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Research article

On inequalities of Hermite-Hadamard type via n-polynomial exponential type s-convex functions

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Abstract: In this paper, a new class of Hermite-Hadamard type integral inequalities using a strong type of convexity, known as n-polynomial exponential type s-convex function, is studied. This class is established by utilizing the Hölder's inequality, which has several applications in optimization theory. Some existing results of the literature are obtained from newly explored consequences. Finally, some novel limits for specific means of positive real numbers are shown as applications.

Keywords: Hermite-Hadamard's type inequalities; n-polynomial exponential convex functions; s-convex function; special means; Hölder's inequality
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1. Introduction

Convex function analysis begins with real-valued functions of a real variable. They serve as a model for deep generalization into the context of numerous variables and a variety of applications; see book [1,2] for more details. Convexity theory gives us a coherent framework for developing extremely efficient, fascinating, and strong numerical tools for tackling and solving a wide range of problems in various domains of mathematics. Zhao et al. [3] utilized the convexity and concavity to modify first kind Bessel functions. Several intriguing generalizations and extensions of classical convexity have been employed in optimization and mathematical inequalities.

Convexity theory was also influential in the development of inequalities theory. Baleanu et al. [4] studied a class of Hermite-Hadamard-Fejer type inequalities using fractional integral. In 2021, Wu et al. [5] explored another class of inequalities via an extended fractional operator. By using this theory, lots of work has been done in the field of inequalities [6–9]. The theory of convexity covers a wide range of convex functions, including exponentially convex function, m-convex functions, and s-convex function. It has vast applications in many disciplines of theoretical and practical mathematics. A powerful and elegant connection between analysis and geometry characterizes the current perspective on convex functions [10].

Definition 1.1. [1] The function $\Omega : K \to \mathbb{R}$, where K is known to be convex on a compact subset Ω of real numbers if for any two points θ and ϑ in K and for any $0 \leq r \leq 1$

$$\Omega(r\theta + (1-r)\vartheta) \leq r\Omega(\theta) + (1-r)\Omega(\vartheta) \tag{1.1}$$

holds.

The classical Hermite-Hadamard's inequality is widely studied via convex functions [11–13]. It is stated in the following definition.

Definition 1.2. Let $\Omega : K \to \mathbb{R}$ be a convex mapping, where $K \subseteq \mathbb{R}$. For any two points θ and ϑ in K with $\theta < \vartheta$

$$\Omega\left(\frac{\theta+\vartheta}{2}\right) \leq \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt \leq \frac{\Omega(\theta)+\Omega(\vartheta)}{2}. \tag{1.2}$$

If Ω is concave, the inequalities are reversed.

The exponentially convex functions are used to modify for statistical learning, sequential prediction, and stochastic optimization; see [14, 15]. Antczak et al. and Noor et al. were the first to propose the class of exponentially convex functions [16–18]. Furthermore, the concept of an exponentially s-convex function was explored by Mehreen, and Anwar [19].

Definition 1.3. [20, Definition 3.1] If $m \in N$ and $s \in [\ln 2.4, 1]$, then the real-valued function $\Omega : K \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial exponentially s-convex function if the inequality

$$\Omega(r\theta + (1-r)\vartheta) \leq \frac{1}{n}\sum_{i=1}^{n}(e^{sr}-1)^{i}\Omega(\theta) + \frac{1}{n}\sum_{i=1}^{n}(e^{s(1-r)}-1)^{i}\Omega(\vartheta) \tag{1.3}$$

holds for all $\theta, \vartheta \in K$ and $r \in [0, 1]$.

Definition 1.4. The beta function is defined by the integral

$$B(u, v) = \int_{0}^{1} r^{u-1}(1-r)^{v-1}dr$$

for $\mathrm{Re}(u) > 0$ and $\mathrm{Re}(v) > 0$.

The following equality can be built in [21] and is stated as:

Lemma 1.5. *Let $K \subseteq \mathbb{R}$ and $\Omega : K \to \mathbb{R}$ be a differentiable function on K^0. If θ and ϑ are any two points in K with $\theta < \vartheta$, then we have*

$$\frac{\Omega(\theta)+\Omega(\vartheta)}{2} - \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt = \frac{(\vartheta-\theta)^2}{2}\int_{0}^{1} r(1-r)\Omega''(r\theta + (1-r)\vartheta)dr. \tag{1.4}$$

In [22], Kirmaci presented the following lemma.

Lemma 1.6. *Let $\Omega : K \to \mathbb{R}$ be a differentiable function on K^0 and $K \subseteq \mathbb{R}$. If θ and ϑ are any two points in K^0 with $\theta < \vartheta$, then we have*

$$\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt - \Omega\left(\frac{\theta+\vartheta}{2}\right) = (\vartheta-\theta)\left[\int_0^{\frac{1}{2}} r\Omega'(\vartheta+(\theta-\vartheta)r)dr + \int_{\frac{1}{2}}^{1}(r-1)\Omega'(\vartheta+(\theta-\vartheta)r)dr\right]. \tag{1.5}$$

Lemma 1.7. *Let $\Omega : K \to \mathbb{R}$ be a differentiable mapping on K^0, where $K \subseteq \mathbb{R}$ and $\theta < \vartheta$. If Ω is a convex mapping, then the following inequalities hold*

$$\Omega\left(\frac{\theta+\vartheta}{2}\right) \le \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt \le \frac{2\Omega\left(\frac{\theta+\vartheta}{2}\right)+\Omega\left(\frac{3\vartheta-\theta}{2}\right)+\Omega\left(\frac{3\theta-\vartheta}{2}\right)}{4} \tag{1.6}$$

and

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt - \frac{\Omega\left(\frac{\theta+\vartheta}{2}\right)}{2}\right| \le \left|\frac{\Omega\left(\frac{3\vartheta-\theta}{2}\right)+\Omega\left(\frac{3\theta-\vartheta}{2}\right)}{4}\right|. \tag{1.7}$$

The proof of above lemma can be found in [21].

Lemma 1.8. [22, Lemma 2.1] *Let $\Omega : K^0 \to \mathbb{R}$ be a differentiable function on K^0, where $K^0 \subset \mathbb{R}$. If $\theta, \vartheta \in K^0$ with $\theta < \vartheta$ and $\Omega' \in L[\theta, \vartheta]$, then the following equality holds*

$$\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt - \Omega\left(\frac{\theta+\vartheta}{2}\right) = (\vartheta-\theta)\left[\int_0^{\frac{1}{2}} r\Omega'(r\theta+(1-r)\vartheta)ds + \int_{\frac{1}{2}}^{1}(r-1)\Omega'(r\theta+(1-r)\vartheta)dr\right]. \tag{1.8}$$

2. Main results

This section contains significant outcomes on Hermite-Hadamard type inequalities that are evaluated using n-polynomial exponentially s-convex functions by utilizing Hölder's inequality. The definition and properties of the Beta function are used to yield the main results.

