot{d}(\xi, \mu) + \dot{d}(\eta, \nu)\right).$$

Definition 2.12. [31] Let $(B, \dot{d})$ be an MM-space. A pair $(\xi, \eta) \in B \times B$ is called coupled FP of a mapping $\Gamma : B \times B \to B$ if $\Gamma(\xi, \eta) = \xi$ and $\Gamma(\eta, \xi) = \eta$.

3. Main results

Now, we are in the position to present our first main result.

Theorem 3.1. *Let* $(B, \dot{d})$ *be a complete MM-space. Let a mapping* $\Gamma : B \times B \to B$ *satisfies;*

$$d(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\max\left\{\begin{array}{c}\dot{d}(\xi,\Gamma(\xi,\eta)),\dot{d}(\mu,\Gamma(\mu,\nu)),\\ \dot{d}(\mu,\Gamma(\xi,\eta)),\dot{d}(\xi,\Gamma(\mu,\nu))\end{array}\right\}\right)^{\lambda}, \tag{3.1}$$

for all $\xi,\eta,\mu,\nu \in B$ *and* $\lambda \in [0,1)$. *Then mapping* Γ *has a unique coupled FP.*

Proof: Let $\xi_0, \eta_0 \in B$ be the arbitrary points and we define the iterative sequences in B such that

$$\xi_{\kappa+1} = \Gamma(\xi_\kappa,\eta_\kappa) \quad \text{and} \quad \eta_{\kappa+1} = \Gamma(\eta_\kappa,\xi_\kappa) \quad \text{for } \kappa \geq 0.$$

Now from (3.1),

$$\begin{aligned}\dot{d}(\xi_\kappa,\xi_{\kappa+1}) &= \dot{d}(\Gamma(\xi_{\kappa-1},\eta_{\kappa-1}),\Gamma(\xi_\kappa,\eta_\kappa))\\ &\leq \left(\max\left\{\begin{array}{c}\dot{d}(\xi_{\kappa-1},\Gamma(\xi_{\kappa-1},\eta_{\kappa-1})),\dot{d}(\xi_\kappa,\Gamma(\xi_\kappa,\eta_\kappa)),\\ \dot{d}(\xi_\kappa,\Gamma(\xi_{\kappa-1},\eta_{\kappa-1})),\dot{d}(\xi_{\kappa-1},\Gamma(\xi_\kappa,\eta_\kappa))\end{array}\right\}\right)^{\lambda}\\ &= \left(\max\left\{\begin{array}{c}\dot{d}(\xi_{\kappa-1},\xi_\kappa),\dot{d}(\xi_\kappa,\xi_{\kappa+1}),\\ \dot{d}(\xi_\kappa,\xi_\kappa),\dot{d}(\xi_{\kappa-1},\xi_{\kappa+1})\end{array}\right\}\right)^{\lambda}\\ &\leq \left(\max\left\{\begin{array}{c}\dot{d}(\xi_{\kappa-1},\xi_\kappa),\dot{d}(\xi_\kappa,\xi_{\kappa+1}),\\ 1,\dot{d}(\xi_{\kappa-1},\xi_\kappa)\cdot\dot{d}(\xi_\kappa,\xi_{\kappa+1})\end{array}\right\}\right)^{\lambda}\\ &= \left(\max\left\{\begin{array}{c}\dot{d}(\xi_{\kappa-1},\xi_\kappa),\dot{d}(\xi_\kappa,\xi_{\kappa+1}),\\ \dot{d}(\xi_{\kappa-1},\xi_\kappa)\cdot\dot{d}(\xi_\kappa,\xi_{\kappa+1})\end{array}\right\}\right)^{\lambda}\\ &= \left(\dot{d}(\xi_{\kappa-1},\xi_\kappa)\cdot\dot{d}(\xi_\kappa,\xi_{\kappa+1})\right)^{\lambda}.\end{aligned}$$

After simplification, we get that

$$\dot{d}(\xi_\kappa,\xi_{\kappa+1}) \leq \left(\dot{d}(\xi_{\kappa-1},\xi_\kappa)\right)^{\hbar}, \qquad \text{where} \quad \hbar = \frac{\lambda}{1-\lambda} < 1. \tag{3.2}$$

Similarly,

$$\dot{d}(\xi_{\kappa-1},\xi_\kappa) \leq \left(\dot{d}(\xi_{\kappa-2},\xi_{\kappa-1})\right)^{\hbar}, \qquad \text{where} \quad \hbar = \frac{\lambda}{1-\lambda} < 1. \tag{3.3}$$

Now from (3.2) and (3.3), and by induction, we have

$$\begin{aligned}\dot{d}(\xi_\kappa,\xi_{\kappa+1}) &\leq \left(\dot{d}(\xi_{\kappa-1},\xi_\kappa)\right)^{\hbar}\\ &\leq \left(\dot{d}(\xi_{\kappa-2},\xi_{\kappa-1})\right)^{\hbar^2}\\ &\leq \cdots \leq \left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^\kappa} \to 1, \quad \text{as } \kappa \to \infty.\end{aligned} \tag{3.4}$$

Hence proved that the sequence $\{\xi_\kappa\}$ is contractive in MM-space. Now $\kappa > j$,

$$\dot{d}(\xi_\kappa,\xi_j) \leq \dot{d}(\xi_\kappa,\xi_{\kappa-1})\cdot\dot{d}(\xi_{\kappa-1},\xi_{\kappa-2})\cdots\dot{d}(\xi_{j+2},\xi_{j+1})\cdot\dot{d}(\xi_{j+1},\xi_j)$$

$$
\begin{aligned}
&\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{\kappa-1}} \cdot \left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{\kappa-2}} \cdots \left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{j+1}} \cdot \left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{j}} \\
&\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\left(\hbar^{\kappa-1}+\hbar^{\kappa-2}+\cdots+\hbar^{j+1}+\hbar^{j}\right)} \\
&\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\left(\hbar^{j}+\hbar^{j+1}+\cdots+\hbar^{\kappa-2}+\hbar^{\kappa-1}\right)} \\
&\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\left(\frac{\hbar^{j}}{1-\hbar}\right)} \to 1, \quad \text{as } j \to \infty.
\end{aligned}
$$

Hence proved that $\{\xi_\kappa\}$ is a Cauchy sequence in MM-space $(B,\dot{d})$. Similarly, for sequence $\{\eta_\kappa\}$ and from (3.1),

$$
\begin{aligned}
\dot{d}(\eta_\kappa,\eta_{\kappa+1}) &= \dot{d}(\Gamma(\eta_{\kappa-1},\xi_{\kappa-1}),\Gamma(\eta_\kappa,\xi_\kappa)) \\
&\preceq \left(\max\left\{\begin{array}{c} \dot{d}(\eta_{\kappa-1},\Gamma(\eta_{\kappa-1},\xi_{\kappa-1})),\dot{d}(\eta_\kappa,\Gamma(\eta_\kappa,\xi_\kappa)) \\ \dot{d}(\eta_\kappa,\Gamma(\eta_{\kappa-1},\eta_{\kappa-1})),\dot{d}(\eta_{\kappa-1},\Gamma(\xi_\kappa,\eta_\kappa)) \end{array}\right\}\right)^\lambda \\
&= \left(\max\left\{\begin{array}{c} \dot{d}(\eta_{\kappa-1},\xi_\kappa),\dot{d}(\eta_\kappa,\eta_{\kappa+1}), \\ \dot{d}(\eta_\kappa,\eta_\kappa),\dot{d}(\eta_{\kappa-1},\eta_{\kappa+1}) \end{array}\right\}\right)^\lambda \\
&\preceq \left(\max\left\{\begin{array}{c} \dot{d}(\eta_{\kappa-1},\eta_\kappa),\dot{d}(\eta_\kappa,\eta_{\kappa+1}), \\ 1,\dot{d}(\eta_{\kappa-1},\eta_\kappa)\cdot\dot{d}(\eta_\kappa,\eta_{\kappa+1}) \end{array}\right\}\right)^\lambda \\
&= \left(\max\left\{\begin{array}{c} \dot{d}(\eta_{\kappa-1},\eta_\kappa),\dot{d}(\eta_\kappa,\eta_{\kappa+1}), \\ \dot{d}(\eta_{\kappa-1},\eta_\kappa)\cdot\dot{d}(\eta_\kappa,\eta_{\kappa+1}) \end{array}\right\}\right)^\lambda \\
&= \left(\dot{d}(\eta_{\kappa-1},\eta_\kappa)\cdot\dot{d}(\eta_\kappa,\eta_{\kappa+1})\right)^\lambda.
\end{aligned}
$$

