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Research article

The geominimal integral curvature

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Abstract: In this paper, the geominimal integral curvature on the convex body is introduced. The existence and uniqueness of the geominimal integral curvature are proved. Some other properties for the geominimal integral curvature, such as continuity, are investigated.

Keywords: geominimal surface area; integral curvature; Petty body; convex body
Mathematics Subject Classification: 52A20, 53A15

1. Introduction

The geominimal surface area belongs to the research category of convex geometric analysis. The classical geominimal surface area, which can be dated back to 1974, was firstly introduced by Petty [21], in his paper, the classical Petty body was obtained and some classical affine isoperimetric inequalities were established. With the development of the L_p-Brunn-Minkowski theory, the classical geominimal surface area has been extended to L_p cases by Lutwak [14] (for $p > 1$) and Ye [26] (for $-n \neq p < 1$) (also see [28, 30]). The dual L_p-Brunn-Minkowski theory was introduced by Lutwak [13, 15]. Wang and Chen [25] introduced the dual L_p geominimal surface area. For more results of the geominimal surface area, one can refer to [9–11, 18, 27, 29, 31] and so on.

We shall work in the n-dimensional Euclidean space $\mathbb{R}^n$ with the standard inner product $x \cdot y$ of x and y in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, write $|x| = \sqrt{x \cdot x}$ for the Euclidean norm of x. We call a set $K \subset \mathbb{R}^n$ is convex if provided that for any two points $x, y \in K$ and $\lambda \in [0, 1]$, one has $\lambda x + (1 - \lambda)y \in K$. A convex subset $K \subset \mathbb{R}^n$ is called a convex body if K is compact with nonempty interior, and the interior point of a convex body K can be written as int$K$. Moreover, if for any two points $x, y \in K$ $(x \neq y)$ and $\lambda \in (0, 1)$, $\lambda x + (1 - \lambda)y \in \text{int}K$ holds, we call the convex body K is a strictly convex.

Throughout this paper, let $\mathcal{K}^n$ be the class of all convex bodies in $\mathbb{R}^n$ and $\mathcal{K}^n_0$ be the class of all convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors and $\mathcal{K}^n_e$ be the class of all origin-symmetric convex bodies in $\mathbb{R}^n$. The standard Lebesgue measure of a set K in $\mathbb{R}^n$ will be denoted by $|K|$, we call $|K|$ is the n-dimensional volume of a set $K \in \mathcal{K}^n$. The volume radius of the convex body K is defined as $vrad(K) = (|K|/\omega_n)^{1/n}$. The origin-symmetric unit ball in n-dimensional Euclidean space is denote by B_2^n, i.e., $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and we use ω_n to denote the volume of the unit ball B_2^n. We use S^{n-1} to denote the unit sphere in $\mathbb{R}^n$, and we let κ_n be the surface area measure of B_2^n. Let $K|u^\perp$ be the orthogonal projection of K onto the subspace orthogonal to u and $V_{n-1}(K|u^\perp)$ be the $(n-1)$-dimensional volume of $K|u^\perp$. Let $C(S^{n-1})$ be the set of continuous functions defined on the unit sphere S^{n-1}.

As we all know, the notion of geominimal surface area was introduced by Petty [21]. For $K \in \mathcal{K}_0^n$, the geominimal surface area of K, is defined by the following optimal problem:

$$\inf\Big\{\int_{S^{n-1}} h_Q(u)dS(K,u) : Q \in \mathcal{K}_s^n \text{and } |Q^*| = \omega_n\Big\}. \tag{1.1}$$

Here Q^* denotes polar body of Q, $h_Q(\cdot)$ is the support function of convex body Q (see Section 2), $\mathcal{K}_s^n$ is the class of convex bodies in $\mathbb{R}^n$ whose Santaló points are at origin point o (see [23]), and $S(K,\cdot)$ is the surface area measure of convex body K, i.e., for any $K \in \mathcal{K}^n$ and any measurable subset $\Omega \subset S^{n-1}$, the surface area measure $S(K,\Omega)$, is defined by

$$S(K,\Omega) = \int_{\nu_K^{-1}(\Omega)} d\mathcal{H}^{n-1},$$

where $\nu_K^{-1} : S^{n-1} \to \partial K$ is the inverse Gauss map and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on ∂K.

In [16], Lutwak extended over body Q restricted to $\mathcal{K}_0^n$. It follows that

$$\inf\Big\{\int_{S^{n-1}} h_Q(u)dS(K,u) : Q \in \mathcal{K}_0^n \text{and } |Q^*| = \omega_n\Big\}. \tag{1.2}$$

For any two convex bodies $K, Q \in \mathcal{K}^n$, the mixed volume $V_1(K,Q)$ of K, Q is defined by

$$V_1(K,Q) = \frac{1}{n}\int_{S^{n-1}} h_Q(u)dS(K,u).$$

Moreover, for any $K \in \mathcal{K}^n$, the volume of K is defined by

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K(u)dS(K,u). \tag{1.3}$$

Together with (1.2), one has,

$$\inf\{nV_1(K,Q) : Q \in \mathcal{K}_0^n \text{and } |Q^*| = \omega_n\}. \tag{1.4}$$

Lutwak [16] proved the uniqueness of the solution to (1.4). This shows that the mixed volume $V_1(\cdot,\cdot)$ is a necessary premise of classical geominimal surface area. Hence, utilizing the relationship between the minimum surface area and the corresponding mixed volume, one can investigate the geominimal integral curvature with the help of the mixed entropy.

We define a new concept of the entropy functional of the convex body, i.e., the entropy functional $\mathcal{E}(K)$ of the convex body $K \in \mathcal{K}_0^n$ can be defined by

$$\mathcal{E}(K) = -\int_{S^{n-1}} \log h_K(u)du,$$

where the integration is with respect to spherical Lebesgue measure (see, Huang et al. [7]).