Theorem 2.1. *Let $\Omega : K \to \mathbb{R}$ be a differentiable function on K^0, where $K \subseteq \mathbb{R}$. If $|\Omega'|$ is bounded, i.e., $|\Omega'(x)| \le L$ and n-polynomial exponentially s-convex on $[\theta, \vartheta]$, where $s \in [\ln 2.4, 1]$, then the following inequality holds:*

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt - \Omega\left(\frac{\theta+\vartheta}{2}\right)\right| \le \frac{(\vartheta-\theta)}{4}\frac{1}{n}\sum_{i=1}^{n}(e^s-1)^i\left(|\Omega'(\theta)|+|\Omega'(\vartheta)|\right) \le \frac{(\vartheta-\theta)}{2}\frac{L}{n}\sum_{i=1}^{n}(e^s-1)^i. \tag{2.1}$$

Proof. By Lemma 1.8, we can write

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\Omega\left(\frac{\theta+\vartheta}{2}\right)\right|$$
$$\leq(\vartheta-\theta)\left[\int_{0}^{\frac{1}{2}}r\left|\Omega'(r\theta+(1-r)\vartheta)\right|dr+\int_{\frac{1}{2}}^{1}(1-r)\left|\Omega'(r\theta+(1-r)\vartheta)\right|dr\right]. \tag{2.2}$$

Since the function $|\Omega'|$ is n-polynomial exponentially s-convex on $[\theta,\vartheta]$ and the facts $e^{sr}\leq e^{s}$ and $e^{s(1-r)}\leq e^{s}$ are true for any $0\leq r\leq 1$, therefore for any $0\leq r\leq 1$, we obtain

$$\begin{aligned}\int_{0}^{\frac{1}{2}}r\left|\Omega'(r\theta+(1-r)\vartheta)\right|dr&\leq|\Omega'(\theta)|\int_{0}^{\frac{1}{2}}\frac{1}{n}\sum_{i=1}^{n}(e^{sr}-1)^{i}rdr+|\Omega'(\vartheta)|\int_{0}^{\frac{1}{2}}\frac{1}{n}\sum_{i=1}^{n}(e^{s(1-r)}-1)^{i}rdr\\&\leq|\Omega'(\theta)|\frac{1}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}\int_{0}^{\frac{1}{2}}rdr+|\Omega'(\vartheta)|\frac{1}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}\int_{0}^{\frac{1}{2}}rdr\\&\leq\frac{1}{8n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left[|\Omega'(\theta)|+|\Omega'(\vartheta)|\right]\\&\leq\frac{L}{4n}\sum_{i=1}^{n}(e^{s}-1)^{i}.\end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned}\int_{\frac{1}{2}}^{1}(1-r)\left|\Omega'(r\theta+(1-r)\vartheta)\right|dr&\leq\frac{1}{8n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left[|\Omega'(\theta)|+|\Omega'(\vartheta)|\right]\\&\leq\frac{L}{4n}\sum_{i=1}^{n}(e^{s}-1)^{i}.\end{aligned} \tag{2.4}$$

By substituting (2.3) and (2.4) in (2.2), we get (2.1). □

Theorem 2.2. *Let $\Omega:K\to\mathbb{R}$ be a differentiable function on K^{0}, where $K\subseteq\mathbb{R}$. If $\gamma>1$, $|\Omega'|^{\sigma}$ is bounded, i.e., $|\Omega'(x)|^{\sigma}\leq L$ for all $\sigma>1$ and n-polynomial exponentially s-convex on $[\theta,\vartheta]$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\Omega\left(\frac{\theta+\vartheta}{2}\right)\right|&\leq(\vartheta-\theta)\left(\frac{1}{2(\gamma+1)}\right)^{\frac{1}{\gamma}}\left(\frac{1}{2n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left(|\Omega'(\theta)|^{\sigma}+|\Omega'(\vartheta)|^{\sigma}\right)\right)^{\frac{1}{\sigma}}\\&\leq(\vartheta-\theta)\left(\frac{1}{2(\gamma+1)}\right)^{\frac{1}{\gamma}}\left(\frac{L}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}\right)^{\frac{1}{\sigma}}.\end{aligned} \tag{2.5}$$

Proof. By using Lemma 1.8 and Hölder's inequality, we deduce

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\Omega\left(\frac{\theta+\vartheta}{2}\right)\right| \leq (\vartheta-\theta)\left[\left(\int_{0}^{\frac{1}{2}} r^{\gamma}dr\right)^{\frac{1}{\gamma}}\left(\int_{0}^{\frac{1}{2}}|\Omega'(r\theta+(1-r)\vartheta)|^{\sigma}dr\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-r)^{\gamma}dr\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^{1}|\Omega'(r\theta+(1-r)\vartheta)|^{\sigma}dr\right)^{\frac{1}{\sigma}}\right]. \tag{2.6}$$

From n-polynomial exponentially s-convexity of $|\Omega'|^{\sigma}$ and the facts $e^{sr} \leq e^{s}$ and $e^{s(1-r)} \leq e^{s}$, for any $0 \leq r \leq 1$ and boundedness of $|\Omega'|^{\sigma}$ for $\sigma > 1$, we get

$$\int_{0}^{\frac{1}{2}}|\Omega'(r\theta+(1-r)\vartheta)|^{\sigma}dr \leq \frac{1}{2n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left(|\Omega'(\theta)|^{\sigma}+|\Omega'(\vartheta)|^{\sigma}\right)$$
$$\leq \frac{L}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}. \tag{2.7}$$

Similarly, we have

$$\int_{\frac{1}{2}}^{1}|\Omega'(r\theta+(1-r)\vartheta)|^{\sigma}dr \leq \frac{1}{2n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left(|\Omega'(\theta)|^{\sigma}+|\Omega'(\vartheta)|^{\sigma}\right)$$
$$\leq \frac{L}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}. \tag{2.8}$$

Using relations (2.7), (2.8) in (2.6) and by simple calculations, we obtain the desired result. □

Theorem 2.3. *Under the assumptions of Theorem 2.2, we have*

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\Omega\left(\frac{\theta+\vartheta}{2}\right)\right| \leq 2L(\vartheta-\theta)\left(\frac{1}{2(\gamma+1)}\right)^{\frac{1}{\gamma}}\left(\frac{1}{2n}\sum_{i=1}^{n}(e^{s}-1)^{i}\right)^{\frac{1}{\sigma}}. \tag{2.9}$$

