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Research article

Solution of fractional kinetic equations involving extended (k,τ)-Gauss hypergeometric matrix functions

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Abstract: In this work, we define an extension of the k-Wright ((k,τ)-Gauss) hypergeometric matrix function and obtain certain properties of this function. Further, we present this function to achieve the solution of the fractional kinetic equations.

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1. Introduction and preliminaries

In the history of hypergeometric functions, Gauss first summarized his studies of the hypergeometric functions which have been of great significance for the mathematical modeling of physical phenomena and other applications (see [1–6]). The Gauss hypergeometric function is defined by the following power series:

$$\mathbf{F}(u_1,u_2,u_3;\zeta)=\sum_{j=0}^{\infty}\frac{(u_1)_j\,(u_2)_j}{(u_3)_j}\,\frac{\zeta^j}{j!},\qquad \zeta\in\mathbb{C},\tag{1.1}$$

which is absolutely and uniformly convergent if $|\zeta|<1$, and where u_1–u_3 are complex parameters with $u_3\in\mathbb{C}\setminus\mathbb{Z}_0^-$, where

$$(u_1)_j=\frac{\Gamma(u_1+j)}{\Gamma(u_1)}=\begin{cases}u_1(u_1+1)\cdots(u_1+j-1), & j\in\mathbb{N},\quad u_1\in\mathbb{C},\\ 1, & j=0,\ u_1\in\mathbb{C}\setminus\{0\}\end{cases}\tag{1.2}$$

is the Pochhammer symbol (or the shifted factorial) and $\Gamma(v)$ is the gamma function defined by

$$\Gamma(v)=\int_0^\infty\theta^{v-1}e^{-\theta}d\theta,\qquad v\in\mathbb{C}\setminus\mathbb{Z}_0^-.\tag{1.3}$$

The generalized (Wright) hypergeometric function was first studied by Virchenko et al. [7], as follows:

$$ {}_2\mathbf{R}_1(\vartheta_1,\vartheta_2;\vartheta_3;\tau;\eta) = \frac{\Gamma(\vartheta_3)}{\Gamma(\vartheta_2)} \sum_{j=0}^{\infty} \frac{(\vartheta_1)_j \Gamma(\vartheta_2+\tau j)}{\Gamma(\vartheta_3+\tau j)} \frac{\eta^j}{j!}, \quad \tau \in \mathbb{R}^+, \quad |\eta| < 1, \tag{1.4} $$

where ϑ_1–ϑ_3 are complex parameters such that $\mathrm{Re}(\vartheta_1) > 0$, $\mathrm{Re}(\vartheta_2) > 0$ and $\mathrm{Re}(\vartheta_3) > 0$.

Recently, various developments and expansions of the Wright (τ-Gauss) hypergeometric function have been archived (see, e.g., [8–13]).

In 1998, Jódar and Cortés [14, 15] gave the matrix version of the gamma and beta functions and the Gauss hypergeometric function. These works have been carried out for many special polynomials and functions; see [16]. In [17, 18], the authors presented interesting expansions of the k-gamma, k-beta, k-Pochhammer and k-hypergeometric matrix functions. Further, extensions of the gamma, beta, Bessel and hypergeometric matrix functions have been given in [19–28]. More recently, Bakhet et al. [29] introduced the Wright hypergeometric functions and discussed some of its properties. In a similar vein, Abdalla investigated some fractional operators for Wright hypergeometric matrix functions in [30]. Motivated by these recent studies on the Wright hypergeometric matrix functions, in this manuscript, we introduce the matrix version of new extended Wright hypergeometric functions and investigate some of its properties.

This manuscript is organized as follows. In Section 2, we define the extended (k,τ)-Wright hypergeometric matrix functions ${}_3\mathcal{W}_2^{(k,\tau)}$ and several special cases. Also, we prove some derivative formulas. In Section 3, we discuss the Mellin transform of the extended (k,τ)-Wright hypergeometric matrix functions. Certain integral representations for the extended (k,τ)-Wright hypergeometric matrix functions are established in Section 4. The k-fractional calculus operators for the matrix functions ${}_3\mathcal{W}_2^{(k,\tau)}$ are investigated in Section 5. In Section 6, we investigate the solutions of fractional kinetic equations involving the extended (k,τ)-Wright hypergeometric matrix function. Finally, in Section 7, concluding remarks are given.

For $B \in \mathbb{C}^{m\times m}$, let $\sigma(B)$ be the set of all eigenvalues of B which is called the spectrum of B. Also, for $B \in \mathbb{C}^{m\times m}$, let

$$ \mu(B) := \max\{\mathrm{Re}(\zeta) : \zeta \in \sigma(B)\} \quad \text{and} \quad \widetilde{\mu}(B) := \min\{\mathrm{Re}(\zeta) : \zeta \in \sigma(B)\}, $$

which imply $\widetilde{\mu}(B) = -\mu(-B)$. Here, $\mu(B)$ is called the spectral abscissa of B, and the matrix B is said to be positive stable if $\widetilde{\mu}(B) > 0$.

For $k \in \mathbb{R}^+$, the k-gamma function $\Gamma_k(\xi)$ is defined by (see [31])

$$ \Gamma_k(\delta) = \int_0^{\infty} \theta^{\delta-1} e^{-\frac{\theta^k}{k}} d\theta, \quad \delta \in \mathbb{C} \setminus k\mathbb{Z}_0^-. \tag{1.5} $$

We note that $\Gamma_k(\delta) \to \Gamma(\delta)$, as $k \to 1$, and $(\delta)_{j,k}$ is the k-Pochhammer symbol given by (see [31])

$$ (\delta)_{j,k} = \frac{\Gamma_k(\delta + jk)}{\Gamma_k(\delta)} = \begin{cases} \delta(\delta+k)\cdots(\delta+(j-1)k), & j \in \mathbb{N},\ \delta \in \mathbb{C}, \\ 1, & j = 0, k \in \mathbb{R}^+, \delta \in \mathbb{C}\setminus\{0\}. \end{cases} \tag{1.6} $$

Clearly, the case $k = 1$ in (1.6) reduces to the Pochhammer symbol defined in (1.2).