For two convex bodies $K, L \in \mathcal{K}_0^n$, the Lp mixed entropy is defined by

$$\mathcal{E}_p(K, L) = \lim_{\varepsilon \to 0} \frac{\mathcal{E}(K \hat{+}_p \varepsilon \diamond L) - \mathcal{E}(K)}{\varepsilon},$$

where $K \hat{+}_p \varepsilon \diamond L$ is the harmonic L_p combination of K, L for $p \geq 1$ and $K \hat{+}_p \varepsilon \diamond L \in \mathcal{K}_0^n$ is a new convex body. In Section 2, we will show that the convex body $K \hat{+}_p \varepsilon \diamond L \in \mathcal{K}_0^n$ can be defined for all $p \in \mathbb{R}$ and even for negative ϵ of sufficiently small absolute value. In this paper, we mainly study the mixed entropy functional when $p = 0$, i.e., for $K, L \in \mathcal{K}_0^n$

$$\mathcal{E}_0(K, L) = \lim_{\varepsilon \to 0} \frac{\mathcal{E}(K \hat{+}_0 \varepsilon \diamond L) - \mathcal{E}(K)}{\varepsilon} = -\int_{S^{n-1}} \log \rho_L(u) dJ(K, u), \tag{1.5}$$

where ρ_L is the radial function of convex body L and $J(K, \cdot)$ is the Aleksandrov integral curvature of the convex body K (see Section 2). For some of the the Aleksandrov integral recent resilts, see [1, 3, 8, 19, 20, 22].

There exists a natural problem: whether there is a convex body $L \in \mathcal{K}_0^n$ with $|L^*| = \omega_n$ such that L is a solution to the following problem:

$$\inf \left\{ \mathcal{E}_0(K, Q) : Q \in \mathcal{K}_0^n \text{ with } |Q^*| = \omega_n \right\}. \tag{1.6}$$

In this paper, let $G_{\mathcal{E}}(K)$ be the geominimal integral curvature of the convex body $K \in \mathcal{K}_0^n$, it can be defined by optimal problem in (1.6). Together with (1.5), it is equivalent to solve the optimal problem as follows:

$$G_{\mathcal{E}}(K) = \sup \left\{ \int_{S^{n-1}} \log \rho_Q(u) dJ(K, u) : Q \in \mathcal{K}_0^n, |Q^*| = \omega_n \right\}.$$

Based on the concept of the Petty body, we will study the entropy form of the Petty body. The following is our main result.

Theorem 1.1. *Let $K \in \mathcal{K}_o^n$, then there exists a convex body $M \in \mathcal{K}_e^n$ with $|M^*| = \omega_n$ such that*

$$\int_{S^{n-1}} \log \rho_M(u) dJ(K, u) = \sup \left\{ \int_{S^{n-1}} \log \rho_L(u) dJ(K, u) : L \in \mathcal{K}_e^n, |L^*| = \omega_n \right\}.$$

In addition, in the plane $\mathbb{R}^2$, these bodies are unique or their polar bodies are parallelograms with parallel sides.

This paper is organized as follows. In Section 2, we collect some basic concepts and various facts that will be used in the proofs of our results. In Section 3, we study the properties of the L_0 mixed entropy. In Section 4, we obtain the existence and uniqueness of the geominimal integral curvature.

2. Background and notation

2.1. Convex bodies

We now introduce some basic facts and standard notations needed in this paper. For more details and concepts in convex geometry, please see [4, 5, 24].

The Minkowski sum of two convex sets K and L is denoted by

$$K + L = \{x + y : x \in K,\ y \in L\}.$$

The scalar product of $\lambda \in \mathbb{R}$ and $K \in \mathcal{K}^n$ is defined by $\lambda K = \{\lambda x : x \in K\}$. For any convex body K in $\mathbb{R}^n$, the support function of K, $h_K : S^{n-1} \to \mathbb{R}$, is defined by

$$h_K(u) = \max\{x \cdot u : x \in K\},\ \text{for all } u \in S^{n-1}.$$

For the convex body K and $u \in S^{n-1}$, the hyperplane

$$H_K(u) = \{x \in \mathbb{R}^n : x \cdot u = h_K(u)\}$$

is called the supporting hyperplane of K with unit normal u. For $x \in \partial K$, if there is only one supporting hyperplane of K passing through point x, we call $x \in \partial K$ is a smooth point. If there exist more than a supporting hyperplane of K passing through point x, we call $x \in \partial K$ is a singlar point. For each $x \in \partial K$ is a smooth point, we call the convex body K is smooth. If a convex body K is strictly convex and smooth, we say that K is a regular convex body. For each $K \in \mathcal{K}^n$, there exists a regular convex body sequence K_i such that K_i converges to K as $i \to \infty$ (see [24]).

Let $l_u = \{tu : t \geq 0\}$ for $u \in S^{n-1}$. If $L \cap l_u$ is a closed line segment for all $u \in S^{n-1}$, we say $L \subset \mathbb{R}^n$ is star-shaped with respect to the origin. Let L be compact and star-shaped with respect to the origin, the radial function $\rho_L : S^{n-1} \to [0, \infty)$ is defined by

$$\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\},\ \text{for all} \quad u \in S^{n-1}.$$

A compact star-shaped set with respect to the origin is uniquely determined by its radial function. If ρ_L is positive and continuous on S^{n-1}, then the star-shaped L is called a star body about the origin. Let $\mathcal{I}_0^n$ be the set of all star bodies about the origin. Clearly, the radial function of a convex body in $\mathcal{K}_0^n$ is continuous and positive, i.e., $\mathcal{K}_0^n \subset \mathcal{I}_0^n$. If $K \in \mathcal{K}_0^n$, then

$$\partial K = \{\rho_K(u)u : u \in S^{n-1}\}.$$

And the volume of $K \in \mathcal{K}_0^n$ can be rewritten by

$$|K| = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)du. \tag{2.1}$$

For Borel set $\eta \subset S^{n-1}$, let $\rho : \eta \to (0, \infty)$ be a continuous function, then the set $\{\rho(u)u : u \in \eta\}$ is a Borel set. For a set $K \subset \mathbb{R}^n$, the convex hull of K, write conv$K$, is the intersection of all convex sets containing K. Hence, the convex hull $\langle\rho\rangle$ generated by ρ,

$$\langle\rho\rangle = conv\{\rho(u)u : u \in \eta\} \tag{2.2}$$

is a compact set (see [24]). Let $\eta \subset S^{n-1}$ always be a closed set and not contained in any great hemisphere of S^{n-1}, then we have $\langle\rho\rangle \in \mathcal{K}_0^n$. By the definition of convex hull of the function ρ, we have

$$\rho_{\langle\rho\rangle}(u) \geq \rho(u), \text{ for all } u \in S^{n-1}. \tag{2.3}$$

If the function ρ is an even function on S^{n-1}, then the convex hull $conv\{\rho(u)u : u \in S^{n-1}\}$ is an origin-symmetric body. Thus a convex body K is origin-symmetric if and only if the radial function ρ_K of the convex body K is an even function.

