AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 14323–14337.
DOI: 10.3934/math.2022789
Received: 05 March 2022
Revised: 27 April 2022
Accepted: 10 May 2022
Published: 01 June 2022

Research article

Fibonacci collocation pseudo-spectral method of variable-order space-fractional diffusion equations with error analysis

A. S. Mohamed*

Department of Mathematics, Faculty of Science, Helwan University, Cairo 11795, Egypt

* **Correspondence:** Email: amany.saad78@yahoo.com; Tel: +201093648069.

Abstract: In this article, we evaluated the approximate solutions of one-dimensional variable-order space-fractional diffusion equations (sFDEs) by using a collocation method. This method depends on operational matrices for fractional derivatives and the integration of generalized Fibonacci polynomials. In this method, a Caputo fractional derivative of variable order is applied. Some properties of these polynomials (using boundary conditions) are presented to simplify and transform sFDEs into a system of equations with the expansion coefficients of the solution. Also, we discuss the convergence and error analysis of the generalized Fibonacci expansion. Finally, we compare the obtained results with those obtained via the other methods.

Keywords: fractional diffusion equation; variable-order fractional equation; collocation method; generalized Fibonacci polynomials; convergence analysis
Mathematics Subject Classification: 11B39, 26A33, 35K57

1. Introduction

As is known, the fractional order is a special case of ordinary and partial derivatives. Fractional diffusion equations (FDEs) (with linear and nonlinear forms) have attracted many scientists. They describe life phenomena by using the fractional order of differentiation and integration. There exist applications in different fields such as mathematics, chemistry, physics, biology, fluids, engineering and mechanics; all of them are described by ordinary and partial fractional differential equations. Fractional calculus investigates the rules and properties of the non-integer orders for the derivatives and integrals. Many researchers use numerical methods such as Adomain decomposition method (see [1–3]), variational iteration method (see [4–6]), the Haar wavelet method (see [7, 8]), finite difference method (see [9, 10]), finite element method (see [11, 12]), the homotopy analysis method (see [13, 14]) and the homotopy perturbation method (see [15, 16]).

Recently, spectral methods have been used to evaluate the approximate solutions of fractional differential equations. These methods are distinguishable from the previous methods with a small error and small number of unknowns. The Chebyshev method is used for solving a class of linear and nonlinear Lane-Emden type equations (see [17]), but the second-order two-point boundary value problems are solved by using the Chebyshev wavelets method (see [18]). The methods Jacobi and shifted Jacobi are applied to solve second-and fourth-order fractional diffusion wave equations and fractional wave equations with damping, as well as linear multi-order fractional differential equations (see [19, 20]). The system of high-order linear differential equations and multi-term fractional differential equations were solved by using Lucas and generalized Lucas methods (see [21–23]). Fibonacci and generalized Fibonacci methods are used to solve multi-term fractional differential equations (see [24, 25]).

The most common spectral methods are Galerkin, collocation and tau methods. Lane-Emden singular-type equations use a Galerkin operational matrix (see [26]); however, the third-and fifth-order differential equations use Petrov-Galerkin methods (see [27]). Tau methods are applied to solve multi-order fractional differential equations and variable coefficients fractional differential equations (see [28, 29]). The fractional differential equations with constant and variable orders mandate the use of a collocation method (see [30, 31]).

In this article, we solve FDEs with a variable-order and apply the generalized Fibonacci collocation (GFC) method. We obtain a system of nonlinear algebraic equations with unknown expansion coefficients; this system is studied by using Newton's method with the help of Mathematica. The obtained results are compared with the forward- substitution (FS) method and the fast divide and conquer (FDAC) method (see [32]). We show that our results are more efficient and yield higher accuracy.

The outline of this paper is as follows. In Section 2, the necessary definitions, properties of fractional calculus and generalized Fibonacci polynomials, which are used in the following sections, are introduced. In Section 3 the algorithm of this method is explored in details for a variable-order space-fractional diffusion equation (sFDE) with homogeneous boundary conditions. In Section 4 we investigate the convergence and error analysis of the proposed generalized Fibonacci expansion. We give examples, their numerical solutions and comparisons in Section 5 to prove the efficiency of the method. In the last section we present our conclusions.

2. Basic properties

In this section, we introduce the definition of a fractional derivative with variable order (see [33–35]). Then the important relations and the derivatives of the generalized Fibonacci polynomials, which are used in the following sections, are stated.

2.1. Important definition for fractional calculus

Definition 1. The Caputo fractional derivative with variable order $\gamma(z)$ is [32, 36]

$$ {}_0^C D_z^{\gamma(z)} g(z) = \frac{1}{\Gamma(\ell - \gamma(z))} \int_0^z (z-t)^{\ell-1-\gamma(z)}\, g^{(\ell)}(t)\, dt, \tag{2.1}$$

where $\ell - 1 < \gamma_{\min} \leq \gamma(z) \leq \gamma_{\max} < \ell$, $\ell \in \mathbb{N}$.

