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Research article

Utilizing fixed point approach to investigate piecewise equations with non-singular type derivative

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Abstract: In this research work, we establish some new results about piecewise equation involving Caputo Fabrizio derivative (CFD). The concerned class has been recently introduced and these results are fundamental for investigation of qualitative theory and numerical interpretation. We derive some necessary results for the existence, uniqueness and various form of Hyers-Ulam (H-U) type stability for the considered problem. For the required results, we need to utilize usual classical fixed point theorems due to Banach and Krasnoselskii's. Moreover, results devoted to H-U stability are derived by using classical tools of nonlinear functional analysis. Some pertinent test problems are given to demonstrate our results.

Keywords: piecewise derivative; piecewise integration; existence theory; stability result
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1. Introduction

In previous few decades the field of fractional calculus has been gotten great attraction from researchers. This is due to the significant applications in the area of nano-technology, rheology, physical and biological sciences as well as engineering disciplines (see [1–3]). The aforesaid area has been interrogated from various aspects in last few years. Recently this area is an hot field of research and many researchers are working on different theories, tools and methodologies to investigate fractional order differential equations (FODEs). In fact differential equations play significant roles in the description of various real world problems and phenomenons. Future prediction and planning can be designed by using the idea of differential and integral equations. The area devoted to classical problems have been enriched by plenty of papers, books and monographs. Further, researchers have expanded the concept of derivatives and integration from integer order to any real or complex order. This concept was old as classical one, but has not properly attracted until eighteenth hundred century. During the mentioned time, Reimann and Liouville introduced properly the concept of fractional order derivatives and integration (for detail see [4]). After that the concept of said operators have been defined in different ways. Recently, we have different kinds of differential operators including Hadamard, Reimann-Liuoville, Caputo, etc. (we provide some detail as [5–8]). Caputo and Fabrizio [9] in 2015 introduced a modified type derivative called non-singular which has been given much attention by the researchers. The considered differential operator involves non singular kernel of exponential type. Also the said operator has been further generalized by Atangana and his co-authors in 2016 by replacing the exponential function by Mittag–Leffler one.

Here, we remark that the aforementioned differential operators have led us to various classes of differential and integral equations which are increasingly applied to deal large numbers of real world problems more successively. One thing, we have to keep in mind that several real world phenomenon have not unique behavior and they keep multiplicity in their dynamical behaviors. For instance fluctuation in economy, earthquake, analogous behaviors of gaseous dynamics, etc. These behaviors mostly are subjected to abrupt changes in their state of rest or uniform motion. This is also called impulsive effect. For the mentioned process researchers increasingly are using different operators mentioned earlier to reach better solutions. But these operators still do not describe the crossover behavior more efficiently. Therefore to more properly investigate the mentioned behaviors, recently authors [10] have introduced the concept of piecewise equations (PEs) of under fractional order derivative. Instead of the classical Reimann-Liouville, Caputo, Caputo-Fabrizio and Atangana-Baleanue derivatives, their piecewise versions work very well to explain the multiple behaviors of a process with more significant ways. Recently some important applicable results by using non-singular type derivatives have been studied, for further detail see [11–14].

Therefore, keeping in mind the aforesaid need and importance, we establish, the existence theory and stability analysis for the following general Cauchy nonlocal implicit problem under the concept of piecewise equations with CFD as

$$\begin{aligned} &{}^{PCF}\mathcal{D}_x^{\delta}u(x) = f(x, u(x), {}^{PCF}\mathcal{D}_x^{\delta}u(x)), \quad x \in [0, T] = \mathcal{J}, \\ &u(0) = u_0 + \varphi(x), \quad u_0 \in \mathcal{R}, \end{aligned} \tag{1.1}$$

such that $\delta \in (0, 1]$, $\varphi \in C(\mathcal{J})$ and $f : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$. The notion ${}^{PCF}\mathcal{D}_x^{\delta}$ stands for piecewise CFD which replicates the power law singular kernel via non singular kernel of exponential kind. For further detailed of CFD, we refer for the readers [15–22].

Here, we first convert the proposed problem to integral form of piecewise via using elementary results of fractional calculus. Further, on utilization of fixed point results, necessary and sufficient conditions are developed to investigate the corresponding existence theory of (1.1). Since stability is an important aspect in optimization theory and numerical analysis. Further, the stability results are fundamental to be investigated for establishing various numerical algorithms and procedures. Therefore, variety of stability concepts have been introduced in literatures including exponential, Laypunov and Mittag–Leffler type. Recently, the H-U type stability theory has been increasingly investigated for usual FODEs (we refer few as [23–27]). Inspired from the aforesaid discussion, we establish some adequate conditions for various forms of H-U type stability including generalized H-U, Hyers-Ulam -Rassias (H-U-R) and generalized H-U-R for the proposed problem by using nonlinear functional analysis. Also for the demonstration of our results, we provide pertinent examples.

The manuscript is organized as: Section first is devoted to literature overview. Second part is related to elementary results. Third part is devoted to existence results. Fourth part is related to stability analysis. Fifth part is enriched by examples. Last part is devoted to brief conclusion.

2. Elementary materials

We need some elementary results which we recall from [10].

Definition 2.1. *Let y be a continues function, then the piecewise integral with fractional order $\delta \in (0,1]$ is defined by*

$$^{PCF}\mathcal{I}_x^{\delta} y(x) = \begin{cases} \displaystyle\int_0^{x_1} y(s)ds, \ \text{if } x \in \mathcal{J}_1 = [0, x_1], \\ \displaystyle\frac{1-\delta}{CF(\delta)} y(x) + \frac{\delta}{CF(\delta)} \int_{x_1}^{x} y(s)ds, \ \text{if } x \in \mathcal{J}_2 = [x_1, T], \end{cases}$$

where $CF(\delta)$ is normalization function.

