AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 14376–14393.
DOI: 10.3934/math.2022792
Received: 12 March 2022
Revised: 10 May 2022
Accepted: 12 May 2022
Published: 06 June 2022

Research article

Qualitative theory and approximate solution to a dynamical system under modified type Caputo-Fabrizio derivative

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Abstract: Qualitative theory, together with approximate solutions to a dynamic system, are investigated. The proposed mathematical model is composed of protected, susceptible, infected and treated classes. The adopted model expresses the mechanism of disease due to Typhoid fever. A modified type Caputo-Fabrizio fractional derivative (CFFD) is considered for the intended results. With the help of fixed point theory, some sufficient conditions for the existence of approximate solutions are developed. Also, to compute an approximate solution with respect to each compartment, we utilize the Laplace Transform and the Adomian decomposition method (ADM). A graphical presentation corresponding to some fundamental data is given.

Keywords: dynamical system; approximate solution; ADM; CFFD
Mathematics Subject Classification: 26A33, 35A02, 35A25

1. Introduction

Infectious diseases are those caused by viruses, bacteria, epiphytes and parasites, such as protozoans or worms [1]. They can easily spread in the human population and cause many infections in humans. Among the most common that spread easily in the human population is Typhoid. Typhoid is also called Typhoid fever. It is caused by the bacterium *Salmonella enterica* serovar Typhi, which usually affects humans. This disease is named due to its symptoms, resemblance to typhus. The common signs and symptoms of this disease include restlessness, headache, constipation or diarrhea, sustained fever, poor appetite and vomiting. It spreads through contaminated food, water or drink. Also, contaminated water or food containing this bacteria causes illness when entering the human body. After entering the blood through the lymph nodes, then the bacteria start to damage the gallbladder, spleen, liver, etc. Abdominal pain, fever and general ill feeling are the common symptoms of Typhoid fever [2]. The incubation period is about 10–14 days, but it can be as short as 3 days or as long as 21 days. This bacterium infiltrates the intestinal wall, and before entering the bloodstream, it multiplies in mononuclear phagocyte cells [3]. This is the reason for its presence in great numbers in the bloodstream. Treatment is based on patient serological testing, blood culture and stool culture. Oral amoxicillin and chloramphenicol may be used for the sensitive strain, whereas the persistent strain may be suppressed by oral therapy using ciprofloxacin. Worldwide, this disease affects millions of people each year. It has been reported that in Africa approximately 0.4 million cases occur every year. Further, the incidence is 50 cases per 100,000 every year [4]. Public awareness and behavior changes are integral to controlling typhoid fever disease, coupled with vaccination of high-risk populations [5]. Oral and injectable vaccines are two types of typhoid fever, vaccine, which may not be 100 percent effective. A serious and prolonged illness may be caused if someone contracts a drug-resistant strain of typhoid fever, which cannot be treated with effective antibiotics. We refer to some detailed studies about the said disease as [6–9].

In developing countries, typhoid fever is an ordinary disease. It is a major cause of illness and death, particularly among children, because of a lack of sanitation. From 2009–2015, the said disease has been transmitted increasingly in Ethiopia. A detailed report on the transmission of this disease can be read in [10]. In Figure 1, the disease cases per year have been presented graphically.

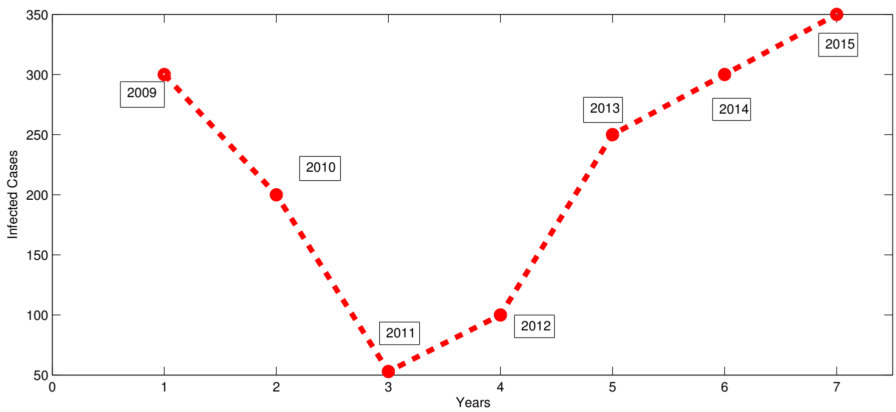

Figure 1. In previous seven years, reported cases in Ethiopia.

Recently, epidemiology has been advanced due to significant progress in medical science and technology. Researchers are increasingly working on investigating different diseases from various aspects, including the transmission mechanism of a disease, reasons and causes. They collect data and then develop various analyses to help the health department in controlling the disease in the community. In this regard, mathematical models play a significant role in describing the dynamics of infectious disease by predicting suitable control strategies and evaluating and ranking them by cost-effectiveness [11–15]. Much effective research based on the transmission dynamics of typhoid has been carried out in the last few decades (see [16]). For instance, some mathematical models for the dynamics of typhoid fever in Kassena–Nankana District of the Upper East Region of Ghana were developed to investigate the transmission of the disease in [17]. In the same line, mathematical models have been formulated to explain the mechanisms of transmission-related infectious diseases of typhoid in the last few years [18–21]. Generally, systems having ordinary differential equations are used to design these models. Also, these models are designed under realistic parameters and assumptions. In the mathematical model developed in [22], the population has been divided into age brackets to get more information on the effects of strategies and activities to control the disease in different age brackets. The data showed that, once the occurrence of the infection dropped down from the threshold level, then it is unlikely to be maintained among the age brackets in the community, as the chronic infections diminish naturally. In the mathematical model developed in [23], the effects of vaccination, both indirect and direct, have been studied. The randomized vaccine trail method was used to validate the model. The data on school-based vaccination strategies suggests that typhoid vaccination alone is not enough for the elimination of the disease. However, it is helpful in the prevention of cases in the short term and causes a decrease in typhoid cases. The transmission of the diseases can happen from both long-term and short-term carriers. However, it is not evident from the data that either type of carrier contributes at the same rate as symptomatic carriers. In [24], mathematical model has been designed for protection through a vaccination program and its benefits. In the mentioned study, the population was divided into controlled (vaccinated) and uncontrolled (unvaccinated) groups. The data analysis showed that vaccination decreases infection and transmission among the entire population. A mathematical model was developed to study the impact of control strategies to effectively control the disease in Kisii town [25]. This model studied the effects of carriers on the dynamics of typhoid and health education on typhoid in Kenya. The model considers that endogenous reactivation and exogenous re-infection are the main reasons behind typhoid fever in exposed individuals. Treatment was given to all infected individual, including exposed individuals. A structure for the kind of individual contacts that may cause the infection's transmission was integrated into the population. The obtained numerical results showed that the Kisii local government in Kenya can achieve a typhoid-free status by 2030 if it can reduce typhoid carriers by 9.5 percent. Using classical differential equations, a model was formulated in [26] as

$$
\begin{aligned}
&\dot{\mathcal{P}}(\mathrm{t}) = a\wedge-(\gamma+\mu)\mathcal{P} = \varphi_1(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&\dot{\mathcal{S}}(\mathrm{t}) = (1-a)\wedge+\gamma\mathcal{P}-(\lambda+\mu)\mathcal{S} = \varphi_2(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&\dot{\mathcal{I}}(\mathrm{t}) = \lambda\mathcal{S}-(\delta+\beta+\mu)\mathcal{I} = \varphi_3(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&\dot{\mathcal{T}}(\mathrm{t}) = \beta\mathcal{I}-\mu\mathcal{T} = \varphi_4(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&\mathcal{P}(0)=\mathcal{P}_0,\ \mathcal{S}(0)=\mathcal{S}_0,\ \mathcal{I}(0)=\mathcal{I}_0,\ \mathcal{T}(0)=\mathcal{T}_0,
\end{aligned}
\tag{1.1}
$$

where $\mathcal{P}_0,\mathcal{S}_0,\mathcal{I}_0,\mathcal{T}_0 \geq 0$. Further, the authors in [26] have divided the entire population into different compartments.