If B is a positive stable matrix in $\mathbb{C}^{m\times m}$ and $k \in \mathbb{R}^+$, then the k-gamma matrix function $\Gamma_k(B)$ is well defined, as follows (see [17]):

$$\Gamma_k(B) = \int_0^\infty e^{-\frac{w^k}{k}}\, w^{B-I}\, \mathrm{d}w = k^{\frac{B^k}{k}-I}\, \Gamma(\frac{B}{k}), \quad w^{B-I} := \exp\left((B-I)\ln w\right). \tag{1.7}$$

If B is a matrix in $\mathbb{C}^{m\times m}$ such that $B + \ell kI$ is an invertible matrix for every $\ell \in \mathbb{N}_0$ and $k \in \mathbb{R}^+$, then $\Gamma_k(B)$ is invertible, its inverse is $\Gamma_k^{-1}(B)$, and one finds (see [17])

$$(B)_{\ell,k} = B(B+kI)\cdots(B+(\ell-1)kI) = \Gamma_k(B+\ell kI)\,\Gamma_k^{-1}(B), \quad \ell \in \mathbb{N}_0,\ k \in \mathbb{R}^+. \tag{1.8}$$

Remark 1.1. *For $k = 1$, (1.7) and (1.8) will reduce to the gamma matrix function $\Gamma(B)$ and Pochhammer matrix symbol, respectively (see [14]).*

Further, let B be a positive stable matrix in $\mathbb{C}^{m\times m}$. Then an extension of the k-gamma of the matrix argument given by (1.7) is defined in [19] as follows:

$$\Gamma_k^\rho(B) = \int_0^\infty w^{B-I}\, e^{\left(-\frac{w^k}{k}-\frac{\rho^k}{kw^k}\right)}\,\mathrm{d}w, \quad \rho \in \mathbb{R}_0^+,\ k \in \mathbb{R}^+. \tag{1.9}$$

For $\alpha,\beta \in \mathbb{C}$, the k-beta function $\mathbf{B}_k(\alpha,\beta)$ is defined by (see [31])

$$\mathbf{B}_k(\alpha,\beta) = \frac{1}{k}\int_0^1 y^{\frac{\alpha}{k}-1}(1-y)^{\frac{\beta}{k}-1}dy, \ \ k \in \mathbb{R}^+,\ \mathrm{Re}(\alpha) > 0,\ \ \mathrm{Re}(\beta) > 0. \tag{1.10}$$

When $k = 1$ in (1.10) reduces to the following beta function $\mathcal{B}(\alpha,\beta)$,

$$\mathcal{B}(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy, \ \ \mathrm{Re}(\alpha) > 0,\ \ \mathrm{Re}(\beta) > 0, \tag{1.11}$$

and

$$\mathbf{B}_k(\alpha,\beta) = \frac{1}{k}\mathcal{B}(\frac{\alpha}{k},\frac{\beta}{k}).$$

The k-beta matrix function is defined by (see [17, 19])

$$\mathbf{B}_k(E,F) = \frac{1}{k}\int_0^1 u^{\frac{E}{k}-I}\,(1-u)^{\frac{F}{k}-I}\,\mathrm{d}u, \quad k \in \mathbb{R}^+, \tag{1.12}$$

where E and F are positive stable matrices in $\mathbb{C}^{m\times m}$. Further, if E and F are diagonalizable matrices in $\mathbb{C}^{m\times m}$ such that $EF = FE$, then (cf. [17, 19])

$$\mathbf{B}_k(E,F) = \Gamma_k(E)\,\Gamma_k(F)\,\Gamma_k^{-1}(E+F). \tag{1.13}$$

When $k = 1$, (1.12) and (1.13) will reduce to the beta matrix function $\mathbf{B}(E,F)$, defined by Jódar and Cortés in [14]. Let $p, q \in \mathbb{N}_0$. Also, let $(A)_p$ and $(B)_q$ be the arrays of p commutative matrices $A_1, A_2, \ldots, A_p$ and q commutative matrices $B_1, B_2, \ldots, B_q$ in $\mathbb{C}^{m\times m}$, respectively, such that $B_s + \ell I$ is invertible for $1 \leq s \leq q$ and all $\ell \in \mathbb{N}_0$; then, the generalized hypergeometric matrix function ${}_p\mathbf{F}_q\left((A)_p;(B)_q;\xi\right)$ ($\xi \in \mathbb{C}$) is defined by (see, e.g., [15, 16])

$$ {}_p\mathbf{F}_q\left((A)_p;(B)_q;\xi\right) = \sum_{s=0}^{\infty} \prod_{j=1}^{p}(A_j)_s \prod_{i=1}^{q}[(B_i)_s]^{-1} \frac{\xi^s}{s!}. \tag{1.14} $$

In particular, the Gauss hypergeometric matrix function ${}_2\mathbf{F}_1(A_1, A_2; A_3; \xi) \equiv \mathbf{H}(A_1, A_2; A_3; \xi)$ is defined by

$$ \mathbf{H}(A_1, A_2; A_3; \xi) = \sum_{s=0}^{\infty} (A_1)_s (A_2)_s [(A_3)_s]^{-1} \frac{\xi^s}{s!}, \tag{1.15} $$

for matrices A_1, A_2 and A_3 in $\mathbb{C}^{m\times m}$ such that $A_3 + \ell I$ is invertible for all $\ell \in \mathbb{N}_0$.

2. Extended (k, τ)-Wright hypergeometric matrix function

In this section, we introduce the extended (k, τ)-Wright hypergeometric matrix functions ${}_3\mathcal{W}_2^{(k,\tau)}$ and some derivative formula as follows:

Definition 2.1. *Assume that D, E, F, G and H are positive stable matrices in $\mathbb{C}^{m\times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0$, $\rho \in \mathbb{R}_0^+$ and $k, \tau \in \mathbb{R}^+$. Then, for $|\xi| < 1$, the extended (k, τ)-Wright hypergeometric matrix function is defined in the following form:*

$$ \begin{aligned} {}_3\mathcal{W}_2^{(k,\tau)}(\xi) &:= {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{cc} (D,k;\rho),(E,k),(F,k) & \\ & ;\xi \\ (G,k),(H,k) & \end{array}\right] \\ &:= \Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H) \\ &\quad \times \sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G + k\tau sI)\Gamma_k(E + k\tau sI) \\ &\quad \times \Gamma_k^{-1}(H + k\tau sI)\Gamma_k(F + k\tau sI)\,\frac{\xi^s}{s!}, \end{aligned} \tag{2.1} $$

where $(D;\rho)_{s,k}$ is the generalized k-Pochhammer matrix symbol defined as

$$ (D;\rho)_{s,k} = \begin{cases} \Gamma_k^{\rho}(D + sI)\,\Gamma_k^{-1}(D), & \widetilde{\mu}(D) > 0,\ \rho, k \in \mathbb{R}^+,\ s \in \mathbb{N}, \\ (D)_{s,k}, & p = 0,\ k \in \mathbb{R}^+,\ s \in \mathbb{N}, \\ I, & s = 0,\ p = 0,\ k = 1. \end{cases} \tag{2.2} $$