Definition 2.2. *Let y be continues function, then the piecewise derivative with classical and exponential decay kernel with fractional order $\delta \in (0,1]$ is defined as*

$$^{PCF}\mathcal{D}_x^{\delta} y(x) = \begin{cases} \displaystyle\frac{dy}{dx}, \ \text{if } x \in \mathcal{J}_1, \\ ^{CF}\mathcal{D}_x^{\delta} y(x), \ \text{if } x \in \mathcal{J}_2, \end{cases}$$

where $^{CF}\mathcal{D}_x^{\delta}$ represents CFD, for $x \in \mathcal{J}_2$ which is defined as

$$^{CF}\mathcal{D}_x^{\delta} y(x) = \frac{CF(\delta)}{1-\delta} \int_0^x \exp\left(\frac{-\delta(x-s)}{1-\delta}\right) y'(s)ds, \ x \geq 0.$$

Lemma 2.1. *Let h be continuous function, then the solution of the given problem under piecewise equation with CFD*

$$^{PCF}\mathcal{D}_x^{\delta} y(x) = h(x), \ \delta \in (0,1],$$

is given by

$$y(x) = \begin{cases} y(0) + \displaystyle\int_0^{x_1} h(s)ds, \ x \in \mathcal{J}_1, \\ y(x_1) + \displaystyle\frac{1-\delta}{CF(\delta)} h(x) + \frac{\delta}{CF(\delta)} \int_{x_1}^{x} h(s)ds, \ \text{if } x \in \mathcal{J}_2. \end{cases}$$

The Banach space is defined by $\mathbb{Z} = \left\{ u : \mathcal{J} \to \mathcal{R} : u \in C(\mathcal{J}_1 \cup \mathcal{J}_2) \right\}$ endowed with a norm

$$\|u\|_{\mathbb{Z}} = \sup_{x \in \mathcal{J}} |u(x)|.$$

Theorem 2.3. *[28] Let $\mathbb{E} \subset \mathbb{Z}$ be closed, convex and non empty subset of $\mathbb{Z}$, then there exist $\mathbf{A}_1$, $\mathbf{A}_2$ two operators, such that*

1) $\mathbf{A}_1 u_1 + \mathbf{A}_2 u_2 \in \mathbb{E}$, *for all* $u_1,\ u_2 \in \mathbb{E}$;
2) $\mathbf{A}_1$ *is contraction and* $\mathbf{A}_2$ *is completely continuous operator,*

then there exist atleast one fixed point $u \in \mathbb{E}$ with $\mathbf{A}_1 u + \mathbf{A}_2 u = u$.

3. Existence results

Lemma 3.1. *On using Lemma 2.1, the solution of the problem with piecewise linear equation*

$$\begin{aligned} &{}^{PCF}\mathcal{D}_x^{\delta} u(x) = h(x),\ \delta \in (0,1], \\ &u(0) = u_0 + \varphi(u) \end{aligned} \tag{3.1}$$

is computed as

$$u(x) = \begin{cases} u_0 + \varphi(u) + \displaystyle\int_0^{x_1} h(s)ds,\ x \in \mathcal{J}_1, \\ u(x_1) + \dfrac{(1-\delta)}{CF(\delta)} h(x) + \dfrac{\delta}{CF(\delta)} \displaystyle\int_{x_1}^{x} h(s)ds,\ x \in \mathcal{J}_2. \end{cases} \tag{3.2}$$

Proof. Applying the piecewise integral on both sides of (3.1), we have

$$u(x) = \begin{cases} u(0) + \displaystyle\int_0^{x_1} h(s)ds,\ x \in \mathcal{J}_1, \\ u(x_1) + \dfrac{(1-\delta)}{CF(\delta)} h(x) + \dfrac{\delta}{CF(\delta)} \displaystyle\int_{x_1}^{x} h(s)ds,\ x \in \mathcal{J}_2. \end{cases} \tag{3.3}$$

Using $u(0) = u_0 + \varphi(u)$ in (3.3), we get

$$u(x) = \begin{cases} u_0 + \varphi(u) + \displaystyle\int_0^{x_1} h(s)ds,\ x \in \mathcal{J}_1, \\ u(x_1) + \dfrac{(1-\delta)}{CF(\delta)} h(x) + \dfrac{\delta}{CF(\delta)} \displaystyle\int_{x_1}^{x} h(s)ds,\ x \in \mathcal{J}_2. \end{cases}$$

□

Corollary 1. *Inview of Lemma 3.1, the solution of our proposed problem* (1.1) *is given by*

$$u(x) = \begin{cases} u_0 + \varphi(u) + \displaystyle\int_0^{x_1} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds,\ x \in \mathcal{J}_1, \\ u(x_1) + \dfrac{(1-\delta)}{CF(\delta)} f(x, u(x), {}^{PCF}\mathcal{D}_x^{\delta} u(x)) + \dfrac{\delta}{CF(\delta)} \displaystyle\int_{x_1}^{x} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds,\ x \in \mathcal{J}_2. \end{cases} \tag{3.4}$$

For our analysis the following hypothesis are needed to be hold:

(H_1) For every $u, v \in \mathbb{Z}$ and constant $C_\varphi > 0$, we have

$$|\varphi(u) - \varphi(v)| \le C_\varphi |u - v|;$$

(H_2) For every $u, v, \bar{u}, \ \bar{v} \in \mathbb{Z}$, and constants $\mathbf{L}_f > 0, \ 0 < \mathbf{M}_f < 1$, one has

$$|f(x,u,v) - f(x,\bar{u},\bar{v})| \leq \mathbf{L}_f|u-\bar{u}| + \mathbf{M}_f|v-\bar{v}|.$$

Let us define the operator $\mathbf{F} : \mathbb{Z} \to \mathbb{Z}$ by

$$\mathbf{F}(u(x)) = \begin{cases} u_0 + \varphi(u) + \int_0^{x_1} f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s))ds, \ x \in \mathcal{J}_1, \\ u(x_1) + \dfrac{(1-\delta)}{CF(\delta)} f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) + \dfrac{\delta}{CF(\delta)} \int_{x_1}^x f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s))ds, \ x \in \mathcal{J}_2. \end{cases} \tag{3.5}$$

Theorem 3.1. *Inview of hypothesis (H_1, H_2), our proposed problem (1.1) has a unique solution if the condition*

$$K = \max\left\{C_\varphi + \frac{x_1 \mathbf{L}_f}{1-\mathbf{M}_f}, \ \frac{1-\delta+\delta(T-x_1)}{CF(\delta)} \frac{\mathbf{L}_f}{1-\mathbf{M}_f}\right\} < 1$$

holds.