- Those individuals who are protected (vaccinated) are included in the protected class $\mathcal{P}$.

- Those individuals who have a high chance of getting an infection of typhoid fever are included in the susceptible class $\mathcal{S}$.
- Those individuals showing the symptoms of typhoid fever are included in the infected class $\mathcal{I}$.
- The treated individuals are included in the treated class $\mathcal{T}$.

Susceptible individuals are recruited into the population at the rate of $(1 - a)\wedge$. Susceptible individuals acquire typhoid infection at a rate of λ. Also, $a\wedge$ and $(1 - a)\wedge$ are the recruitment rates into the class of individuals protected against typhoid and the class of individuals susceptible to typhoid, respectively, μ is the natural mortality rate, δ is the mortality rate due to disease, and β is the rate at which the disease is treated. In this model, it is assumed that the individual is not re-infected once treated. Here, $\mathcal{N}$ denotes the size of the entire population at any time t and is taken as $\mathcal{N} = \mathcal{P} + \mathcal{S} + \mathcal{I} + \mathcal{T}$.

Mathematical models have been well investigated by using ordinary differential equations. Nevertheless, due to the rapid development in the field of fractional calculus, various researchers have extended the area of mathematical modeling with the concepts of fractional calculus in the last few years. Fractional calculus has recently been an attractive branch of research due to its wide range of applications in modeling real-world phenomena more comprehensively. For some fundamental concepts and valuable work, refer to [27, 28]. Researchers have investigated fractional differential equations from various aspects, such as existence theory, stability and numerical analysis. They have established various techniques, methods and theories for calculating exact or numerical solutions to fractional order problems as it is quite difficult to solve every fractional order differential equation for its exact or analytical solution. Because fractional order derivatives have been defined in various ways, there is no unique definition. Also, fractional order operators are complex. Therefore, in many cases for complex problems, it is very difficult or even impossible to find an exact solution. For instance, the authors of [29] used the local fractional homotopy perturbation method for solving fractional partial differential equations arising in mathematical physics. In the same line, the authors of [30] used the Laplace Adomian decomposition method (LADM) for the semi-analytical solution of fractional partial differential equations. ADM has been discussed in detail for fractional differential equations in [31]. Also, LADM and ADM have been used for various problems of fractional calculus in [32–34]. Using numerical methods, some problems of fractional order differential equations have been studied in [35] and [36]. The authors of [37] studied a fractional cantilever beam model in the q-difference inclusion settings via special multi-valued operators for numerical and theoretical results. Similarly, the authors of [38] studied some fractional multi-term sequential problems via some special categories of functions.

Epidemiological models have been investigated very well under the Caputo fractional order derivative. Also, various researchers have shown proved that fractional-order models describe real-world phenomena in a more accurate, systematic and precise way than the classic integer-order counterparts with ordinary time-derivatives. Although these studies have provided significant results as compared to classic integer order models, a satisfactory precision may not be obtained in the whole time duration. This is due to the appearance of a singularity in the definition of usual fractional order derivatives. This appearance of singular kernels sometimes causes difficulties in the numerical analysis of the mentioned area. Therefore, the said reason makes those operators impractical for the description of nonlocal dynamics in various problems (for details see [39]). Therefore, to overcome said difficulties, in 2015 and 2016 two new nonsingular fractional derivatives were proposed. The first one is the Caputo-Fabrizio fractional derivative (CFFD) [40], and the second one is the Atangana-Baeanu-Caputo fractional derivative [41]. Some authors developed various results, including inequalities satisfying different properties and comparison principles, in [42, 43]. The advantages of the mentioned operators have been discussed in detail by various authors. The CFFD has been extensively used in mathematical models of various infectious diseases, as well as in other real-world problems (see [44–47]). Recent good results where various advantages of the CFFD are discussed in [48–51]. For dealing with various problems with the CFFD for analytical or numerical results, various tools have been used in the literature. One of the important tools increasingly used for traditional problems under the Caputo power-law derivative is LADM. The mentioned method has been used to deal with various problems with the CFFD (see [52–56]).

2. Formulation of proposed model

Motivated by the above work, the proposed model has not been investigated for existence, uniqueness and semi-analytical results with the CFFD. Therefore, to fill this gap, in this paper we handle the model (1.1) under CFFD for existence theory and semi-analytical results as

$$\begin{aligned}
&{}^{CF}D_{\mathrm{t}}^{\omega}\mathcal{P}(\mathrm{t}) = a\wedge -(\gamma+\mu)\mathcal{P} = \varphi_1(t,\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&{}^{CF}D_{\mathrm{t}}^{\omega}\mathcal{S}(\mathrm{t}) = (1-a)\wedge +\gamma\mathcal{P} - (\lambda+\mu)\mathcal{S} = \varphi_2(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&{}^{CF}D_{\mathrm{t}}^{\omega}\mathcal{I}(\mathrm{t}) = \lambda\mathcal{S} - (\delta+\beta+\mu)\mathcal{I} = \varphi_3(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&{}^{CF}D_{\mathrm{t}}^{\omega}\mathcal{T}(\mathrm{t}) = \beta\mathcal{I} - \mu\mathcal{T} = \varphi_4(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}),\\
&\mathcal{P}(0)=\mathcal{P}_0,\ \mathcal{S}(0)=\mathcal{S}_0,\ \mathcal{I}(0)=\mathcal{I}_0,\ \mathcal{T}(0)=\mathcal{T}_0,
\end{aligned} \tag{2.1}$$

where $\mathcal{P}_0,\ \mathcal{S}_0,\ \mathcal{I}_0,\ \mathcal{T}_0 \geq 0$, and $\omega \in (0,1]$. Here, λ is a force infection and has been defined in [26] as $\lambda = \frac{\pi\theta\mathcal{I}}{N}$, where π is defined as the probability of getting typhoid fever, and θ is the contact rate of infection. For our simplicity, we write it as $\lambda = \bar{\lambda}\mathcal{I},$ where $\bar{\lambda} = \frac{\pi\theta}{N}$, and N is a constant population such that $N = \mathcal{P}+\mathcal{S}+\mathcal{I}+\mathcal{T}$.

The right side of the system (2.1) vanishes at $t = 0$, to receive the initialization conditions [42]. The operator ${}^{CF}D_{\mathrm{t}}^{\omega}$ stands for the modified type Caputo-Fabrizio derivative, adjusted by Abdeljawad [42]. Since the original CFFD operator [40] has been modified by Abdeljawad, converges to the classical differential operator if $\omega \to 1$. Conversely, the ordinary CFFD does not have the aforementioned property. For a description of the above model, a flow chart is given in Figure 2. First of all, some qualitative results, such as the existence and uniqueness of the solution corresponding to the considered model, are established. It is necessary that a dynamic system, which we are investigating, should be tested for the existence of a solution. For this purpose, various methods have been used. Fixed point theory is one of the most powerful recent tools for the investigation of the aforesaid analysis. Therefore, to obtain these results, fixed point theorems of Krassnoselskii [57] and Banach are used to develop sufficient conditions for the existence and uniqueness of solutions to the proposed model. Further, we compute some semi-analytical results for approximate solution to various compartments of the model. Then, Laplace transforms together with the ADM are used to investigate some approximate analytical results. In the end, by using Matlab, graphical representations of approximate results are provided.

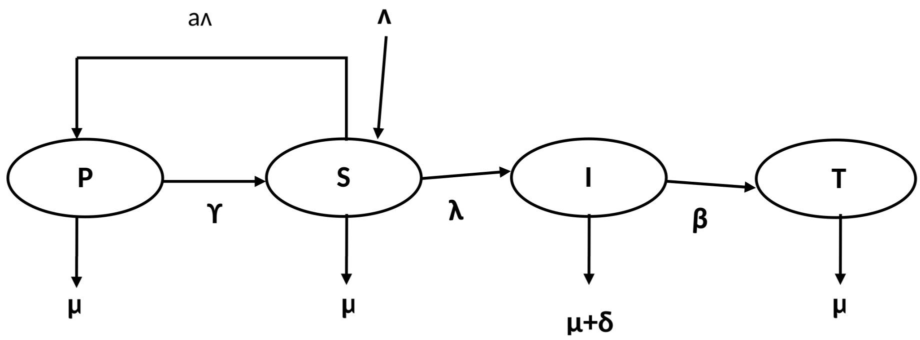

Figure 2. A flow chart of the proposed model.