2.2. *Overview and relations of generalized Fibonacci polynomials*

If α and β are non zero real numbers, the recurrence relation for the generalized Fibonacci polynomials is [37]

$$\Upsilon_{j}^{\alpha,\beta}(z) = \alpha z\, \Upsilon_{j-1}^{\alpha,\beta}(z) + \beta\ \Upsilon_{j-2}^{\alpha,\beta}(z),\ j \geq 2, \tag{2.2}$$

with the following initial values:

$$\Upsilon_{0}^{\alpha,\beta}(z) = 1, \quad \Upsilon_{1}^{\alpha,\beta}(z) = \alpha z.$$

Its Binet form is

$$\Upsilon_{j}^{\alpha,\beta}(z) = \frac{\left(\alpha z + \sqrt{\alpha^2 z^2 + 4\beta}\right)^j - \left(\alpha z - \sqrt{\alpha^2 z^2 + 4\beta}\right)^j}{2^j \sqrt{\alpha^2 z^2 + 4\beta}},\ j \geq 0 \tag{2.3}$$

when there are no zeros in the denominator. The power form is:

$$\Upsilon_{j}^{\alpha,\beta}(z) = \sum_{i=0}^{j} \alpha^i\, \beta^{\frac{j-i}{2}}\, \zeta_{j+i} \binom{\frac{j+i}{2}}{\frac{j-i}{2}} z^i, \tag{2.4}$$

where

$$\zeta_j = \begin{cases} 1 & j\ even \\ 0 & j\ \ odd \end{cases}.$$

2.3. *Integer derivatives of Fibonacci polynomial*

If

$$\Phi(z) = \left[\Upsilon_{0}^{\alpha,\beta}(z), \Upsilon_{1}^{\alpha,\beta}(z), ..., \Upsilon_{N}^{\alpha,\beta}(z)\right]^T,$$

the first derivative of $\Phi(z)$ can be written as (see, [37])

$$\frac{d(\Phi)(z)}{dz} = \chi^{(1)}\,(\Phi)(z),$$

where $\chi^{(1)} = \left(\varkappa_{nm}^{(1)}\right)$ is the $(M+1) \times (M+1)$ matrix of derivatives.

$$\varkappa_{nm}^{(1)} = \begin{cases} (-1)^{\frac{n-m-1}{2}}(m+1)\,\alpha\,\beta^{\frac{n-m-1}{2}}, & n > m, (n+m)\text{odd} \\ 0 & otherwise \end{cases}$$

Also, the integer derivatives of $(\Phi)(z)$ can be easily written in the form (see [37])

$$\frac{d^{\iota}(\Phi)(z)}{dz^{\iota}} = \chi^{(\iota)}\,(\Phi)(z) = (\chi^{(1)})^{\iota}(\Phi)(z).$$

2.4. *Fractional derivatives of Fibonacci polynomial*

If λ is not an integer, then the fractional derivative of $(\Phi)(z)$ takes the form (see [37])

$$D^{\lambda}(\Phi)(z) = z^{-\lambda}\chi^{(\lambda)}\,(\Phi)(z),$$

where $\chi^{(\lambda)} = \left(\varkappa_{nm}^{\lambda}\right)$ is the $(M+1)\times(M+1)$ Fibonacci operational matrix of fractional derivatives of order λ, which has the form

$$\chi^{(\lambda)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \theta_{\lambda}(\lceil\lambda\rceil,0) & \theta_{\lambda}(\lceil\lambda\rceil,1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \theta_{\lambda}(i,0) & \cdots & \theta_{\lambda}(i,i) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \theta_{\lambda}(M,0) & \theta_{\lambda}(M,1) & \theta_{\lambda}(M,2) & \cdots & \theta_{\lambda}(M,M) \end{bmatrix}.$$

$\varkappa_{nm}^{\lambda}$ can be written in the form

$$\varkappa_{nm}^{\lambda} = \begin{cases} \theta_{\lambda}(n,m), & n \geq \lceil\lambda\rceil,\ n \geq m \\ 0 & otherwise \end{cases},$$

where

$$\theta_{\lambda}(n,m) = m\,\xi_m \sum_{l=\lceil\lambda\rceil}^{n} \frac{(-1)^{\frac{1+l-m}{2}}(\beta)^{\frac{n-m}{2}}\,l!\,\left(\frac{n+l-1}{2}\right)!}{\left(\frac{n-l-1}{2}\right)!\left(\frac{1+l-m}{2}\right)!\left(\frac{1+m+l}{2}\right)!\Gamma(1+l-\lambda)},$$

$$\xi_k = \begin{cases} \frac{1}{2}, & k=0, \\ 1, & otherwise \end{cases}.$$

So the fractional derivative has the relation

$$D^{\lambda}(\Phi)_n(z) = z^{-\lambda}\sum_{m=0}^{n}\theta_{\lambda}(n,m)\ (\Phi)_m(z).$$

3. A variable-order sFDE and the algorithm for the method

In this section, we explain the method for solving a variable-order sFDE with homogeneous boundary conditions by using generalized Fibonacci polynomials.