Proof. Consider $u, \bar{u} \in \mathbb{Z}$, then one has

$$\begin{aligned} \left|{}^{PCF}\mathcal{D}_x^\delta u(x) - {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x)\right| &= \left|f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) - f(x, \bar{u}(x), {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x))\right| \\ &\leq \mathbf{L}_f \left|u(x) - \bar{u}(x)\right| + \mathbf{M}_f \left|{}^{PCF}\mathcal{D}_x^\delta u(x) - {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x)\right|, \end{aligned} \tag{3.6}$$

hence, we have from (3.6)

$$\left|{}^{PCF}\mathcal{D}_x^\delta u(x) - {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x)\right| \leq \frac{\mathbf{L}_f}{1-\mathbf{M}_f} \left|u(x) - \bar{u}(x)\right|. \tag{3.7}$$

Therefore, we consider $u, \bar{u} \in \mathbb{Z}$, and using (3.7)

$$\|\mathbf{F}(u) - \mathbf{F}(\bar{u})\|_{\mathbb{Z}} \leq \sup_{x \in \mathcal{J}} \begin{cases} |\varphi(u) - \varphi(\bar{u})| + \int_0^{x_1} \left|f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s)) - f(s, \bar{u}(s), {}^{PCF}\mathcal{D}_s^\delta \bar{u}(s))\right| ds, \ x \in \mathcal{J}_1, \\ \left(\dfrac{1-\delta}{CF(\delta)}\right) \left|f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) - f(x, \bar{u}(x), {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x))\right| \\ + \dfrac{\delta}{CF(\delta)} \int_{x_1}^x \left|f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s)) - f(s, \bar{u}(s), {}^{PCF}\mathcal{D}_s^\delta \bar{u}(s))\right| ds, \ x \in \mathcal{J}_2. \end{cases} \tag{3.8}$$

Thus, (3.8) yields

$$\|\mathbf{F}(u) - \mathbf{F}(\bar{u})\|_{\mathbb{Z}} \leq \sup_{x \in \mathcal{J}} \begin{cases} C_\varphi |u - \bar{u}| + \int_0^{x_1} \left[\mathbf{L}_f |u(s) - \bar{u}(s)| + \mathbf{M}_f \left|{}^{PCF}\mathcal{D}_s^\delta u(s) - {}^{PCF}\mathcal{D}_s^\delta \bar{u}(s)\right|\right] ds, \ \ x \in \mathcal{J}_1, \\ \left(\dfrac{1-\delta}{CF(\delta)}\right) \left[\mathbf{L}_f |u(x) - \bar{u}(x)| + \mathbf{M}_f \left|{}^{PCF}\mathcal{D}_x^\delta u(x)) - {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x))\right|\right] \\ + \dfrac{\delta}{CF(\delta)} \int_{x_1}^x \left[\mathbf{L}_f |u(s) - \bar{u}(s)| + \mathbf{M}_f \left|{}^{PCF}\mathcal{D}_s^\delta u(s)) - {}^{PCF}\mathcal{D}_s^\delta \bar{u}(s))\right|\right] ds, \ x \in \mathcal{J}_2. \end{cases} \tag{3.9}$$

On further simplification, (3.9) gives

$$\|F(u) - F(\bar{u})\|_{\mathbb{Z}} \leq \sup_{x \in \mathcal{J}} \begin{cases} C_\varphi |u - \bar{u}| + \int_0^{x_1} \dfrac{\mathbf{L}_f}{1-\mathbf{M}_f} |u(s) - \bar{u}(s)| ds, \ x \in \mathcal{J}_1, \\ \left(\dfrac{1-\delta}{CF(\delta)} \dfrac{\mathbf{L}_f}{(1-\mathbf{M}_f)} |u(x) - \bar{u}(x)| + \dfrac{\delta}{CF(\delta)} \dfrac{\mathbf{L}_f}{(1-\mathbf{M}_f)} \int_{x_1}^x |u(s) - \bar{u}(s)| ds, \ x \in \mathcal{J}_2. \right. \end{cases} \tag{3.10}$$

Hence, we have from(3.10) that

$$\|\mathbf{F}(u)-\mathbf{F}(\bar{u})\|_{\mathbb{Z}} \leq \begin{cases} \left(C_\varphi + \dfrac{x_1 \mathbf{L}_f}{1-\mathbf{M}_f}\right)\|u-\bar{u}\|_{\mathbb{Z}}, \ x \in \mathcal{J}_1, \\ \left(\dfrac{1-\delta+\delta(T-x_1)}{CF(\delta)}\right)\dfrac{\mathbf{L}_f}{1-\mathbf{M}_f}\|u-\bar{u}\|_{\mathbb{Z}}, \ x \in \mathcal{J}_2. \end{cases} \tag{3.11}$$

Therefore, (3.12) can be written as

$$\|\mathbf{F}(u)-\mathbf{F}(\bar{u})\|_{\mathbb{Z}} \leq K\|u-\bar{u}\|_{\mathbb{Z}}.$$

Hence $\mathbf{F}$ is contraction operator. Therefore invew of Banach contraction theorem, the proposed problem has a unique solution. □

To derive existence criteria for atleast one solution, we utilize Theorem 2.3. The given assumptions to be holds true:

(H_3) Let for constants $\mathbf{a} > 0$, $\mathbf{C}_f > 0$ and $0 < \mathbf{D}_f < 1$, we have

$$|f(x,u(x),v(x))| \leq \mathbf{a}_f(x) + \mathbf{C}_f(x)|u(x)| + \mathbf{D}_f(x)|v(x)|.$$

Further, assume

$$\mathbf{a}^* = \sup_{x\in\mathcal{J}}|\mathbf{a}_f(x)|, \ \mathbf{b}^* = \sup_{x\in\mathcal{J}}|\mathbf{C}_f(x)|, \ \mathbf{c}^* = \sup_{x\in\mathcal{J}}|\mathbf{D}_f(x)| < 1.$$

Theorem 3.2. *Reference to the hypothesis $(H_1)-(H_3)$, the proposed problem has atleast one solution if the condition* $\max\left\{C_\varphi, \frac{(1-\delta)}{CF(\delta)}\frac{\mathbf{L}_f}{(1-\mathbf{M}_f)}\right\} < 1$ *holds.*

Proof. Here we first define the operators as

$$\mathbf{A}_1 u(x) = \begin{cases} \varphi(u), \ x \in \mathcal{J}_1, \\ u(x_1) + \dfrac{(1-\delta)}{CF(\delta)} f(x,u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) \end{cases} \tag{3.12}$$

and

$$\mathbf{A}_2 u(x) = \begin{cases} \displaystyle\int_0^{x_1} f(s,u(s), {}^{PCF}\mathcal{D}_s^\delta u(s))ds, \ x \in \mathcal{J}_1, \\ \dfrac{\delta}{CF(\delta)} \displaystyle\int_{x_1}^{x} f(s,u(s), {}^{PCF}\mathcal{D}_s^\delta u(s))ds, \ x \in \mathcal{J}_2. \end{cases} \tag{3.13}$$

We now perform the following steps.