Proof. Consider

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\Omega\left(\frac{\theta+\vartheta}{2}\right)\right| \leq (\vartheta-\theta)\left(\frac{1}{2(\gamma+1)}\right)^{\frac{1}{\gamma}}\left(\frac{1}{2n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left(|\Omega'(\theta)|^{\sigma}+|\Omega'(\vartheta)|^{\sigma}\right)\right)^{\frac{1}{\sigma}}.$$

By using the assumptions

$$\theta_1 = \frac{1}{2n}\sum_{i=1}^{n}(e^{s}-1)^{i}|\Omega'(\theta)|^{\sigma}$$

$$\vartheta_1 = \frac{1}{2n}\sum_{i=1}^{n}(e^s - 1)^i |\Omega'(\vartheta)|^\sigma$$

and utilizing the fact

$$\sum_{p=1}^{n}(\theta_p + \vartheta_p)^u \leq \sum_{p=1}^{n}\theta_p^u + \sum_{p=1}^{n}\vartheta_p^u$$

for $0 \leq u < 1$ and $\theta_i, \vartheta_i \geq 0$ for $i = 1, 2, \ldots, n$, we obtain the inequality (2.9). □

Theorem 2.4. *Let* $\Omega : K \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* K^0 *with* $\Omega' \in C\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$ *such that* $\Omega'(t) \in \mathbb{R}$ *for all* $t \in \left(\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right)$*. If* $|\Omega'|^\sigma$ *is bounded, i.e.,* $|\Omega'(x)|^\sigma \leq L$ *for all* $\sigma \geq 1$ *and n-polynomial exponentially s-convex mapping on* $\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$*, then we have the following inequality holds:*

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt - \Omega\left(\frac{\theta+\vartheta}{2}\right)\right| \leq \frac{\vartheta-\theta}{4}\left(\frac{1}{n}\sum_{i=1}^{n}(e^s-1)^i\left|\Omega'\left(\frac{3\theta-\vartheta}{2}\right)\right|^\sigma + \left|\Omega'\left(\frac{3\vartheta-\theta}{2}\right)\right|^\sigma\right)^{\frac{1}{\sigma}}$$
$$\leq \frac{\vartheta-\theta}{4}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}. \tag{2.10}$$

Proof. By using Lemma 1.6, we have

$$\frac{1}{2(\vartheta-\theta)}\int_{\frac{3\theta-\vartheta}{2}}^{\frac{3\vartheta-\theta}{2}}\Omega(t)dt - \Omega\left(\frac{\theta+\vartheta}{2}\right)$$
$$= 2(\vartheta-\theta)\left(\int_0^{\frac{1}{2}} r\Omega'\left(\frac{3\vartheta-\theta}{2} + 2(\theta-\vartheta)r\right)dr + \int_{\frac{1}{2}}^{1}(r-1)\Omega'\left(\frac{3\vartheta-\theta}{2} + 2(\theta-\vartheta)r\right)dr\right).$$

By using Lemma 1.7, we obtain

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt - \Omega\left(\frac{\theta+\vartheta}{2}\right)\right| \leq (\vartheta-\theta)\left(\int_0^{\frac{1}{2}} r\left|\Omega'\left(\frac{3\vartheta-\theta}{2} + 2(\vartheta-\theta)r\right)\right|dr\right.$$
$$\left. + \int_{\frac{1}{2}}^{1}(1-r)\left|\Omega'\left(\frac{3\vartheta-\theta}{2} + 2(\vartheta-\theta)r\right)\right|dr\right). \tag{2.11}$$

Since $|\Omega'|^\sigma$ is n-polynomial exponentially s-convex, so there arise two cases.

Case (i). For $\sigma = 1$. From n-polynomial exponentially s-convexity of $|\Omega'|$ on $\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$ and using the facts $e^{sr} \leq e^s$ and $e^{s(1-r)} \leq e^s$ for any $0 \leq r \leq 1$, we obtain

$$\int_0^{\frac{1}{2}} r\left|\Omega'\left(\frac{3\vartheta-\theta}{2} + 2(\theta-\vartheta)r\right)\right|dr = \int_0^{\frac{1}{2}} r\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right) + (1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr$$

$$\leq \frac{L}{4n}\sum_{i=1}^{n}(e^{s}-1)^{i}. \tag{2.12}$$

Similarly, we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}(1-r)\left|\Omega'\left(\frac{3\vartheta-\theta}{2}+2(\vartheta-\theta)r\right)\right|dr\\
&=\int_{\frac{1}{2}}^{1}(1-r)\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr\\
&\leq \frac{L}{4n}\sum_{i=1}^{n}(e^{s}-1)^{i}.
\end{aligned} \tag{2.13}$$

By substituting inequalities (2.12) and (2.13) in (2.11), we get

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\Omega\left(\frac{\theta+\vartheta}{2}\right)\right|\leq \frac{\vartheta-\theta}{2}\frac{L}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}.$$

Case (ii). For $\sigma>1$. By using the Hölder's inequality for $\sigma>1$ and the facts $e^{sr}\leq e^{s}$ and $e^{s(1-r)}\leq e^{s}$ for any $0\leq r\leq 1$, we get

$$\begin{aligned}
\int_{0}^{\frac{1}{2}}r\left|\Omega'\left(\frac{3\vartheta-\theta}{2}+2(\theta-\vartheta)r\right)\right|dr&=\int_{0}^{\frac{1}{2}}r\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr\\
&=\int_{0}^{\frac{1}{2}}r^{1-\frac{1}{\sigma}}\left(r^{\frac{1}{\sigma}}\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|\right)dr\\
&\leq\left(\int_{0}^{\frac{1}{2}}rdr\right)^{1-\frac{1}{\sigma}}\left(\int_{0}^{\frac{1}{2}}r\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|^{\sigma}dr\right)^{\frac{1}{\sigma}}\\
&\leq\left(\frac{1}{8}\right)^{1-\frac{1}{\sigma}}\left(\frac{1}{8n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left[\left|\Omega'\left(\frac{3\theta-\vartheta}{2}\right)\right|^{\sigma}+\left|\Omega'\left(\frac{3\vartheta-\theta}{2}\right)\right|^{\sigma}\right]\right)^{\frac{1}{\sigma}}\\
&\leq\left(\frac{1}{8}\right)^{1-\frac{1}{\sigma}}\left(\frac{L}{4n}\sum_{i=1}^{n}(e^{s}-1)^{i}\right)^{\frac{1}{\sigma}}.
\end{aligned} \tag{2.14}$$