The definitions of radial function and support function immediately give that for $\lambda > 0$ and $K, L \in \mathcal{K}_0^n$, one has $h_{\lambda K} = \lambda h_K$, $\rho_{\lambda K} = \lambda \rho_K$ and $h_{K+L} = h_K + h_L$. Moreover

$$K \subset L \ \Leftrightarrow \ h_K(u) \le h_L(u) \text{ and } K \subset L \Leftrightarrow \rho_K(u) \le \rho_L(u), \text{ for all } u \in S^{n-1}.$$

If $K \in \mathcal{K}_0^n$, the following formulas hold for all $u \in S^{n-1}$,

$$h_{K^*}(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_{K^*}(u) = \frac{1}{h_K(u)}, \tag{2.4}$$

where K^* is the polar body of K, and it is given by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1, \text{ for all } y \in K\}.$$

On the set $\mathcal{K}^n$, we consider the topology generated by the Hausdorff metric $d_H(\cdot,\cdot)$. For $K, K' \in \mathcal{K}^n$, the Hausdorff metric $d_H(K, K')$ is defined by

$$d_H(K, K') = \|h_K - h_{K'}\|_\infty = \sup_{u \in S^{n-1}} |h_K(u) - h_{K'}(u)|.$$

A sequence $\{K_i\}_{i\ge1} \subset \mathcal{K}^n$ converges to a convex body $K_0 \in \mathcal{K}^n$ if $d_H(K_i, K_0) \to 0$ as $i \to \infty$, i.e.,

$$d_H(K_i, K_0) \to 0 \text{ if and only if } h_{K_i} \to h_{K_0} \text{ uniformly as } i \to \infty.$$

The radial metric is defined by

$$d_\rho(K, L) = \|\rho_K - \rho_L\|_\infty = \sup_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|$$

for $K, L \in \mathcal{I}_0^n$. We use the fact that on $\mathcal{K}_0^n$, the Hausdorrf metric and the radial metric are topologically equivalent, i.e.,

$$d_H(K_i, K_0) \to 0 \text{ if and only if } d_\rho(K_i, K_0) \to 0$$

for $\{K_i\}_{i\ge1} \subset \mathcal{K}_0^n$ and $K_0 \in \mathcal{K}_0^n$.

If the function $f : \eta \to (0, \infty)$ is continuous, the Wulff shape $[f] \in \mathcal{K}_0^n$ determined by f is a convex body defined by

$$[f] = \bigcap_{u \in \eta} \{x \in \mathbb{R}^n : x \cdot u \le f(u)\}.$$

Note that, if $f = h_K$ is a support function of convex body $K \in \mathcal{K}_0^n$, one has

$$[f] = K.$$

Furthermore, if the function $f : \eta \to (0, \infty)$ is continuous, we have (see e.g., [17] p.95)

$$[f]^* = \langle 1/f \rangle. \tag{2.5}$$

The L_p Minkowski combination is a basic concept in the L_p-Brunn-Minkowski theory. For each $p \geq 1$, the Minkowski-Firey L_p-combination $K +_p L$ introduced by Firey (see, e.g., [24]) can be defined by the support function as follows, i.e., for $K, L \in \mathcal{K}_0^n$ and $a, b > 0$

$$h^p_{a\cdot K +_p b\cdot L} = ah^p_K + bh^p_L.$$

Now we fix $p \neq 0$. For $K, L \in \mathcal{K}_0^n$ and $a, b > 0$, we define the general L_p Minkowski combination, $a \cdot K +_p b \cdot L \in \mathcal{K}_0^n$, via the Wulff shape,

$$a \cdot K +_p b \cdot L = [(ah^p_K + bh^p_L)^{1/p}].$$

When $p = 0$, we define $a \cdot K +_0 b \cdot L \in \mathcal{K}_0^n$ via the Wulff shape,

$$a \cdot K +_0 b \cdot L = [h^a_K h^b_L]. \tag{2.6}$$

Note that "$\cdot$" is written without its subscript p.

For any $p \in \mathbb{R}$, Huang et al. [7] gave the definition of the L_p-harmonic combination $(1-\lambda) \diamond K \widehat{+}_p \lambda \diamond L \in \mathcal{K}_0^n$, i.e.,

$$(1-\lambda) \diamond K \widehat{+}_p \lambda \diamond L = ((1-\lambda) \cdot K^* +_p \lambda \cdot L^*)^*. \tag{2.7}$$

Hence, together with (2.4)–(2.6), we obtain that

$$[h^a_{K^*} h^b_{L^*}]^* = \langle \rho^a_K \rho^b_L \rangle. \tag{2.8}$$

Let $\{K_i\}_{i=1}^\infty \subset \mathcal{K}_0^n$ and $K \in \mathcal{K}_0^n$, combined with (2.4), this implies that

$$K_i \to K \text{ if and only if } K_i^* \to K^*. \tag{2.9}$$

2.2. The integral curvature

For a convex body K in $\mathbb{R}^n$, the *Gauss image* of $\sigma \subset \partial K$ is defined by

$$\boldsymbol{\nu}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} \subset S^{n-1}.$$

The *reverse Gauss image* of $\eta \subset S^{n-1}$ is defined by

$$\boldsymbol{\nu}^*_K(\eta) = \{x \in \partial K : x \in H_K(v) \text{ for some } v \in \eta\} \subset \partial K.$$

Let $\sigma_K = \{x : x \in \partial K \text{ is a singlar point}\} \subset \partial K$. It is known that $\mathcal{H}^{n-1}(\sigma_K) = 0$ (see, p.84 of Schneider [24]). The *Gauss map* of the convex body K is defined by

$$\nu_K : \partial K \setminus \sigma_K \to S^{n-1}.$$

From Lemma 2.2.12 of Schneider [24] we know that the Gauss map ν_K is continuous. The set $\eta_K \subset S^{n-1}$ consisting of all $v \in S^{n-1}$, for which the set $\boldsymbol{\nu}^*_K(\{v\})$, abbreviated as $\boldsymbol{\nu}^*_K(v)$, contains more than a single element, is of $\mathcal{H}^{n-1}$-measure 0. The *inverse Gauss map* of the convex body K is defined by

$$\nu_K^{-1} : S^{n-1} \setminus \eta_K \to \partial K,$$

From Lemma 2.2.12 of Schneider [24] we also know that the function ν_K^{-1} is continuous.