Step 1: We describe a set by $\Omega = \{u \in \mathbb{Z} : \|u\|_{\mathbb{Z}} \leq r\}$, as φ and f are continuous, so is $\mathbf{A}_1$. Now to show that $\mathbf{A}_1$ is contraction operator, taking $u, \bar{u} \in \Omega$, and use (3.12), we have

$$\|\mathbf{A}_1 u - \mathbf{A}_1 \bar{u}\| \leq \sup_{x\in\mathcal{J}} \begin{cases} |\varphi(u)-\varphi(\bar{u})|, \ x \in \mathcal{J}_1, \\ \left|\dfrac{(1-\delta)}{CF(\delta)} f(x,u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) - \dfrac{(1-\delta)}{CF(\delta)} f(x,\bar{u}(x), {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x))\right|, \ x \in \mathcal{J}_2. \end{cases} \tag{3.14}$$

Excising the aforesaid Hypothesis (H_1), (H_2) and using the condition

$$\max\left\{C_\varphi,\ \frac{(1-\delta)}{CF(\delta)}\frac{\mathbf{L}_f}{(1-\mathbf{M}_f)}\right\} = \mathbf{H}_1 < 1,$$

one has from (3.14)

$$\|\mathbf{A}_1 u - \mathbf{A}_1\bar{u}\|_{\mathbb{Z}} \leq \mathbf{H}_1\|u - v\|_{\mathbb{Z}}.$$

Thus $\mathbf{A}_1$ is condensing operator.

Step 2: To show that $\mathbf{A}_2$ is completely continuous operator, let $u \in \Omega$, one has from (3.13)

$$\|\mathbf{A}_2 u\|_{\mathbb{Z}} \leq \sup_{x\in\mathcal{J}}\begin{cases}\displaystyle\int_0^{x_1}\left|f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))\right|ds,\ x\in\mathcal{J}_1,\\ \displaystyle\frac{\delta}{CF(\delta)}\int_{x_1}^{x}\left|f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))\right|ds,\ x\in\mathcal{J}_2.\end{cases} \tag{3.15}$$

Thank to Hypothesis (H_3), we have from (3.15) that

$$\|\mathbf{A}_2 u\|_{\mathbb{Z}} \leq \sup_{x\in\mathcal{J}}\begin{cases}\displaystyle\int_0^{x_1}\left[|\mathbf{a}_f(s)| + |\mathbf{C}_f(s)| + |\mathbf{D}_f(s)|\right]ds,\ x\in\mathcal{J}_1,\\ \displaystyle\frac{\delta}{CF(\delta)}\int_{x_1}^{x}\left[|\mathbf{a}_f(s)| + |\mathbf{C}_f(s)| + |\mathbf{D}_f(s)|\right]ds,\ x\in\mathcal{J}_2,\end{cases}$$

which further yields that

$$\|\mathbf{A}_2 u\|_{\mathbb{Z}} \leq \begin{cases}\dfrac{x_1(\mathbf{a}^* + r\mathbf{b}^*)}{(1-\mathbf{c}^*)},\\ \dfrac{\delta}{CF(\delta)}\dfrac{(T-x_1)(\mathbf{a}^* + r\mathbf{b}^*)}{(1-\mathbf{c}^*)}.\end{cases} \tag{3.16}$$

Putting

$$\max\left\{\frac{x_1(\mathbf{a}^* + r\mathbf{b}^*)}{(1-\mathbf{c}^*)},\ \frac{\delta}{CF(\delta)}\frac{(T-x_1)(\mathbf{a}^* + r\mathbf{b}^*)}{(1-\mathbf{c}^*)}\right\} = \mathbf{H}^*,$$

(3.16) takes the form

$$\|\mathbf{A}_2 u\|_{\mathbb{Z}} \leq \mathbf{H}^*. \tag{3.17}$$

Therefore, operator $\mathbf{A}_2$ is bounded.

Step 3: Now to deduce equi-continuity, let $x_2 < x_3 \in \mathcal{J}$, then one has

$$\begin{aligned}|\mathbf{A}_2 u(x_3) - \mathbf{S}_2 u(x_2)| &= \left|\frac{\delta}{CF(\delta)}\left[\int_{x_1}^{x_3} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds - \int_{x_1}^{x_2} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds\right]\right| \\ &= \left|\frac{\delta}{CF(\delta)}\left[\int_{x_1}^{x_2} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds + \int_{x_2}^{x_3} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds \right.\right. \\ &\left.\left. - \int_{x_1}^{x_2} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds\right]\right| \\ &\leq \frac{\delta}{CF(\delta)}\left|\int_{x_2}^{x_3} f(s, u(s), {}^{PCF}\mathcal{D}_s^{\delta} u(s))ds\right| \\ &\leq \frac{\delta}{CF(\delta)}\int_{x_2}^{x_3}\left[|\mathbf{a}_f(s)| + |\mathbf{C}_f(s)||u(s)| + |\mathbf{D}_f(s)||{}^{PCF}\mathcal{D}_s^{\delta} u(s))|\right]ds \\ &\leq \frac{\delta}{CF(\delta)}\frac{(\mathbf{a}^* + \mathbf{b}^* r)}{1 - \mathbf{c}^*}(x_3 - x_2) \to 0, \text{ as } x_3 \to x_2.\end{aligned}$$

Further as $\mathbf{A}_2$ is bounded and continuous on $\mathcal{J}$ so is uniformly continuous. Thus one has

$$\|\mathbf{A}_2 u(x_3) - \mathbf{S}_2 u(x_2)\|_{\mathbb{Z}} \to 0, \text{ as } x_3 \to x_2.$$

Hence $\mathbf{A}_2$ is equi-continuous. Therefore, the operator $\mathbf{A}2$ is relatively compact. Hence inview of Arzelá-Ascoli theorem $\mathbf{A}_2$ is completely continuous operator. Thus all the conditions of theorem are satisfied, hence the proposed problem has atleast one solution. □

4. Stability analysis

Here we describe the results about H-U stability and its various version for the proposed problem. We recall some definition and results given in [29] as:

Definition 4.1. *The proposed problem* (1.1) *is H-U stable if for every* $\epsilon > 0$, *and for the inequality*

$$|{}^{PCF}\mathcal{D}_x^{\delta} u(x) - f(x, u(x), {}^{PCF}\mathcal{D}_x^{\delta} u(x))| < \epsilon, \textit{ for all, } x \in \mathcal{J},$$

there exists a unique solution $\bar{u} \in \mathbb{Z}$ *and a constant* $\mathcal{H}_f > 0$, *such that*