Similarly, we have

$$\int_{\frac{1}{2}}^{1}(1-r)\left|\Omega'\left(\frac{3\vartheta-\theta}{2}+2(\vartheta-\theta)r\right)\right|dr$$

$$\leq \left(\frac{1}{8}\right)^{1-\frac{1}{\sigma}} \left(\frac{1}{8n}\sum_{i=1}^{n}(e^s-1)^i \left(\left|\Omega'\left(\frac{3\theta-\vartheta}{2}\right)\right|^{\sigma} + \left|\Omega'\left(\frac{3\vartheta-\theta}{2}\right)\right|^{\sigma}\right)\right)^{\frac{1}{\sigma}}$$

$$\leq \left(\frac{1}{8}\right)^{1-\frac{1}{\sigma}} \left(\frac{L}{4n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}. \tag{2.15}$$

So the inequalities (2.11), (2.14), and (2.15) gives the proof of required result. □

Theorem 2.5. *Let $\Omega : K \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on K^0 with $\Omega' \in C\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$, such that $\Omega'(t) \in \mathbb{R}$ for all $t \in \left(\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right)$. If $|\Omega'|^{\sigma}$ is bounded, i.e., $|\Omega'(x)|^{\sigma} \leq L$ for all $\sigma > 1$ and n-polynomial exponentially s-convex mapping on $\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$, then the following inequality holds:*

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt - \Omega\left(\frac{\theta+\vartheta}{2}\right)\right|$$

$$\leq (\vartheta-\theta)\left(\frac{1}{2(\gamma+1)}\right)^{\frac{1}{\gamma}} \left(\frac{1}{2n}\sum_{i=1}^{n}(e^s-1)^i \left(\left|\Omega'\left(\frac{3\theta-\vartheta}{2}\right)\right|^{\sigma} + \left|\Omega'\left(\frac{3\vartheta-\theta}{2}\right)\right|^{\sigma}\right)\right)^{\frac{1}{\sigma}}$$

$$\leq (\vartheta-\theta)\left(\frac{1}{2(\gamma+1)}\right)^{\frac{1}{\gamma}} \left(\frac{L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}} \tag{2.16}$$

with $\frac{1}{\gamma} + \frac{1}{\sigma} = 1$.

Proof. By using first the Hölder's inequality and the fact $e^{sr} \leq e^s$ and $e^{s(1-r)} \leq e^s$ for any $0 \leq r \leq 1$, on first part of inequality (2.11), we obtain

$$\int_0^{\frac{1}{2}} r\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right) + (1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr$$

$$\leq \left(\int_0^{\frac{1}{2}} r^{\gamma}dr\right)^{\frac{1}{\gamma}} \left(\int_0^{\frac{1}{2}}\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right) + (1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|^{\sigma}dr\right)^{\frac{1}{\sigma}}$$

$$= \left(\int_0^{\frac{1}{2}} r^{\gamma}dr\right)^{\frac{1}{\gamma}} \left(\int_0^{\frac{1}{2}}\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right) + (1-r)\left(\frac{3\theta-\vartheta}{2}\right)\right)\right|^{\sigma}dr\right)^{\frac{1}{\sigma}}$$

$$\leq \left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}} \left(\frac{1}{2n}\sum_{i=1}^{n}(e^s-1)^i \left(\left|\Omega'\left(\frac{3\theta-\vartheta}{2}\right)\right|^{\sigma} + \left|\Omega'\left(\frac{3\vartheta-\theta}{2}\right)\right|^{\sigma}\right)\right)^{\frac{1}{\sigma}}$$

$$\leq \left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}} \left(\frac{L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}. \tag{2.17}$$

Similarly, the second part of (2.11) can be written as

$$\int_{\frac{1}{2}}^{1}(1-r)\left|\Omega'\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr$$
$$\leq\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}\left(\frac{1}{2n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left(\left|\Omega'\left(\frac{3\theta-\vartheta}{2}\right)\right|^{\sigma}+\left|\Omega'\left(\frac{3\vartheta-\theta}{2}\right)\right|^{\sigma}\right)\right)^{\frac{1}{\sigma}}$$
$$\leq\left(\frac{1}{(\gamma+1)2^{\gamma+1}}\right)^{\frac{1}{\gamma}}\left(\frac{L}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}\right)^{\frac{1}{\sigma}}. \tag{2.18}$$

Thus by combining the inequalities (2.17) and (2.18), we get the required result. □

Corollary 1. *Under the assumption of Theorems 2.4 and 2.5, we have the following inequality for* $\sigma>1$

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\Omega\left(\frac{\theta+\vartheta}{2}\right)\right|$$
$$\leq\min\{K_1,K_2\}(\vartheta-\theta)\left[\frac{1}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}\left(\left|\Omega'\left(\frac{3\theta-\vartheta}{2}\right)\right|^{\sigma}+\left|\Omega'\left(\frac{3\vartheta-\theta}{2}\right)\right|^{\sigma}\right)\right]^{\frac{1}{\sigma}}, \tag{2.19}$$

where $K_1=\frac{1}{4}$ *and* $K_2=\left(\frac{1}{(\gamma+1)2^{1+\frac{1}{\gamma\sigma}}}\right)^{\frac{1}{\gamma}}$ *with* $\frac{1}{\gamma}+\frac{1}{\sigma}=1$.

Theorem 2.6. *Let* $K\subseteq\mathbb{R}$ *and* $\Omega:K\to\mathbb{R}$ *be a function such that* Ω'' *exists on* K^0 *and suppose that* $\Omega'':\left[\frac{3\theta-\vartheta}{2},\frac{3\vartheta-\theta}{2}\right]\to\mathbb{R}$ *is a continuous function. If* $|\Omega''|^{\sigma}$ *is bounded, i.e.,* $|\Omega''(x)|^{\sigma}\leq L$ *for all* $\sigma\geq 1$ *and n-polynomial exponentially s-convex function on* $\left[\frac{3\theta-\vartheta}{2},\frac{3\vartheta-\theta}{2}\right]$*, then the following inequality holds:*

$$\left|\frac{1}{(\vartheta-\theta)}\int_{\theta}^{\vartheta}\Omega(t)dt-\frac{\Omega\left(\frac{3\theta-\vartheta}{2}\right)+\Omega\left(\frac{3\vartheta-\theta}{2}\right)+2\Omega\left(\frac{\theta+\vartheta}{2}\right)}{4}\right|\leq\frac{(\vartheta-\theta)^2}{3}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^{s}-1)^{i}\right)^{\frac{1}{\sigma}}. \tag{2.20}$$