Suppose that the function $W(z)$ expanded in terms of generalized Fibonacci polynomials:

$$W(z) = \sum_{j=0}^{\infty} c_j\, \Upsilon_j^{\alpha,\beta}(z).$$

If we approximate this function as:

$$W(z) \approx W_N(z) = \sum_{j=0}^{N} c_j\ \Upsilon_j^{\alpha,\beta}(z) = C^T\ \Phi(z), \tag{3.1}$$

where

$$C^T = [c_0, c_1, ..., c_N], \tag{3.2}$$

$$\Phi(z) = \left[\Upsilon_0^{\alpha,\beta}(z), \Upsilon_1^{\alpha,\beta}(z), ..., \Upsilon_N^{\alpha,\beta}(z)\right]^T. \tag{3.3}$$

The variable-order sFDE is given by [32]

$$- W''(z) - f(z)\, {}_{0}^{C}D_{z}^{\gamma(z)} W(z) = g(z), \tag{3.4}$$

with the boundary conditions

$$W(0) = W(1) = 0, \tag{3.5}$$

and where $f(z)$ and $g(z)$ are continuous functions. Substituting Eq (2.1) with $\ell = 2$ into Eq (3.4), we have

$$- W''(z) - \frac{f(z)}{\Gamma(2-\gamma(z))} \int_0^z (z-t)^{1-\gamma(z)}\; W''(t)\, dt = g(z).$$

Suppose that $\upsilon(z) = W''(z)$; then,

$$- \upsilon(z) - \frac{f(z)}{\Gamma(2-\gamma(z))} \int_0^z (z-t)^{1-\gamma(z)}\; \upsilon(t)\, dt = g(z). \tag{3.6}$$

Substituting the expansion of Eq (3.1) into Eq (3.6), we have

$$-\left(\sum_{j=0}^{N} c_j\;\; \Upsilon_j^{\alpha,\beta}(z)\right) - \frac{f(z)}{\Gamma(2-\gamma(z))} \int_0^z (z-t)^{1-\gamma(z)} \left(\sum_{j=0}^{N} c_j\;\; \Upsilon_j^{\alpha,\beta}(t)\right) dt$$
$$= g(z). \tag{3.7}$$

Let

$$h_j(z) = -\Upsilon_j^{\alpha,\beta}(z) - \frac{f(z)}{\Gamma(2-\gamma(z))} \int_0^z (z-t)^{1-\gamma(z)}\, \Upsilon_j^{\alpha,\beta}(t)\, dt.$$

So Eq (3.7) can be written as:

$$\sum_{j=0}^{N} c_j\;\; h_j(z) = g(z).$$

Collocating this equation at the generalized Fibonacci roots, we obtain the following system of equations

$$\sum_{j=0}^{N} c_j\;\; h_j(z_i) = g(z_i). \tag{3.8}$$

The matrix form for this equation is:

$$H^T C = G.$$

So

$$C = \left(H^T\right)^{-1} G,$$

where $H = \left(h_j(z_i)\right), i, j = 0, 1, ..., N$ and $G = (g(z_i)) = [g(z_0), g(z_1), ..., g(z_N)]^T$. With the boundary conditions given by Eq (3.5), we have

$$C^T\; \Phi(0) = C^T\; \Phi(1) = 0. \tag{3.9}$$

Equations (3.8) and (3.9) give a linear system of equations with the coefficients c_j, $j = 0, 1, ...N$. These coefficients must be determined. Lastly, we integrate $\upsilon(z)$ two times to evaluate $W(z)$.

4. Investigation of convergence and error analysis

In this section, we explain the convergence and error analysis of generalized Fibonacci expansion.
Theorem 1. If $W(z)$ is defined on $[0, \iota]$ and $\left|W^{(i)}(s)\right| \leq L^i$, $i \geq 0$ where L is a positive constant, s is any point in $[0, \iota]$, and if $W(z)$ has the expansion:

$$W(z) = \sum_{j=0}^{\infty} c_j \ \Upsilon_j^{\alpha,\beta}(z),$$

then one has:

$$1) \left|c_j\right| < \frac{\sigma\left(\frac{L}{|\alpha|}\right)^{j+1}}{j!},$$

2) the series converges absolutely,

where

$$\sigma = \frac{6\,\Omega}{L\,s^2}\left|\frac{\beta}{\alpha}\right| \cosh\left(\frac{2L\ \sqrt{|\beta|}}{|\alpha|}\right),$$

and

$$\Omega = Li_6\left(\frac{\alpha^2 s^2}{3\,|\beta|}\right),$$

where $Li_6(z)$ is the polylogarithmic function.

Proof. See [37] □

Theorem 2. Let $W(z)$ satisfy the assumptions stated in Theorem 1, where $\varepsilon_N(z) = \sum_{i=N+1}^{\infty} c_j \ \Upsilon_j^{\alpha,\beta}(z)$; then we have the following truncation error estimate:

$$E_N = \max_{0 \leq z \leq \ell} \varepsilon_N(z) < \bar{E}\frac{\varsigma^N}{(N-1)!},$$

where $\bar{E} = \frac{\sigma L\, e^{\varsigma}}{|\alpha|}$, $\varsigma = \frac{L\,p}{|\alpha|}$ and $p = \sqrt{\alpha^2 \ell^2 + 2\,|\beta|}$.