For $K \in \mathcal{K}_0^n$, define the radial map of the convex body K

$$r_K(\cdot) : S^{n-1} \to \partial K \ \text{by} \ r_K(u) = \rho_K(u)u \in \partial K,$$

for $u \in S^{n-1}$. Note that $r_K^{-1}(\cdot) : \partial K \to S^{n-1}$ is the map $r_K^{-1}(x) = \bar{x} = x/|x|$. For $\omega \subset S^{n-1}$, define the *radial Gauss image* of ω by

$$\boldsymbol{\alpha}_K(\omega) = \boldsymbol{\nu}_K(r_K(\omega)) \subset S^{n-1}.$$

Thus, for $u \in S^{n-1}$

$$\boldsymbol{\alpha}_K(u) = \{v \in S^{n-1} : r_K(u) \in H_K(v)\}.$$

Define the *radial Gauss map* of the convex body $K \in \mathcal{K}_0^n$

$$\alpha_K : S^{n-1} \setminus \omega_K \to S^{n-1} \text{ by } \alpha_K = \nu_K \circ r_K,$$

where $\omega_K = \{x/|x| : x \in \sigma_K\}$.

Define the *reverse radial Gauss image* of $\eta \subset S^{n-1}$ by

$$\boldsymbol{\alpha}_K^* : S^{n-1} \to S^{n-1} \text{ by } \boldsymbol{\alpha}_K^{-1}(\eta) = r_K^{-1}(\boldsymbol{\nu}_K^*(\eta)).$$

The *inverse radial Gauss map* of the convex body $K \in \mathcal{K}_0^n$ is defined by

$$\alpha_K^{-1} : S^{n-1} \setminus \eta_K \to S^{n-1} \text{ by } \alpha_K^{-1} = r_K^{-1} \circ \nu_K^{-1}.$$

Note that since both r_K^{-1} and ν_K^{-1} are continuous, α_K^{-1} is continuous.

The integral curvature $J(K, \cdot)$ of convex body $K \in \mathcal{K}_0^n$ is defined by,

$$J(K, \omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)), \tag{2.10}$$

for each Borel set $\omega \subset S^{n-1}$. The total integral curvature of convex body K, is the surface area of the unit sphere S^{n-1}, thus $J(K, S^{n-1}) = \kappa_n$. The concept of integral curvature was introduced by Aleksandrov.

Following formula (2.10), and characteristic function I on S^{n-1}, then

$$\int_{S^{n-1}} I_\omega(u) dJ(K, u) = \int_{S^{n-1}} I_{\boldsymbol{\alpha}_K(\omega)}(u) du = \int_{S^{n-1}} I_\omega(\alpha_K^{-1}(u)) du, \tag{2.11}$$

the last identity holds from the fact that $v \in \boldsymbol{\alpha}_K(\omega)$ if and only if $\alpha_K^{-1} \in \omega$ for almost all u with respect to the spherical Lebesgue measure (see (2.20) in [6]). Furthermore, from formula (2.11), we have that

$$\int_{S^{n-1}} f(u) dJ(K, u) = \int_{S^{n-1}} f(\alpha_K^*(u)) du \tag{2.12}$$

for each continuous function f on S^{n-1}.

Lemma 2.1. ([6] Lemma 2.2) *Let $K_i \in \mathcal{K}_0^n$ be such that* $\lim_{i\to\infty} K_i = K_0 \in \mathcal{K}_0^n$. *Let $\omega = \bigcup_{i=0}^{\infty} \omega_{K_i}$ be the set (of $\mathcal{H}^{n-1}$-measure zero) off of which all of the α_{K_i} are defined. If $u_i \in S^{n-1} \backslash \omega$ are such that* $\lim_{i\to\infty} u_i = u_0 \in S^{n-1} \backslash \omega$, *then* $\lim_{i\to\infty} \alpha_{K_i}(u_i) = \alpha_{K_0}(u_0)$.

Lemma 2.2. ([6] Lemma 2.5) *Let $K \in \mathcal{K}_0^n$, then*

$$\boldsymbol{\alpha}_K^*(\eta) = \boldsymbol{\alpha}_{K^*}(\eta)$$

for each $\eta \subset S^{n-1}$.

Note that if $K_i \to K_0$ in the Hausdorff metric, then for all $f \in C(S^{n-1})$, by formulas (2.9), (2.12), Lemmas 2.1 and 2.2, one has

$$\lim_{i\to\infty} \int_{S^{n-1}} f(u)\, dJ(K_i, u) = \int_{S^{n-1}} f(u) dJ(K_0, u).$$

This proves that the integral curvature $J(K, \cdot)$ is weakly convergence measure.

Lemma 2.3. ([12] Lemma 2.1) *If a sequence of measures $\{\mu_i\}_{i=1}^{\infty}$ on S^{n-1} converges weakly to a finite measure μ on S^{n-1} and a sequence of functions $\{f_i\}_{i\geq 1} \subset C(S^{n-1})$ converges uniformly to a function $f_0 \in C(S^{n-1})$, then*

$$\lim_{i\to\infty} \int_{S^{n-1}} f_i(u)\, d\mu_i = \int_{S^{n-1}} f_0(u)\, d\mu.$$

Thus, by Lemma 2.3, if $\{f_i\}_{i\geq 1} \subset C(S^{n-1})$ is uniformly convergent to $f_0 \in C(S^{n-1})$ and $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ converges to $K_0 \in \mathcal{K}_0^n$ in the Hausdorff metric, together with the weak convergence of $J(K, \cdot)$, we have

$$\lim_{i\to\infty} \int_{S^{n-1}} f_i(u)\, dJ(K_i, u) = \int_{S^{n-1}} f_0(u)\, dJ(K_0, u). \tag{2.13}$$

The Blaschke selection theorem is a powerful tool in convex geometry (see [5, 24]) and will be often used in this paper. It reads: *Every bounded sequence of convex bodies has a subsequence that converges to a convex set.*

We will also use the following lemmas in the proofs of our main results.