Proof. By using Lemma 1.5, we have

$$\frac{1}{2(\vartheta-\theta)}\int_{\frac{3\theta-\vartheta}{2}}^{\frac{3\vartheta-\theta}{2}}\Omega(t)dt=\frac{\Omega\left(\frac{3\vartheta-\theta}{2}\right)+\Omega\left(\frac{3\theta-\vartheta}{2}\right)}{2}$$
$$-2(\vartheta-\theta)^2\int_{0}^{1}r(1-r)\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)dr. \tag{2.21}$$

Thus by using Lemma 1.7 in (2.21), we obtain

$$\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\frac{\Omega\left(\frac{3\theta-\vartheta}{2}\right)+\Omega\left(\frac{3\vartheta-\theta}{2}\right)+2\Omega\left(\frac{\theta+\vartheta}{2}\right)}{4}\right|$$

$$\leq 2(\vartheta-\theta)^2\int_0^1 r(1-r)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr. \tag{2.22}$$

Corresponding to $\sigma = 1$ the function $|\Omega''|$ is bounded and n-polynomial exponentially s-convex on $\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$. Also, using the facts $e^{sr} \leq e^s$ and $e^{s(1-r)} \leq e^s$, therefore for any $0 \leq r \leq 1$, we get

$$\begin{aligned}
&\int_0^1 r(1-r)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr \\
&\leq \frac{1}{n}\sum_{i=1}^{n}(e^s-1)^i\int_0^1 r(1-r)dr\left(\left|\Omega''\left(\frac{3\theta-\vartheta}{2}\right)\right|+\left|\Omega''\left(\frac{3\vartheta-\theta}{2}\right)\right|\right) \\
&= \frac{1}{6n}\sum_{i=1}^{n}(e^s-1)^i\left(\left|\Omega''\left(\frac{3\theta-\vartheta}{2}\right)\right|+\left|\Omega''\left(\frac{3\vartheta-\theta}{2}\right)\right|\right) \\
&= \frac{L}{3n}\sum_{i=1}^{n}(e^s-1)^i.
\end{aligned} \tag{2.23}$$

By using this value in (2.22), we conclude that the inequality (2.20) is true for $\sigma = 1$.

Now, assume that $\sigma > 1$, by using Hölder's inequality along with the facts $e^{sr} \leq e^s$ and $e^{s(1-r)} \leq e^s$ for any $0 \leq r \leq 1$, we get

$$\begin{aligned}
&\int_0^1 (r-r^2)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr \\
&= \int_0^1 \left((r-r^2)^{1-\frac{1}{\sigma}}(r-r^2)^{\frac{1}{\sigma}}\right)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr \\
&\leq \left(\int_0^1 (r-r^2)dr\right)^{1-\frac{1}{\sigma}}\left(\int_0^1 (r-r^2)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|^{\sigma}dr\right)^{\frac{1}{\sigma}} \\
&\leq \left(\frac{1}{6}\right)^{1-\frac{1}{\sigma}}\left(\frac{1}{n}\sum_{i=1}^{n}(e^s-1)^i\int_0^1 (r-r^2)dr\left(\left|\Omega''\left(\frac{3\theta-\vartheta}{2}\right)\right|^{\sigma}+\left|\Omega''\left(\frac{3\vartheta-\theta}{2}\right)\right|^{\sigma}\right)\right)^{\frac{1}{\sigma}} \\
&\leq \left(\frac{1}{6}\right)^{1-\frac{1}{\sigma}}\left(\frac{2L}{6n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}} \\
&\leq \left(\frac{1}{6}\right)\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}.
\end{aligned}$$

This completes the desired result. □

Theorem 2.7. *Let $\Omega : K^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on K^0 and $\Omega'' \in C\left[\frac{3\theta-\vartheta}{2}\frac{3\vartheta-\theta}{2}\right]$ such that $\Omega''(t) \in \mathbb{R}$ for all $t \in \left(\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right)$. If $|\Omega''|^\sigma$ is bounded, i.e., $|\Omega''(x)|^\sigma \leq L$ for $\sigma > 1$ and n-polynomial exponentially s-convex function on $\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$, then the following inequality holds:*

$$\left|\frac{1}{\vartheta-\theta}\int_\theta^\vartheta \Omega(t)dt - \frac{\Omega\left(\frac{3\theta-\vartheta}{2}\right)+\Omega\left(\frac{3\vartheta-\theta}{2}\right)+2\Omega\left(\frac{\theta+\vartheta}{2}\right)}{4}\right|$$
$$\leq \frac{(\vartheta-\theta)^2}{2}\left(\frac{\sqrt{\pi}\Gamma(\gamma+1)}{2\Gamma(\gamma+\frac{3}{2})}\right)^{\frac{1}{\gamma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}},$$

where $\frac{1}{\gamma}+\frac{1}{\sigma}=1$.

Proof. By using first the Hölder's inequality and then n-polynomial exponentially s-convexity of the function $|\Omega''|^\sigma$. Also, using the facts $e^{sr} \leq e^s$ and $e^{s(1-r)} \leq e^s$, therefore for any $0 \leq r \leq 1$, we obtain

$$\int_0^1 (r-r^2)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr$$
$$\leq \left(\int_0^1 (r-r^2)^\gamma dr\right)^{\frac{1}{\gamma}}\left(\int_0^1 \left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|^\sigma dr\right)^{\frac{1}{\sigma}}$$
$$\leq \left(\int_0^1 (r-r^2)^\gamma dr\right)^{\frac{1}{\gamma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}$$
$$\leq [B(\gamma+1,\gamma+1)]^{\frac{1}{\gamma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}$$
$$= \left(\frac{\sqrt{\pi}\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{3}{2})2^{1+2\gamma}}\right)^{\frac{1}{\gamma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}$$
$$= \left(\frac{\sqrt{\pi}\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{3}{2})2^{1+2\gamma}}\right)^{\frac{1}{\gamma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}, \tag{2.24}$$

since

$$B(y,y) = 2^{1-2y}B\left(\frac{1}{2},y\right) \quad \text{and} \quad B(z,y) = \frac{\Gamma(z)\Gamma(y)}{\Gamma(z+y)}.$$