Proof. See [37] □

Lemma 1. The derivatives of $\Upsilon_j^{(\gamma(z))}$ and Υ_j'' are denoted by the following estimates:

$$i) \left|\Upsilon_j^{(\gamma(z))}\right| \leq j^2;$$

$$ii) \left|\Upsilon_j''\right| \leq j^2.$$

Proof. From Lemma 5 [37]

$$\left|\Upsilon_j^{\alpha,\beta}(z)\right| \leq p^j.$$

By applying the differential operators to the right-hand side of Eq (2.4) two times, we have

$$\left|(\Upsilon_j^{\alpha,\beta}(z))''\right| = \left|\sum_{i=2}^{j} i\,(i-1)\alpha^i\,\beta^{\frac{j-i}{2}}\ \zeta_{j+i}\binom{\frac{j+i}{2}}{\frac{j-i}{2}} z^{i-2}\right| \leq j^2 p^j.$$

We obtain $i(i-1)$ and the end is $j^2 p^j$. Finally by induction on j, noting that $z < 1$ and $\ell - 1 \leq \gamma(z) \leq \ell$, So, we get the desired results. $\square$

Theorem 3. If $\upsilon(z) = W''(z) = \sum_{j=0}^{\infty} c_j \left(\Upsilon_j^{\alpha,\beta}(z)\right)''$ is the exact solution of Eq (3.6) and $\upsilon(z)$ is approximated by $\upsilon_N(z) = \sum_{j=0}^{N} c_j \left(\Upsilon_j^{\alpha,\beta}(z)\right)''$, let

$$R_N(z) = \left| -\upsilon(z) - \frac{f(z)}{\Gamma(2-\gamma(z))} \int_0^z (z-t)^{1-\gamma(z)} \, \upsilon(t)\, dt - g(z) \right|,$$

$$\mathfrak{R}_N = \max_{0 \leq z \leq \ell} R_N(z),$$

where $f(z)$ and $g(z)$ are continuous functions such that $|f(z)| \leq \lambda$, where λ is a positive constant. Then we have the following global error estimate:

$$\mathfrak{R}_N \leq \frac{1}{2} A \sigma \varsigma^{1+N} \left\{ 1 + \frac{\lambda \ell^{2-\ell}}{\Gamma(3-\ell)} \right\} . \left\{ \frac{(1+N+\varsigma)}{\Gamma(1+N)} + \frac{(1+\varsigma)\varsigma}{\Gamma(2+N)} e^{\varsigma} \right\},$$

where $A = \frac{2L}{|\alpha|}$.

Proof. From Eq (3.6)

$$g(z) = -\upsilon(z) - \frac{f(z)}{\Gamma(2-\gamma(z))} \int_0^z (z-t)^{1-\gamma(z)} \, \upsilon(t)\, dt.$$

So

$$R_N(z) = \left| (\upsilon(z) - \upsilon_N(z)) + \frac{f(z)}{\Gamma(2-\gamma(z))} \int_0^z (z-t)^{1-\gamma(z)} \, (\upsilon(z) - \upsilon_N(z))\, dt \right|.$$

By integrating and taking $\ell - 1 \leq \gamma(z) \leq \ell$, so we have

$$R_N(z) \leq |\upsilon(z) - \upsilon_N(z)| \left\{ 1 + \frac{\lambda}{\Gamma(2-\ell)} \frac{z^{2-\ell}}{2-\ell} \right\}$$

From Theorems 1 and 2

$$R_N(z) \leq \sum_{j=N+1}^{\infty} \left(\frac{\sigma \left(\frac{L}{|\alpha|}\right)^{j+1}}{j!} \right) \left\{ 1 + \frac{\lambda}{\Gamma(2-\ell)} \frac{z^{2-\ell}}{2-\ell} \right\} \left| \left(\Upsilon_j^{\alpha,\beta}(z)\right)'' \right|.$$

By Lemma 1, we have

$$R_N(z) \leq \frac{1}{2} A \sigma \left\{ 1 + \frac{\lambda}{\Gamma(2-\ell)} \frac{z^{2-\ell}}{2-\ell} \right\} \sum_{j=N+1}^{\infty} \frac{\varsigma^j}{j!} j^2.$$

Using some calculations, we obtain

$$R_N(z) \leq \frac{1}{2} \frac{A\sigma\varsigma}{\Gamma(1+N)} \left\{ 1 + \frac{\lambda z^{2-\ell}}{\Gamma(3-\ell)} \right\} \times$$

$$\left\{\varsigma^{N}\left(1+N+\varsigma\right)+\left(1+\varsigma\right)e^{\varsigma}\left(\Gamma(1+N)-\Gamma(1+N,\varsigma)\right)\right\},$$

where $\Gamma(\ )$ and $\Gamma(\ ,\ )$ are gamma and incomplete gamma functions respectively; therefore

$$\mathfrak{R}_N \leq \frac{1}{2}\frac{A\sigma\varsigma}{\Gamma(1+N)}\left\{1+\frac{\lambda\ell^{2-\ell}}{\Gamma(3-\ell)}\right\}.\left\{\varsigma^{N}\left(1+N+\varsigma\right)+e^{\varsigma}\frac{\left(1+\varsigma\right)\varsigma^{N+1}}{N+1}\right\}.$$

So the proof is completed ◻

Theorem 4. If $W(z) = \sum_{j=0}^{\infty} c_j\ \Upsilon_j^{\alpha,\beta}(z)$ is the exact solution of Eq (3.4) and it satisfies the hypotheses of Theorem 1 and if $W(z)$ is approximated by $W_N(z) = \sum_{j=0}^{N} c_j\ \Upsilon_j^{\alpha,\beta}(z)$, let

$$R_N(z) = \left|W_N''(z) + f(z)\,{}_0^C D_z^{\gamma(z)} W_N(z) + g(z)\right|,$$

$$\mathfrak{R}_N = \max_{0\leq z\leq \ell} R_N(z).$$

Then we have the following global error estimate:

$$\mathfrak{R}_N \leq \frac{1}{2}(1+\lambda)A\sigma\varsigma^{N+1}\left\{\frac{(1+N+\varsigma)}{\Gamma(1+N)}+e^{\varsigma}\frac{\left(1+\varsigma\right)\varsigma}{\Gamma(2+N)}\right\}.$$