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \mathcal{H}_f \epsilon, \textit{ for every, } x \in \mathcal{J}.$$

Further, if there exist a nondecreasing function $\phi : [0, \infty) \to \mathcal{R}^+$ *for the given inequality*

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \mathcal{H}_f \phi(\epsilon), \textit{ at every, } x \in \mathcal{J}$$

such that $\phi(0) = 0$, *then the concerned solution is generalized H-U stable.*

Definition 4.2. *Our proposed problem* (1.1) *is H-U-R stable corresponding to a function* $\psi : [0, \infty) \to \mathcal{R}^+$, *if for every* $\epsilon > 0$ *and for the inequality*

$$|{}^{PCF}\mathcal{D}_x^{\delta} u(x) - f(x, u(x), {}^{PCF}\mathcal{D}_x^{\delta} u(x))| < \epsilon\psi(x), \quad x \in \mathcal{J},$$

there exists a unique solution $\bar{u} \in \mathbb{Z}$ *of problem* (1.1) *and constant* $\mathcal{H}_{f,\psi} > 0$, *such that*

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \mathcal{H}_{f,\psi}\epsilon\psi(x), \; x \in \mathcal{J}.$$

Again, if there exists $\psi : [0, \infty) \to \mathcal{R}^+$, *for the inequality*

$$|^{PCF}\mathcal{D}_x^\delta u(x) - f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x))| < \psi(x), \quad x \in \mathcal{J},$$

there exists a unique solution $\bar{u} \in \mathbb{Z}$ *and constant* $\mathcal{H}_{f,\psi} > 0$, *such that*

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \mathcal{H}_{f,\psi}\psi(x), \; \text{at ever, } x \in \mathcal{J},$$

then the solution is generalized H-U-R stable.

Before to derive main result of stability, we present some remarks as:

Remark 1. *Consider a function* $\alpha \in C(J)$ *independent of* $u \in \mathbb{Z}$, *such that* $\alpha(0) = 0$, *then*

$$\begin{aligned} |\alpha(x)| &\leq \epsilon, \; x \in \mathrm{J}; \\ {}^{PCF}\mathcal{D}_x^\delta u(x) &= f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) + \alpha(x), \; x \in \mathcal{J}. \end{aligned}$$

Lemma 4.1. *Consider the perturbed problem*

$$\begin{aligned} {}^{PCF}\mathcal{D}_x^\delta u(x) &= f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) + \alpha(x), \; \text{at every, } x \in \mathcal{J}, \\ u(0) &= u_0 + \varphi(u). \end{aligned} \tag{4.1}$$

The solution of (4.1) *is computed as*

$$u(x) = \begin{cases} u_0 + \varphi(u) + \displaystyle\int_0^{x_1} f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s))ds + \int_0^{x_1} \alpha(s)ds, \; x \in \mathcal{J}_1, \\ u(x_1) + \dfrac{(1-\delta)}{CF(\delta)}\Big[f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) + \alpha(x)\Big] \\ \quad + \dfrac{\delta}{CF(\delta)}\Big[\displaystyle\int_{x_1}^{x} f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s))ds + \int_{x_1}^{x_2} \alpha(s)ds\Big], \; x \in \mathcal{J}_2. \end{cases} \tag{4.2}$$

Moreover the solution fulfils the criteria by using (3.5)

$$\|u - \mathbf{F}(\bar{u})\|_{\mathbb{Z}} \leq \begin{cases} x_1\epsilon, \; x \in \mathcal{J}_1 \\ \Big[\dfrac{1 - \delta + \delta(T - x_1)}{CF(\delta)}\Big]\epsilon = \Lambda\epsilon, \; x \in \mathcal{J}_2. \end{cases} \tag{4.3}$$

Proof. Like the proof of Lemma 3.1, the solution of the problem (4.1) given in (4.2) can be computed easily. Further on usual analysis and using Remark 1, the relation (4.3) can be obtained. □

Theorem 4.3. *Inview of Lemma 4.1 and if the condition* $\frac{\mathbf{L}_f}{1-\mathbf{M}_f} < 1$ *holds, then the solution of the considered problem* (1.1) *is H-U stable and further generalized H-U stable.*

Proof. Consider any solution $u \in \mathbb{Z}$ of (1.1) and unique solution $\bar{u} \in \mathbb{Z}$ of (1.1), then we have
Case I. for $x \in \mathcal{J}_1$, one has

$$\begin{aligned}
\|u-\bar{u}\|_{\mathbb{Z}} &= \sup_{x\in\mathcal{J}}\left|u-\left(u_0+\varphi(\bar{u})+\int_0^{x_1} f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))ds+\int_0^{x_1}\alpha(s)ds\right)\right| \\
&\le \sup_{x\in\mathcal{J}}\left|u-\left[u_0+\varphi(u)+\int_0^{x_1} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds\right.\right| \\
&+ \sup_{x\in\mathcal{J}}\left|\varphi(u)-\varphi(\bar{u})+\int_0^{x_1} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds-\int_0^{x_1} f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))ds\right| \\
&\le x_1\epsilon+\frac{\mathbf{L}_f}{1-\mathbf{M}_f}\|u-\bar{u}\|_{\mathbb{Z}}.
\end{aligned} \tag{4.4}$$

Therefore on simplification of (4.4), one has

$$\|u-\bar{u}\|_{\mathbb{Z}} \le \left(\frac{x_1}{1-\frac{\mathbf{L}_f}{1-\mathbf{M}_f}}\right)\epsilon. \tag{4.5}$$

Case II. When $x \in \mathcal{J}_2$, one has

$$\begin{aligned}
\|u-\bar{u}\|_{\mathbb{Z}} &\le \sup_{x\in\mathcal{J}}\left|u-\left[u(x_1)+\frac{(1-\delta)}{CF(\delta)}\left[f(x,u(x),{}^{PCF}\mathcal{D}_x^{\delta}u(x))\right]\right.\right. \\
&+ \left.\left.\frac{\delta}{CF(\delta)}\left[\int_{x_1}^{x} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds\right]\right]\right| \\
&+ \sup_{x\in\mathcal{J}}\frac{(1-\delta)}{CF(\delta)}\left|f(x,u(x),{}^{PCF}\mathcal{D}_x^{\delta}u(x))-f(x,\bar{u}(x),{}^{PCF}\mathcal{D}_x^{\delta}\bar{u}(x))\right| \\
&+ \sup_{x\in\mathcal{J}}\frac{\delta}{CF(\delta)}\int_{x_1}^{x}\left|f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds-f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))\right|ds.
\end{aligned} \tag{4.6}$$