Finally, from (2.22) and (2.24), we obtained the desired result. □

Theorem 2.8. *Under the assumptions of Theorem 2.7, we have the following inequality holds:*

$$\left|\frac{1}{\vartheta-\theta}\int_\theta^\vartheta \Omega(t)dt - \frac{\Omega\left(\frac{3\theta-\vartheta}{2}\right)+\Omega\left(\frac{3\vartheta-\theta}{2}\right)+2\Omega\left(\frac{\theta+\vartheta}{2}\right)}{4}\right|$$

$$\leq (\vartheta-\theta)^2 K(\gamma,\sigma)\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}, \tag{2.25}$$

where

$$K(\gamma,\sigma)=2\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{\sigma+1}\right)^{\frac{1}{\sigma}}.$$

Proof. By using first the Hölder's inequality and then n-polynomial exponentially s-convexity along with the facts $e^{sr}\leq e^s$ and $e^{s(1-r)}\leq e^s$ for any $0\leq r\leq 1$, we get

$$\begin{aligned}
&\int_0^1 (r-r^2)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|dr\\
&\leq\left(\int_0^1 r^{\gamma}dr\right)^{\frac{1}{\gamma}}\left(\int_0^1(1-r)^{\sigma}\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right)+(1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|^{\sigma}dr\right)^{\frac{1}{\sigma}}\\
&\leq\left(\int_0^1 r^{\gamma}dr\right)^{\frac{1}{\sigma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\int_0^1 r^{1-1}(1-r)^{\sigma+1-1}dr\right)^{\frac{1}{\sigma}}\\
&=\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}\left(B(1,\sigma+1)\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}\\
&=\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}\left(\frac{\Gamma(1)\Gamma(\sigma+1)}{\Gamma(\sigma+2)}\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}\\
&=\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{\sigma+1}\right)^{\frac{1}{\sigma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}\\
&=\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{\sigma+1}\right)^{\frac{1}{\sigma}}\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}}.
\end{aligned}\tag{2.26}$$

Keeping in mind (2.22) and (2.26), we obtained (2.25). □

Theorem 2.9. *Let $\Omega : K^0\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on K^0 and assume that $\Omega''\in C\left[\frac{3\theta-\vartheta}{2},\frac{3\vartheta-\theta}{2}\right]$ such that $\Omega''(t)\in\mathbb{R}$ for all $t\in\left(\frac{3\theta-\vartheta}{2},\frac{3\vartheta-\theta}{2}\right)$. If $|\Omega''|^{\sigma}$ is a bounded, i.e., $|\Omega''(x)|^{\sigma}\leq L$ for all $\sigma\geq 1$ and n-polynomial exponentially s-convex mapping on $\left[\frac{3\theta-\vartheta}{2},\frac{3\vartheta-\theta}{2}\right]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\Omega(t)dt-\frac{\Omega\left(\frac{3\theta-\vartheta}{2}\right)+\Omega\left(\frac{3\vartheta-\theta}{2}\right)+2\Omega\left(\frac{\theta+\vartheta}{2}\right)}{4}\right|\\
&\leq(\vartheta-\theta)^2K_2(\sigma)\left(\frac{2L}{n}\sum_{i=1}^{n}(e^s-1)^i\right)^{\frac{1}{\sigma}},
\end{aligned}\tag{2.27}$$

where $K_2(\sigma) = \left(\frac{2}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}$.

Proof. Let $\sigma > 1$, by using first the Hölder's inequality and then n-polynomial exponentially s-convexity along with the fact $e^{sr} \leq e^s$ and $e^{s(1-r)} \leq e^s$ for any $0 \leq r \leq 1$, we get

$$\begin{aligned}
&\int_0^1 (r - r^2)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right) + (1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right| dr \\
&= \int_0^1 r^{1-\frac{1}{\sigma}}\left(r^{\frac{1}{\sigma}}(1-r)\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right) + (1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|\right) dr \\
&\leq \left(\int_0^1 r dr\right)^{1-\frac{1}{\sigma}}\left(\int_0^1 r(1-r)^\sigma\left|\Omega''\left(r\left(\frac{3\theta-\vartheta}{2}\right) + (1-r)\left(\frac{3\vartheta-\theta}{2}\right)\right)\right|^\sigma dr\right)^{\frac{1}{\sigma}} \\
&\leq \left(\int_0^1 r dr\right)^{1-\frac{1}{\sigma}}\left(\frac{1}{n}\sum_{i=1}^n (e^s-1)^i \int_0^1 r(1-r)^\sigma dr\left[\left|\Omega''\left(\frac{3\theta-\vartheta}{2}\right)\right|^\sigma + \left|\Omega''\left(\frac{3\vartheta-\theta}{2}\right)\right|^\sigma\right]\right)^{\frac{1}{\sigma}} \\
&= \left(\frac{1}{2}\right)^{1-\frac{1}{\sigma}}\left(B(2,\sigma+1)\frac{2L}{n}\sum_{i=1}^n (e^s-1)^i\right)^{\frac{1}{\sigma}} \\
&= \left(\frac{1}{2}\right)^{1-\frac{1}{\sigma}}\left(\frac{1}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}\left(\frac{2L}{n}\sum_{i=1}^n (e^s-1)^i\right)^{\frac{1}{\sigma}} \\
&= \left(\frac{1}{2}\right)^{1-\frac{1}{\sigma}}\left(\frac{1}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}}\left(\frac{2L}{n}\sum_{i=1}^n (e^s-1)^i\right)^{\frac{1}{\sigma}}. \qquad (2.28)
\end{aligned}$$

In view of (2.22) and (2.28), we deduce that (2.27) holds, when $\sigma > 1$. From (2.23), we deduce that (2.27) is true when $\sigma = 1$. This completes the proof. □

3. Applications to means and trapezoid formulae

In this section, we use the main results of Section 2 to give some applications to special means of positive real numbers. We first need to recall the following basic definitions of different means and techniques of numerical integration.

For $\theta, \vartheta \in \mathbb{R}$ with $\theta, \vartheta > 0$, the following equations

$$\mathcal{A} = \mathcal{A}(\theta,\vartheta) = \frac{\theta+\vartheta}{2};$$
$$\mathcal{G} = \mathcal{G}(\theta,\vartheta) = \sqrt{\theta\vartheta};$$
$$\mathcal{L}(\theta,\vartheta) = \frac{\vartheta-\theta}{\log(\vartheta)-\log(\theta)}$$

and

$$\mathcal{L}_m(\theta,\vartheta) = \left[\frac{\vartheta^{m+1}-\theta^{m+1}}{(\vartheta-\theta)(m+1)}\right]^{\frac{1}{m}}; \quad m \in Y \setminus \{-1, 0\}$$

are known as arithmetic, geometric, logarithmic and generalized logarithmic means, respectively.