Proof. From Eq (3.4)

$$g(z) = -W''(z) - f(z)\,{}_0^C D_z^{\gamma(z)} W(z).$$

So

$$\begin{aligned} R_N(z) &= \left|(W_N'' - W'') + f(z)\,{}_0^C D_z^{\gamma(z)}\left(W(z) - W_N(z)\right)\right| \\ &\leq \left|W_N'' - W''\right| + \left|f(z)\,{}_0^C D_z^{\gamma(z)}\left(W(z) - W_N(z)\right)\right|. \end{aligned}$$

From Theorems 1 and 2

$$R_N(z) \leq \sum_{j=N+1}^{\infty}\left(\frac{\sigma\left(\frac{L}{|\alpha|}\right)^{j+1}}{j!}\right)\left(\left|(\Phi(z))''\right| + |f(z)|\left|(\Phi(z))^{(\gamma(z))}\right|\right).$$

By Lemma 1, we have

$$R_N(z) \leq \sum_{j=N+1}^{\infty}\frac{\sigma\left(\frac{L}{|\alpha|}\right)^{j+1}}{j!}j^2 p^j\left(1+\lambda\right).$$

Therefore

$$R_N(z) \leq \frac{(1+\lambda)}{2\Gamma(1+N)}A\sigma\varsigma\left\{\varsigma^{N}\left(1+N+\varsigma\right)+\left(1+\varsigma\right)e^{\varsigma}\left(\Gamma(1+N)-\Gamma(1+N,\varsigma)\right)\right\}.$$

In the same way as with the previous theorem, we obtain

$$\mathfrak{R}_N \leq \frac{1}{2}\frac{(1+\lambda)}{\Gamma(1+N)}A\sigma\varsigma^{N+1}\left\{\left(1+N+\varsigma\right)+e^{\varsigma}\frac{\left(1+\varsigma\right)\varsigma}{N+1}\right\}.$$

So the proof is completed ◻

5. Numerical examples

In this section, we solve some examples by applying Eqs (3.4) and (3.5) using the generalized Fibonacci polynomials.

Example 1. Suppose we have Eq (3.4) with $f(z) = 1$ *and*

$$\gamma(z) = (\gamma_0 - \gamma_1)\left(1 - z - \frac{\sin(2\pi(1-z))}{2\pi}\right) + \gamma_1,$$

with $\gamma_0 = 1.2$ and $\gamma_1 = 1.6$. The exact solution is $W(z) = z^4(1-z)$.

In Tables 1 and 2, we observe that the absolute error obtained via the GFC method is better than that obtained via the FS method and FDAC method [32]. Figure 1 illustrates the results of the present method at $N = 5, 8$ and 10. The figure shows that the convergence is exponential.

Table 1. Maximum absolute error E with GFC method.

α	β	N	E	N	E	N	E
1	1	5	2.7×10^{-15}	8	3.3×10^{-15}	10	4.8×10^{-15}
2	1		1.3×10^{-15}		5.4×10^{-16}		9.2×10^{-16}
3	-2		2.8×10^{-16}		4.8×10^{-16}		4.4×10^{-16}
2	-1		2.2×10^{-16}		2.5×10^{-16}		2.1×10^{-16}

Table 2. Maximum absolute error E with FS and FDAC methods.

N	FS [32]	$FDAC$ [32]
2^8	4.4×10^{-6}	4.3×10^{-6}
2^9	1.2×10^{-6}	1.1×10^{-6}
2^{10}	2.7×10^{-7}	2.7×10^{-7}
2^{11}	6.8×10^{-8}	6.6×10^{-8}
2^{12}	1.7×10^{-8}	1.7×10^{-8}
2^{13}	4.2×10^{-9}	4.4×10^{-9}

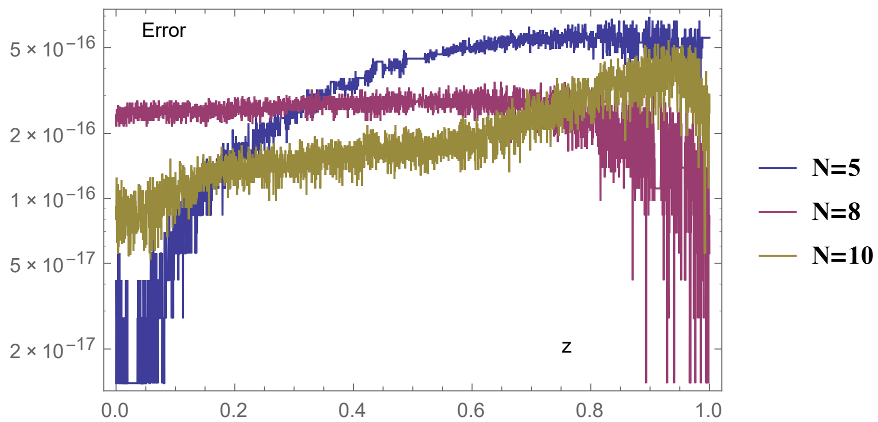

Figure 1. Graph of the absolute error obtained for N=5, 8 and 10 and different values of α and β.