3. Preliminaries

To obtain all these results, we need some basic definitions and results of the modified model, which are given as follows.

Definition 1. [42] Let $v \in \mathcal{H}^1(0,\phi)$, $\phi > 0$, $\omega \in (0,1)$; then, the CFFD is recalled as

$$^{CF}\mathcal{D}_{\mathrm{t}}^{\omega}(v(\mathrm{t})) = \frac{\mathscr{M}(\omega)}{1-\omega}\int_0^{\mathrm{t}} v'(\mathrm{t})\exp\Big[-\omega\frac{\mathrm{t}-\xi}{1-\omega}\Big]d\xi.$$

However, if v does not belong to $\mathcal{H}^1(0,\phi)$, then the derivative is given by

$$^{CF}\mathcal{D}_{\mathrm{t}}^{\omega}(v(t)) = \frac{\mathscr{M}(\omega)}{1-\omega}\int_0^{\mathrm{t}} (v(\mathrm{t})-v(\xi))\exp\Big[-\omega\frac{F-\xi}{1-\omega}\Big]d\xi.$$

Definition 2. [42] For $v \in \mathcal{H}^1(0,\phi)$, $\phi > 0$, the CF integral is given as

$$^{CF}\mathcal{I}_{\mathrm{t}}^{\omega}[v(\mathrm{t})] = \frac{(1-\omega)}{\mathscr{M}(\omega)}v(\mathrm{t}) + \frac{\omega}{\mathscr{M}(\omega)}\int_0^{\mathrm{t}} v(\xi)d\xi,\ \omega \in (0,1].$$

Definition 3. [44] The LT of $^{CF}\mathcal{D}_{\mathrm{t}}^{\omega}v(t)$ with $\mathscr{M}(\omega) = 1$ is given as

$$\mathscr{L}[^{CF}\mathcal{D}_{\mathrm{t}}^{\omega}v(\mathrm{t})] = \frac{s\mathscr{L}[v(\mathrm{t})] - v(0)}{s+\omega(1-s)}, s \geq 0,\ \omega \in (0,1].$$

Note: Corresponding to existence theory, let $\mathcal{J} = [0,\phi]$ and $0 \leq \mathrm{t} \leq \phi < \infty$ and we define space as $\mathbf{A} = B([0,\phi]\times\mathcal{R}^2,\mathcal{R})$, with norm given as $\|(x,y)\| = \sup_{t\in\mathcal{J}}[|x(\mathrm{t})| + |y(\mathrm{t})|]$.

Theorem 1. *[57] Let $\mathbb{X}$ be a convex subset of $\mathbf{Y}$, and we have two operators P_1, P_2 with the following properties:*

1) $P_1U + P_2U \in \mathbb{X}$ for every $U \in \mathbb{X}$;
2) P_1 is a contraction;
3) P_2 is continuous and compact.

Then, at least one solution for the operator equation $P_1U + P_2U = U$ exists.

4. Main result for proposed fractional order model

In this portion, the first part of our main results is discussed. Some results about the existence and uniqueness of the solution for the proposed model (2.1) are derived here.

Applying integral operator ${}^{CF}\mathcal{I}_{\mathrm{t}}^{\omega}$ on both sides of (2.1) and including initial conditions, one gets

$$\begin{cases}\mathcal{P}(\mathrm{t}) = \mathcal{P}(0) + \frac{(1-\omega)}{\mathscr{M}(\omega)}\varphi_1(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}) + \frac{\omega}{\mathscr{M}(\omega)\Gamma(\omega)}\int_0^{\mathrm{t}}(\mathrm{t}-\xi)^{\omega-1}\varphi_1(\xi,\mathcal{P}(\xi),\mathcal{S}(\xi),\mathcal{I}(\xi),\mathcal{T}(\xi))d\xi,\\ \mathcal{S}(\mathrm{t}) = \mathcal{S}(0) + \frac{(1-\omega)}{\mathscr{M}(\omega)}\varphi_2(t,\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}) + \frac{\omega}{\mathscr{M}(\omega)\Gamma(\omega)}\int_0^{\mathrm{t}}(\mathrm{t}-\xi)^{\omega-1}\varphi_2(\xi,\mathcal{P}(\xi),\mathcal{S}(\xi),\mathcal{I}(\xi),\mathcal{T}(\xi))d\xi,\\ \mathcal{I}(\mathrm{t}) = \mathcal{I}(0) + \frac{(1-\omega)}{\mathscr{M}(\omega)}\varphi_3(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}) + \frac{\omega}{\mathscr{M}(\omega)\Gamma(\omega)}\int_0^{\mathrm{t}}(t-\xi)^{\omega-1}\varphi_3(\xi,\mathcal{P}(\xi),\mathcal{S}(\xi),\mathcal{I}(\xi),\mathcal{T}(\xi))d\xi,\\ \mathcal{T}(\mathrm{t}) = \mathcal{T}(0) + \frac{(1-\omega)}{\mathscr{M}(\omega)}\varphi_4(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}) + \frac{\omega}{\mathscr{M}(\omega)\Gamma(\omega)}\int_0^{\mathrm{t}}(\mathrm{t}-\xi)^{\omega-1}\varphi_4(\xi,\mathcal{P}(\xi),\mathcal{S}(\xi),\mathcal{I}(\xi),\mathcal{T}(\xi))d\xi.\end{cases} \tag{4.1}$$

This further can be simplified as

$$\Theta(\mathrm{t}) = \Theta_0 + \mho(\mathrm{t},\Theta(\mathrm{t}))\frac{(1-\omega)}{\mathscr{M}(\omega)} + \frac{\omega}{\mathscr{M}(\omega)}\int_0^{\mathrm{t}}\Psi(\xi,\Theta(\xi))d\xi, \tag{4.2}$$

where

$$\Theta(\mathrm{t}) = \begin{cases}\mathcal{P}(\mathrm{t})\\ \mathcal{S}(\mathrm{t})\\ \mathcal{I}(\mathrm{t})\\ \mathcal{T}(\mathrm{t})\end{cases}, \quad \Theta_0 = \begin{cases}\mathcal{P}_0\\ \mathcal{S}_0\\ \mathcal{I}_0\\ \mathcal{T}_0\end{cases}, \quad \mho(\mathrm{t},\Theta(\mathrm{t})) = \begin{cases}\varphi_1(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T})\\ \varphi_2(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T})\\ \varphi_3(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T})\\ \varphi_4(\mathrm{t},\mathcal{P},\mathcal{S},\mathcal{I},\mathcal{T}).\end{cases} \tag{4.3}$$

The following assumptions are taken into account for further analysis.

(D1) For constants $\mathbf{L}_{\mho} > 0$ with $\Theta, \bar{\Theta} \in \mathbf{A}$, one has

$$|\mho(\mathrm{t},\Theta(\mathrm{t})) - \mho(\mathrm{t},\bar{\Theta}(\mathrm{t}))| \leq \mathbf{L}_{\mho}[|\Theta - \bar{\Theta}|].$$

(D2) For fixed real values $C_{\mho}, C_{\mho} > 0$ and $\mathbf{M}_{\mho} > 0$, one has

$$|\mho(\mathrm{t},\Theta(\mathrm{t}))| \leq C_{\mho}|\Theta| + M_{\mho}.$$

Using (4.2) and (4.3), the two operators are defined as follows:

$$\begin{aligned} P_1(\Theta) &= \Theta_0(\mathrm{t}) + \mho(\mathrm{t},\Theta(\mathrm{t}))\frac{(1-\omega)}{\mathscr{M}(\omega)},\\ P_2(\Theta) &= \frac{\omega}{\mathscr{M}(\omega)}\int_0^{\mathrm{t}}\mho(\xi,\Theta(\xi))d\xi. \end{aligned} \tag{4.4}$$

Theorem 2. *Using assumptions* $(D1)$ *and* $(D2)$*, the integral system* (4.2) *has at least one solution with the restriction* $\frac{\mathbf{L}_{\mho}}{\mathscr{M}(\omega)} < 1$.