On simplification (4.6) yields and using $\Lambda = \left[\frac{1-\delta+\delta(T-x_1)}{CF(\delta)}\right]$, we have

$$\|u-\bar{u}\|_{\mathbb{Z}} \le \Lambda\epsilon+\Lambda\frac{\mathbf{L}_f}{1-\mathbf{M}_f}\|u-\bar{u}\|_{\mathbb{Z}}. \tag{4.7}$$

Hence, we get from (4.7) that

$$\|u-\bar{u}\|_{\mathbb{Z}} \le \left(\frac{\Lambda}{1-\frac{\Lambda\mathbf{L}_f}{1-\mathbf{M}_f}}\right)\epsilon\|u-\bar{u}\|_{\mathbb{Z}}. \tag{4.8}$$

Using

$$\mathcal{H} = \max\left\{\frac{x_1}{1-\frac{\mathbf{L}_f}{1-\mathbf{M}_f}},\ \frac{\Lambda}{1-\frac{\Lambda\mathbf{L}_f}{1-\mathbf{M}_f}}\right\},$$

then from (4.5) and (4.7), one has

$$\|u-\bar{u}\|_{\mathbb{Z}} \le \mathcal{H}\epsilon, \text{ at each } x \in \mathcal{J}. \tag{4.9}$$

Hence the solution of (1.1) is H-U stable. Further replacing $\phi(\epsilon)=\epsilon$, then, from (4.17), we get

$$\|u-\bar{u}\|_{\mathbb{Z}} \leq \mathcal{H}\phi(\epsilon), \text{ at each } x \in \mathcal{J}.$$

Since we see that $\phi(0)=0$ which means that the solution of (1.1) is also generalized H-U stable. □

To deduce the results of Rassias stability and its generalized form, we state the following remark.

Remark 2. *Let the function $\alpha \in C(\mathcal{J})$ is independent of $u \in \mathbb{Z}$, with $\alpha(0)=0$, then*

$$\begin{aligned} |\alpha(x)| &\leq \psi(x)\epsilon, \ x \in \mathcal{J}; \\ {}^{PCF}\mathcal{D}_x^{\delta}u(x) &= f(x,u(x),{}^{PCF}\mathcal{D}_x^{\delta}u(x)) + \alpha(x), \ x \in \mathcal{J}; \\ \int_0^x \psi(s)ds &\leq C_\psi\ \psi(x), \ \ x \in \mathcal{J}. \end{aligned}$$

Lemma 4.2. *The solution of the problem*

$$\begin{aligned} {}^{PCF}\mathcal{D}_x^{\delta}u(x) &= f(x,u(x),{}^{PCF}\mathcal{D}_x^{\delta}u(x)) + \alpha(x), \ \textit{at every}, \ x \in \mathcal{J}, \\ u(0) &= u_0 + \varphi(u), \end{aligned} \tag{4.10}$$

satisfies the relation given by

$$\|u-\mathbf{F}(\bar{u})\|_{\mathbb{Z}} \leq \begin{cases} x_1 C_\psi \psi(x)\epsilon, \ \ x \in \mathcal{J}_1, \\ \Big[\dfrac{1-\delta+\delta(T-x_1)}{CF(\delta)}\Big]\mathcal{H}_{f,\psi}C_\psi\psi(x)\epsilon = \mathcal{H}_{f,C_\psi,\Lambda}\ \psi(x)\ \epsilon, \ \ x \in \mathcal{J}_2, \end{cases} \tag{4.11}$$

where $\mathcal{H}_{f,\psi,\Lambda} = \Lambda\ \mathcal{H}_{f,\psi}$.

Proof. Using Lemma 3.1, the solution of (4.10) can be computed easily. Further on usual analysis and using Remark 2, the relation (4.11) can be obtained. □

Theorem 4.4. *Inview of* $(H_1), (H_2)$ *and Lemma 4.2, the solution of the proposed problem* (1.1) *is H-U-R stable if* $\mathbf{M}_f < 1$.

Proof. We deduce this results in two cases as:
Case I. For $x \in \mathcal{J}_1$, we have

$$\begin{aligned} \|u-\bar{u}\|_{\mathbb{Z}} &= \sup_{x\in\mathcal{J}}\left|u-\left(u_0+\varphi(\bar{u})+\int_0^{x_1} f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))ds+\int_0^{x_1}\alpha(s)ds\right)\right| \\ &\leq \sup_{x\in\mathcal{J}}\left|u-\left[u_0+\varphi(u)+\int_0^{x_1} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds\right.\right| \\ &+ \sup_{x\in\mathcal{J}}\left|\varphi(u)-\varphi(\bar{u})+\int_0^{x_1} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds-\int_0^{x_1} f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))ds\right| \\ &\leq x_1C_\psi\psi(x)\epsilon+\frac{\mathbf{L}_f}{1-\mathbf{M}_f}\|u-\bar{u}\|_{\mathbb{Z}}. \end{aligned} \tag{4.12}$$

Therefore, on simplification of (4.12), one has

$$\|u-\bar{u}\|_{\mathbb{Z}} \leq \left(\frac{x_1C_\psi}{1-\frac{\mathbf{L}_f}{1-\mathbf{M}_f}}\right)\psi(x)\epsilon = \mathcal{H}_{f,C_\psi,x_1}\ \psi(x)\ \epsilon. \tag{4.13}$$

Case II. For $x \in \mathcal{J}_2$, one has

$$\begin{aligned}\|u-\bar{u}\|_{\mathbb{Z}} &\leq \sup_{x\in\mathcal{J}}\Big|u-\Big[u(x_1)+\frac{(1-\delta)}{CF(\delta)}\Big[f(x,u(x),{}^{PCF}\mathcal{D}_x^{\delta}u(x))\Big]\\ &+ \frac{\delta}{CF(\delta)}\Big[\int_{x_1}^{x} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds\Big]\Big|\\ &+ \sup_{x\in\mathcal{J}}\frac{(1-\delta)}{CF(\delta)}\Big|f(x,u(x),{}^{PCF}\mathcal{D}_x^{\delta}u(x))-f(x,\bar{u}(x),{}^{PCF}\mathcal{D}_x^{\delta}\bar{u}(x))\Big|\\ &+ \sup_{x\in\mathcal{J}}\frac{\delta}{CF(\delta)}\int_{x_1}^{x}\Big|f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds-f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))\Big|ds. \end{aligned} \tag{4.14}$$