Or, equivalently, by means of the integral formula given by (1.9), as follows:

$$ (D;\rho)_{s,k} = \Gamma_k^{-1}(D) \int_0^{\infty} \theta^{D+(s-1)I} e^{\left(-\frac{\theta^k}{k} - \frac{\rho^k}{k\theta^k}\right)} d\theta, \quad k \in \mathbb{R}^+,\ \rho \in \mathbb{R}_0^+,\ \widetilde{\mu}(D + sI) > 0. $$

Remark 2.1. *The following are some of the special cases of the extended (k, τ)-Wright hypergeometric matrix functions ${}_3\mathcal{W}_2^{(k,\tau)}$ given by (2.1):*

(i) *When $k = 1$, (2.1) reduces to the following set of extended τ-Wright hypergeometric matrix functions (see [29, 30]):*

$$\begin{aligned}
{}_3\mathcal{W}_2^{(\tau)}(\xi) &:= {}_3\mathcal{W}_2^{(\tau)}\left[\begin{array}{c}(D;\rho),(E),(F)\\ \\ (G),(H)\end{array};\xi\right]\\
&:= \Gamma^{-1}(E)\Gamma(G)\Gamma^{-1}(F)\Gamma(H)\\
&\quad\times\sum_{s=0}^{\infty}(D;\rho)_s\,\Gamma^{-1}(G+\tau sI)\Gamma(E+\tau sI)\\
&\quad\times\Gamma^{-1}(H+\tau sI)\Gamma(F+\tau sI)\,\frac{\xi^s}{s!},
\end{aligned}\tag{2.3}$$

where D, E, F, G and H are positive stable matrices in $\mathbb{C}^{m\times m}$, such that $G+\ell I$ and $H+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$, $\rho\in\mathbb{R}_0^+$ and $\tau\in\mathbb{R}^+$.

(ii) *When $\tau = 1$ in (2.1), and by using some properties of k-Pochhammer matrix symbols, we obtain the following extended k-Gauss hypergeometric matrix function (see [18]):*

$$\begin{aligned}
{}_3\mathcal{W}_2^{(k)}(\xi) &:= {}_3\mathcal{W}_2^{(k)}\left[\begin{array}{c}(D;k,\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\xi\right]\\
&= \sum_{s=0}^{\infty}(D;\rho)_{s,k}(E)_{s,k}\,(F)_{s,k}\,[(G)_{s,k}]^{-1}[(H)_{s,k}]^{-1}\frac{\xi^s}{s!},
\end{aligned}\tag{2.4}$$

where D, E, F, G and H are positive stable matrices in $\mathbb{C}^{m\times m}$, such that $G+\ell I$ and $H+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$, $\rho\in\mathbb{R}_0^+$ and $k\in\mathbb{R}^+$.

(iii) *When $F = H$, (2.1) reduces to the extended (k,τ)-Wright hypergeometric matrix function ${}_\mathbf{2}\mathbf{R}_1^{(k,\tau;\rho)}(\xi)$ defined by*

$$\begin{aligned}
{}_\mathbf{2}\mathbf{R}_1^{(k,\tau;\rho)}(\xi) &:= {}_\mathbf{2}\mathbf{R}_1^{(\tau)}\Big((D,k;\rho),(E,k);(G,k);\xi\Big)\\
&:= \Gamma_k^{-1}(E)\Gamma_k(G)\sum_{s=0}^{\infty}(D;\rho)_{s,k}\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\frac{\xi^s}{s!},
\end{aligned}\tag{2.5}$$

where D, E and G are positive stable matrices in $\mathbb{C}^{m\times m}$, such that $E+\ell I$ and $G+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$, $\rho\in\mathbb{R}_0^+$ and $\tau\in\mathbb{R}^+$.

(iv) *If we set $\rho = 0$ and $F = H$, then (2.1) reduces to the (k,τ)-Gauss hypergeometric matrix function ${}_\mathbf{2}\mathbf{R}_1^{(k,\tau)}(\xi)$ given by (see [32])*

$$\begin{aligned}
{}_\mathbf{2}\mathbf{R}_1^{(k,\tau)}(\xi) &:= {}_\mathbf{2}\mathbf{R}_1^{(\tau)}\Big((D,k),(E,k);(G,k);\xi\Big)\\
&:= \Gamma_k^{-1}(E)\Gamma_k(G)\sum_{s=0}^{\infty}(D)_{s,k}\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\frac{\xi^s}{s!},
\end{aligned}\tag{2.6}$$

where D, E and G are positive stable matrices in $\mathbb{C}^{m\times m}$, such that $E+\ell I$ and $G+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$ and $k,\tau\in\mathbb{R}^+$.

(v) *When $\tau = 1$ in (2.6), and by using some properties of k-Pochhammer matrix symbols, we obtain the following k-hypergeometric matrix function (see [18]):*

$$\boldsymbol{H}^k(D,E;G;\xi) = \sum_{s=0}^{\infty}(D)_{s,k}(F)_{s,k}[(G)_{s,k}]^{-1}\frac{\xi^s}{s!}, \tag{2.7}$$

where $k \in \mathbb{R}^+$ and D, E and G are positive stable matrices in $\mathbb{C}^{m\times m}$, such that $G + \ell I$ is invertible for all $\ell \in \mathbb{N}_0$.