Proof. Let us consider a closed as well as convex subset of $\mathbf{A}$ as $\mathbb{X} = \{\Theta \in \mathbf{A} : \ \|\Theta\| \leq \rho, \ \rho > 0\}$. It is required to prove that $P_1 : \mathbb{X} \to \mathbb{X}$ is a contraction. As given that $\Theta, \bar{\Theta} \in \mathbb{B}$, and we have

$$\|P_1\Theta - P_1\bar{\Theta}\| = \sup_{\mathrm{t}\in\mathcal{J}}\left|\Big(\mho(\mathrm{t},\Theta(\mathrm{t})) - (\mho(\mathrm{t},\bar{\Theta}(\mathrm{t}))\Big)\frac{(1-\omega)}{\mathscr{M}(\omega)}\right|$$

$$
\begin{aligned}
&\leq \frac{(1-\omega)}{\mathscr{M}(\omega)} \mathbf{L}_{\mho} \sup_{t \in \mathcal{J}} |\Theta(t) - \bar{\Theta}(t)| \\
&\leq \frac{\mathbf{L}_{\mho}}{\mathscr{M}(\omega)} \|\Theta - \bar{\Theta}\|.
\end{aligned}
$$

Hence, P_1 is a contraction. Now, to show P_2 is a compact and continuous operator, given that $\Theta \in \mathbb{X}$, we have

$$
\begin{aligned}
\|P_2(\Theta)\| &= \sup_{t \in \mathcal{J}} \left| \frac{\omega}{\mathscr{M}(\omega)} \int_0^t \mho(\xi, \Theta(\xi)) d\xi \right| \\
&\leq \frac{\phi}{\mathscr{M}(\omega)} [C_{\mho}\rho + \mathbf{M}_{\mho}] := \Delta. \qquad (4.5)
\end{aligned}
$$

From (4.5), we conclude that P_2 is bounded. Because of the continuity of $\mho$, P_2 is also continuous. In the same manner, it can be proved that P_2 is equi-continuous by taking $t_1 < t_2 \in \mathcal{J}$. Therefore, in view of the Theorem 1, the problem 2.1 has at least one solution. □

Theorem 3. *By hypothesis* $(D1)$*, the system* (4.2) *has a unique solution under the condition* $\frac{(1+\phi)\mathbf{L}}{\mathscr{M}(\omega)} < 1$.

Proof. Let us define $\mathbf{P} : \mathbf{A} \to \mathbf{A}$ by

$$
\mathbf{P}(\Theta) = \Theta_0 + \mho(t, \Theta(t)) \frac{(1-\omega)}{\mathscr{M}(\omega)} + \frac{\omega}{\mathscr{M}(\omega)} \int_0^t \mho(\xi, \Theta(\xi)) d\xi.
$$

Let $\Theta, \bar{\Theta} \in \mathbf{A}$, and we have

$$
\begin{aligned}
\|\mathbf{P}(\Theta) - \mathbf{P}(\bar{\Theta})\| &\leq \sup_{t \in \mathcal{J}} \frac{(1-\omega)}{\mathscr{M}(\omega)} \left| \mho(t, \Theta(t)) - \mho(t, \bar{\Theta}(t)) \right| \\
&+ \frac{\omega}{\mathscr{M}(\omega)} \sup_{t \in \mathcal{J}} \int_0^t |\mho(\xi, \Theta(\xi)) - \mho(\xi, \bar{\Theta}(\xi))| d\xi \\
&\leq \frac{(1+\phi)\mathbf{L}}{\mathscr{M}(\omega)} \|\Theta - \bar{\Theta}\|. \qquad (4.6)
\end{aligned}
$$

This shows that $\mathbf{P}$ is a contraction, so the concerned problem (4.2) has a unique solution. Therefore, the proposed model (2.1) has a unique solution. □

5. Algorithm for approximate solution of fractional order model (2.1)

To compute the algorithm for the required approximate solution of the proposed model, we use the Laplace transform on both sides of the system (2.1) and for simplicity take $\mathscr{M}(\omega) = 1$ and $\Xi(\omega, s) = \frac{s+\omega(1-s)}{s}$. Then, we have

$$
\begin{cases}
\mathscr{L}[\mathcal{P}(t)] = \dfrac{\mathcal{P}(0)}{s} + \Xi(\omega, s)\mathscr{L}[a \wedge -(\gamma + \mu)\mathcal{P}], \\
\mathscr{L}[\mathcal{S}(t)] = \dfrac{\mathcal{S}(0)}{s} + \Xi(\omega, s)\mathscr{L}[(1-a) \wedge +\gamma\mathcal{P} - (\bar{\lambda}\mathcal{I} + \mu)\mathcal{S}], \\
\mathscr{L}[\mathcal{I}(t)] = \dfrac{\mathcal{I}(0)}{s} + \Xi(\omega, s)\mathscr{L}[\bar{\lambda}\mathcal{I}\mathcal{S} - (\delta + \beta + \mu)\mathcal{I}], \\
\mathscr{L}[\mathcal{T}(t)] = \dfrac{\mathcal{T}(0)}{s} + \Xi(\omega, s)\mathscr{L}[\beta\mathcal{I} - \mu\mathcal{T}].
\end{cases} \qquad (5.1)
$$

The needed solution can be obtained in as

$$\begin{aligned}\mathcal{P}(\mathrm{t}) &= \sum_{q=0}^{\infty}\mathcal{P}_q(\mathrm{t}),\ \mathcal{S}(\mathrm{t}) = \sum_{q=0}^{\infty}\mathcal{S}_q(\mathrm{t}),\\ \mathcal{I}(\mathrm{t}) &= \sum_{q=0}^{\infty}\mathcal{I}_q(\mathrm{t}),\ \mathcal{T}(\mathrm{t}) = \sum_{q=0}^{\infty}\mathcal{T}_q(\mathrm{t}),\end{aligned} \tag{5.2}$$

where the nonlinear term can be decomposed as $\mathcal{I}\mathcal{S} = \sum_{q=0}^{\infty}\mathbf{A}_q$ such that

$$\mathbf{A}_q = \frac{1}{q!}\frac{d^q}{d\xi^q}\Big[\sum_{k=0}^{q}\xi^k\mathcal{S}_q\sum_{k=0}^{q}\xi^k\mathcal{I}_q\Big]\Big|_{\xi=0}.$$

A few Adomian polynomials are computed as

$$\mathbf{A}_0 = \mathcal{S}_0\mathcal{I}_0,\ \mathbf{A}_1 = \mathcal{S}_0\mathcal{I}_1 + \mathcal{S}_1\mathcal{I}_0,\ \text{etc.}$$

Hence, using (5.2), the system (5.1) becomes

$$\begin{cases}\mathscr{L}\Big[\sum_{q=0}^{\infty}\mathcal{P}_q(\mathrm{t})\Big] = \dfrac{\mathcal{P}(0)}{s} + \Xi(\omega,s)\mathscr{L}\Big[a\wedge-(\gamma+\mu)\sum_{q=0}^{\infty}\mathcal{P}_q(\mathrm{t})\Big],\\ \mathscr{L}\Big[\sum_{q=0}^{\infty}\mathcal{S}_q(\mathrm{t})\Big] = \dfrac{\mathcal{S}(0)}{s} + \Xi(\omega,s)\mathscr{L}\Big[(1-a)\wedge+\gamma\sum_{q=0}^{\infty}\mathcal{P}_q(\mathrm{t}) - \bar{\lambda}\sum_{q=0}^{\infty}\mathbf{A}_q(\mathrm{t}) - \mu\sum_{q=0}^{\infty}\mathcal{S}_q(\mathrm{t})\Big],\\ \mathscr{L}\Big[\sum_{q=0}^{\infty}\mathcal{I}_q(\mathrm{t})\Big] = \dfrac{\mathcal{I}(0)}{s} + \Xi(\omega,s)\mathscr{L}\Big[\bar{\lambda}\sum_{q=0}^{\infty}\mathbf{A}_q(\mathrm{t}) - (\delta+\beta+\mu)\sum_{q=0}^{\infty}\mathcal{I}_q(\mathrm{t})\Big],\\ \mathscr{L}\Big[\sum_{q=0}^{\infty}\mathcal{T}_q(\mathrm{t})\Big] = \dfrac{\mathcal{T}(0)}{s} + \Xi(\omega,s)\mathscr{L}\Big[\beta\sum_{q=0}^{\infty}\mathcal{I}_q(\mathrm{t}) - \mu\sum_{q=0}^{\infty}\mathcal{T}_q(\mathrm{t})\Big].\end{cases} \tag{5.3}$$