Lemma 2.4. (see [12]) *If $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ is a bounded sequence and $\{|K_i^*|\}_{i\geq 1}$ is also a bounded sequence, there is a subsequence $\{K_{i_j}\}_{j\geq 1}$ of the sequence $\{K_i\}_{i\geq 1}$ and a body $K \in \mathcal{K}_0^n$ such that $K_{i_j} \to K$. In addition, if $|K_i^*| = \omega_n$, then $|K^*| = \omega_n$.*

Lemma 2.5. (see [16]) *Let $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ be a convergent sequence with limit K_0, i.e., $K_i \to K_0$ in the Hausdorff distance. If the sequence $\{|K_i^*|\}_{i\geq 1}$ is bounded, then $K_0 \in \mathcal{K}_0^n$.*

3. Properties for the L_0 mixed entropy $\mathcal{E}_0(K, L)$

In this section, we mainly prove some properties for the L_0 mixed entropy. We now prove the continuity of the L_0 mixed entropy as follows.

Proposition 3.1. *Let $\{K_i\}_{i=0}^{\infty} \subset \mathcal{K}_0^n$ and $\{L_i\}_{i=0}^{\infty} \subset \mathcal{K}_0^n$ be two sequences of convex bodies such that $K_i \to K_0 \in \mathcal{K}_0^n$ and $L_i \to L_0 \in \mathcal{K}_0^n$ as $i \to \infty$ in the Hausdorff metric, then*

$$\mathcal{E}_0(K_i, L_i) \to \mathcal{E}_0(K_0, L_0) \quad as \quad i \to \infty.$$

Proof. Since $K_i \to K_0 \in \mathcal{K}_0^n$ and $L_i \to L_0 \in \mathcal{K}_0^n$ as $i \to \infty$ in the Hausdorff metric, then $J(K_i, \cdot)$ converges weakly to $J(K_0, \cdot)$ and the radial functions ρ_{L_i} converges uniformly to ρ_{L_0}, as $i \to \infty$. And there are two constants $r,\ R > 0$ such that for all $i \geq 1$,

$$rB_2^n \subset L_i \subset RB_2^n.$$

We have $r \leq \rho_{L_i} \leq R$, for all $i \geq 1$. Furthermore, together with the continuity of the logarithmic function on $[r, R]$, we get

$$\log \rho_{L_i}(u) \to \log \rho_{L_0}(u) \text{ uniformly on } S^{n-1}.$$

By formula (2.13), one has

$$\begin{aligned}\lim_{i\to\infty} \mathcal{E}_0(K_i, L_i) &= \lim_{i\to\infty} -\int_{S^{n-1}} \log \rho_{L_i} dJ(K_i, u)\\ &= -\int_{S^{n-1}} \log \rho_{L_0} dJ(K_0, u)\\ &= \mathcal{E}_0(K_0, L_0).\end{aligned}$$

□

Proposition 3.2. *Let $\{K_i\}_{i=1}^\infty \subset \mathcal{K}_0^n$ and $K \in \mathcal{K}_0^n$ be regular convex bodies such that $K_i \to K$ as $i \to \infty$ in the Hausdorff metric. For $\{L_i\}_{i=1}^\infty \subset \mathcal{K}_e^n$, then $\{\mathcal{E}_0(K_i, L_i)\}_{i=1}^\infty$ is bounded and $\{L_i\}_{i=1}^\infty$ is uniformly bounded if and only if there exist $\alpha, r > 0$ such that for all $i \geq 1$*

$$rB_2^n \subset L_i \text{ and } |L_i^*| \geq \alpha.$$

Proof. The boundedness of $\{\mathcal{E}_0(K_i, L_i)\}_{i=1}^\infty$ is equivalent to the boundedness of $\{-\mathcal{E}_0(K_i, L_i)\}_{i=1}^\infty$, which shows that there are constants c and C such that $c \leq -\mathcal{E}_0(K_i, L_i) \leq C$ for all $i \geq 1$.

We first show that the sequence $\{L_i\}_{i=1}^\infty$ contains a small ball. For $u_i \in S^{n-1}$, let

$$R_i(u_i) = \max\{\rho_{L_i}(u) : u \in S^{n-1}\}.$$

Since the sequence $\{L_i\}_{i=1}^\infty$ is bounded, there is a constant $\beta > 0$, such that $R_i(u_i) \leq \beta$ for all i, we have $\beta^{-1} B_2^n \subset L_i^*$ for all i. Hence

$$|L_i^*| \geq \frac{\omega_n}{\beta^n}$$

for all i.

On the other hand, by the Blaschke selection theorem, there is a subsequence of $\{L_i\}_{i=1}^\infty$, for convenience, we still record it as $\{L_i\}_{i=1}^\infty$, and a compact convex set L_0, such that $L_i \to L_0$ as $i \to \infty$ in the Hausdorff metric, the radial function sequence ρ_{L_i} is uniformly continuous, we have that ρ_{L_0} is continuous, now we prove that L_0 contains a small ball rB_2^n, if not, then there is a nonzero set ω and a sufficiently small real $\epsilon > 0$ such that $\omega = \{u : \rho_{L_0}(u) < \epsilon\}$, according to the regularity of the convex body K, we have $J(K, \omega) > 0$ and $J(K, S^{n-1}\backslash\omega) \log R < \infty$. Thus, by Proposition 3.1,

$$\begin{aligned}c \leq -\lim_{i\to\infty} \mathcal{E}_0(K_i, L_i) = -\mathcal{E}_0(K, L_0) &= \int_{S^{n-1}} \log \rho_{L_0}(u) dJ(K, u)\\ &\leq \int_\omega \log \epsilon dJ(K, u) + \int_{S^{n-1}\backslash\omega} \log \rho_{L_0}(u) dJ(K, u)\\ &\leq J(K, \omega) \log \epsilon + J(K, S^{n-1}\backslash\omega) \log R.\end{aligned}$$

Let $\epsilon \to 0^+$, hence $J(K,\omega)\log\epsilon \to -\infty$, This is a contradiction to the boundedness of the mixed entropy $\mathcal{E}_0(K_i, L_i)$.