The midpoint rule for numerical integration is stated as follows: Let D be a partition with points $\theta = t_0 < t_1 < \cdots < t_{m-1} < t_m = \vartheta$ of the interval $[\theta, \vartheta]$ and consider the quadrature formula

$$\int_{\theta}^{\vartheta} \Omega(t)dt = T_j(\Omega, D) + E_j(\Omega, D), \quad j = 1, 2,$$

where

$$T_1(\Omega, D) = \sum_{j=0}^{m-1} \frac{\Omega(t_j) + \Omega(t_{j+1})}{2} \left(t_{j+1} - t_j\right)$$

for the trapezoidal version and

$$T_2(\Omega, D) = \sum_{j=0}^{m-1} \Omega\left(\frac{t_j + t_{j+1}}{2}\right)\left(t_{j+1} - t_j\right)$$

for the midpoint version and $E_j(\Omega, D)$ represent the approximate error.

Proposition 1. *Let $m \in Y \setminus \{-1, 0\}$, where $\theta, \vartheta \in \mathbb{R}$ with $0 < \theta < \vartheta$. Then the following inequality holds:*

$$\left|\mathcal{A}^m(\theta, \vartheta) - \mathcal{L}_m^m(\theta, \vartheta)\right| \le \min\{K_1, K_2\} \left(2^{\frac{1}{\sigma}}|m|(\vartheta - \theta)\right)\left(\frac{1}{n}\sum_{i=1}^{n}(e^s - 1)^i\right)^{\frac{1}{\sigma}}$$
$$\times \left[\mathcal{A}\left(\left|\frac{3\theta - \vartheta}{2}\right|^{(m-1)\sigma}, \left|\frac{3\vartheta - \theta}{2}\right|^{(m-1)\sigma}\right)\right]^{\frac{1}{\sigma}}. \tag{3.1}$$

Proof. Using Corollary 1 with the substitution $\Omega(t) = t^m$ and by simple mathematical calculation, we get (3.1). □

Proposition 2. *Let $\theta, \vartheta \in \mathbb{R}$ with $0 < \theta < \vartheta$. Then the following inequality holds:*

$$\left|\mathcal{G}^{-2}(\theta, \vartheta) - \mathcal{A}^{-2}(\theta, \vartheta)\right| \le \min\{K_1, K_2\} \left(4^{\frac{1}{\sigma}}(\vartheta - \theta)\right)\left(\frac{1}{n}\sum_{i=1}^{n}(e^s - 1)^i\right)^{\frac{1}{\sigma}}$$
$$\times \left[\mathcal{A}\left(\left|\frac{3\theta - \vartheta}{2}\right|^{-3\sigma}, \left|\frac{3\vartheta - \theta}{2}\right|^{-3\sigma}\right)\right]^{\frac{1}{\sigma}}. \tag{3.2}$$

Proof. By using Corollary 1 with the substitution $\Omega(t) = \frac{1}{t^2}$ and by simple mathematical calculation, we get (3.2). □

Proposition 3. *If $\sigma \ge 1$ and $\theta, \vartheta \in \mathbb{R}$ with $0 < \theta < \vartheta$, then the following inequality holds:*

$$\left|\mathcal{A}^{-1}(\theta, \vartheta) - \mathcal{L}^{-1}(\theta, \vartheta)\right| \le \min\{K_1, K_2\} \left(2^{\frac{1}{\sigma}}(\vartheta - \theta)\right)\left(\frac{1}{n}\sum_{i=1}^{n}(e^s - 1)^i\right)^{\frac{1}{\sigma}}$$
$$\times \left[\mathcal{A}\left(\left|\frac{3\theta - \vartheta}{2}\right|^{-2\sigma}, \left|\frac{3\vartheta - \theta}{2}\right|^{-2\sigma}\right)\right]^{\frac{1}{\sigma}}. \tag{3.3}$$

Proof. By utilizing the Corollary 1 with the substitution $\Omega(t) = \frac{1}{t}$ and by simple mathematical calculation, we get (3.3). □

Proposition 4. *If $|\Omega'|^{\sigma}$ is n-polynomial exponentially s-convex function for $\sigma \geq 1$, then for every partition of $\left[\frac{3\theta-\vartheta}{2}, \frac{3\vartheta-\theta}{2}\right]$ the midpoint error satisfies*

$$\begin{aligned}|E_2(\Omega; D)| &\leq \min(K_1, K_2) \sum_{j=0}^{m-1} \left(t_{j+1} - t_j\right)^2 \left(\frac{1}{n} \sum_{i=1}^{n} (e^s - 1)^i\right)^{\frac{1}{\sigma}} \\ &\quad \times \left[\left|\Omega'\left(\frac{3t_j - t_{j+1}}{2}\right)\right|^{\sigma} + \left|\Omega'\left(\frac{3t_{j+1} - t_j}{2}\right)\right|^{\sigma}\right]^{\frac{1}{\sigma}} \\ &\leq 2\min(K_1, K_2) \sum_{j=0}^{m-1} \left(t_{j+1} - t_j\right)^2 \left(\frac{1}{n} \sum_{i=1}^{n} (e^s - 1)^i\right)^{\frac{1}{\sigma}} \\ &\quad \times \max\left[\left|\Omega'\left(\frac{3t_j - t_{j+1}}{2}\right)\right|, \left|\Omega'\left(\frac{3t_{j+1} - t_j}{2}\right)\right|\right].\end{aligned}$$

Proof. From Corollary 1, we obtain

$$\begin{aligned}&\left|\int_{t_j}^{t_{j+1}} \Omega(t)dt - \left(t_{j+1} - t_j\right) \Omega\left(\frac{t_j + t_{j+1}}{2}\right)\right| \\ &\leq \min(K_1, K_2) \left(t_{j+1} - t_j\right)^2 \left(\frac{1}{n} \sum_{i=1}^{n} (e^s - 1)^i\right)^{\frac{1}{\sigma}} \\ &\quad \times \left[\left|\Omega'\left(\frac{3t_j - t_{j+1}}{2}\right)\right|^{\sigma} + \left|\Omega'\left(\frac{3t_{j+1} - t_j}{2}\right)\right|^{\sigma}\right]^{\frac{1}{\sigma}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|\left\{\int_{\theta}^{\vartheta} \Omega(t)dt - T_2(\Omega, D)\right\}\right| = \left|\sum_{j=0}^{m-1} \left\{\int_{t_j}^{t_{j+1}} \Omega(t)dt - \left(t_{j+1} - t_j\right) \Omega\left(\frac{t_j + t_{j+1}}{2}\right)\right\}\right| \\ &\leq \min(K_1, K_2) \sum_{j=0}^{m-1} \left(t_{j+1} - t_j\right)^2 \left(\frac{1}{n} \sum_{i=1}^{n} (e^s - 1)^i\right)^{\frac{1}{\sigma}} \\ &\quad \times \left[\left|\Omega'\left(\frac{3t_j - t_{j+1}}{2}\right)\right|^{\sigma} + \left|\Omega'\left(\frac{3t_{j+1} - t_j}{2}\right)\right|^{\sigma}\right]^{\frac{1}{\sigma}} \\ &\leq 2\min(K_1, K_2) \sum_{j=0}^{m-1} \left(t_{j+1} - t_j\right)^2 \left(\frac{1}{n} \sum_{i=1}^{n} (e^s - 1)^i\right)^{\frac{1}{\sigma}} \\ &\quad \times \max\left[\left|\Omega'\left(\frac{3t_j - t_{j+1}}{2}\right)\right|, \left|\Omega'\left(\frac{3t_{j+1} - t_j}{2}\right)\right|\right].\end{aligned}$$