Example 2. Suppose we have Eq (3.4) with $f(z) = 1$ and

$$\gamma(z) = (\gamma_1 - \gamma_0)\, z + \gamma_0.$$

Tables 3–5 compare the methods at different values of γ_0 and γ_1. We notice that the error calculated using this method is the best because it is the smallest. Also, we prove the accuracy and efficiency of the method. The absolute error at these values is plotted in Figures 2 and 3. We observe from the Figures that the convergence is exponential and the errors are better for different values of α and β.

With $\gamma_0 = 1.2$, $\gamma_1 = 1.6$

Table 3. Maximum absolute error E with GFC method.

α	β	N	E	N	E	N	E
1	1	5	4.4×10^{-15}	8	4.4×10^{-15}	10	6.4×10^{-15}
2	1		8.9×10^{-16}		1×10^{-15}		1.1×10^{-15}
3	-2		2.8×10^{-16}		3.4×10^{-16}		3.7×10^{-16}
2	-1		1.9×10^{-16}		2×10^{-16}		4×10^{-16}

Table 4. Maximum absolute error E with FS and FDAC methods.

N	FS [32]	$FDAC$ [32]
2^8	4.4×10^{-6}	4.3×10^{-6}
2^9	1.2×10^{-6}	1.1×10^{-6}
2^{10}	2.7×10^{-7}	2.7×10^{-7}
2^{11}	6.8×10^{-8}	6.6×10^{-8}
2^{12}	1.7×10^{-8}	1.7×10^{-8}
2^{13}	4.2×10^{-9}	4.4×10^{-9}

With $\gamma_0 = 1$, $\gamma_1 = 1.5$

Table 5. Maximum absolute error E with GFC method.

α	β	N	E	N	E	N	E
1	1	5	4.4×10^{-15}	8	5.5×10^{-15}	10	4×10^{-15}
2	1		6.7×10^{-16}		7.4×10^{-16}		1.7×10^{-15}
3	-2		3.1×10^{-16}		2.3×10^{-16}		4.5×10^{-16}
2	-1		1.7×10^{-16}		5×10^{-16}		2.5×10^{-16}

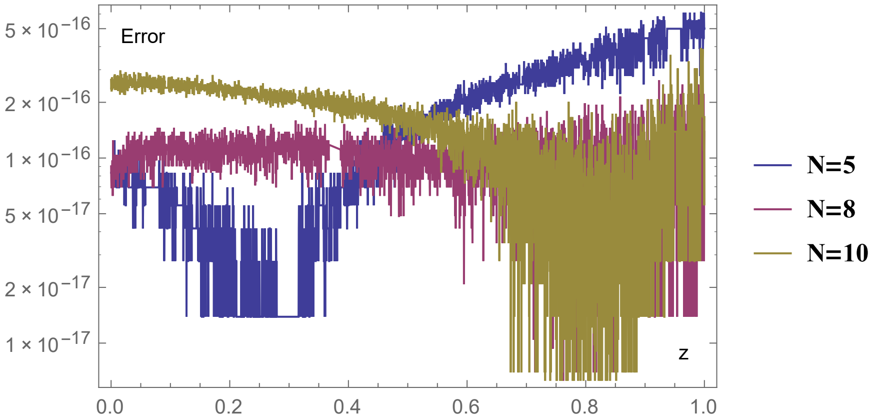

Figure 2. Graph of the absolute error obtained for N=5, 8 and 10 and different values of α and β.

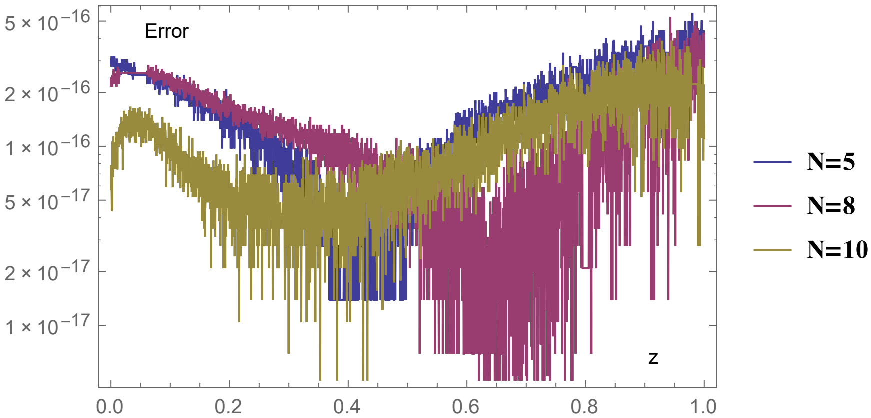

Figure 3. Graph of the absolute error obtained for N=5, 8 and 10 and different values of α and β

.

Example 3. Suppose we have Eq (3.4) with $f(z) = z$ and $\gamma(z) = \frac{1+z}{3}$.

The exact solution is $W(z) = z(1-z)$. The results of the present method are shown for $N = 5, 8$ and 10 in Table 6 and Figure 4. It is clear from the figure that the absolute errors decrease drastically as a result of increasing the number of steps and the values of α and β. Table 5 lists the numerical results obtained via the proposed method for N= 8, 12 and 16 and different values of a and b. The absolute errors of this method are plotted in Figure 2. We observe from the figure that the convergence is exponential.

Table 6. Maximum absolute error E with GFC method.