From (5.3), we equate terms as

$$\begin{cases}\mathscr{L}[\mathcal{P}_0(\mathrm{t})] = \dfrac{\mathcal{P}_0}{s} + \Xi(\omega,s)\mathscr{L}(a\wedge),\\ \mathscr{L}[\mathcal{S}_0(\mathrm{t})] = \dfrac{\mathcal{S}_0}{s} + \Xi(\omega,s)\mathscr{L}\Big((1-a)\wedge\Big),\\ \mathscr{L}[\mathcal{I}_0(\mathrm{t})] = \dfrac{I_0}{s},\ \mathscr{L}[\mathcal{T}_0(\mathrm{t})] = \dfrac{\mathcal{T}_0}{s},\end{cases} \tag{5.4}$$

$$\begin{cases}\mathscr{L}[\mathcal{P}_1(\mathrm{t})] = \Xi(\omega,s)\mathscr{L}\Big(-(\gamma+\mu)\mathcal{P}_0\Big),\\ \mathscr{L}[\mathcal{S}_1(\mathrm{t})] = \Xi(\omega,s)\mathscr{L}\Big(\gamma\mathcal{P}_0 - \bar{\lambda}\mathcal{S}_0\mathcal{I}_0 - \mu\mathcal{S}_0\Big),\\ \mathscr{L}[\mathcal{I}_1(\mathrm{t})] = \Xi(\omega,s)\mathscr{L}\Big(\bar{\lambda}\mathcal{S}_0\mathcal{I}_0 - (\delta+\beta+\mu)\mathcal{I}_0\Big),\\ \mathscr{L}[\mathcal{T}_1(\mathrm{t})] = \Xi(\omega,s)\mathscr{L}\Big(\beta\mathcal{I}_0 - \mu\mathcal{T}_0\Big),\end{cases} \tag{5.5}$$

$$\begin{cases}\mathscr{L}[\mathcal{P}_2(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(-(\gamma+\mu)\mathcal{P}_1\Big),\\ \mathscr{L}[\mathcal{S}_2(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(\gamma\mathcal{P}_1 - \bar{\lambda}[\mathcal{S}_1\mathcal{I}_0 + \mathcal{I}_1\mathcal{S}_0] - \mu\mathcal{S}_1\Big),\\ \mathscr{L}[\mathcal{I}_2(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(\bar{\lambda}[\mathcal{S}_1\mathcal{I}_0 + \mathcal{I}_1\mathcal{S}_0] - (\delta+\beta+\mu)\mathcal{I}_1\Big),\\ \mathscr{L}[\mathcal{T}_2(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(\beta\mathcal{I}_1 - \mu\mathcal{T}_1\Big),\\ \vdots \end{cases} \tag{5.6}$$

$$\begin{cases}\mathscr{L}[\mathcal{P}_{q+1}(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(-(\gamma+\mu)\mathcal{P}_q\Big),\\ \mathscr{L}[\mathcal{S}_{q+1}(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(\gamma\mathcal{P}_q - \bar{\lambda}\mathbf{A}_q - \mu\mathcal{S}_q\Big),\\ \mathscr{L}[\mathcal{I}_{q+1}(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(\bar{\lambda}\mathbf{A}_q - (\delta+\beta+\mu)\mathcal{I}_q\Big),\\ \mathscr{L}[\mathcal{T}_{q+1}(\mathrm{t})] = \Xi(\omega, s)\mathscr{L}\Big(\beta\mathcal{I}_q - \mu\mathcal{T}_q\Big),\ q \geq 0. \end{cases} \tag{5.7}$$

After simplifying, we get

$$\begin{cases}\mathcal{P}_0(\mathrm{t}) = \mathcal{P}_0 + \Big(1 + (\mathrm{t}-1)\omega\Big)a\wedge,\\ \mathcal{S}_0(\mathrm{t}) = \mathcal{S}_0 + \Big(1 + (\mathrm{t}-1)\omega\Big)(1-a)\wedge,\\ \mathcal{I}_0(\mathrm{t}) = \mathcal{I}_0,\ \mathcal{T}_0(\mathrm{t}) = \mathcal{T}_0, \end{cases} \tag{5.8}$$

$$\begin{cases}\mathcal{P}_1(\mathrm{t}) = -(\gamma+\mu)\Big(\mathcal{P}_0(1 + (\mathrm{t}-1)\omega) + a\wedge(1 + 2(\mathrm{t}-1)\omega) + a\wedge(1 - 2\mathrm{t} + \frac{\mathrm{t}^2}{2})\omega^2\Big),\\ \mathcal{S}_1(\mathrm{t}) = \Big(\gamma\mathcal{P}_0 - \bar{\lambda}\mathcal{S}_0\mathcal{I}_0 - \mu\mathcal{S}_0\Big)(1 + (\mathrm{t}-1)\omega) + \Big(a\wedge + (1-a)\wedge\Big)(1 + 2(\mathrm{t}-1)\omega)\\ +\Big(a\wedge + (1-a)\wedge\Big)(1 - 2\mathrm{t} + \frac{\mathrm{t}^2}{2!})\omega^2),\\ \mathcal{I}_1(\mathrm{t}) = \bar{\lambda}\Big(\mathcal{S}_0\mathcal{I}_0(1 + (\mathrm{t}-1)\omega) + (1-a)a\wedge(1 + 2(\mathrm{t}-1)\omega) + (1-a)\wedge(1 - 2\mathrm{t} + \frac{\mathrm{t}^2}{2!})\omega^2\Big)\\ -(\delta+\beta+\mu)\Big(\mathcal{I}_0(1 + (\mathrm{t}-1)\omega)\Big),\\ \mathcal{T}_1(\mathrm{t}) = (\beta\mathcal{I}_0 - \mu\mathcal{T}_0)(1 + (\mathrm{t}-1)\omega), \end{cases} \tag{5.9}$$

$$\begin{cases}\mathcal{P}_2(\mathrm{t}) = (\gamma+\mu)^2\Big(P_0\Big(1+2(\mathrm{t}-1)\omega+(1-2\mathrm{t}+\frac{\mathrm{t}^2}{2!})\omega^2\Big)\\ +a\wedge\Big(1+3(\mathrm{t}-1)\omega+3(1-2\mathrm{t}+\frac{\mathrm{t}^2}{2!})\omega^2-(1-3\mathrm{t}+3\frac{\mathrm{t}^2}{2!}-\frac{\mathrm{t}^3}{3!})\omega^3\Big)\Big),\\ \mathcal{S}_2(\mathrm{t}) = \Big(-\gamma(\gamma+\mu)P_0+(\bar{\lambda}\mathcal{I}_0\mathcal{S}_0+\mu)\Big((\bar{\lambda}\mathcal{I}_0+\mu)S_0-\gamma P_0\Big)\Big)\Big(1+2(\mathrm{t}-1)\omega+(1-2\mathrm{t}+\frac{\mathrm{t}^2}{2!})\omega^2\Big)\\ +(-\gamma(\gamma+\mu)a\wedge+\Big((\bar{\lambda}\mathcal{I}_0\mathcal{S}_0+\mu)(\lambda+\mu)(1-a)\wedge-\gamma a\wedge\Big)\\ \Big(1+3(t-1)\omega+3(1-2\mathrm{t}+\frac{\mathrm{t}^2}{2!})\omega^2+(3t-3\frac{\mathrm{t}^2}{2!}+\frac{\mathrm{t}^3}{3!})\omega^3\Big),\end{cases} \tag{5.10}$$