□

4. Conclusions

In this paper, we examined key generalizations of convexity known as n-polynomial exponentially s-convex functions. We have utilized the well-known Hölders inequality to explore new identities for Hermite-Hadamard inequalities. We demonstrated how our newly developed results are utilized to establish certain means of two positive real numbers in various ways. In the future, the methodologies used in this study can generate new inequalities for n-polynomial exponentially s-convex functions.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. C. P. Niculescu, L. E. Persson, *Convex functions and their applications: A contemporary approach*, 1 Ed., New York: CMC Books in Mathematics, 2004. https://doi.org/10.1007/978-3-319-78337-6
2. J. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings and statistical applications*, 1 Ed., New York: Academic Press, 1992. http://dx.doi.org/10.1016/s0076-5392(08)x6162-4
3. T. H. Zhao, L. Shi, Y. M. Chu, Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means, *RACSAM Rev. R. Acad. A*, **114** (2020), 96. https://doi.org/10.1007/s13398-020-00825-3
4. D. Baleanu, M. Samraiz, Z. Perveen, S. Iqbal, K. S. Nisar, G. Rahman, Hermite-Hadamard-Fejer type inequalities via fractional integral of a function concerning another function, *AIMS Math.*, **6** (2021), 4280–4295. https://doi.org/10.3934/math.2021253
5. S. Wu, S. Iqbal, M. Aamir, M. Samraiz, A. Younus, On some Hermite-Hadamard inequalities involving k-fractional operators, *J. Inequal. Appl.*, **2021** (2021), 32. https://doi.org/10.1186/s13660-020-02527-1
6. M. K. Wang, Z. Y. He, Y. M. Chu, Sharp power mean inequalities for the generalized elliptic integral of the first kind, *Comput. Meth. Funct. Th.*, **20** (2020), 111–124. https://doi.org/10.1007/s40315-020-00298-w
7. X. M. Hu, J. F. Tian, Y. M. Chu, Y. X. Lu, On Cauchy-Schwarz inequality for N-tuple diamond-alpha integral, *J. Inequal. Appl.*, **2020** (2020), 8. https://doi.org/10.1186/s13660-020-2283-4
8. M. A. Latif, S. Rashid, S. S. Dragomir, Y. M. Chu, Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications, *J. Inequal. Appl.*, **2019** (2019), 317. https://doi.org/10.1186/s13660-019-2272-7
9. S. Khan, M. A. Khan, S. I. Butt, Y. M. Chu, A new bound for the Jensen gap pertaining twice differentiable functions with applications, *Adv. Differ. Equ.*, **2020** (2020), 333. https://doi.org/10.1186/s13662-020-02794-8

10. R. J. Gardner, The Brunn-Minkowski inequaity, *Bull. Amer. Math. Soc.*, **39** (2002), 355–405. https://doi.org/10.1090/S0273-0979-02-00941-2

11. S. I. Butt, A. Kashuri, M. Tariq, J. Nasir, A. Aslam, W. Gao, Hermite-Hadamard type inequalities via n-polynomial exponential type convexity and their applications, *Adv. Differ. Equ.*, **2020** (2020), 508. https://doi.org/10.1186/s13662-020-02967-5

12. W. Gao, A. Kashuri, S. I. Butt, M. Tariq, A. Aslam, M. Nadeem, New inequalities via n-polynomial harmonically exponential type convex functions, *AIMS Math.*, **5** (2020), 6856–6873. https://doi.org/10.3934/math.2020440

13. S. I. Butt, A. Kashuri, M. Umar, A. Aslam, W. Gao, Hermite-Jensen-Mercer type inequalities via ψ-Riemann-Liouville k-fractional integrals, *AIMS Math.,* **5** (2020), 5193–5220. https://doi.org/10.3934/math.2020334

14. G. Alirezaei, R. Mathar, On exponentially concave functions and their impact in information theory, *IEEE Xplore*, **2018** (2018), 1–10. https://doi.org/10.1109/ITA.2018.8503202

15. S. Pal, T. K. L. Wong, Exponentially concave functions and a new information geometry, *Ann. Probab.*, **46** (2018), 1070–1113. https://www.jstor.org/stable/26402375

16. T. Antczak, On (p, r)-invex sets and functions, *J. Math. Anal. Appl.*, **263** (2001), 355–379. https://doi.org/10.1006/jmaa.2001.7574

17. S. S. Dragomir, I. Gomm, Some Hermite-Hadamard type inequalities for functions whose exponentials are convex, *Stud. Univ. Babes-Bolyai Math.*, **60** (2015), 527–534.

18. M. A. Noor, K. I. Noor, On exponentially convex functions, *J. Orissa Math. Soc.*, **38** (2019), 33–51.

19. N. Mehreen, M. Anwar, Hermite-Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications, *J. Inequal. Appl.*, **2019** (2019), 92. https://doi.org/10.1186/s13660-019-2047-1

20. M. Tariq, S. K. Sahoo, J. Nasir, H. Aydi, H. Alsamir, Some Ostrowski type inequalities via n-polynomial exponentially s-convex functions and their applications, *AIMS Math.*, **6** (2021), 13272–13290. https://doi.org/10.3934/math.2021768

21. K. Mehrez, P. Agarwal, New Hermite-Hadamard type integral inequalities for convex functions and their applications, *J. Comput. Appl. Math.*, **350** (2019), 274–285. https://doi.org/10.1016/j.cam.2018.10.022

22. U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.*, **147** (2004), 137–146. https://doi.org/10.1016/S0096-3003(02)00657-4

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