On further simplification yields by using $\Lambda = \left[\frac{1-\delta+\delta(T-x_1)}{CF(\delta)}\right]$

$$\|u-\bar{u}\|_{\mathbb{Z}} \leq \Lambda C_{\psi}\psi(x)\epsilon + \Lambda\frac{\mathbf{L}_f}{1-\mathbf{M}_f}\|u-\bar{u}\|_{\mathbb{Z}}. \tag{4.15}$$

Hence we get from (4.15)

$$\|u-\bar{u}\|_{\mathbb{Z}} \leq \left(\frac{\Lambda C_{\psi}}{1-\frac{\Lambda\mathbf{L}_f}{1-\mathbf{M}_f}}\right)\psi(x)\epsilon. \tag{4.16}$$

Using

$$\mathcal{H}_{\Lambda,C_{\psi}} = \max\left\{\frac{x_1}{1-\frac{\mathbf{L}_f}{1-\mathbf{M}_f}},\ \left(\frac{\Lambda C_{\psi}}{1-\frac{\Lambda\mathbf{L}_f}{1-\mathbf{M}_f}}\right)\right\}$$

then from (4.13) and (4.16), we have

$$\|u-\bar{u}\|_{\mathbb{Z}} \leq \mathcal{H}_{\Lambda,C_{\psi}}\psi(x)\,\epsilon,\ \text{at each, } x\in\mathcal{J}. \tag{4.17}$$

Therefore the solution of the proposed problem (1.1) is H-U-R stable. □

Remark 3. *Let the function α be independent of $u \in \mathbb{Z}$, such that $\alpha(0) = 0$, then*

1) $|\alpha(x)| \leq \psi(x), \quad x \in \mathcal{J}$;

Theorem 4.5. *Inview of* (H_1), (H_2), *Remark 3 and Lemma 4.2, the solution of the proposed problem* (1.1) *is generalized H-U-R, if* $\mathbf{M}_f < 1$.

Proof. We derive the required result in two cases as:
Case I. For $x \in \mathcal{J}_1$, we have

$$\begin{aligned}\|u-\bar{u}\|_{\mathbb{Z}} &= \sup_{x\in\mathcal{J}}\Big|u-\Big(u_0+\varphi(\bar{u})+\int_0^{x_1} f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))ds+\int_0^{x_1}\alpha(s)ds\Big)\Big|\\ &\leq \sup_{x\in\mathcal{J}}\Big|u-\Big[u_0+\varphi(u)+\int_0^{x_1} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds\Big|\\ &+ \sup_{x\in\mathcal{J}}\Big|\varphi(u)-\varphi(\bar{u})+\int_0^{x_1} f(s,u(s),{}^{PCF}\mathcal{D}_s^{\delta}u(s))ds-\int_0^{x_1} f(s,\bar{u}(s),{}^{PCF}\mathcal{D}_s^{\delta}\bar{u}(s))ds\Big|\end{aligned}$$

$$\leq \quad x_1 C_\psi \, \psi(x) + \frac{\mathbf{L}_f}{1-\mathbf{M}_f} \|u - \bar{u}\|_{\mathbb{Z}} \,. \tag{4.18}$$

Further from (4.18), one has

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \left(\frac{x_1 C_\psi}{1 - \frac{\mathbf{L}_f}{1-\mathbf{M}_f}} \right) \psi(x) = \mathcal{H}_{f, C_\psi, x_1} \psi(x). \tag{4.19}$$

Case II. For $x \in \mathcal{J}_2$, one has

$$\begin{aligned} \|u - \bar{u}\|_{\mathbb{Z}} \quad &\leq \quad \sup_{x \in \mathcal{J}} \Big| u - \Big[u(x_1) + \frac{(1-\delta)}{CF(\delta)} f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) \\ &+ \quad \frac{\delta}{CF(\delta)} \int_{x_1}^x f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s)) ds \Big] \Big| \\ &+ \quad \sup_{x \in \mathcal{J}} \frac{(1-\delta)}{CF(\delta)} \Big| f(x, u(x), {}^{PCF}\mathcal{D}_x^\delta u(x)) - f(x, \bar{u}(x), {}^{PCF}\mathcal{D}_x^\delta \bar{u}(x)) \Big| \\ &+ \quad \sup_{x \in \mathcal{J}} \frac{\delta}{CF(\delta)} \int_{x_1}^x \Big| f(s, u(s), {}^{PCF}\mathcal{D}_s^\delta u(s)) ds - f(s, \bar{u}(s), {}^{PCF}\mathcal{D}_s^\delta \bar{u}(s)) \Big| ds. \end{aligned} \tag{4.20}$$

On further simplification of (4.20) and using $\Lambda = \left[\frac{1-\delta+\delta(T-x_1)}{CF(\delta)} \right]$, we have

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \Lambda C_\psi \, \psi(x) + \Lambda \frac{\mathbf{L}_f}{1-\mathbf{M}_f} \|u\bar{u}\|_{\mathbb{Z}} \,. \tag{4.21}$$

Hence, we get from (4.21)

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \left(\frac{\Lambda C_\psi}{1 - \frac{\Lambda \mathbf{L}_f}{1-\mathbf{M}_f}} \right) \psi(x). \tag{4.22}$$

Using

$$\mathcal{H}_{\Lambda, C_\psi} = \max \left\{ \frac{x_1}{1 - \frac{\mathbf{L}_f}{1-\mathbf{M}_f}}, \left(\frac{\Lambda C_\psi}{1 - \frac{\Lambda \mathbf{L}_f}{1-\mathbf{M}_f}} \right) \right\},$$

from (4.19) and (4.22), we have

$$\|u - \bar{u}\|_{\mathbb{Z}} \leq \mathcal{H}_{\Lambda, C_\psi} \, \psi(x), \text{ at each, } \; x \in \mathcal{J}.$$

Therefore the solution of the proposed problem (1.1) is generalized H-U-R stable. □

5. Pertinent examples

We provide some examples to verify our results.