(vi) *When $k = 1$, (2.6) reduces to the following Wright hypergeometric matrix function (see [23]):*

$${}_2\mathbf{R}_1^{(\tau)}(D,E;G;\xi) := \Gamma^{-1}(E)\Gamma(G)\sum_{s=0}^{\infty}(D)_s\Gamma^{-1}(G+\tau sI)\Gamma(E+\tau sI)\frac{\xi^s}{s!}, \tag{2.8}$$

where $\tau \in \mathbb{R}^+$ and D, E and G are positive stable matrices in $\mathbb{C}^{m\times m}$, such that $G + \ell I$ is invertible for all $\ell \in \mathbb{N}_0$.

(vii) *If we set $k = 1$, (2.7) will yield the hypergeometric matrix function defined in (1.15).*

Now, we will present some derivative formulas of the extended (k,τ)-Wright hypergeometric matrix function defined by (2.1).

Theorem 2.1. *Under the conditions of the hypothesis in Definition 2.1, the following derivative formulas for ${}_3\mathcal{W}_2^{(k,\tau)}(\xi)$ hold true:*

$$\begin{aligned}
&\frac{d^n}{d\xi^n}\left\{{}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\xi\right]\right\}\\
=&(D)_{n,k}\ \Gamma_k(G)\Gamma_k(E+\tau knI)\Gamma_k^{-1}(E)\Gamma_k^{-1}(G+\tau knI)\\
&\times\ \Gamma_k(H)\Gamma_k(F+\tau knI)\Gamma_k^{-1}(F)\Gamma_k^{-1}(H+\tau knI)\\
&\times\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D+nkI,k;\rho),(E+\tau nkI,k),(F+\tau nkI,k)\\ \\ (G+\tau nkI,k),(H+\tau nkI,k)\end{array};\xi\right],
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&k^n\frac{d^n}{d\xi^n}\left[\xi^{\frac{G}{k}-I}\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\omega\xi^\tau\right]\right]\\
=&\xi^{\frac{(G-nkI)}{k}-I}\ \Gamma_k(G)\Gamma^{-1}(G-nkI)\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G-nkI,k),(H,k)\end{array};\omega\xi^\tau\right],
\end{aligned} \tag{2.10}$$

where $\omega \in \mathbb{C}$, $\rho \in \mathbb{R}_0^+$ and $k,\tau \in \mathbb{R}^+$.

Proof. Differentiating n times both sides of (2.1) with respect to ξ, we can easily obtain the derivative formula for the set of extended (k,τ)-Wright hypergeometric matrix functions ${}_3\mathcal{W}_2^{(k,\tau)}(\xi)$ asserted by (2.9).

Next, we will prove the derivative formula given by (2.10) according to the uniform convergence of the series given by (2.1), differentiating term by term under the sign of summation before using (2.1) to get the right-hand side of (2.10) after minimal simplifications. □

Theorem 2.2. *Assume that $\omega \in \mathbb{C}$ and $\alpha, \xi \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}(\xi) > \mathrm{Re}(\alpha)$, $\rho \in \mathbb{R}_0^+$ and $k, \tau \in \mathbb{R}^+$. Also, let $\mu \in \mathbb{C} \setminus \{-1\}$, and $n \in \mathbb{N}$. Further, let D, E, G and H be positive stable matrices in $\mathbb{C}^{m\times m}$, such that $G + \ell I$ and $H + \ell I$ are invertible for all $\ell \in \mathbb{N}_0$. Then, we have*

$$\begin{aligned}
&\Big(\frac{1}{\xi^\mu}\frac{d}{d\xi}\Big)^n\Big\{(\xi^{\mu+1}-\alpha^{\mu+1})^{\frac{H}{k}-I}\,{}_2\mathbf{R}_1^{(k,\tau)}\Big((D,k;\rho),(E,k);(G,k);\omega\,(\xi^{\mu+1}-\alpha^{\mu+1})^\tau\Big)\Big\}\\
=&k^{-n}\,(\mu+1)^n\,\Gamma_k(H)\,\Gamma_k^{-1}(H-nkI)\,(\xi^{\mu+1}-\alpha^{\mu+1})^{\frac{H}{k}-(n+1)I}\\
&\times\,{}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(H,k)\\ \\ (G,k),(H-nkI,k)\end{array};\omega(\xi^{\mu+1}-\alpha^{\mu+1})^\tau\right].
\end{aligned} \tag{2.11}$$

Proof. For convenience, we denote the left-hand side of (2.11) by Ł. By invoking (2.5) and interchanging the order of summation and differentiation, we find that

$$\begin{aligned}
\text{Ł} =&\Gamma_k(G)\Gamma^{-1}(E)\sum_{s=0}^{\infty}(D,k;\rho)_{s,k}\,\Gamma_k(E+sk\tau I)\,\Gamma_k^{-1}(G+sk\tau I)\,\frac{\omega^s}{s!}\\
&\times\Big\{\Big(\frac{1}{\xi^\mu}\frac{d}{d\xi}\Big)^n\,(\xi^{\mu+1}-\alpha^{\mu+1})^{\frac{H}{k}+(\tau s-1)I}\Big\}\\
=&\Gamma_k(G)\Gamma^{-1}(E)\sum_{s=0}^{\infty}(D,k;\rho)_{s,k}\,\Gamma_k(E+sk\tau I)\,\Gamma_k^{-1}(G+sk\tau I)\,\frac{\omega^s}{s!}\\
&\times\Big\{(\mu+1)^n\,\Gamma(\frac{H}{k}+\tau sI)\,\Gamma^{-1}(\frac{H}{k}+(\tau s-n)I)\,(\xi^{\mu+1}-\alpha^{\mu+1})^{\frac{H}{k}+(\tau s-n-1)I}\Big\}.
\end{aligned}$$

Making use of the relation given by (1.7), we arrive to

$$\begin{aligned}
\text{Ł} =&\frac{(\mu+1)^n}{k^n}\,(\xi^{\mu+1}-\alpha^{\mu+1})^{\frac{H}{k}-(n+1)I}\\
&\times\Gamma_k(G)\Gamma^{-1}(E)\sum_{s=0}^{\infty}(D,k;\rho)_{s,k}\,\Gamma_k(E+sk\tau I)\,\Gamma_k^{-1}(G+sk\tau I)\\
&\times\,\Gamma_k(H+\tau skI)\,\Gamma_k^{-1}(H+k(\tau s-n)I)\,\frac{\Big\{\omega(\xi^{\mu+1}-\alpha^{\mu+1})^\tau\Big\}^s}{s!},
\end{aligned}$$

which, in view of (2.1), leads to the right-hand side of (2.11) in Theorem 2.2. □

Remark 2.2. *If we take Remark 2.1 into account, then we can get several special cases of Theorems 2.1 and 2.2.*

3. Mellin transform

The Mellin transform of a suitable integrable function $\Psi(u)$ is defined, as usual, by

$$\mathcal{M}\Big\{\Psi(u) : u \to \varepsilon\Big\} = \int_0^\infty u^{\varepsilon-1}\, \Psi(u)\, du, \qquad \varepsilon \in \mathbb{R}^+, \tag{3.1}$$

provided that the improper integral in (3.1) exists.