After simplification, we get that

$$
\dot{d}(\eta_\kappa,\eta_{\kappa+1}) \preceq \left(\dot{d}(\eta_{\kappa-1},\eta_\kappa)\right)^{\frac{\lambda}{1-\lambda}} = \left(\dot{d}(\eta_{\kappa-1},\eta_\kappa)\right)^{\hbar}, \qquad \text{where} \quad \hbar = \frac{\lambda}{1-\lambda} < 1. \tag{3.5}
$$

Similarly,

$$
\dot{d}(\eta_{\kappa-1},\eta_\kappa) \preceq \left(\dot{d}(\eta_{\kappa-2},\eta_{\kappa-1})\right)^{\frac{\lambda}{1-\lambda}} = \left(\dot{d}(\eta_{\kappa-2},\eta_{\kappa-1})\right)^{\hbar}, \qquad \text{where} \quad \hbar = \frac{\lambda}{1-\lambda} < 1. \tag{3.6}
$$

Now from (3.5) and (3.6), and by induction, we have

$$
\begin{aligned}
\dot{d}(\eta_\kappa,\eta_{\kappa+1}) &\preceq \left(\dot{d}(\eta_{\kappa-1},\eta_\kappa)\right)^{\hbar} \\
&\preceq \left(\dot{d}(\eta_{\kappa-2},\eta_{\kappa-1})\right)^{\hbar^2} \\
&\preceq \cdots \preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^\kappa} \to 1, \quad \text{as } \kappa \to \infty.
\end{aligned} \tag{3.7}
$$

Hence proved that the sequence $\{\eta_\kappa\}$ is contractive in MM-space. Now $\kappa > j$,

$$
\begin{aligned}
\dot{d}(\eta_\kappa,\eta_j) &\preceq \dot{d}(\eta_\kappa,\eta_{\kappa-1})\cdot\dot{d}(\eta_{\kappa-1},\eta_{\kappa-2})\cdots\dot{d}(\eta_{j+2},\eta_{j+1})\cdot\dot{d}(\eta_{j+1},\eta_j) \\
&\preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{\kappa-1}} \cdot \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{\kappa-2}} \cdots \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{j+1}} \cdot \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{j}} \\
&\preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{\left(\hbar^{\kappa-1}+\hbar^{\kappa-2}+\cdots+\hbar^{j+1}+\hbar^{j}\right)}
\end{aligned}
$$

$$\leq \left(\dot{d}(\eta_0,\eta_1)\right)^{\left(\hbar^j+\hbar^{j+1}+\cdots+\hbar^{\kappa-2}+\hbar^{\kappa-1}\right)}$$
$$\leq \left(\dot{d}(\eta_0,\eta_1)\right)^{\left(\frac{\hbar^j}{1-\hbar}\right)} \to 1, \quad \text{as } j \to \infty.$$

Hence proved that $\{\xi_\kappa\}$ and $\{\eta_\kappa\}$ are Cauchy sequences in MM-space $(B, \dot{d})$. Since, by the completeness of MM-space, there exist $\xi^\star, \eta^\star \in B$ such that $\lim_{\kappa\to\infty} \xi_\kappa = \xi^\star$ and $\lim_{\kappa\to\infty} \eta_\kappa = \eta^\star$, therefore

$$\lim_{\kappa\to\infty} \dot{d}(\xi_\kappa, \xi^\star) = 1, \quad \text{and} \quad \lim_{\kappa\to\infty} \dot{d}(\eta_\kappa, \eta^\star) = 1. \tag{3.8}$$

Now, we prove that $\Gamma(\xi^*, \eta^*) = \xi^*$. Then, by the view of (3.1) and (3.8), and by using the triangular property of MM-space, we have that

$$\begin{aligned}
\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star) &\leq \left(\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi_{\kappa+1})\right)\cdot\left(\dot{d}(\xi_{\kappa+1},\xi^\star)\right)\\
&= \left(\dot{d}(\Gamma(\xi^\star,\eta^\star),\Gamma(\xi_\kappa,\eta_\kappa))\right)\cdot\left(\dot{d}(\xi_{\kappa+1},\xi^\star)\right)\\
&\leq \left(\max\left\{\begin{array}{l}\dot{d}(\xi^\star,\Gamma(\xi^\star,\eta^\star)),\dot{d}(\xi_\kappa,\Gamma(\xi_\kappa,\eta_\kappa))\\ \dot{d}(\xi_\kappa,\Gamma(\xi^\star,\eta^\star)),\dot{d}(\xi^\star,\Gamma(\xi_\kappa,\eta_\kappa))\end{array}\right\}\right)^\lambda\cdot\left(\dot{d}(\xi_{\kappa+1},\xi^\star)\right)\\
&= \left(\max\left\{\begin{array}{l}\dot{d}(\xi^\star,\Gamma(\xi^\star,\eta^\star)),\dot{d}(\xi_\kappa,\xi_{\kappa+1})\\ \dot{d}(\xi_\kappa,\Gamma(\xi^\star,\eta^\star)),\dot{d}(\xi^\star,\xi_{\kappa+1})\end{array}\right\}\right)^\lambda\cdot\left(\dot{d}(\xi_{\kappa+1},\xi^\star)\right).
\end{aligned}$$

Taking limit $\kappa \to \infty$, we get

$$\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star) \leq \left(\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star)\right)^\lambda.$$

So,

$$\left(\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star)\right)^{1-\lambda} \leq 1.$$

Hence, $\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star) = 1$. Thus $\Gamma(\xi^\star,\eta^\star) = \xi^\star$. Similarly, we can prove that $\dot{d}(\Gamma(\eta^\star,\xi^\star),\eta^\star) = 1 \Rightarrow \Gamma(\eta^\star,\xi^\star) = \eta^\star$. Hence, we get that $(\Gamma(\xi^\star,\eta^\star),\Gamma(\eta^\star,\xi^\star)) = (\xi^\star,\eta^\star)$, this implies that $(\xi^\star,\eta^\star)$ is a coupled FP of Γ in MM-space $(B, \dot{d})$.

Uniqueness: let (ξ', η') is another coupled FP of the mapping Γ in MM-space $(B, \dot{d})$ such that

$$\Gamma(\xi',\eta') = \xi' \quad \textit{and} \quad \Gamma(\eta',\xi') = \eta'. \tag{3.9}$$

First, we prove $\xi^\star = \xi'$, then by the view of (3.1) and (3.9), we have

$$\begin{aligned}
\dot{d}(\xi',\xi^\star) &= \dot{d}(\Gamma(\xi',\eta'),\Gamma(\xi^\star,\eta^\star))\\
&\leq \left(\max\left\{\begin{array}{l}\dot{d}(\xi',\Gamma(\xi',\eta')),\dot{d}(\xi^\star,\Gamma(\xi^\star,\eta^\star))\\ \dot{d}(\xi^\star,\Gamma(\xi',\eta')),\dot{d}(\xi',\Gamma(\xi^\star,\eta^\star))\end{array}\right\}\right)^\lambda\\
&= \left(\max\left\{\begin{array}{l}\dot{d}(\xi',\xi'),\dot{d}(\xi^\star,\xi^\star)\\ \dot{d}(\xi^\star,\xi'),\dot{d}(\xi',\xi^\star)\end{array}\right\}\right)^\lambda\\
&= \left(\dot{d}(\xi',\xi^\star)\right)^\lambda.
\end{aligned}$$