$$\begin{cases}\mathcal{I}_2(\mathrm{t}) = \lambda\Big(\gamma(\mathcal{P}_0+a\wedge)-(\lambda+\mu)(\mathcal{S}_0+(1-a)\wedge)-(\delta+\beta+\mu)\Big(\bar{\lambda}\mathcal{I}_0\mathcal{S}_0(\mathcal{S}_0+(1-a)\wedge)-(\delta+\beta+\mu)\mathcal{I}_0\Big)\Big)\\ +\omega\Big(\Big(\lambda(2\gamma\mathcal{P}_0+3a\wedge\bar{\lambda}\mathcal{I}_0\mathcal{S}_0+2\mathcal{S}_0(\lambda+\mu)-3(\bar{\lambda}\mathcal{I}_0\mathcal{S}_0+\mu)(1-a)\wedge\Big)\\ -(\delta+\beta+\mu)\Big(2\lambda\mathcal{S}_0+3\lambda(1-a)\wedge-(\delta+\beta+\mu)2\mathcal{I}_0\Big)(\mathrm{t}-1)\Big)\\ +\omega^2\Big(\bar{\lambda}\mathcal{I}_0\mathcal{S}_0\Big(\gamma\mathcal{P}_0-\mathcal{S}_0(\bar{\lambda}\mathcal{I}_0+\mu)-(\delta+\beta+\mu)\lambda\mathcal{S}_0+3\lambda(1-a)\wedge-(\delta+\beta+\mu)\mathcal{I}_0\Big)\\ (1-2\mathrm{t}+\frac{\mathrm{t}^2}{2!})+\gamma a\wedge(3-4\mathrm{t}+3\frac{\mathrm{t}^2}{2!})\\ -(\bar{\lambda}\mathcal{I}_0\mathcal{S}_0+\mu)(1-a)\wedge(3+6\mathrm{t}-4\frac{\mathrm{t}^2}{2!})\Big)\\ +\omega^3\Big(\bar{\lambda}\mathcal{I}_0\Big(\gamma a\wedge(-1+\mathrm{t}-3\frac{\mathrm{t}^2}{2!}+\frac{\mathrm{t}^3}{3!})+(\lambda+\mu)(1-a)\wedge(1+3\mathrm{t}+4\frac{\mathrm{t}^2}{2!}-\frac{\mathrm{t}^3}{3!})\Big)\\ -(\delta+\beta+\mu)\Big(-\bar{\lambda}\mathcal{I}_0(1-a)\wedge(1-3\mathrm{t}+3\frac{\mathrm{t}^2}{2!}-\frac{\mathrm{t}^3}{3!})\Big)\Big),\end{cases} \tag{5.11}$$

$$\begin{cases}\mathcal{T}_2(\mathrm{t}) = \beta\bar{\lambda}\mathcal{I}_0\mathcal{S}_0+(1-a)\wedge)-\beta I_0((\delta+\beta+\mu)+\mu)-\mu^2T_0\\ +\omega\Big(\Big(2\beta\bar{\lambda}\mathcal{I}_0\mathcal{S}_0+3\beta\bar{\lambda}\mathcal{I}_0\mathcal{S}_0(1-a)\wedge+2\beta(\delta+\beta+\mu)I_0-2\mu\beta I_0-2\mu^2T_0\Big)(\mathrm{t}-1)\Big)\\ +\omega^2\Big(\Big(\beta\lambda\mathcal{S}_0+\beta\bar{\lambda}\mathcal{I}_0\mathcal{S}_0(1-a)\wedge-\beta(\delta+\beta+\mu)I_0-\mu\beta I_0-\mu^2T_0\Big)(1-2\mathrm{t}+\frac{\mathrm{t}^2}{2!})\Big)\\ +\omega^3\Big(\beta\bar{\lambda}\mathcal{I}_0\mathcal{S}_0(1-a)\wedge(-1+\mathrm{t}+\frac{\mathrm{t}^2}{2!}+\frac{\mathrm{t}^3}{3!})\Big),\end{cases}, \tag{5.12}$$

and so on. Similarly, the other terms are calculated. The required solutions will be written as

$$\begin{cases}\mathcal{P}(\mathrm{t}) = \mathcal{P}_0+\mathcal{P}_1(\mathrm{t})+\mathcal{P}_2(\mathrm{t})+\mathcal{P}_3(\mathrm{t})+\dots.\\ \mathcal{S}(\mathrm{t}) = \mathcal{S}_0+\mathcal{S}_1(\mathrm{t})+\mathcal{S}_2(\mathrm{t})+\mathcal{S}_3(\mathrm{t})+\dots.\\ \mathcal{I}(\mathrm{t}) = \mathcal{I}_0+\mathcal{I}_1(\mathrm{t})+\mathcal{I}_2(\mathrm{t})+\mathcal{I}_3(\mathrm{t})+\dots.\\ \mathcal{T}(\mathrm{t}) = \mathcal{T}_0+\mathcal{T}_1(\mathrm{t})+\mathcal{T}_2(\mathrm{t})+\mathcal{T}_3(\mathrm{t})+\dots.\end{cases} \tag{5.13}$$

6. Results and discussion

Here, we use the values given in Table 1. Using Matlab, we plot the solutions of (5.13) up to few terms. The numerical values for parameters used are given in Table 1. For various fractional orders, the solutions are displayed in Figures 3–6.

From Figure 3, we see that the protected populations are decreasing at different rates due to variation in fractional order. Consequently, the susceptible class population is also declining with variation in dynamics due to different fractional orders, as shown in Figure 4. In the same line, the infected population dynamics are also different at different fractional orders, as presented in Figure 5. In the Figure 6, the dynamics of the treated class are shown using various fractional orders. Fractional orders derivatives explain the dynamics in more detail.

Table 1. Interpretations and numerical values of model parameters.

Parameters	Description of parameters	Numerical value	Source
$\wedge$	Rate of recruitment	0.0044	[26]
a	Adjustment parameter	0.8	[26]
μ	Rate of natural mortality	0.016	[26]
δ	Rate of disease induced mortality	0.005	[26]
γ	Loss of protection rate	0.001	[26]
β	Treatment rate	0.9	[26]
π	Probability rate of getting typhoid fever disease	0.0011	[26]
ϕ	Typhoid transmission probability rate	0.0011	[26]
θ	Infection contact rate	0.0002	[26]
$\mathcal{P}_0$	Protected class initial value	500	assumed
$\mathcal{S}_0$	Susceptible class initial value	90	[26]
$\mathcal{I}_0$	Infected class initial value	20	[26]
$\mathcal{T}_0$	Treated class initial value	0	assumed

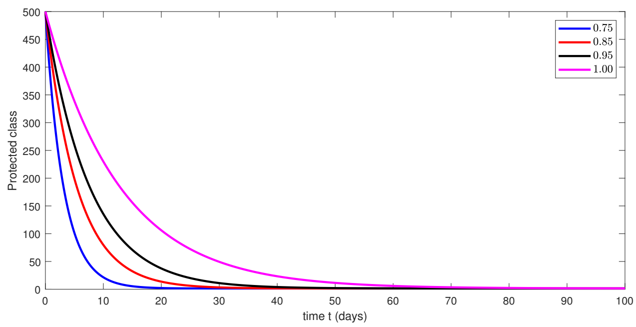

Figure 3. Representation of a few terms of approximate solutions for protected class at various fractional orders.

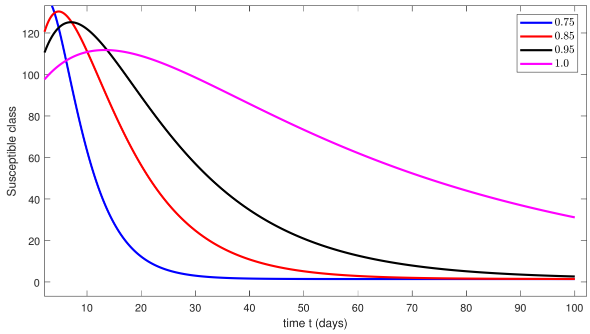

Figure 4. Representation of a few terms of approximate solutions for susceptible class at various fractional orders.