The following lemma will be useful in the sequel.

Lemma 3.1. *For a matrix F in $\mathbb{C}^{m\times m}$, $\rho \in \mathbb{R}_0^+$ and $k, \varepsilon \in \mathbb{R}^+$, we have*

$$\mathcal{M}\Big\{\Gamma_k^\rho(F) : \rho \to \varepsilon\Big\} = \Gamma_k(\varepsilon I)\, \Gamma_k(F + \varepsilon I) \qquad (\widetilde{\mu}(F + \varepsilon I) > 0 \text{ when } k = 1), \tag{3.2}$$

where $\Gamma_k^\rho(F)$ is the extended k-gamma of a matrix argument defined in (1.9).

Proof. From (3.1), the Mellin transform of $\Gamma_k^\rho(F)$ in ρ is

$$\mathcal{M}\Big\{\Gamma_k^\rho(F) : \rho \to \varepsilon\Big\} = \int_0^\infty \rho^{\varepsilon-1} \int_0^\infty w^{F-I}\, e^{\left(-\frac{w^k}{k} - \frac{\rho^k}{kw^k}\right)}\, \mathrm{d}w\, \mathrm{d}\rho.$$

An application of the Fubini theorem [33], with few calculations, yields

$$\mathcal{M}\Big\{\Gamma_k^\rho(F) : \rho \to \varepsilon\Big\} = k^{\frac{\varepsilon}{k}-1}.\Gamma(\frac{\varepsilon}{k}) \int_0^\infty w^{F+(\varepsilon-1)I}\, e^{-\frac{w^k}{k}}\, \mathrm{d}w.$$

Upon using the relation given by (1.7), we can complete the proof of (3.2). □

Remark 3.1. *If $k = 1$ in (3.2), we have a matrix version of the result of Chaudhry and Zubair [33, p. 16, Eq. (1.110)] in the following form:*

$$\int_0^\infty \rho^{\varepsilon-1}\, \Gamma^\rho(F)\, d\rho = \Gamma(\varepsilon I)\, \Gamma(F + \varepsilon I), \qquad \widetilde{\mu}(F + \varepsilon I) > 0. \tag{3.3}$$

Theorem 3.1. *Under the conditions of the hypothesis in Definition 2.1, the Mellin transform of the set of extended (k,τ)-Wright hypergeometric matrix functions ${}_3\mathcal{W}_2^{(k,\tau)}(\xi)$, defined by (2.1), is given as*

$$\begin{aligned}
&\mathcal{M}\left\{ {}_3\mathcal{W}_2^{(k,\tau)} \left[\begin{array}{c} (D,k;\rho),(E,k),(F,k) \\ \\ (G,k),(H,k) \end{array} ;\xi \right] : \rho \to \varepsilon \right\} \\
=&\Gamma_k(\varepsilon)\, (D)_{\varepsilon,k}\; {}_3\mathcal{W}_2^{(k,\tau)} \left[\begin{array}{c} (D+\varepsilon I,k;\rho),(E,k),(F,k) \\ \\ (G,k),(H,k) \end{array} ;\xi \right],
\end{aligned} \tag{3.4}$$

where $\mathrm{Re}(\varepsilon) > 0$ *and* $\widetilde{\mu}(D + \varepsilon I) > 0$ *when* $\rho = 0$ *and* $k = 1$.

Proof. According to Definitions (2.1) and (3.1), we find that

$$
\begin{aligned}
&\mathcal{M}\left\{ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{matrix} (D,k;\rho),(E,k),(F,k) \\ \\ (G,k),(H,k) \end{matrix} ;\xi \right] : \rho \to \varepsilon \right\} \\
&= \int_0^{\infty} \rho^{\varepsilon-1} \left\{ \Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H) \times \sum_{s=0}^{\infty} (D;\rho)_{s,k}\, \Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI) \right. \\
&\quad \left. \times \Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\, \frac{\xi^s}{s!} \right\} d\rho \\
&= \Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H) \times \sum_{s=0}^{\infty} \Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI) \\
&\quad \times \Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\, \frac{\xi^s}{s!} \times \Gamma_k^{-1}(D) \int_0^{\infty} \rho^{\varepsilon-1}\, \Gamma_k^{\rho}(D+sI)\, d\rho.
\end{aligned}
$$

Applying Lemma 3.1, we arrive to

$$
\begin{aligned}
&\mathcal{M}\left\{ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{matrix} (D,k;\rho),(E,k),(F,k) \\ \\ (G,k),(H,k) \end{matrix} ;\xi \right] : \rho \to \varepsilon \right\} \\
&= \Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H) \times \sum_{s=0}^{\infty} \Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI) \\
&\quad \times \Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\, \frac{\xi^s}{s!} \times \Gamma_k^{-1}(D)\, \Gamma_k(\varepsilon I)\, \Gamma_k(D+(s+\varepsilon I) \\
&= \Gamma_k(\varepsilon I)\, (D)_{\varepsilon,k}\, \Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H) \times \sum_{s=0}^{\infty} (D+\varepsilon I;\rho)_{s,k}\, \Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI) \\
&\quad \times \Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\, \frac{\xi^s}{s!},
\end{aligned}
$$

which, upon expression in terms of (2.1), leads to the desired formula given by (3.4). □

Remark 3.2. *If we take the results (2.3)–(2.5) in Remark 2.1 into account, then we can obtain some special cases of Theorem 3.1. Further, the result proved in (3.4), which involves certain matrices in* $\mathbb{C}^{m\times m}$*, may reduce to the corresponding classical one when* $m = 1$ *and* $k = 1$ *(see, e.g., [12, 13]).*

4. Integral representations

In this section, we show certain integral representations for the extended (k,τ)-Wright hypergeometric matrix functions.