Thus,

$$\left(\dot{d}(\xi',\xi^\star)\right)^{1-\lambda} \leq 1.$$

Hence, we get that

$$\dot{d}(\xi', \xi^\star) = 1 \Rightarrow \xi' = \xi^\star. \tag{3.10}$$

Next, we prove $\eta^* = \eta'$, then by the view of (3.1) and (3.9), we have

$$\begin{aligned}
\dot{d}(\eta', \eta^\star) &= \dot{d}(\Gamma(\eta', \xi'), \Gamma(\eta^\star, \xi^\star)) \\
&\leq \left(\max\left\{\begin{array}{c} \dot{d}((\eta', \Gamma(\eta', \xi'), \dot{d}(\eta^\star, \Gamma(\eta^\star, \xi^\star), \\ \dot{d}(\eta^\star, \Gamma(\eta', \xi'), \dot{d}(\eta', \Gamma(\eta^\star, \xi^\star) \end{array}\right\}\right)^\lambda \\
&= \left(\max\left\{\begin{array}{c} \dot{d}(\eta', \eta'), \dot{d}(\eta^\star, \eta^\star), \\ \dot{d}(\eta^\star, \eta'), \dot{d}(\eta', \eta^\star) \end{array}\right\}\right)^\lambda \\
&= \left(\dot{d}(\eta', \eta^\star)\right)^\lambda.
\end{aligned}$$

Thus,

$$\left(\dot{d}(\eta', \eta^\star)\right)^{1-\lambda} \leq 1.$$

Hence, we get that

$$\dot{d}(\eta', \eta^\star) = 1 \Rightarrow \eta' = \eta^\star. \tag{3.11}$$

Now, from (3.10) and (3.11), we get that $(\xi', \eta') = (\xi^\star, \eta^\star)$. Hence proved that the mapping Γ has a unique coupled FP in B. This completes the proof.

Corollary 3.2. *Let $(B, \dot{d})$ be a complete MM-space and let a mapping $\Gamma : B \times B \to B$ satisfies,*

$$\dot{d}(\Gamma(\xi, \eta), \Gamma(\mu, \nu)) \leq \left(\max\left\{\dot{d}(\xi, \Gamma(\xi, \eta)), \dot{d}(\mu, \Gamma(\mu, \nu))\right\}\right)^\lambda,$$

for all $\xi, \eta, \mu, \nu \in B$ and $\lambda \in [0, 1)$. Then mapping Γ has a unique coupled FP.

Corollary 3.3. *Let $(B, \dot{d})$ be a complete MM-space and let a mapping $\Gamma : B \times B \to B$ satisfies,*

$$\dot{d}(\Gamma(\xi, \eta), \Gamma(\mu, \nu)) \leq \left(\max\left\{\dot{d}(\mu, \Gamma(\xi, \eta)), \dot{d}(\xi, \Gamma(\mu, \nu))\right\}\right)^\lambda,$$

for all $\xi, \eta, \mu, \nu \in B$ and $\lambda \in [0, 1)$. Then mapping Γ has a unique coupled FP.

Example 3.4. Let $B = (0, \infty)$ and a mapping $\dot{d} : B \times B \to \mathbb{R}$ be a complete MM-space which is defined as $\dot{d}(\xi, \eta) = 2^{|\xi - \eta|}\ \forall\ \xi, \eta \in B$. Now we define a mapping $\Gamma : B \times B \to B$ by

$$\Gamma(\xi, \eta) = \frac{\xi}{400} \quad \text{for } \xi, \eta \in (0, 1] \quad \text{and} \quad \Gamma(\xi, \eta) = \frac{2\xi + 4\eta}{12} + 4 \quad \text{for } \xi, \eta \in (1, \infty).$$

Now from (3.1), we have

$$\dot{d}(\Gamma(\xi, \eta), \Gamma(\mu, \nu)) \leq \left(\max\left\{\begin{array}{c} \dot{d}(\xi, \Gamma(\xi, \eta)), \dot{d}(\mu, \Gamma(\mu, \nu)), \\ \dot{d}(\mu, \Gamma(\xi, \eta)), \dot{d}(\xi, \Gamma(\mu, \nu)) \end{array}\right\}\right)^\lambda,$$

for all $\xi, \eta, \mu, \nu \in B$ and $\lambda \in [0, 1)$.

$$\dot{d}(\Gamma(\xi, \eta, \Gamma(\mu, \nu)) = \dot{d}\left(\frac{\xi}{400}, \frac{\mu}{400}\right) = 2^{\frac{1}{400}|\xi - \mu|}. \tag{3.12}$$

And

$$\left(\max\left\{\begin{array}{c}\dot{d}(\xi,\Gamma(\xi,\eta),\dot{d}(\mu,\Gamma(\mu,\nu)),\\ \dot{d}(\mu,\Gamma(\xi,\eta),\dot{d}(\xi,\Gamma(\mu,\nu))\end{array}\right\}\right)^{\lambda}=\left(\max\left\{\begin{array}{c}\dot{d}\left(\xi,\frac{\xi}{400}\right),\dot{d}\left(\mu,\frac{\mu}{400}\right),\\ \dot{d}\left(\mu,\frac{\xi}{400}\right),\dot{d}\left(\xi,\frac{\mu}{400}\right)\end{array}\right\}\right)^{\lambda},$$

$$=\left(\max\left\{\begin{array}{c}2^{\frac{399}{400}|\xi|},2^{\frac{399}{400}|\mu|},\\ 2^{\left|\mu-\frac{\xi}{400}\right|},2^{\left|\xi-\frac{\mu}{400}\right|}\end{array}\right\}\right)^{\lambda}. \tag{3.13}$$

Now the four cases arise for $\xi,\mu\in(0,1]$:

Case.1 if $\dot{d}(\xi,\Gamma(\xi,\eta))=2^{\frac{399}{400}|\xi|}$ is the maximum term in (3.13), then we have the following subcases for $\xi,\mu\in(0,1]$ are discussed with $\lambda=0.3$.

1-(i). If $\xi=0.1$ and $\mu=0.1$, then from (3.1), (3.12), and (3.13), we have that

$$\begin{aligned}&2^{\frac{1}{400}|0.1-0.1|}\leq\left(2^{\frac{399}{400}|0.1|}\right)^{0.3}\Rightarrow 1<1.0209,\\ &\Rightarrow\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu))\leq\left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda}\quad\text{with }\lambda=0.3.\end{aligned}$$

1-(ii). If $\xi=0.1$ and $\mu=0.2$, then from (3.1), (3.12), and (3.13), we have that

$$\begin{aligned}&2^{\frac{1}{400}|0.1-0.2|}\leq\left(2^{\frac{399}{400}|0.1|}\right)^{0.3}\Rightarrow 1.0001<1.0209,\\ &\Rightarrow\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu))\leq\left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda}\quad\text{with }\lambda=0.3.\end{aligned}$$

1-(iii). If $\xi=0.2$ and $\mu=0.1$, then from (3.1), (3.12), and (3.13), we have that

$$\begin{aligned}&2^{\frac{1}{400}|0.2-0.1|}\leq\left(2^{\frac{399}{400}|0.2|}\right)^{0.3}\Rightarrow 1.0001<1.0423,\\ &\Rightarrow\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu))\leq\left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda}\quad\text{with }\lambda=0.3.\end{aligned}$$

1-(iv). If $\xi=0.1$ and $\mu=0.5$, then from (3.1), (3.12), and (3.13), we have that

$$\begin{aligned}&2^{\frac{1}{400}|0.1-0.5|}\leq\left(2^{\frac{399}{400}|0.1|}\right)^{0.3}\Rightarrow 1.0006<1.0209,\\ &\Rightarrow\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu))\leq\left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda}\quad\text{with }\lambda=0.3.\end{aligned}$$

1-(v). If $\xi=0.5$ and $\mu=0.1$, then rom (3.1), (3.12), and (3.13), we have that

$$\begin{aligned}&2^{\frac{1}{400}|0.1-0.5|}\leq\left(2^{\frac{399}{400}|0.5|}\right)^{0.3}\Rightarrow 1.0006<1.1092,\\ &\Rightarrow\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu))\leq\left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda}\quad\text{with }\lambda=0.3.\end{aligned}$$

1-(vi). If $\xi=0.1$ and $\mu=1$, then from (3.1), (3.12), and (3.13), we have that

$$\begin{aligned}&2^{\frac{1}{400}|0.1-1|}\leq\left(2^{\frac{399}{400}|0.1|}\right)^{0.3}\Rightarrow 1.0015<1.0209,\\ &\Rightarrow\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu))\leq\left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda}\quad\text{with }\lambda=0.3.\end{aligned}$$

1-(vii). If $\xi = 1$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|1-0.1|} \leq \left(2^{\frac{399}{400}|1|}\right)^{0.3} \Rightarrow 1.0015 < 1.2305,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

Hence, (3.1) is satisfied for all the subcases (1-(i-vii)) for the first maximum term, that is, $\dot{d}(\xi,\Gamma(\xi,\eta)) = 2^{\frac{399}{400}|\xi|}$.