α	β	N	E	N	E	N	E
1	1	5	2.2×10^{-16}	8	6.3×10^{-16}	10	4.3×10^{-16}
2	1		2.2×10^{-16}		6.6×10^{-16}		5.3×10^{-16}
3	-2		2.8×10^{-16}		5.2×10^{-16}		3.2×10^{-16}

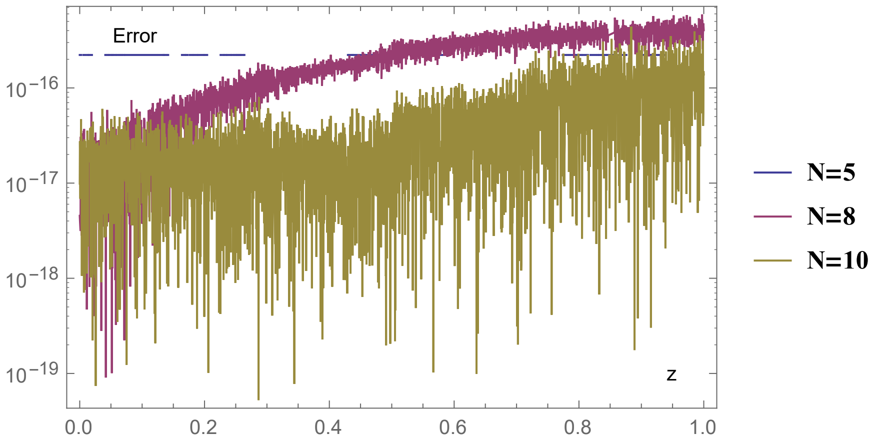

Figure 4. Graph of the absolute error obtained for N=5, 8 and 10 and different values of α and β.

6. Conclusions

This paper details an attempt to solve the variable-order sFDE by using a collocation method based on the operational matrix of fractional derivatives of the generalized Fibonacci polynomials. We modified the variable-order sFDE for compatibility with a system of linear algebraic equations which were solved by using Mathematica software. Then we evaluated the errors. The spectral results which were used throughout, indicate strong adequacy, viability and ease of application. Finally, we discussed the convergence and error analysis of our method.

Conflict of interest

The authors declare no conflict of interest.

References

1. V. Daftardar-Gejji, H. Jafari, Solving a multi-order fractional differential equation using Adomain decomposition method, *Appl. Math. Comput.,* **89** (2007), 541–548.
2. N. T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, *Appl. Math. Comput.,* **131** (2002), 517–529. https://doi.org/10.1016/S0096-3003(01)00167-9
3. A. M. El-Sayed, S. H. Behiry, W. E. Raslan, Adomain decomposition method for solving an intermediate fracional advection-dispersion equation, *Comput. Math. Appl.,* **59** (2010), 1759–1765. https://doi.org/10.1016/j.camwa.2009.08.065
4. A. Yanga, A. Xiao, H. Su, Convergence of the variational iteration method for solving multi-order fractional differential equations, *Comput. Math. Appl.,* **60** (2010), 2871–2879. https://doi.org/10.1016/j.camwa.2010.09.044
5. S. Das, Analytical solution of a fractional diffusion equation by variational iteration method, *Comput. Math. Appl.,* **57** (2009), 483–487. https://doi.org/10.1016/j.camwa.2008.09.045

6. B. K. Singh, P. Kumar, Fractional variational iteration method for solving fractional partial differential equations with proportional delay, *Int. J. Differ. Equ.,* **2017** (2017). https://doi.org/10.1155/2017/5206380

7. Y. L. Li, Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, *Appl. Math. Comput.,* **216** (2010), 2276–2285. https://doi.org/10.1016/j.amc.2010.03.063

8. Y. Chen, M. Yi, C. Yu, Error analysis for numerical solution of fractional differential equation by Haar wavelet method, *J. Comput. Sci.,* **3** (2012), 376–373. https://doi.org/10.1016/j.jocs.2012.04.008

9. M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.,* **56** (2006), 80–90. https://doi.org/10.1016/j.apnum.2005.02.008

10. N. H. Sweilama, M. M. Khaderb, A. M. Nagyb, Numerical solution of two-sided space-fractional wave equation using finite difference method, *J. Comput. Appl. Math.,* **235** (2011), 2832–2841. https://doi.org/10.1016/j.cam.2010.12.002

11. Y. L. Jiang, J. T. Ma, High-order finite element methods for time-fractional partial differential equations, *J. Comput. Appl. Math.,* **235** (2011), 3285–3290. https://doi.org/10.1016/j.cam.2011.01.011

12. X. G. Zhu, J. G. Wang, Y. F. Nie, Z. Z. Yang, Finite element method for time-space-fractional Schrdinger equation, *Electron. J. Differ. Equ.,* **2017** (2017), 1–18.

13. I. Hashim, O. Abdulaziz, S. Momani, Homotopy analysis method for fractional IVPs, *Commun. Nonlinear Sci.,* **14** (2009), 674–684. https://doi.org/10.1016/j.cnsns.2007.09.014

14. S. Das, P. C. Ray, R. K. Bera, P. Sarkar, Solution of non-linear fractional differential equation (NFDE) by homotopy analysis method, *Int. J. Sci. Res. Edu.,* **3** (2015), 3084–3103.