Theorem 4.1. *Let* $\xi, \omega \in \mathbb{C}$, $\mathrm{Re}(\omega) > 0$, $k, \tau \in \mathbb{R}^+$, $\rho \in \mathbb{R}_0^+$ *and* $|\xi v^\tau| < 1$. *Also, let D, E, F, G, H and*

$G-E$ be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $GE = EG$. Then

$$
\begin{aligned}
&{}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\xi\right]\\
=&\frac{1}{k}\Gamma_k(G)\Gamma_k^{-1}(E)\Gamma_k^{-1}(G-E)\int_0^1 v^{\frac{E}{k}-I}(1-v)^{\frac{G-E}{k}-I}\,{}_2\mathbf{R}_1^{(\tau)}\Big((D,k;\rho),(F,k);(H,k);\xi v^{\tau}\Big)dv.
\end{aligned}
\tag{4.1}
$$

Proof. Loading the following elementary identity involving the k-beta matrix function

$$
\begin{aligned}
(E)_{k,n\tau}[(G)_{k,n\tau}]^{-1} &= \Gamma_k^{-1}(E)\Gamma_k^{-1}(G+kn\tau I)\Gamma_k(G)\Gamma_k(E+kn\tau I)\\
&= \frac{1}{k}\Gamma_k^{-1}(E)\Gamma_k^{-1}(G-E)\Gamma_k(G)\int_0^1 v^{\frac{E}{k}+(n\tau-1)I}(1-v)^{\frac{G-E}{k}-I}dv,
\end{aligned}
\tag{4.2}
$$

in (2.1), and by using the series representation in (2.5), then we obtain the required integral representation given by (4.1). □

Theorem 4.2. *Let $\xi,\alpha\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$, $k,\tau\in\mathbb{R}^+$, $\rho\in\mathbb{R}_0^+$ and $|\alpha\xi|<1$. Let D, E, F, G, H, T and $G+T$ be positive stable matrices in $\mathbb{C}^{m\times m}$ such that $GE = EG$. Then, we have*

$$
\begin{aligned}
&\Gamma_k^{-1}(T)\Gamma_k^{-1}(G)\Gamma_k(G+T)\int_0^{\xi} u^{\frac{G}{k}-I}(\xi-u)^{\frac{T}{k}-I}\,{}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\alpha u\right]du\\
=&\xi^{\frac{G+T}{k}-I}\;{}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G+T,k),(H,k)\end{array};\alpha\xi\right].
\end{aligned}
\tag{4.3}
$$

Proof. Suppose that Υ is the left-hand side of (4.3). By invoking (2.1), we have

$$
\begin{aligned}
\Upsilon =&\Gamma_k^{-1}(T)\Gamma_k^{-1}(G)\Gamma_k(G+T)\int_0^{\xi} u^{\frac{G}{k}-I}(\xi-u)^{\frac{T}{k}-I}\\
&\times\Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H)\\
&\times\sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\\
&\times\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\,\frac{(u\alpha)^s}{s!}\,du.
\end{aligned}
$$

Substituting $u=\xi v$, we find that

$$
\begin{aligned}
\Upsilon =&\xi^{\frac{G+T}{k}-I}\,\Gamma_k^{-1}(E)\Gamma_k^{-1}(T)\Gamma_k(G+T)\,\Gamma_k^{-1}(F)\Gamma_k(H)\\
&\times\int_0^1 v^{\frac{G}{k}+(s-1)I}(1-v)^{\frac{T}{k}-I}dv\\
&\times\sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\\
&\times\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\,\frac{(\xi\alpha)^s}{s!}.
\end{aligned}
$$

Employing (1.12) and after simple computations, we obtain the right-hand side of (4.3). □

Remark 4.1. *From the special cases in Remark 2.1, we can obtain many special cases of (4.1) and (4.3).*

5. k-fractional calculus approach

In recent years, various studies on k-fractional calculus operators were archived by many researchers (see, for example, [34–37]). Here, $\mathbf{I}^{\mu}_{\alpha^+,k}$ is the k-Riemann-Liouville fractional integral operator and $\mathbf{D}^{\mu}_{\alpha^+,k}$ is the k-Riemann-Liouville fractional differential operator of order $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) > 0$, which are defined as (see [32, 36])

$$\Big(\mathbf{I}^{\mu}_{\alpha^+,k}\Phi\Big)(\xi) = \frac{1}{k\Gamma_k(\mu)}\int_{\alpha}^{\xi}\frac{\Phi(v)}{(\xi - v)^{1-\frac{\mu}{k}}}dv, \quad \mu \in \mathbb{C},\ \mathrm{Re}(\mu) > 0, \tag{5.1}$$

and

$$\Big(\mathbf{D}^{\mu}_{\alpha^+,k}\Phi\Big)(\xi) = \Big(\frac{d}{d\xi}\Big)^{n}\Big(k^{n}\mathbf{I}^{nk-\mu}_{\alpha^+,k}\Phi\Big)(\xi), \quad \mu \in \mathbb{C},\ \mathrm{Re}(\mu) > 0,\ n = [\mathrm{Re}(\mu)] + 1, \tag{5.2}$$

respectively.

The following lemma will be required in this section.

Lemma 5.1. *[32] Let E be a positive stable matrix in $\mathbb{C}^{N\times N}$. Then, the k-Riemann-Liouville fractional integrals of order μ, such that* $\mathrm{Re}(\mu) > 0$ *is given as*

$$\mathbf{I}^{\mu}_{\alpha^+,k}\Big[(\xi-\alpha)^{\frac{E}{k}-I}\Big](\xi) = \Gamma_k(E)\Gamma_k^{-1}(E+\mu I)\Big(\xi-\alpha\Big)^{\frac{E+\mu I}{k}-I},\ \ \xi > \alpha.$$