Case.2 If $\dot{d}(\mu,\Gamma(\mu,\nu)) = 2^{\frac{399}{400}|\mu|}$ is the maximum term in (3.13), then we have the following subcases for $\xi,\mu \in (0,1]$ are discussed with $\lambda = 0.3$:

2-(i). For $\xi = 0.1$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.1|} \leq \left(2^{\frac{399}{400}|0.1|}\right)^{0.3} \Rightarrow 1 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

2-(ii). If $\xi = 0.1$ and $\mu = 0.2$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.2|} \leq \left(2^{\frac{399}{400}|0.2|}\right)^{0.3} \Rightarrow 1.0001 < 1.0423,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

2-(iii). If $\xi = 0.2$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.2-0.1|} \leq \left(2^{\frac{399}{400}|0.1|}\right)^{0.3} \Rightarrow 1.0002 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

2-(iv). If $\xi = 0.1$ and $\mu = 0.5$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.5|} \leq \left(2^{\frac{399}{400}|0.5|}\right)^{0.3} \Rightarrow 1.0006 < 1.1092,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

2-(v). If $\xi = 0.5$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.5-0.1|} \leq \left(2^{\frac{399}{400}|0.1|}\right)^{0.3} \Rightarrow 1.0006 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

2-(vi). If $\xi = 0.1$ and $\mu = 1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-1|} \leq \left(2^{\frac{399}{400}|1|}\right)^{0.3} \Rightarrow 1.0015 < 1.2305,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

2-(vii). If $\xi = 1$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|1-0.1|} \leq \left(2^{\frac{399}{400}|0.1|}\right)^{0.3} \Rightarrow 1.0015 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

Hence, (3.1) is satisfied for all the subcases (2-(i-vii)) for the second maximum term in (3.13), that is, $\dot{d}(\mu,\Gamma(\mu,\nu)) = 2^{\frac{399}{400}|\mu|}$.

Case.3 If $\dot{d}(\mu,\Gamma(\xi,\eta)) = 2^{\left|\mu-\frac{\xi}{400}\right|}$ is the maximum term in (3.13), then we have the following subcases for $\xi,\mu \in (0,1]$ are discussed with $\lambda = 0.3$.

3-(i). For $\xi = 0.1$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.1|} \leq \left(2^{\left|0.1-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 1 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

3-(ii). If $\xi = 0.1$ and $\mu = 0.2$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.2|} \leq \left(2^{\left|0.2-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 1.0001 < 1.0424,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

3-(iii). If $\xi = 0.2$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.2-0.1|} \leq \left(2^{\left|0.1-\frac{0.2}{400}\right|}\right)^{0.3} \Rightarrow 1.0001 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

3-(iv). If $\xi = 0.1$ and $\mu = 0.5$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.5|} \leq \left(2^{\left|0.5-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 1.0006 < 1.1095,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

3-(v). If $\xi = 0.5$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.5-0.1|} \leq \left(2^{\left|0.1-\frac{0.5}{400}\right|}\right)^{0.3} \Rightarrow 1.0006 < 1.0207,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

3-(vi). If $\xi = 0.1$ and $\mu = 1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-1|} \leq \left(2^{\left|1-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 1.0015 < 1.0205,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

3-(vii). If $\xi = 1$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|1-0.1|} \leq \left(2^{\left|0.1-\frac{1}{400}\right|}\right)^{0.3} \Rightarrow 1.0015 < 1.2310,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

Hence, (3.1) is satisfied for all the subcases (3-(i-vii)) for the third maximum term in (3.13), that is, $\dot{d}(\mu,\Gamma(\xi,\eta)) = 2^{\left|\mu-\frac{\xi}{400}\right|}$.

Case.4 If $\dot{d}(\xi,\Gamma(\mu,\nu)) = 2^{\left|\xi-\frac{\mu}{400}\right|}$ is the maximum term in (3.13), then we have the following subcases for $\xi,\mu \in (0,1]$ are discussed with $\lambda = 0.3$.

4-(i). If $\xi = 0.1$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.1|} \leq \left(2^{\left|0.1-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 1 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

4-(ii). If $\xi = 0.1$ and $\mu = 0.2$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.2|} \leq \left(2^{\left|0.1-\frac{0.2}{400}\right|}\right)^{0.3} \Rightarrow 1.0001 < 1.0209,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

4-(iii). If $\xi = 0.2$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.2-0.1|} \leq \left(2^{\left|0.2-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 2^{0.00025} \leq \left(2^{0.19975}\right)^{0.3} \Rightarrow 1.0001 \leq 1.0424,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

4-(iv). If $\xi = 0.1$ and $\mu = 0.5$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-0.5|} \leq \left(2^{\left|0.1-\frac{0.5}{400}\right|}\right)^{0.3} \Rightarrow 1.0006 < 1.0207,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

4-(v). If $\xi = 0.5$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.5-0.1|} \leq \left(2^{\left|0.5-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 1.0006 < 1.1095,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

4-(vi). If $\xi = 1$ and $\mu = 0.1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|1-0.1|} \leq \left(2^{\left|1-\frac{0.1}{400}\right|}\right)^{0.3} \Rightarrow 1.0015 < 1.2310,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

4-(vii). If $\xi = 0.1$ and $\mu = 1$, then from (3.1), (3.12), and (3.13), we have that

$$2^{\frac{1}{400}|0.1-1|} \leq \left(2^{\left|0.1-\frac{1}{400}\right|}\right)^{0.3} \Rightarrow 1.0015 < 1.0204,$$
$$\Rightarrow \dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) \leq \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda} \quad \text{with } \lambda = 0.3.$$

Hence, (3.1) is satisfied for all the subcases (4-(i-vii)) for the fourth maximum term in (3.13), that is, $\dot{d}(\xi,\Gamma(\mu,\nu)) = 2^{\left|\xi-\frac{\mu}{400}\right|}$.

Thus, from the above cases, we conclude that all the conditions of Theorem 3.1 are satisfied with $\lambda = 0.3$ and the mapping Γ has a unique coupled FP, that is, $\Gamma(\xi,\eta) = \frac{2\xi+4\eta}{12} + 4 \Rightarrow \Gamma(8,8) = 8$.

Theorem 3.5. *Let $(B,\dot{d})$ be a complete MM-space, let a mapping $\Gamma : B \times B \to B$, satisfies,*

$$\dot{d}(\Gamma(\xi,\eta),\Gamma(u,v)) \leq \begin{pmatrix} (\dot{d}(\xi,\mu))^{\lambda_1} \\ \cdot(\dot{d}(\xi,\Gamma(\xi,\eta)))^{\lambda_2} \cdot (\dot{d}(\xi,\Gamma(\mu,\nu)))^{\lambda_3} \\ \cdot(\dot{d}(\mu,\Gamma(\xi,\eta)))^{\lambda_4} \cdot (\dot{d}(\mu,\Gamma(\mu,\nu)))^{\lambda_5} \end{pmatrix}, \tag{3.14}$$

for all $\xi,\eta,\mu,\nu \in B$, $\lambda_1 \in (0,1)$, and $\lambda_2,\lambda_3,\lambda_4,\lambda_5 \in [0,1)$ with $(\lambda_1+\lambda_2+2\lambda_3+\lambda_4+\lambda_5) < 1$. Then a mapping Γ has a unique coupled FP.