Example 1. *Consider the problem as*

$$\begin{aligned} {}^{PCF}\mathcal{D}_x^{0.5} u(x) &= \frac{\exp(-\pi x) \sin |u(x)| + \sin |{}^{PCF}\mathcal{D}_x^{0.5} u(x)|}{50 + x^4}, \quad x \in \mathcal{J}, \\ u(0) &= 1 + \frac{\exp(-|u|)}{30}. \end{aligned} \tag{5.1}$$

Taking $T = 1$ *and* $x_1 = 0.5$ *and we see that* $\mathbf{L}_f = \frac{1}{50} = \mathbf{M}_f$, $C_\varphi = \frac{1}{30}$. *Then we see that on using calculation by Theorem 3.1*

$$\begin{aligned} K &= \max\left\{C_\varphi + \frac{x_1\mathbf{L}_f}{1-\mathbf{M}_f}, \ \frac{1-\delta+\delta(T-x_1)}{CF(\delta)}\frac{\mathbf{L}_f}{1-\mathbf{M}_f}\right\} \\ &= \max\{0.914, \ 0.015\} \\ &= 0.914 < 1. \end{aligned}$$

Hence by using Theorem 3.1, the given problem has a unique solution. Further upon calculation we see that $\mathbf{a}^* = 0$, $\mathbf{b}^* = \frac{1}{50}$, $\mathbf{c}^* = \frac{1}{50}$. *Now inview of Theorem 3.2, we see that*

$$\mathbf{H}_1 = \max\left\{\frac{1}{30}, \frac{1}{98}\right\} = \frac{1}{30} < 1.$$

Hence the conditions of Theorem 3.2 fulfill so the given problem has atleast one solution. Moreover the condition for H-U stability and generalized are obvious as we see that $\Lambda = 0.75$, $\frac{\mathbf{L}_f}{1-\mathbf{M}_f} = \frac{1}{49} < 1$. *Further if we take* $\psi(x) = x$, *then the conditions of H-U-R and generalized H-U-R stabilities are obviously verified given in Theorems 4.4 and 4.5 respectively. Here the graphical behavior of solution is given in Figure 1.*

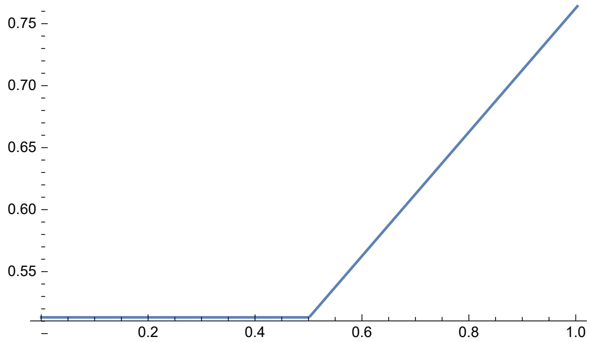

Figure 1. Graphical presentation of solution at given jump of 0.5 for Example 1.

Example 2. *Consider another example as*

$$\begin{aligned} {}^{PCF}\mathscr{D}_x^{0.7}u(x) &= \frac{\exp(-\cos x)\exp(-|u(x)|) + \sin|{}^{PCF}\mathscr{D}_x^{0.7}u(x)|}{100+\cos x}, \quad x \in \mathscr{J}, \\ u(0) &= 0.5 + \frac{\sin(|u|)}{60}. \end{aligned} \tag{5.2}$$

Taking $T = 1.5$ *and* $x_1 = 0.8$ *and we see that* $\mathbf{L}_f = \frac{1}{100} = \mathbf{M}_f$, $C_\varphi = \frac{1}{60}$. *Applying Theorem 3.1 to get*

$$\begin{aligned} K &= \max\left\{C_\varphi + \frac{x_1\mathbf{L}_f}{1-\mathbf{M}_f}, \ \frac{1-\delta+\delta(T-x_1)}{CF(\delta)}\frac{\mathbf{L}_f}{1-\mathbf{M}_f}\right\} \\ &= \max\{0.024747, \ 0.00797\} \\ &= 0.024747 < 1. \end{aligned}$$

Hence by using Theorem 3.1, the given problem has a unique solution. Also see that $\mathbf{a}^* = 0$, $\mathbf{b}^* = \frac{1}{100}$, $\mathbf{c}^* = \frac{1}{100}$. *Thank to Theorem 3.2, we see that*

$$\mathbf{H}_1 = \max\left\{\frac{1}{60}, \frac{1}{330}\right\} = \frac{1}{60} < 1.$$

Thus conditions of Theorem 3.2 are satisfied the given problem has atleast one solution. Also H-U and generalized H-U stability results are satisfied as we see that $\Lambda = 0.79$, $\frac{\mathbf{L}_f}{1-\mathbf{M}_f} = \frac{1}{99} < 1$. *Consider nondecreasing function* $\psi(x) = x$, *then the conditions of H-U-R and generalized H-U-R stabilities can easily be verified given by using Theorems 4.4 and 4.5 respectively. Further, the graphical behavior of solution is given in Figure 2.*

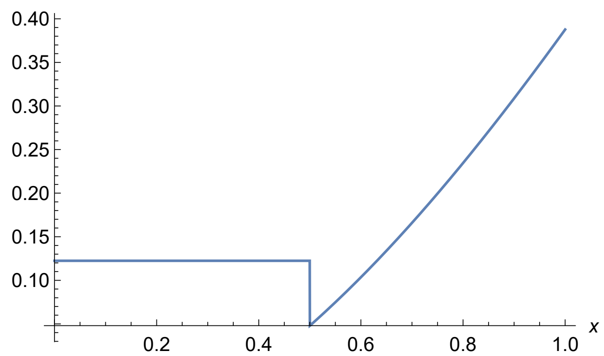

Figure 2. Graphical presentation of solution at given jump of 0.8 for Example 2.

6. Conclusions

Some new concept of piecewise equations under CFD have been introduced in this work. Keeping in mind the importance of fractional calculus in recent time, we have established some results devoted to the existence, uniqueness and stability analysis for a nonlocal Cauchy type problem. The concerned results have been established by using the concept of fixed point approach and nonlinear functional analysis. Sufficient conditions have been developed which guaranteed the existence of atleast one solution and its uniqueness to the proposed nonlocal Cauchy problem. Further its stability has been deduced via nonlinear analysis tools. Pertinent test problems have been provided to illustrate the results. Some graphical presentation have also given. We see that these kinds derivatives more excellently express the sudden change in behavior of dynamical systems. Hence we conclude that this type of calculus in near future will open new area of research. In future more further investigation that how to deal boundary value problems of piecewise equations under various fractional order derivative will be treated.

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Conflict of interest

There is no competing interest regarding this work.

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