Theorem 5.1. *Assume that D, E, F, G and H are positive stable matrices in $\mathbb{C}^{m\times m}$ and $k, \tau \in \mathbb{R}^+$, $\rho \in \mathbb{R}^+_0$, $\alpha \in \mathbb{R}^+_0$ and $\mu, \omega \in \mathbb{C}$ such that* $\mathrm{Re}(\mu) > 0$. *Then, for $\xi > \alpha$ and $|(\xi-\alpha)^{\tau}| < 1$, we have k-Riemann-Liouville fractional integral and derivative representations of order μ of the extended (k,τ)-Wright hypergeometric matrix functions ${}_3\mathcal{W}_2^{(k,\tau)}(\xi)$ as follows:*

$$\begin{aligned}
&\mathbf{I}^{\mu}_{\alpha^+,k}\left[(v-\alpha)^{\frac{G}{k}-I}\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{matrix}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{matrix}\ ;\omega(v-\alpha)^{\tau}\right]\right]\\
=&\Big(\xi-\alpha\Big)^{\frac{G+\mu I}{k}-I}\Gamma_k(G)\Gamma^{-1}(G+\mu I)\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{matrix}(D,k;\rho),(E,k),(F,k)\\ \\ (G+\mu I,k),(H,k)\end{matrix}\ ;\omega(\xi-\alpha)^{\tau}\right]
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\mathbf{D}^{\mu}_{\alpha^+,k}\left[\Big(v-\alpha\Big)^{\frac{G}{k}-I}\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{matrix}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{matrix}\ ;\omega(v-\alpha)^{\tau}\right]\right]\\
=&\Gamma_k(G)\Big(\xi-\alpha\Big)^{\frac{G-\mu I}{k}-I}\Gamma_k^{-1}(G-\mu I)\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{matrix}(D,k;\rho),(E,k),(F,k)\\ \\ (G-\mu I,k),(H,k)\end{matrix}\ ;\omega(\xi-\alpha)^{\tau}\right].
\end{aligned} \tag{5.4}$$

Proof. By virtue of the formulas given by (5.1) and (2.1), and via application of Lemma 5.1, we obtain

$$\begin{aligned}
&\mathbf{I}^{\mu}_{\alpha^+,k}\left[(v-\alpha)^{\frac{G}{k}-I}\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\omega(v-\alpha)^{\tau}\right]\right](\xi)\\
=&\frac{1}{k\Gamma_k(\mu)}\int_{\alpha}^{\xi}\frac{(v-\alpha)^{\frac{G}{k}-I}}{(v-\alpha)^{1-\frac{\mu}{k}}}\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\omega(v-\alpha)^{\tau}\right]dv\\
=&\Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H)\times\sum_{s=0}^{\infty}(D;\rho)_{s,k}\ \Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\\
&\times\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\ \frac{\omega^s}{s!}\mathbf{I}^{\mu}_{\alpha^+,k}\left[(v-\alpha)^{\frac{G}{k}+\tau s-I}\right]\\
=&\left(\xi-\alpha\right)^{\frac{G+\mu I}{k}-I}\Gamma_k(G)\Gamma_k^{-1}(G+\mu I)\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G+\mu I,k),(H,k)\end{array};\omega(\xi-\alpha)^{\tau}\right].
\end{aligned}$$

Next, from (2.1) and (5.2), we have

$$\begin{aligned}
&\mathbf{D}^{\mu}_{\alpha^+,k}\left[\left(v-\alpha\right)^{\frac{G}{k}-I}\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\omega(v-\alpha)^{\tau}\right]\right]\\
=&\left(\frac{d}{d\xi}\right)^n\left\{k^n\mathbf{I}^{nk-\mu}_{\alpha^+,k}\left[(v-\alpha)^{\frac{G}{k}-I}\ {}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G,k),(H,k)\end{array};\omega(v-\alpha)^{\tau}\right]\right](\xi)\right\}\\
=&\left(\frac{d}{d\xi}\right)^n\left\{k^n\left(\xi-\alpha\right)^{\frac{G-\mu I}{k}+(n-1)I}\Gamma_k(G)\Gamma_k^{-1}(G+(nk-\mu)I)\right.\\
&\left.\times{}_3\mathcal{W}_2^{(k,\tau)}\left[\begin{array}{c}(D,k;\rho),(E,k),(F,k)\\ \\ (G+(nk-\mu)I,k),(H,k)\end{array};\omega(\xi-\alpha)^{\tau}\right]\right\}.
\end{aligned}$$

Upon using (2.10), we thus arrive to the desired result given by (5.4) in Theorem 5.1. □

Remark 5.1. *For $\rho=0$ and $F=H$ in Theorem 5.1, we get interesting results concerning the k-fractional calculus of the (k,τ)-Wright hypergeometric matrix function (cf. [32]).*

Remark 5.2. *For $k=1$, $\rho=0$ and $F=H$ in Theorem 5.1, we get interesting results concerning the fractional calculus of the Wright hypergeometric matrix function (see [29, 30]).*

6. Applications: Fractional kinetic equations

Recently, fractional kinetic equations have attracted the attention of many researchers due to their importance in diverse areas of applied science such as astrophysics, dynamical systems, control systems and mathematical physics. The kinetic equations of fractional order have been used to determine certain physical phenomena. Especially, the kinetic equations describe the continuity of the motion of substances. Therefore, a large number of articles in the solution of these equations have been published in the literature (see [38–43]).

The fractional kinetic equation

$$N(t) - N_0 = -C\,{}_0D_t^{-\nu}N(t), \quad C > 0,\, t > 0, \tag{6.1}$$

is the fractional version of the classical kinetic equation

$$N(t) - N_0 = -C\,{}_0D_t^{-1}N(t), \quad C > 0,\, t > 0, \tag{6.2}$$

or equivalently, the destruction-production time dependence equation derived in 2002 by Haubold and Mathai [38, 39]:

$$\frac{dN}{dt} = -\delta(N) + p(N),$$

where $N = N(t)$ is the rate of reaction, $\delta(Nt)$ is the rate of destruction, and $p = p(N)$ is the rate of production. In (6.1), ${}_0D_t^{-\nu}$ is the well-known Riemann-Liouville fractional integral operator, defined as

$${}_0D_t^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(s)ds, \ \mathrm{Re}(\nu) > 0.$$

${}_0D_t^{-1}$, in (6.2), is the classical integral operator with respect to t, and a special case of ${}_0D_t^{-\nu}$.