Proof: Let $\xi_0,\eta_0 \in B$ be the arbitrary points and we define the iterative sequences in B such that

$$\xi_{\kappa+1} = \Gamma(\xi_\kappa,\eta_\kappa) \quad \text{and} \quad \eta_{\kappa+1} = \Gamma(\eta_\kappa,\xi_\kappa) \quad \text{for } \kappa \geq 0.$$

Now from (3.14)

$$\begin{aligned}
\dot{d}(\xi_\kappa,\xi_{\kappa+1}) &= \dot{d}(\Gamma(\xi_{\kappa-1},\eta_{\kappa-1}),\Gamma(\xi_\kappa,\eta_\kappa)), \\
&\leq \begin{pmatrix} (\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\xi_{\kappa-1},\Gamma(\xi_{\kappa-1},\eta_{\kappa-1})))^{\lambda_2} \cdot (\dot{d}(\xi_{\kappa-1},\Gamma(\xi_\kappa,\eta_\kappa)))^{\lambda_3} \\ \cdot(\dot{d}(\xi_\kappa,\Gamma(\xi_{\kappa-1},\eta_{\kappa-1})))^{\lambda_4} \cdot (\dot{d}(\xi_\kappa,\Gamma(\xi_\kappa,\eta_\kappa)))^{\lambda_5} \end{pmatrix} \\
&= \begin{pmatrix} (\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_2} \cdot \dot{d}(\xi_{\kappa-1},\xi_{\kappa+1})^{\lambda_3} \\ \cdot(\dot{d}(\xi_\kappa,\xi_\kappa))^{\lambda_4} \cdot (\dot{d}(\xi_\kappa,\xi_{\kappa+1}))^{\lambda_5} \end{pmatrix} \\
&= \begin{pmatrix} (\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_2} \cdot (\dot{d}(\xi_{\kappa-1},\xi_{\kappa+1}))^{\lambda_3} \\ \cdot(\dot{d}(\xi_\kappa,\xi_{\kappa+1}))^{\lambda_5} \end{pmatrix} \\
&\leq \begin{pmatrix} (\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_2} \cdot (\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_3} \\ \cdot(\dot{d}(\xi_\kappa,\xi_{\kappa+1}))^{\lambda_3} \cdot (\dot{d}(\xi_\kappa,\xi_{\kappa+1}))^{\lambda_5} \end{pmatrix} \\
&= \left((\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\lambda_1+\lambda_2+\lambda_3} \cdot (\dot{d}(\xi_\kappa,\xi_{\kappa+1}))^{\lambda_3+\lambda_5}\right).
\end{aligned}$$

After simplification, we get that

$$\dot{d}(\xi_\kappa,\xi_{\kappa+1}) \leq (\dot{d}(\xi_{\kappa-1},\xi_\kappa))^{\hbar}, \quad \text{where} \quad \hbar = \frac{\lambda_1+\lambda_2+\lambda_3}{1-(\lambda_3+\lambda_5)} < 1. \tag{3.15}$$

Similarly,

$$\dot{d}(\xi_{\kappa-1},\xi_\kappa) \preceq (\dot{d}(\xi_{\kappa-2},\xi_{\kappa-1}))^{\hbar}, \quad \text{where} \quad \hbar = \frac{\lambda_1+\lambda_2+\lambda_3}{1-(\lambda_3+\lambda_5)} < 1. \tag{3.16}$$

Now from (3.15) and (3.16), and by induction, we have

$$\begin{aligned}\dot{d}(\xi_\kappa,\xi_{\kappa+1}) &\preceq \left(\dot{d}(\xi_{\kappa-1},\xi_\kappa)\right)^{\hbar} \\ &\preceq \left(\dot{d}(\xi_{\kappa-2},\xi_{\kappa-1})\right)^{\hbar^2} \\ &\preceq \cdots \preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^\kappa} \to 1, \quad \text{as } \kappa \to \infty.\end{aligned} \tag{3.17}$$

Hence proved that the sequence $\{\xi_\kappa\}$ is contractive in MM-space. Now $\kappa > j$,

$$\begin{aligned}\dot{d}(\xi_\kappa,\xi_j) &\preceq \dot{d}(\xi_\kappa,\xi_{\kappa-1})\cdot\dot{d}(\xi_{\kappa-1},\xi_{\kappa-2})\cdots\dot{d}(\xi_{j+2},\xi_{j+1})\cdot\dot{d}(\xi_{j+1},\xi_j) \\ &\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{\kappa-1}}\cdot\left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{\kappa-2}}\cdots\left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{j+1}}\cdot\left(\dot{d}(\xi_0,\xi_1)\right)^{\hbar^{j}} \\ &\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\left(\hbar^{\kappa-1}+\hbar^{\kappa-2}+\cdots+\hbar^{j+1}+\hbar^{j}\right)} \\ &\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\left(\hbar^{j}+\hbar^{j+1}+\cdots+\hbar^{\kappa-2}+\hbar^{\kappa-1}\right)} \\ &\preceq \left(\dot{d}(\xi_0,\xi_1)\right)^{\left(\frac{\hbar^j}{1-\hbar}\right)} \to 1, \quad \text{as } j \to \infty.\end{aligned}$$

Hence proved that $\{\xi_\kappa\}$ is a Cauchy sequence in MM-space $(B,\dot{d})$. Similarly, for sequence $\{\eta_\kappa\}$ and from (3.14),

$$\begin{aligned}\dot{d}(\eta_\kappa,\eta_{\kappa+1}) &= \dot{d}(\Gamma(\eta_{\kappa-1},\xi_{\kappa-1}),\Gamma(\eta_\kappa,\xi_\kappa)), \\ &\preceq \left(\begin{array}{c}(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\eta_{\kappa-1},\Gamma(\eta_{\kappa-1},\xi_{\kappa-1})))^{\lambda_2}\cdot(\dot{d}(\eta_{\kappa-1},\Gamma(\xi_\kappa,\eta_\kappa)))^{\lambda_3} \\ \cdot(\dot{d}(\eta_\kappa,\Gamma(\eta_{\kappa-1},\xi_{\kappa-1})))^{\lambda_4}\cdot(\dot{d}(\eta_\kappa,\Gamma(\eta_\kappa,\xi_\kappa)))^{\lambda_5}\end{array}\right) \\ &= \left(\begin{array}{c}(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_2}\cdot\dot{d}(\eta_{\kappa-1},\eta_{\kappa+1})^{\lambda_3} \\ \cdot(\dot{d}(\eta_\kappa,\eta_\kappa))^{\lambda_4}\cdot(\dot{d}(\eta_\kappa,\eta_{\kappa+1}))^{\lambda_5}\end{array}\right) \\ &= \left(\begin{array}{c}(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_2}\cdot(\dot{d}(\eta_{\kappa-1},\eta_{\kappa+1}))^{\lambda_3} \\ \cdot(\dot{d}(\eta_\kappa,\eta_{\kappa+1}))^{\lambda_5}\end{array}\right) \\ &\preceq \left(\begin{array}{c}(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_2}\cdot(\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_3} \\ \cdot(\dot{d}(\eta_\kappa,\eta_{\kappa+1}))^{\lambda_3}\cdot(\dot{d}(\eta_\kappa,\eta_{\kappa+1}))^{\lambda_5}\end{array}\right) \\ &= \left((\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\lambda_1+\lambda_2+\lambda_3}\cdot(\dot{d}(\eta_\kappa,\eta_{\kappa+1}))^{\lambda_3+\lambda_5}\right).\end{aligned}$$