Now we prove that the sequence $\{L_i\}_{i=1}^{\infty}$ is bounded. We let $R_i(u_i) = \max\{\rho_{L_i}(u) : u \in S^{n-1}\}$ for some $u_i \in S^{n-1}$. Since the sequence $\{L_i\}_{i=1}^{\infty}$ contains a small ball, there is a constant $r > 0$ such that $rB_2^n \subset L_i$ for all $i \geq 1$, let Q_i be the convex hull of point $R_i(u_i)u_i$ and $rB_2^n|u_i^{\perp}$, i.e.,

$$Q_i = conv\{R_i(u_i)u_i, rB_2^n|u_i^{\perp}\}.$$

Obviously, $Q_i \subset L_i$, together with the monotonicity of volume and (1.3),

$$|L_i| \geq |Q_i| = \frac{1}{n}R_i(u_i)V_{n-1}(rB_2^n|u_i^{\perp}).$$

By the Blaschke-Stantaló inequality, i.e., for $L_i \in \mathcal{K}_e^n$,

$$|L_i||L_i^*| \leq \omega_n^2.$$

Combined with $|L_i^*| \geq \alpha$, this implies that

$$R_i(u_i) = \frac{n}{V_{n-1}(rB_2^n|u_i^{\perp})}|Q_i| \leq \frac{n}{V_{n-1}(rB_2^n|u_i^{\perp})}|L_i| \leq \frac{n\omega_n^2}{\alpha r^{n-1}\omega_{n-1}}, \tag{3.1}$$

for all i. Thus the sequence $\{L_i\}_{i=1}^{\infty}$ is bounded. There are two constants $r, R > 0$ such that $rB_2^n \subset L_i \subset RB_2^n$ for all i, together with (2.10), we know that $J(K_i, S^{n-1}) = \kappa_n$, then for all i,

$$-\kappa_n \log R \leq \mathcal{E}_0(K_i, L_i) = -\int_{S^{n-1}} \log\rho_{L_i} dJ(K_i, u) \leq -\kappa_n \log r.$$

This shows that the sequence $\{\mathcal{E}_0(K_i, L_i)\}_{i=1}^{\infty}$ is bounded. □

Remark 1. According to the above proof of Proposition 3.2, if $\{L_i\}_{i=1}^{\infty} \subset \mathcal{K}_e^n$ and some $\alpha > 0$ such that $rB_2^n \subset L_i$ and $|L_i^*| \geq \alpha$, then we can remove the condition that $K \in \mathcal{K}_0^n$ is a regular convex body, we also obtain the results that $\{\mathcal{E}_0(K_i, L_i)\}_{i=1}^{\infty}$ is bounded and $\{L_i\}_{i=1}^{\infty}$ is uniformly bounded.

4. The geominimal integral curvature

Throughout this section, we suppose that $K \in \mathcal{K}_0^n$, we mainly prove the existence and uniqueness of the Entropy-Petty body. For further discussion, we introduce the continuity of the geominimal integral curvature $G_{\mathcal{E}}(K)$. We first define the geominimal integral curvature $G_{\mathcal{E}}(K)$ as follows:

Definition 4.1. *Suppose $K \in \mathcal{K}_0^n$ is a convex body, the geominimal integral curvature of K is defined by*

$$\begin{aligned} G_{\mathcal{E}}(K) &= \sup_{L\in \mathcal{K}_e^n}\Big\{\int_{S^{n-1}} \log\rho(vrad(L^*)L, u)dJ(K,u)\Big\} \\ &= \sup\Big\{\int_{S^{n-1}} \log\rho(L,u)dJ(K,u) : L \in \mathcal{K}_e^n \text{ with } |L^*| = \omega_n\Big\}. \end{aligned}$$

Remark 2. We show that the above definition is well defined. In fact, since $|L^*| = \omega_n$ and $L \in \mathcal{K}_e^n$, by the Blaschke-Stantaló inequality, we have $|L| \leq \omega_n$. Hence, by (2.1) and the Jensen inequality

$$\omega_n \geq \frac{1}{n}\int_{S^{n-1}} \rho_L^n(u)du \geq \frac{\kappa_n}{n}\Big(\frac{1}{\kappa_n}\int_{S^{n-1}} \rho_L(u)du\Big)^n.$$

Hence $\int_{S^{n-1}} \rho_L(u)du$ is uniformly bounded. In the next, we assume $K \in \mathcal{K}_0^n$ is a regular convex body, by the concavity of Logarithmic function, we obtain that

$$\int_{S^{n-1}} \log\rho(L,u)dJ(K,u) \leq \kappa_n \log\Big(\frac{1}{\kappa_n}\int_{S^{n-1}} \rho(L,u)dJ(K,u)\Big) = \kappa_n \log\Big(\frac{1}{\kappa_n}\int_{S^{n-1}} \rho(L,u)du\Big) < \infty.$$

For any $K \in \mathcal{K}_0^n$, we choose a regular convex body sequence K_i such that $K_i \to K$ as $i \to \infty$, combined with Proposition 3.1, this implies that

$$\int_{S^{n-1}} \log\rho(L,u)dJ(K,u) < \infty.$$

Hence Definition 4.1 is well defined.