Theorem 6.1. *Let C be a positive stable and invertible matrix in $\mathbb{C}^{m\times m}$, and let the hypothesis assumed in Definition 2.1 still hold true . Then the solution to the generalized fractional kinetic matrix equation*

$$N(t)I - N_0\,{}_3W_2^{(k,\tau)}(t) = -C^\nu\,{}_0D_t^{-\nu}N(t), \tag{6.3}$$

is given as

$$\begin{aligned} N(t)I =& N_0\,\Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H)\\ &\times \sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\\ &\times\ t^s\mathbb{E}_{\nu,s+1}\left(-C^\nu t^\nu\right), \end{aligned} \tag{6.4}$$

where $\mathbb{E}_{\nu,s+1}(-C^\nu t^\nu)$ is the generalized Mittag-Leffler matrix function, defined as (cf. [44, 45])

$$\mathbb{E}_{\nu,s+1}\left(-C^\nu t^\nu\right) = \sum_{r=0}^{\infty}(-1)^r C^{\nu r}\frac{t^{\nu r}}{\Gamma(\nu r + s + 1)}. \tag{6.5}$$

Proof. First, recall that the Laplace transform of a Riemann-Liouville fractional integral is [46]

$$\mathcal{L}\left[{}_0D_t^{-\nu}f(t)\right](p) = p^{-\nu}\hat{f}(p),$$

where $\hat{f}(p)$ is the Laplace transform of $f(t)$. Applying the Laplace transform to (6.3) gives

$$\begin{aligned} &\left(I + p^{-\nu}C^\nu\right)\hat{N}(p) = N_0\mathcal{L}\left[{}_3W_2^{(k,\tau)}(t)\right](p)\\ =&N_0\,\Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H)\sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\\ &\times\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)p^{-(s+1)}. \end{aligned}$$

Hence,

$$\begin{aligned}\hat{N}(p)I =& N_0\,\Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H)\\ &\times\sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\\ &\times\sum_{r=0}^{\infty}(-1)^r C^{\nu r}p^{-(\nu r+s+1)}.\end{aligned}$$

Taking the inverse Laplace transform of the above result, and by using the fact that

$$\mathcal{L}^{-1}\left[p^{-\mu}\right]=\frac{t^{\mu-1}}{\Gamma(\mu)},\ \mathrm{Re}(\mu)>0,$$

we get

$$\begin{aligned}N(t)I =& N_0\,\Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H)\\ &\times\sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\\ &\times\sum_{r=0}^{\infty}(-1)^r C^{\nu r}\frac{t^{\nu r+s}}{\Gamma(\nu r+s+1)},\end{aligned}$$

which is the targeted result given by (6.4). □

Theorem 6.2. *Let C be a positive stable matrix in $\mathbb{C}^{m\times m}$, where $\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$, and let the hypothesis given in Definition 2.1 be satisfied. Then the generalized fractional kinetic matrix equation*

$$N(t)I - N_0\,{}_3W_2^{(k,\tau)}(\alpha^\nu t) = -C^\nu\,{}_0D_t^{-\nu}N(t) \tag{6.6}$$

is solvable, and its solution is

$$\begin{aligned}N(t)I =& N_0\,\Gamma_k^{-1}(E)\Gamma_k(G)\Gamma_k^{-1}(F)\Gamma_k(H)\\ &\times\sum_{s=0}^{\infty}(D;\rho)_{s,k}\,\Gamma_k^{-1}(G+k\tau sI)\Gamma_k(E+k\tau sI)\Gamma_k^{-1}(H+k\tau sI)\Gamma_k(F+k\tau sI)\\ &\times\alpha^{\nu s}t^s\mathbb{E}_{\nu,\nu s+1}\left(-C^\nu t^\nu\right),\end{aligned}$$

where $\mathbb{E}_{\nu,r}\left(-C^\nu t^\nu\right)$ is the generalized Mittag-Leffler matrix function defined in (6.5).

Upon using Remark 2.1, several special cases can be obtained from Theorems 6.1 and 6.2, such as the following corollaries.

Corollary 6.1. *Let C be a positive stable and invertible matrix in $\mathbb{C}^{m\times m}$ and $\boldsymbol{H}(D,F;G;t)$ be the hypergeometric matrix function defined by* (1.15)*; then the solution to the generalized fractional kinetic matrix equation*

$$N(t)I - N_0\,\boldsymbol{H}(D,F;G;t) = -C^\nu\,{}_0D_t^{-\nu}N(t), \tag{6.7}$$

is given as

$$N(t)I = N_0 \sum_{s=0}^{\infty} (D)_s (F)_s [(G)_s]^{-1} t^s \, \mathbb{E}_{\nu,s+1}(-C^\nu t^\nu), \tag{6.8}$$

where $\mathbb{E}_{\nu,s+1}(-C^\nu t^\nu)$ *is the generalized Mittag-Leffler matrix function defined by* (6.5).

Corollary 6.2. *Let C be a positive stable and invertible matrix in $\mathbb{C}^{m\times m}$, where $\alpha \in \mathbb{C}$ with* $\mathrm{Re}(\alpha) > 0$*, and $\boldsymbol{H}(D, F; G; t)$ be the hypergeometric matrix function defined by* (1.15)*. Then the solution to the generalized fractional kinetic matrix equation*

$$N(t)I - N_0 \boldsymbol{H}(D, F; G; \alpha^\nu t) = -C^\nu \, {}_0D_t^{-\nu} N(t), \tag{6.9}$$

is given as

$$N(t)I = N_0 \sum_{s=0}^{\infty} (D)_s (F)_s [(G)_s]^{-1} \alpha^{\nu s} t^s \mathbb{E}_{\nu,s+1}(-C^\nu t^\nu), \tag{6.10}$$

where $\mathbb{E}_{\nu,s+1}(-C^\nu t^\nu)$ *is the generalized Mittag-Leffler matrix function defined by* (6.5).

7. Conclusions

Motivated by recent researches [29, 30, 40–43] in the current work, we introduce an extension of the k-Wright ((k, τ)-Gauss) hypergeometric matrix function in Definition 2.1. Several properties which have been archived in the article include integral representations, the Mellin transform and the k-Riemann-Liouville fractional integral and derivative of the new extended (k, τ)-Gauss matrix function. Also, many specific cases are considered. As an application, we demonstrated the solvability of fractional kinetic matrix equations involving the new function. We also obtained many special cases for these fractional equations.

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Conflicts of interest

This work does not have any conflict of interest.

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