15. O. Abdulaziz, I. Hashim, S. Momani, Application of homotopy-perturbation method to fractional IVPs, *J. Comput. Appl. Math.,* **216** (2008), 574–584. https://doi.org/10.1016/j.cam.2007.06.010

16. B. Ghazanfari, A. Sepahvandzadeh, Homotopy perturbation method for solving fractional Bratu-type equation, *J. Math. Model,* **2** (2015), 143–155.

17. E. H. Doha, W. M. Abd- Elhameed, Y. H. Youssri, Second kind Chebyshev operational matrix for solving fractional differential equations of Lane-Emden type, *New Astron.,* **23** (2013), 113–117. https://doi.org/10.1016/j.newast.2013.03.002

18. W. M. Abd-Elhameed, E. H. Doha, Y. H. Youssri, New spectral second kind Chebyshev wavelets algorithm for solving linear and nonlinear second-order differential equations involving singular ang Bratu type equations, *Abstr. Appl. Anal.,* **2013** (2013), 1–9. https://doi.org/10.1155/2013/715756

19. A. H. Bhrawy, E. H. Doha, D. Baleanu, S. S. Ezz-Eldien, A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations, *J. Comput. Phys.,* **293** (2015), 142–156. https://doi.org/10.1016/j.jcp.2014.03.039

20. E. H. Doha, A. H. Bhrawy, D. Baleanu, S. S. Ezz-Eldien, On shifted Jacobi spectral approximations for solving fractional differential equations, *Appl. Math. Comput.,* **219** (2013), 8042–8056. https://doi.org/10.1016/j.amc.2013.01.051

21. M. Cetin, M. Sezer, C. Guler, Lucas polynomial approach for system of high-order linear differential equations and residual error estimation, *Math. Probl. Eng.,* **2015** (2015). https://doi.org/10.1155/2015/625984

22. W. M. Abd-Elhameed, Y. H. Youssri, Spectral solutions for fractional differential equations via a novel Lucas operational matrix of fractional derivatives, *Rom. J. Phys.,* **61** (2016), 795–813.

23. W. M. Abd-Elhameed, Y. H. Youssri, Generalized Lucas polynomials sequence approach for fractional differential equations, *Nonlinear Dyn.,* **89** (2017), 1341–1355. https://doi.org/10.1007/s11071-017-3519-9

24. Y. H. Youssri, W. M. Abd-Elhameed, Spectral solutions for multi-term fractional initial value problems using a new Fibonacci operational matrix of fractional integration, *Progr. Fract. Differ. Appl.,* **2** (2016), 141–151. https://doi.org/10.18576/pfda/020207

25. A. G. Ataa, G. M. Moatimid, Y. H. Youssri, Generalized Fibonacci operational collocation approach for fractional initial value problems, *Int. J. Appl. Comput. Math.,* **5** (2019), 1–9. https://doi.org/10.1007/s40819-018-0597-4

26. W. M. Abd-Elhameed, New Galerkin operational matrix of derivatives for solving Lane-Emden singular-type equations, *Eur. Phys. J. Plus,* **130** (2015), 1–12. https://doi.org/10.1140/epjp/i2015-15052-2

27. W. M. Abd-Elhameed, E. H. Doha, Y. H. Youssri, Efficient spectral Petrov-Galerkin methods for third and fifth-order differential equations using general parameters generalized Jacobi polynomials, *Quaest. Math.,* **36** (2013), 15–38. https://doi.org/10.2989/16073606.2013.779945

28. F. Ghoreishi, S. Yazdani, An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis, *Comput. Math. Appl.,* **61** (2011), 30–43. https://doi.org/10.1016/j.camwa.2010.10.027

29. A. H. Bhrawy, A. S. Alofi, S. S. Ezz-Eldien, A quadrature tau method for variable coefficients fractional differential equations, *Appl. Math. Lett.,* **24** (2011), 2146–2152. https://doi.org/10.1016/j.aml.2011.06.016

30. S. Esmaeili, M. Shamsi, Y. Luchko, Numerical solution of fractional differential equations with a collocation method based on Muntz polynomials, *Comput. Math. Appl.,* **62** (2011), 918–929. https://doi.org/10.1016/j.camwa.2011.04.023

31. R. M. Hafez, Y. H. Youssri, Legendre-collocation spectral solver for variable-order fractional differential equations, *Comput. Meth. Differ. Equ.,* **8** (2020), 99–110. https://dx.doi.org/10.22034/cmde.2019.9465

32. J. Jia, X. Zheng, H. Wang, A fast method for variable-order space-fractional diffusion equations, *Numer. Algorithms,* 2020. https://doi.org/10.1007/s11075-020-00875-z

33. S. M. Kenneth, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Academic Press, 1993.

34. I. Podlubny, *Fractional differential equations*, San Diego.California: Academic Press, 1999.

35. E. D. Rainville, *Special functions*, Chelsea, New York, 1960.

36. R. M. Hafez, Y. H. Youssri, Jacobi collocation scheme for variable-order fractional reaction-subdiffusion equation, *J. Comput. Appl. Math.,* **37** (2018), 5315–5333. https://doi.org/10.1007/s40314-018-0633-3

37. W. M. Abd-Elhameed, Y. H. Youssri, Spectral tau algorithm for certain coupled system of fractional differential equations via generalized Fibonacci polynomial sequence, *Iran. J. Sci. Technol. A,* **43** (2019), 543–554. https://doi.org/10.1007/s40995-017-0420-9