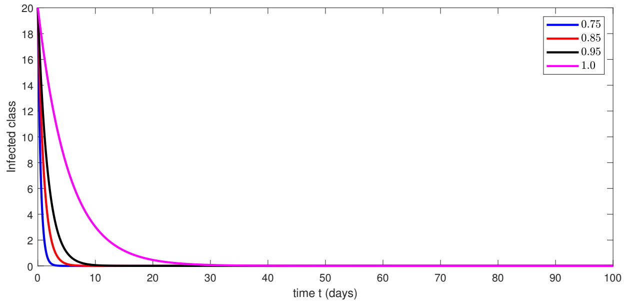

Figure 5. Representation of a few terms of approximate solutions for infected class at various fractional orders.

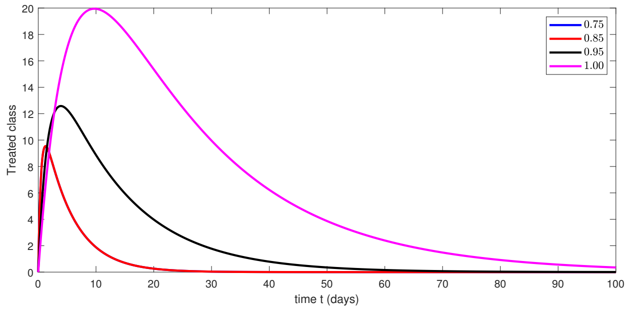

Figure 6. Representation of a few terms of approximate solutions for treated class at various fractional orders.

7. Conclusions

In this work, we have investigated a four-compartmental mathematical model of typhoid fever of fractional order. The four compartments are protected, susceptible, infected and treated individuals. The CFFD has been used to investigate the qualitative and analytical aspects of the considered model.

The existence of a solution of the model has been checked by using fixed point theorems due to Banach and Krassnoselskii. We have used the Laplace transform coupled with the ADM for computation of the semi-analytical solution of the proposed model. The proposed method has some advantages, for example, it does not need any prior discretization of initial data. Also, it does not depend on auxiliary parameters, like the homotopy method, to control it. Also, the method provides a series of types of solutions. In most cases, the series solution converges to the exact value (solution) of the problem. Further, the proposed method is easy to implement for the computation of solutions to various problems of fractional order differential equations. In the future, we will extend these results for piecewise equations of the fractional-order derivative model of typhoid fever.

Acknowledgements

This research was funded by King Mongkut's University of Technology North Bangkok (Contract no. KMUTNB-64-KNOW-42).

Conflict of interest

There does not exist any conflict of interest.

References

1. World Health Organization, Typhoid fever fact sheet, 2000, Available from: https://www.who.int/news-room/fact-sheets/detail/typhoid.
2. World Health Organization, Background paper on vaccination against typhoid fever using New Generation Vaccines presented at the SAGE November 2007 meeting, 2007. Available from: https://silo.tips/download/background-paper-on-vaccination-against-typhoid-fever-using-new-generation-vacci
3. C. Jenkins, S. H. Gillespie, *Salmonella infection in: GC Cook and AL Zumla Mansons tropical diseases*, Elsevier, 2009.
4. R. Milligan, M. Paul, M. Richardson, A. Neuberger, Vaccines for preventing typhoid fever, *Cochrane Db. Syst. Rev.*, 2018, CD001261. https://doi.org/10.1002/14651858.CD001261.pub4
5. World Health Organization, Background document: The diagnosis, treatment and prevention of typhoid fever, 2003, 1–26.
6. M. M. Gibani, E. Jones, A. Barton, C. Jin, J. Meek, S. Camara, et al., Investigation of the role of typhoid toxin in acute typhoid fever in a human challenge model, *Nat. Med.*, **25** (2019), 1082–1088. https://doi.org/10.1038/s41591-019-0505-4
7. O. J. Peter, M. O. Ibrahim, O. B. Akinduko, M. Rabiu, Mathematical model for the control of typhoid fever, *IOSR-JM*, **13** (2017), 60–66. https://doi.org/10.9790/5728-1304026066
8. J. A. Harrison, B. Villarreal-Ramos, P. Mastroeni, R. D. de Hormaeche, C. E. Hormaeche, Correlates of protection induced by live Aro- Salmonella typhimurium vaccines in the murine typhoid model, *Immunology*, **90** (1997), 618–625. https://doi.org/10.1046/j.1365-2567.1997.00158.x

9. J. Gonzalez-Guzman, An epidemiological model for direct and indirect transmission of typhoid fever, *Math. Biosci.*, **96** (1989), 33–46. https://doi.org/10.1016/0025-5564(89)90081-3

10. S. Kariuki, Typhoid fever in sub-saharan Africa: Challenges of diagnosis and management of infections, *JIDC*, **2** (2008), 443–447. https://doi.org/10.3855/jidc.159

11. O. D. Makinde, K. O. Okosun, Impact of chemo–therapy on optimal control of malaria disease with infected immigrants, *Biosystems*, **104** (2011), 32–41. https://doi.org/10.1016/j.biosystems.2010.12.010

12. S. Cobey, Modeling infectious disease dynamics, *Science*, **368** (2020), 713–714.

13. K. O. Okosun, O. D. Makinde, Optimal control analysis of malaria in the presence of non–linear incidence rate, *Appl. Comput. Math.,* **12** (2013), 20–32.

14. K. O. Okosun, O. D. Makinde, Optimal control analysis of hepatitis C virus with acute and chronic stages in the presence of treatment and infected immigrants, *Int. J. Biomath.*, **7** (2014), 1450019. https://doi.org/10.1142/S1793524514500193

15. C. I. Siettos, L. Russo, Mathematical modeling of infectious disease dynamics, *Virulence*, **4** (2013), 295–306. https://doi.org/10.4161/viru.24041

16. S. Kariuki, G. Revathi, J. Kiiru, D. M. Mengo, J. Mwituria, J. Muyodi, et al., Typhoid in Kenya is associated with a dominant multidrug resistant Salmonellaenterica serovar typhi Haplotype that is also widespread Southeast Asia, *J. Clin. Microbiol.*, **48** (2010), 2171–2176. https://doi.org/10.1128/JCM.01983-09

17. I. A. Adetunde, Mathematical models for the dynamics of typhoid fever in Kassena–Nankana district of upper East region of Ghana, *J. Mod. Math. Stat.*, **2** (2008), 45–49.

18. M. N. Chamuchi, J. K. Sigey, J. A. Okello, J. M. Okwoyo, SIICR model and simulation of the effects of carriers on the transmission dynamics of typhoid fever in kisii town kenya, *CSEA*, **2** (2014), 109–116.

19. K. R. Adeboye, M. Haruna, A mathematical model for the transmission and control of malaria and typhoid co–infection using sirs approach, *Nigeria Res. J. Math.*, **2** (2015), 1–24.

20. A. Omame, R.A. Umana, N.O. Iheonu, S. Chioma, On the existence of a stochastic model of typhoid fever, *Math. Theory Model.*, **5** (2015), 104–113.

21. M. A. Khan, M. Parvez, S. Islam, I. Khan, S. Shafie, T. Gul, Mathematical analysis of typhoid model with saturated incidence rate, *Adv. Stud. Biol.*, **7** (2015), 65–78. http://doi.org/10.12988/asb.2015.41059

22. B. Cvjetanovic, B. Grab, K. Eumura, Epidemiological model of typhoid fever and its use in planning and evaluation of antityphoid immunization and sanitation programmes, *B. World Health Organ*, **45** (1971), 53–75.

23. V. E. Pitzer, C. C. Bowles, S. Baker, G. Kang, V. Balaji, J. Farrar, et al., Predicting the impact of vaccination on the transmission dynamics of typhoid in South Asia: A mathematical modeling study, *PLoS Negl. Trop Dis.*, **8** (2014), e2642. https://doi.org/10.1371/journal.pntd.0002642

24. J. H. Cook, Are cholera and typhoid vaccines a good investment for slums in Kolkata, *India. Pediatr. Infect. Dis. J.*, **9** (2010), 485–496.