After simplification, we get that

$$\dot{d}(\eta_\kappa,\eta_{\kappa+1}) \preceq (\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\frac{\lambda_1+\lambda_2+\lambda_3}{1-(\lambda_3+\lambda_5)}} = (\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\hbar}, \quad \text{where} \quad \hbar = \frac{\lambda_1+\lambda_2+\lambda_3}{1-(\lambda_3+\lambda_5)} < 1. \tag{3.18}$$

Similarly,

$$\dot{d}(\eta_{\kappa-1},\eta_\kappa) \preceq (\dot{d}(\eta_{\kappa-2},\eta_{\kappa-1}))^{\frac{\lambda_1+\lambda_2+\lambda_3}{1-(\lambda_3+\lambda_5)}} = (\dot{d}(\eta_{\kappa-1},\eta_\kappa))^{\hbar}, \quad \text{where} \quad \hbar = \frac{\lambda_1+\lambda_2+\lambda_3}{1-(\lambda_3+\lambda_5)} < 1. \tag{3.19}$$

Now from (3.18) and (3.19), and by induction,

$$\begin{aligned} \dot{d}(\eta_\kappa,\eta_{\kappa+1}) &\preceq \left(\dot{d}(\eta_{\kappa-1},\eta_\kappa)\right)^{\hbar} \\ &\preceq \left(\dot{d}(\eta_{\kappa-2},\eta_{\kappa-1})\right)^{\hbar^2} \\ &\preceq \cdots \preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^\kappa} \to 1, \quad \text{as } \kappa \to \infty. \end{aligned} \tag{3.20}$$

Hence proved that the sequence $\{\eta_\kappa\}$ is contractive in MM-space. Now $\kappa > j$,

$$\begin{aligned} \dot{d}(\eta_\kappa,\eta_j) &\preceq \dot{d}(\eta_\kappa,\eta_{\kappa-1}) \cdot \dot{d}(\eta_{\kappa-1},\eta_{\kappa-2}) \cdots \dot{d}(\eta_{j+2},\eta_{j+1}) \cdot \dot{d}(\eta_{j+1},\eta_j) \\ &\preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{\kappa-1}} \cdot \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{\kappa-2}} \cdots \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{j+1}} \cdot \left(\dot{d}(\eta_0,\eta_1)\right)^{\hbar^{j}} \\ &\preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{(\hbar^{\kappa-1}+\hbar^{\kappa-2}+\cdots+\hbar^{j+1}+\hbar^{j})} \\ &\preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{(\hbar^{j}+\hbar^{j+1}+\cdots+\hbar^{\kappa-2}+\hbar^{\kappa-1})} \\ &\preceq \left(\dot{d}(\eta_0,\eta_1)\right)^{\left(\frac{\hbar^j}{1-\hbar}\right)} \to 1, \quad \text{as } j \to \infty. \end{aligned}$$

Hence proved that $\{\xi_\kappa\}$ and $\{\eta_\kappa\}$ are Cauchy sequences in MM-space $(B,\dot{d})$. Since, by the completeness of MM-space, there exist $\xi^\star,\eta^\star \in B$ such that $\lim_{\kappa\to\infty}\xi_\kappa = \xi^\star$ and $\lim_{\kappa\to\infty}\eta_\kappa = \eta^\star$, therefore

$$\lim_{\kappa\to\infty}\dot{d}(\xi_\kappa,\xi^\star) = 1, \quad \text{and} \quad \lim_{\kappa\to\infty}\dot{d}(\eta_\kappa,\eta^\star) = 1. \tag{3.21}$$

Now, we prove that $\Gamma(\xi^*,\eta^*) = \xi^*$. Then, by the view of (3.14) and (3.21), and by using the triangular property of MM-space, we have that

$$\begin{aligned} \dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star)) &\preceq \dot{d}(\Gamma(\xi^\star,\eta^\star),\xi_{\kappa+1})) \cdot \dot{d}(\xi_{\kappa+1},\xi^\star) \\ &= \dot{d}(\Gamma(\xi^\star,\eta^\star),\Gamma(\xi_\kappa,\eta_\kappa)) \cdot \dot{d}(\xi_{\kappa+1},\xi^\star) \\ &\preceq \begin{pmatrix} (\dot{d}(\xi^\star,\xi_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\xi^\star,\Gamma(\xi^\star,\eta^\star)))^{\lambda_2} \cdot (\dot{d}(\xi^\star,\Gamma(\xi_\kappa,\eta_\kappa)))^{\lambda_3} \\ \cdot(\dot{d}(\xi_\kappa,\Gamma(\xi^\star,\eta^\star)))^{\lambda_4} \cdot (\dot{d}(\xi_\kappa,\Gamma(\xi_\kappa,\eta_\kappa)))^{\lambda_5} \end{pmatrix} \cdot \left(\dot{d}(\xi_{\kappa+1},\xi^\star)\right) \\ &= \begin{pmatrix} (\dot{d}(\xi^\star,\xi_\kappa))^{\lambda_1} \\ \cdot(\dot{d}(\xi^\star,\Gamma(\xi^\star,\eta^\star)))^{\lambda_2} \cdot (\dot{d}(\xi^\star,\xi_{\kappa+1}))^{\lambda_3} \\ \cdot(\dot{d}(\xi_\kappa,\Gamma(\xi^\star,\eta^\star)))^{\lambda_4} \cdot (\dot{d}(\xi_\kappa,\xi_{\kappa+1}))^{\lambda_5} \end{pmatrix} \cdot \left(\dot{d}(\xi_{\kappa+1},\xi^\star)\right). \end{aligned}$$

Taking limit $\kappa \to \infty$, we get

$$\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star) \preceq \left(\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star)\right)^{\beta}, \quad \text{where } \beta = (\lambda_2+\lambda_4) < 1.$$

So,

$$\left(\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star)\right)^{1-\beta} \preceq 1.$$

Hence, $\dot{d}(\Gamma(\xi^\star,\eta^\star),\xi^\star) = 1$. Thus $\Gamma(\xi^\star,\eta^\star) = \xi^\star$. Similarly, we can prove that $\dot{d}(\Gamma(\eta^\star,\xi^\star),\eta^\star) = 1 \Rightarrow \Gamma(\eta^\star,\xi^\star) = \eta^\star$. Hence, we get that $(\Gamma(\xi^\star,\eta^\star),\Gamma(\eta^\star,\xi^\star)) = (\xi^\star,\eta^\star)$, this implies that $(\xi^\star,\eta^\star)$ is a coupled FP of Γ in MM-space $(B,\dot{d})$.
Uniqueness: let (ξ',η') is another coupled FP of Γ in MM-space $(B,\dot{d})$ such that

$$\Gamma(\xi',\eta') = \xi' \quad and \quad \Gamma(\eta',\xi') = \eta'. \tag{3.22}$$

First, we prove $\xi^* = \xi'$, then from (3.14) and (3.22), we have

$$\begin{aligned}
\dot{d}(\xi',\xi^\star) &= \dot{d}(\Gamma(\xi',\eta'),\Gamma(\xi^\star,\eta^\star)) \\
&\preceq \begin{pmatrix} (\dot{d}(\xi',\xi^\star))^{\lambda_1} \\ \cdot(\dot{d}(\xi',\Gamma(\xi',\eta')))^{\lambda_2}\cdot(\dot{d}(\xi',\Gamma(\xi^\star,\eta^\star)))^{\lambda_3} \\ \cdot(\dot{d}(\xi^\star,\Gamma(\xi',\eta')))^{\lambda_4}\cdot(\dot{d}(\xi^\star,\Gamma(\xi^\star,\eta^\star)))^{\lambda_5} \end{pmatrix} \\
&= \begin{pmatrix} (\dot{d}(\xi',\xi^\star))^{\lambda_1} \\ \cdot(\dot{d}(\xi',\xi'))^{\lambda_2}\cdot(\dot{d}(\xi',\xi^\star)^{\lambda_3} \\ \cdot(\dot{d}(\xi^\star,\xi'))^{\lambda_4}\cdot(\dot{d}(\xi^\star,\xi^\star))^{\lambda_5} \end{pmatrix} \\
&= \left((\dot{d}(\xi',\xi^\star))^{\lambda_1}\cdot(\dot{d}(\xi',\xi^\star)^{\lambda_3}\cdot(\dot{d}(\xi^\star,\xi'))^{\lambda_4}\right).
\end{aligned}$$