In the next, we prove our mainly result Theorem 1.1.

proof of Theorem 1.1. Firstly, we prove the existence. By Definition 4.1, there is a sequence $\{M_i\}_{i=1}^\infty \subset \mathcal{K}_e^n$ with $|M_i^*| = \omega_n$ such that

$$0 = \int_{S^{n-1}} \log\rho_{B_2^n}(u)dJ(K,u) \leq \int_{S^{n-1}} \log\rho_{M_i}(u)dJ(K,u) < \infty, \text{for all } i \geq 1.$$

Let $K \in \mathcal{K}_0^n$ be a regular convex body, we have

$$0 \leq \int_{S^{n-1}} \log\rho_{M_i}(u)dJ(K,u) = \int_{S^{n-1}} \log\rho_{M_i}(u)du < \infty.$$

By formula (2.4), we get

$$-\infty < \int_{S^{n-1}} \log h_{M_i^*}(u)du \leq 0.$$

Let $R_i(u_i) = \max\{\rho_{M_i^*}(u) : u \in S^{n-1}\}$, and since $M_i \in \mathcal{K}_e^n$, we have $[-R_i(u_i)u_i, R_i(u_i)u_i] \subset M_i^*$. Hence $h(M_i^*, u) \geq R_i(u_i)|u \cdot u_i|$ for all $u \in S^{n-1}$. Therefore

$$\kappa_n \log R_i(u_i) + \int_{S^{n-1}} \log|u \cdot u_i|du \leq \int_{S^{n-1}} \log h_{M_i^*}(u)du \leq 0.$$

Now, assume $K \in \mathcal{K}_0^n$ is not a regular convex body, we can choose a regular convex body sequence K_i such that $K_i \to K$ as $i \to \infty$, combined with Proposition 3.1, this implies that

$$\kappa_n \log R_i(u_i) + \int_{S^{n-1}} \log|u \cdot u_i|du \leq \int_{S^{n-1}} \log h_{M_i^*}(u)dJ(K,u) \leq 0.$$

Since the integral on the left is independent of u_i, this implies that $R_i(u_i)$ is uniformly bounded. Hence, there exists $r > 0$ such that $rB_2^n \subset M_i$ for all $i \geq 1$. By Proposition 3.2, the sequence $\{M_i\}_{i=1}^\infty$ is bounded. By the Blaschke selection theorem, there is a subsequence, for convenience, it is still recorded it as

$\{M_i\}_{i=1}^{\infty}$, which converges to a compact convex set M. Since $|M_i^*| = \omega_n$, by Lemma 2.5, we have $M \in \mathcal{K}_0^n$. Therefore $M_i \in \mathcal{K}_e^n$, gives $M \in \mathcal{K}_e^n$. By Proposition 3.1, we obtain that

$$G_{\mathcal{E}}(K) = \lim_{i\to\infty} \int_{S^{n-1}} \log \rho_{M_i}(u) dJ(K,u) = \int_{S^{n-1}} \log \rho_M(u) dJ(K,u) \text{ with } |M^*| = \omega_n.$$

Next, we prove the uniqueness of theorem in the plane $\mathbb{R}^2$. Assume that there are two convex bodies $M_1, M_2 \in \mathcal{K}_e^2$ with $|M_1^*| = |M_2^*| = \pi$ such that

$$G_{\mathcal{E}}(K) = \int_{S^{n-1}} \log \rho_{M_1}(u) dJ(K,u) = \int_{S^{n-1}} \log \rho_{M_2}(u) dJ(K,u).$$

Now we define a new set $M \subset \mathbb{R}^n$ about M_1, M_2 and together with (2.7), we have

$$M = \frac{1}{2} \diamond M_1 \widehat{+}_0 \frac{1}{2} \diamond M_2 = (\frac{1}{2} \cdot M_1^* +_0 \frac{1}{2} \cdot M_2^*)^*.$$

Combining (2.3) and (2.8), we obtain $M = \langle \rho_{M_1}^{1/2} \rho_{M_2}^{1/2} \rangle$, this together with $M_1, M_2 \in \mathcal{K}_e^2$ implies that the function $\rho(u) = \rho_{M_1}^{1/2}(u) \rho_{M_2}^{1/2}(u)$ is even function on S^{n-1}. Hence $M \in \mathcal{K}_e^2$ and

$$\rho(M,u) \geq \rho(M_1,u)^{\frac{1}{2}} \rho(M_2,u)^{\frac{1}{2}}, \text{ for all } u \in S^{n-1}. \tag{4.1}$$

Furthermore, by the log Brunn-Minkowski inequality in the plane (see [2]),

$$|M^*| = |\frac{1}{2} \cdot M_1^* +_0 \frac{1}{2} \cdot M_2^*| \geq \sqrt{|M_1^*| \cdot |M_2^*|} = \pi,$$

with equality if and only if M_1^* and M_2^* are dilates or they are parallelograms with parallel sides. If M_1^* and M_2^* are dilates, we let $M_1^* = sM_2^*$ for real number $s > 0$ and together with $|M_1^*| = s^2|M_2^*|$, we have $s = 1$, thus we see $M_1 = M_2$, which can be checked that $\mathrm{vrad}(M^*) \geq 1$, with equality if and only if $M_1 = M_2$ or their polar bodies are parallelograms with parallel sides. By (4.1) and Definition 4.1, we have

$$\begin{aligned} G_{\mathcal{E}}(K) &\geq \int_{S^1} \log \rho(vrad(M^*)M, u) dJ(K,u) \\ &\geq \int_{S^1} \log \rho(M,u) dJ(K,u) \\ &\geq \int_{S^1} \log[\rho^{\frac{1}{2}}(M_1,u)\rho^{\frac{1}{2}}(M_2,u)] dJ(K,u) \\ &= \int_{S^1} \frac{1}{2}[\log \rho_{M_1}(u) + \log \rho_{M_2}(u)] dJ(K,u) \\ &= G_{\mathcal{E}}(K). \end{aligned}$$

Hence, this forces $\mathrm{vrad}(M^*) = 1$ and then $M_1 = M_2$ or their polar bodies are parallelograms with parallel sides.

We will prove the continuity of $G_{\mathcal{E}}(K)$ as follows:

Theorem 4.1. *Let $\{K_i\}_{i=1}^{\infty} \subset \mathcal{K}_0^n$ and $K \in \mathcal{K}_0^n$ be such that $K_i \to K$ as $i \to \infty$ in the Hausdorff metric, then* $\lim_{i\to\infty} G_{\mathcal{E}}(K_i) = G_{\mathcal{E}}(K)$.