25. N. C. Moffat, K. Johanna, A. Jeconiah, J. M. Okwoyo, SIIR model and simulation of the effects of carrier on the transmission dynamics of tyhoid fever in Kisii town, *Kenya*, **2** (2014), 109–116.
26. J. K. Nthiiri, O. G. Lawi, C. O. Akinyi, D. O. Oganga, C. M. Wachira, M. J. Musyoka, et al., Mathematical modelling of Typhoid fever disease incorporating protection against infection, *J. Adv. Math. Comput. Sci.*, **14** (2016), 1–10. https://doi.org/10.9734/BJMCS/2016/23325
27. I. Podlubny, *Fractional differential equations: Mathematics in science and engineering*, New York: Academic Press, 1999.
28. A. A. Kilbas, H. Srivastava, J. J. Trujillo, *Theory and application of fractional differential equations*, Amsterdam: Elseveir, 2006.
29. X. J. Yang, H. M. Srivastava, C. Cattani, Local fractional homotopy perturbation method for solving fractal partial differential equations arising in mathematical physics, *Rom. Rep. Phys.*, **67** (2015), 752–761
30. R. Shah, H. Khan, M. Arif, P. Kumam, Application of Laplace-Adomian decomposition method for the analytical solution of third-order dispersive fractional partial differential equations, *Entropy*, **21** (2019), 335. https://doi.org/10.3390/e21040335
31. J. S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, *Commun. Fract. Calc.*, **3** (2012), 73–99.
32. O. Kiymaz, An algorithm for solving initial value problems using Laplace Adomian decomposition method. *Appl. Math. Sci.*, **3** (2009), 1453–1459.
33. J. Biazar, Solution of the epidemic model by Adomian decomposition method, *Appl. Math. Comput.*, **173** (2006), 1101–1106. https://doi.org/10.1016/j.amc.2005.04.036
34. F. Haq, K. Shah, G. ur Rahman, M. Shahzad, Numerical solution of fractional order smoking model via Laplace Adomian decomposition method. *Alex. Eng. J.* **57** (2018), 1061–1069. https://doi.org/10.1016/j.aej.2017.02.015
35. Z. Odibat, D. Baleanu, Numerical simulation of initial value problems with generalized Caputo-type fractional derivatives, *Appl. Numer. Math.*, **156** (2020), 94–105. https://doi.org/10.1016/j.apnum.2020.04.015
36. P. Veeresha, H. M. Baskonus, D. G. Prakasha, W. Gao, G. Yel, Regarding new numerical solution of fractional Schistosomiasis disease arising in biological phenomena, *Chaos Soliton. Fract.*, **133** (2020), 109661. https://doi.org/10.1016/j.chaos.2020.109661
37. S. Etemad, A. Hussain, A. Imran, J. Alzabut, S. Rezapour, A. G. Selvam, On a fractional cantilever beam model in the q-difference inclusion settings via special multi-valued operators, *J. Inequal. Appl.*, **2021** (2021), 174. https://doi.org/10.1186/s13660-021-02708-6
38. D. Baleanu, M. Q. Iqbal, A. Hussain, S. Etemad, S. Rezapour, On solutions of fractional multi-term sequential problems via some special categories of functions and (AEP)-property, *Adv. Differ. Equ.*, **2021** (2021), 197. https://doi.org/10.1186/s13662-021-03356-2
39. D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative, *Chaos Soliton. Fract.*, **134** (2020), 109705. https://doi.org/10.1016/j.chaos.2020.109705

40. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. https://doi.org/10.12785/pfda/010201

41. A. Atangana, I. Koca, Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order, *Chaos Soliton. Fract.*, **89** (2016), 447–454. https://doi.org/10.1016/j.chaos.2016.02.012

42. T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Differ. Equ.*, **2017** (2017), 313. https://doi.org/10.1186/s13662-017-1285-0

43. M. Al-Refai, D. Baleanu, Comparison principles of fractional differential equations with non-local derivative and their applications, *AIMS Mathematics*, **6** (2021), 1443–1451. https://doi.org/10.3934/math.2021088

44. J. Singh, D. Kumar, J. J. Nieto, Analysis of an El Nino-Southern Oscillation model with a new fractional derivative, *Chaos Soliton. Fract.*, **99** (2017), 109–115. https://doi.org/10.1016/j.chaos.2017.03.058

45. V. E. Tarasov, Caputo-Fabrizio operator in terms of integer derivatives: Memory or distributed lag? *Comput. Appl. Math.*, **38** (2019), 113. https://doi.org/10.1007/s40314-019-0883-8

46. H. Mohammadi, M. K. A. Kaabar, J. Alzabut, A. G. M. Selvam, S. Rezapour, A complete model of Crimean-Congo Hemorrhagic Fever (CCHF) transmission cycle with nonlocal fractional derivative, *J. Funct. Spaces*, **2021** (2021), 1273405, https://doi.org/10.1155/2021/1273405

47. D. Baleanu, H. Mohammadi, S. Rezapour, A mathematical theoretical study of a particular system of Caputo-Fabrizio fractional differential equations for the Rubella disease model, *Adv. Differ. Equ.*, **2020** (2020), 184. https://doi.org/10.1186/s13662-020-02614-z

48. S. Rezapour, H. Mohammadi, A. Jajarmi, A new mathematical model for Zika virus transmission, *Adva. Differ. Equ.*, **2020** (2020), 589. https://doi.org/10.1186/s13662-020-03044-7

49. A. Atangana, Blind in a commutative world: Simple illustrations with functions and chaotic attractors, *Chaos Soliton. Fract.*, **114** (2018), 347–363. https://doi.org/10.1016/j.chaos.2018.07.022

50. A. Atangana, Fractional discretization: The African's tortoise walk, *Chaos Soliton. Fract.*, **130** (2020), 109399. https://doi.org/10.1016/j.chaos.2019.109399

51. S. Rezapour, S. Etemad, H. Mohammadi, A mathematical analysis of a system of Caputo-Fabrizio fractional differential equations for the anthrax disease model in animals, *Adv. Differ. Equ.*, **2020** (2020), 481. https://doi.org/10.1186/s13662-020-02937-x

52. K. Shah, M. A. Alqudah, F. Jarad, T. Abdeljawad, Semi-analytical study of Pine Wilt Disease model with convex rate under Caputo-Febrizio fractional order derivative, *Chaos Soliton. Fract.*, **135** (2020), 109754. https://doi.org/10.1016/j.chaos.2020.109754

53. D. Baleanu, S. M. Aydogn, H. Mohammadi, S. Rezapour, On modelling of epidemic childhood diseases with the Caputo-Fabrizio derivative by using the Laplace Adomian decomposition method, *Alex. Eng. J.*, **59** (2020), 3029–3039. https://doi.org/10.1016/j.aej.2020.05.007

54. K. Shah, F. Jarad, T. Abdeljawad, On a nonlinear fractional order model of dengue fever disease under Caputo-Fabrizio derivative, *Alex. Eng. J.*, **59** (2020), 2305–2313. https://doi.org/10.1016/j.aej.2020.02.022

55. G. Nazir, K. Shah, A. Debbouche, R. A. Khan, Study of HIV mathematical model under nonsingular kernel type derivative of fractional order, *Chaos Soliton. Fract.*, **139** (2020), 110095. https://doi.org/10.1016/j.chaos.2020.110095
56. A. S. Shaikh, K. S. Nisar, Transmission dynamics of fractional order Typhoid fever model using Caputo–Fabrizio operator, *Chaos Soliton. Fract.*, **128** (2019), 355–365. https://doi.org/10.1016/j.chaos.2019.08.012
57. A. B. Amar, A. Jeribi, M. Mnif, Some fixed point theorems and application to biological model, *Numer. Funct. Anal. Optim.*, **29** (2008), 1–23. https://doi.org/10.1080/01630560701749482