Hence,

$$\left(\dot{d}(\xi',\xi^\star)\right) \preceq \left(\dot{d}(\xi',\xi^\star)\right)^{\Upsilon} \quad \text{where } \Upsilon = (\lambda_1+\lambda_3+\lambda_4) < 1.$$

Thus,

$$\left(\dot{d}(\xi',\xi^\star)\right)^{1-\Upsilon} \preceq 1.$$

We get that

$$\dot{d}(\xi',\xi^\star) = 1 \Rightarrow \xi' = \xi^\star. \tag{3.23}$$

Next, we prove $\eta^* = \eta'$, then from (3.14) and (3.22), we have

$$\begin{aligned}
\dot{d}(\eta',\eta^\star) &= \dot{d}(\Gamma(\eta',\xi'),\Gamma(\eta^\star,\xi^\star)) \\
&\preceq \begin{pmatrix} (\dot{d}(\eta',\eta^\star))^{\lambda_1} \\ \cdot(\dot{d}(\eta',\Gamma(\eta',\xi')))^{\lambda_2}\cdot(\dot{d}(\eta',\Gamma(\eta^\star,\xi^\star)))^{\lambda_3} \\ \cdot(\dot{d}(\eta^\star,\Gamma(\eta',\xi')))^{\lambda_4}\cdot(\dot{d}(\eta^\star,\Gamma(\eta^\star,\xi^\star)))^{\lambda_5} \end{pmatrix} \\
&= \begin{pmatrix} (\dot{d}(\eta',\eta^\star))^{\lambda_1} \\ \cdot(\dot{d}(\eta',\eta'))^{\lambda_2}\cdot(\dot{d}(\eta',\eta^\star)^{\lambda_3} \\ \cdot(\dot{d}(\eta^\star,\eta'))^{\lambda_4}\cdot(\dot{d}(\eta^\star,\eta^\star))^{\lambda_5} \end{pmatrix} \\
&= \left((\dot{d}(\eta',\eta^\star))^{\lambda_1}\cdot(\dot{d}(\eta',\eta^\star)^{\lambda_3}\cdot(\dot{d}(\eta^\star,\eta'))^{\lambda_4}\right).
\end{aligned}$$

Hence,

$$\left(\dot{d}(\eta',\eta^\star)\right) \preceq \left(\dot{d}(\eta',\eta^\star)\right)^{\Upsilon} \quad \text{where } \Upsilon = (\lambda_1+\lambda_3+\lambda_4) < 1.$$

Thus,

$$\left(\dot{d}(\eta',\eta^\star)\right)^{1-\Upsilon} \preceq 1.$$

We get that

$$\dot{d}(\eta', \eta^\star) = 1 \Rightarrow \eta' = \eta^\star. \tag{3.24}$$

Now, from (3.23) and (3.24), we get that $(\xi', \eta') = (\xi^\star, \eta^\star)$. Hence proved that the mapping Γ has a unique coupled FP in B. This completes the proof.

Corollary 3.6. *Let $(B, \dot{d})$ be a complete MM-space and let a mapping $\Gamma : B \times B \to B$ satisfies,*

$$\dot{d}(\Gamma(\xi, \eta), \Gamma(\mu, \nu)) \leq (\dot{d}(\xi, \mu))^{\lambda_1} \cdot (\dot{d}(\xi, \Gamma(\xi, \eta)))^{\lambda_2} \cdot (\dot{d}(\xi, \Gamma(\mu, \nu)))^{\lambda_3} \cdot (\dot{d}(\mu, \Gamma(\mu, \nu)))^{\lambda_4},$$

for all $\xi, \eta, \mu, \nu \in B$, $\lambda_1 \in (0, 1)$, and $\lambda_2, \lambda_3, \lambda_4 \in [0, 1)$ with $(\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4) < 1$. Then mapping Γ has a unique coupled FP.

Corollary 3.7. *Let $(B, \dot{d})$ be a complete MM-space and let a mapping $\Gamma : B \times B \to B$ satisfies,*

$$\dot{d}(\Gamma(\xi, \eta), \Gamma(\mu, \nu)) \leq (\dot{d}(\xi, \Gamma(\xi, \eta)))^{\lambda_1} \cdot (\dot{d}(\xi, \Gamma(\mu, \nu)))^{\lambda_2} \cdot (\dot{d}(\mu, \Gamma(\mu, \nu)))^{\lambda_3},$$

for all $\xi, \eta, \mu, \nu \in B$, $\lambda_1 \in (0, 1)$, and $\lambda_2, \lambda_3 \in [0, 1)$ with $(\lambda_1 + 2\lambda_2 + \lambda_3) < 1$. Then a mapping Γ has a unique coupled FP.

Example 3.8. Let $B = [0, +\infty[$ and a mapping $\dot{d} : B \times B \to \mathbb{R}$ be a complete MM-space which is defined as $\dot{d}(\xi, \eta) = 2^{|\xi - \eta|}$ for all $\xi, \eta \in B$. Now we define a mapping $\Gamma : B \times B \to B$ by $\Gamma(\xi, \eta) = \frac{\xi}{7}$ for $\xi, \eta \in B = [0, +\infty)$. Now from Theorem (3.14), we have that

$$\begin{aligned}
\dot{d}(\Gamma(\xi, \eta), \Gamma(\mu, \nu)) &= 2^{\frac{1}{7}|\xi - \mu|} \\
&\leq 2^{\frac{15}{70}|\xi + \mu|} \\
&= 2^{\left|\left(\frac{3\xi + 6\xi - \xi}{70} + \frac{\xi}{10}\right) + \left(\frac{-3\mu - \mu + 12\mu}{70} + \frac{\mu}{70}\right)\right|} \\
&= 2^{\left|\left(\frac{3\xi}{70} + \frac{6\xi}{70} - \frac{\xi}{70} + \frac{\xi}{10}\right) + \left(-\frac{3\mu}{70} - \frac{\mu}{70} + \frac{6\mu}{35} + \frac{\mu}{10}\right)\right|} \\
&= 2^{\left|\left(\frac{3\xi}{70} - \frac{3\mu}{70}\right) + \left(\frac{6\xi}{70}\right) + \left(\frac{\xi}{10} - \frac{\mu}{70}\right) + \left(\frac{\mu}{10} - \frac{\xi}{70}\right) + \left(\frac{6\mu}{35}\right)\right|} \\
&= 2^{\left|\frac{3}{10}\left(\frac{\xi}{7} - \frac{\mu}{7}\right) + \frac{1}{10}\left(\xi - \frac{\xi}{7}\right) + \frac{1}{10}\left(\xi - \frac{\mu}{7}\right) + \frac{1}{10}\left(\mu - \frac{\xi}{7}\right) + \frac{1}{5}\left(\mu - \frac{\mu}{7}\right)\right|} \\
&\leq 2^{\left|\frac{3}{10}\left(\frac{\xi}{7} - \frac{\mu}{7}\right)\right| + \left|\frac{1}{10}\left(\xi - \frac{\xi}{7}\right)\right| + \left|\frac{1}{10}\left(\xi - \frac{\mu}{7}\right)\right| + \left|\frac{1}{10}\left(\mu - \frac{\xi}{7}\right)\right| + \left|\frac{1}{5}\left(\mu - \frac{\mu}{7}\right)\right|} \\
&= \left(2^{\left|\left(\frac{\xi}{7} - \frac{\mu}{7}\right)\right|}\right)^{\frac{3}{10}} \cdot \left(2^{\left|\left(\xi - \frac{\xi}{7}\right)\right|}\right)^{\frac{1}{10}} \cdot \left(2^{\left|\left(\xi - \frac{\mu}{7}\right)\right|}\right)^{\frac{1}{10}} \\
&\quad \cdot \left(2^{\left|\left(\mu - \frac{\xi}{7}\right)\right|}\right)^{\frac{1}{10}} \cdot \left(2^{\left|\left(\mu - \frac{\mu}{7}\right)\right|}\right)^{\frac{1}{5}} \\
&= \left(\dot{d}(\xi, \mu)\right)^{\frac{3}{10}} \cdot \left(\dot{d}(\xi, \Gamma(\xi, \eta))\right)^{\frac{1}{10}} \cdot \left(\dot{d}(\xi, \Gamma(\mu, \nu))\right)^{\frac{1}{10}} \\
&\quad \cdot \left(\dot{d}(\mu, \Gamma(\xi, \eta))\right)^{\frac{1}{10}} \cdot \left(\dot{d}(\mu, \Gamma(\mu, \nu))\right)^{\frac{1}{5}}.
\end{aligned}$$