Proof. Let $\{K_i\}_{i=1}^{\infty} \subset \mathcal{K}_0^n$ be such that $K_i \to K \in \mathcal{K}_0^n$ as $i \to \infty$. For any fixed small $\varepsilon > 0$, by Definition 4.1 and Proposition 3.1, there is a convex body M_ε with $|M_\varepsilon^*| = \omega_n$, we have

$$G_{\mathcal{E}}(K) - \varepsilon \leq -\mathcal{E}_0(K, M_\varepsilon) = -\lim_{i\to\infty} \mathcal{E}_0(K_i, M_\varepsilon) = -\liminf_{i\to\infty} \mathcal{E}_0(K_i, M_\varepsilon) \leq \liminf_{i\to\infty} G_{\mathcal{E}}(K_i).$$

Since $\varepsilon > 0$ is arbitrary small, one has

$$G_{\mathcal{E}}(K) \leq \liminf_{i\to\infty} G_{\mathcal{E}}(K_i). \tag{4.2}$$

We now assume that $M_i \in \mathcal{K}_e^n$ with $|M_i^*| = \omega_n$ such that $G_{\mathcal{E}}(K_i) = -\mathcal{E}_0(K_i, M_i)$,

$$0 = \int_{S^{n-1}} \log \rho_{B_2^n}(u) dJ(K_i, u) \leq \int_{S^{n-1}} \log \rho_{M_i}(u) dJ(K_i, u) < \infty, \text{for all } \ i \geq 1.$$

Since both the sequence $\mathcal{E}_0(K_i, M_i)$ and $\{M_i\}_{i=1}^{\infty}$ are bounded, the Blaschke selection theorem now yields that a subsequence $\{M_{i_j}\}_{j=1}^{\infty}$ of $\{M_i\}_{i=1}^{\infty}$ converges to some compact convex set M'. But $|M_{i_j}^*| = \omega_n$, by LemmaS 2.4, 2.5, and $M_i \in \mathcal{K}_e^n$, the set $M' \in \mathcal{K}_e^n$ is an origin-symmetric convex body and $|(M')^*| = \omega_n$. Together with the Definition 4.1, Proposition 3.1 and Theorem 1.1, we obtain

$$G_{\mathcal{E}}(K) \geq -\mathcal{E}_0(K, M') = -\lim_{i\to\infty} \mathcal{E}_0(K_i, M_i) = \lim_{i\to\infty} G_{\mathcal{E}}(K_i) = \limsup_{i\to\infty} G_{\mathcal{E}}(K_i). \tag{4.3}$$

Combining (4.2) and (4.3), we complete the proof, i.e.,

$$\lim_{i\to\infty} G_{\mathcal{E}}(K_i) = G_{\mathcal{E}}(K).$$

□

In the following corollary, we show that if the convex body K is an origin-symmetric polytope, then the optimal problem has an origin-symmetric polytope solution.

Corollary 4.1. *Let $K \in \mathcal{K}_e^n$ be a polytope with vertices $u_1, u_2, \cdots, u_m$. If $M \in \mathcal{K}_e^n$ such that*

$$G_{\mathcal{E}}(K) = -\mathcal{E}_0(K, M) \text{ with } |M^*| = \omega_n,$$

then M is a polytope with vertices $v_1, v_2, \cdots, v_m$. Moreover, $v_i = \lambda_i u_i$ for $\lambda_i > 0$, $i \in \{1, 2, \cdots, m\}$.

Proof. Let $K \in \mathcal{K}_e^n$ be a polytope with vert$K = \{u_1, u_2, \cdots, u_m\}$ ($m = 2N \geq n + 1$). Obviously, $\{\frac{u_1}{|u_1|}, \cdots, \frac{u_m}{|u_m|}\} \subset S^{n-1}$ are not concentrated in any closed hemisphere of S^{n-1}. Then the integral curvature measure $J(K, \cdot)$ about convex body K is the discrete measure concentrated on $\{\frac{u_1}{|u_1|}, \cdots, \frac{u_m}{|u_m|}\} \subset S^{n-1}$. Let P be a polytope

$$P = conv\{\rho(M, \bar{u}_1)\bar{u}_1, \rho(M, \bar{u}_2)\bar{u}_2, \cdots, \rho(M, \bar{u}_m)\bar{u}_m\}. \tag{4.4}$$

where $\bar{u}_i = \frac{u_i}{|u_i|} \in S^{n-1}$ for $i = 1, ...m$. Let $u_{P,i} = \rho(M, \bar{u}_i)\bar{u}_i \in \partial P$ be vertices of polytope P, then these are $\lambda_i > 0$ such that $u_{P,i} = \lambda_i u_i$ for $i \in \{1, 2, \cdots, m\}$.

In the next, we only need prove $P = M$. By (4.4), we have $\rho(P, \bar{u}_i) = \rho(M, \bar{u}_i)$ ($1 \leq i \leq m$) and $P \subset M$. Thus

$$vrad(P^*) \geq vrad(M^*) = 1.$$

We obtain

$$
\begin{aligned}
G_{\mathcal{E}}(K) &= \sup_{L\in\mathcal{K}_e^n}\left\{\int_{S^{n-1}} \log[vrad(L^*)\rho(L,u)]dJ(K,u)\right\}\\
&\geq \int_{S^{n-1}} \log[vrad(P^*)\rho(P,u)]dJ(K,u)\\
&\geq \int_{S^{n-1}} \log\rho(P,u)dJ(K,u)\\
&= \sum_{i=1}^{m} \log\rho(P,\bar{u}_i)\cdot J(K,\{\bar{u}_i\})\\
&= \sum_{i=1}^{m} \log\rho(M,\bar{u}_i)\cdot J(K,\{\bar{u}_i\})\\
&= \int_{S^{n-1}} \log\rho(M,u)dJ(K,u)\\
&= \sup_{L\in\mathcal{K}_e^n}\left\{\int_{S^{n-1}} \log[vrad(L^*)\rho(L,u)]dJ(K,u)\right\}\\
&= G_{\mathcal{E}}(K).
\end{aligned}
$$

This shows that $vrad(P^*) = vrad(M^*) = 1$. Hence we know that $|M| = |P|$. Thus $P = M$. □

5. Conclusions

In this paper, the geominimal integral curvature on the convex body is introduced. The existence and uniqueness of the geominimal integral curvature are proved. Some other properties for the geominimal integral curvature , such as continuity, are investigated.

Conflict of interest

The author declares that they have no competing interests.

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