This implies that

$$\dot{d}(\Gamma(\xi, \eta), \Gamma(u, v)) \leq \begin{pmatrix} (\dot{d}(\xi, \mu))^{\lambda_1} \\ \cdot(\dot{d}(\xi, \Gamma(\xi, \eta)))^{\lambda_2} \cdot (\dot{d}(\xi, \Gamma(\mu, \nu)))^{\lambda_3} \\ \cdot(\dot{d}(\mu, \Gamma(\xi, \eta)))^{\lambda_4} \cdot (\dot{d}(\mu, \Gamma(\mu, \nu)))^{\lambda_5} \end{pmatrix}.$$

Hence all conditions of Theorem 3.5 are satisfied for particular $\lambda_1 = \frac{3}{10}$, $\lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{10}$, and $\lambda_5 = \frac{1}{5}$, and a mapping Γ has a unique coupled FP, that is, $\Gamma(0, 0) = 0$.

4. Application

In this section, we present an integral type application, that is the two Urysohn integral equations (UIEs) for the existing result of a common solution. Chen *et al.* [32], Gupta *et al.* [33], Shamas *et al.* [34,35], and Waheed *et al.* [36] proved some FP, common FP and coupled FP theorems for different contractive type mappings and spaces with the applications of integral equations and differential equations.

Let $B = C([a,b],\mathbb{R})$ be the Banach space of all continuous functions defined on $[a,b]$ with supremum norm

$$\|\xi\| = \sup_{r\in[a,b]} |\xi(r)|, \quad \text{where } \xi \in C([a,b],\mathbb{R}).$$

and the induced metric is defined as

$$\dot{d}(\xi,\eta) = \sup_{r\in[a,b]} |\xi(r)-\eta(r)|, \quad \text{where} \quad \xi,\eta \in C([a,b],\mathbb{R}).$$

Now, we are in the position to give the two UIEs for common solution to support our result.

Theorem 4.1. *The two UIEs are*

$$\xi(l) = \int_a^b K_1(l,s,\xi(s))ds + f_1(l) \quad \text{and} \quad \mu(l) = \int_a^b K_1(l,s,\mu(s))ds + f_2(l), \tag{4.1}$$

where $l \in [a,b] \subset \mathbb{R}$ *and* $\xi,\mu,f_1,f_2 \in B$. *Let* $K_1,K_2 : [a,b]^2\times\mathbb{R} \to \mathbb{R}$ *such that* $G_{(\xi,\eta)}, G_{(\mu,\nu)} \in B$, *therefore*

$$G(\xi,\eta)(l) = \int_a^b K_1(l,s,(\xi,\eta)(s))ds \quad \text{and} \quad G(\mu,\nu)(l) = \int_a^b K_2(l,s,(\mu,\nu)(s))ds,$$

where $l \in [a,b]$. If there exists $\lambda \in (0,1)$ such that ,

$$\|(G_{(\xi,\eta)}+f_1)-(G_{(\mu,\nu)}+f_2)\| \le (N((\xi,\eta),(\mu,\nu)))^{\lambda},$$

where

$$N((\xi,\eta),(\mu,\nu)) = \max\left\{\begin{array}{c}\|G_{(\xi,\eta)}+f_1-\xi\|, \|G_{(\mu,\nu)}+f_2-\mu\|,\\ \|G_{(\xi,\eta)}+f_1-\mu\|, \|G_{(\mu,\nu)}+f_2-\xi\|\end{array}\right\}. \tag{4.2}$$

Then the two UIEs (4.1), have a unique common solution.

Proof: Define a mapping $\Gamma : B\times B \to B$

$$\Gamma(\xi,\eta) = G_{(\xi,\eta)}+f_1 \quad \text{and} \quad \Gamma(\mu,\nu) = G_{(\mu,\nu)}+f_2, \quad \forall \xi,\eta,\mu,\nu, G_{(\xi,\eta)}, G_{(\mu,\nu)}, f_1, f_2 \in B.$$

Then, we may have the following four cases;

(a) If $\|G_{(\xi,\eta)}+f_1-\xi\|$ is maximum in (4.2), then

$$N((\xi,\eta),(\mu,\nu)) = \|G_{(\xi,\eta)}+f_1-\xi\|.$$

Hence, we obtain

$$\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) = \|\Gamma(\xi,\eta)-\Gamma(\mu,\nu)\| \le (\|\Gamma(\xi,\eta)-\xi\|)^{\lambda} = \left(\dot{d}(\xi,\Gamma(\xi,\eta))\right)^{\lambda}.$$

(b) If $\|G_{(\mu,\nu)} + f_2 - \mu\|$ is maximum in (4.2), then

$$N((\xi,\eta),(\mu,\nu)) = \|G_{(\mu,\nu)} + f_2 - \mu\|.$$

Hence, we obtain

$$\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) = \|\Gamma(\xi,\eta) - \Gamma(\mu,\nu)\| \leq (\|\Gamma(\mu,\nu) - \mu\|)^{\lambda} = \left(\dot{d}(\mu,\Gamma(\mu,\nu))\right)^{\lambda}.$$

(c) If $\|G_{(\xi,\eta)} + f_1 - \mu\|$ is maximum in (4.2), then

$$N((\xi,\eta),(\mu,\nu)) = \|G_{(\xi,\eta)} + f_1 - \mu\|.$$

Hence, we obtain

$$\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) = \|\Gamma(\xi,\eta) - \Gamma(\mu,\nu)\| \leq (\|\Gamma(\xi,\eta) - \mu\|)^{\lambda} = \left(\dot{d}(\mu,\Gamma(\xi,\eta))\right)^{\lambda}.$$

(d) If $\|G_{(\mu,\nu)} + f_2 - \xi\|$ is maximum in (4.2), then

$$N((\xi,\eta),(\mu,\nu)) = \|G_{(\mu,\nu)} + f_2 - \xi\|.$$

Hence, we obtain

$$\dot{d}(\Gamma(\xi,\eta),\Gamma(\mu,\nu)) = \|\Gamma(\xi,\eta) - \Gamma(\mu,\nu)\| \leq (\|\Gamma(\mu,\nu) - \xi\|)^{\lambda} = \left(\dot{d}(\xi,\Gamma(\mu,\nu))\right)^{\lambda},$$

for every $\xi,\eta,\mu,\nu \in B$. Hence from all the cases, we conclude that all the conditions of Theorem 3.1 are satisfied with $\lambda \in (0,1)$. Thus the two UIEs (4.1) have a unique common solution in B.

5. Conclusions

In this paper, we studied and proved some generalized coupled FP theorems by using nonlinear contractive type mapping in MM-space. Our presented work consists of the maximum type and some other expressions in the framework of MM-space. We also provided illustrative examples and an application of the two UIEs in support of our new generalized results. Our results expand and develop a variety of latest outcomes in the existing literature. In this direction, one can prove more different contractive type coupled FP theorems on complete MM-space with different types of applications.

Conflict of interest

The authors declare that they have no conflict of interests.

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