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Research article

On star and acyclic coloring of generalized lexicographic product of graphs

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Abstract: A *star coloring* of a graph G is a proper vertex coloring of G such that any path of length 3 in G is not bicolored. The *star chromatic number* $\chi_s(G)$ of G is the smallest integer k for which G admits a star coloring with k colors. A *acyclic coloring* of G is a proper coloring of G such that any cycle in G is not bicolored. The *acyclic chromatic number* of G, denoted by $a(G)$, is the minimum number of colors needed to acyclically color G. In this paper, we present upper bound for the star and acyclic chromatic numbers of the generalized lexicographic product $G[h_n]$ of graph G and disjoint graph sequence h_n, where G exists a $k-$colorful neighbor star coloring or $k-$colorful neighbor acyclic coloring. In addition, the upper bounds are tight.

Keywords: star coloring; acyclic coloring; generalized lexicographic product; colorful neighbor star coloring; colorful neighbor acyclic coloring
Mathematics Subject Classification: 05C15

1. Introduction

A *proper vertex coloring* of a graph G is a coloring of the vertices of G such that no two neighbors in G are assigned the same color. A *star coloring* of a graph G is a proper vertex coloring of G such that any path of length 3 in G is not bicolored. The *star chromatic number* $\chi_s(G)$ of G is the smallest integer k for which G admits a star coloring with k colors. A *acyclic coloring* of a graph G is a proper vertex coloring of G such that any cycle in G is not bicolored. The *acyclic chromatic number* of G, denoted by $a(G)$, is the minimum number of colors needed to acyclically color G. From the definitions of star and acyclic coloring, we can see that the star chromatic number of a graph is the upper bound of the acyclic chromatic number of the graph.

In recent decades, many scholars had made extensive researches on star coloring and acyclic coloring of graphs, and obtained many valuable results, which further enrich the vertex coloring theory of graphs. Albertson et al. [1] proved that planar graphs have star colorings with at most 20 colors and exhibited a planar graph which requires 10 colors. Timmons [2] researched the upper bounds of the star chromatic number of planar graphs with girth at least 9 and 14 respectively. Shalu and Sandhya [3] researched the upper bound of the star chromatic number of graphs with girth at least 5. Mary and Rayen [4] studied the star coloring of some graph families formed from the Cartesian product of some simple graphs. Han et al. [5] obtained the star chromatic numbers for some infinite subgraphs of Cartesian product of paths and cycles. Lyons [6] researched the star coloring of joins of graphs. Venkatesan et al. [7] found the star chromatic number for the corona graph of path with complete graph on the same order, path with cycle on the same order, path on order n with star graph on order $n+1$, path on order n with bipartite graph on order n_1+n_2, and corona graph of star graph on order $n+1$ with complete graph on order n_1+n_2 respectively. Subramanian and Joseph [8] gave the exact value of star chromatic number of degree splitting of comb product of complete graph with complete graph, complete graph with path, complete graph with cycle, complete graph with star graph, cycle with complete graph, path with complete graph and cycle with path graph. Kaliraj and Sivakami [9] found the exact values of the star chromatic number of modular product of complete graph with complete graph, path with complete graph and star graph with complete graph. Kowsalya [10] researched the star chromatic number of tensor products of path and complete graphs have been investigated in this article.

Grünbaum [11] proved that the acyclic chromatic number of every planar graph is not more than 9, and proposed the acyclic chromatic number conjecture: The acyclic chromatic number of every planar graphs does not exceed 5. Fertin et al. [12] obtained the upper or lower bounds of acyclic chromatic number of graphs of planar graphs, outerplanar graphs, $1-$planar graphs, $k-$trees, etc. Fertin et al. [12] gave the upper bound of the acyclic chromatic number of $d-$dimensional grids is $d+1$. In literatures [13–16], the upper bounds of acyclic chromatic number of graphs with maximum degree 3, 4, 5, 6, 7 are given respectively. Zhu et al. [17] gave the acyclic chromatic number of generalized Petersen graphs except $P(4,1)$ and $P(5,2)$ is 3. In literatures [18–20], the exact values or bounds of acyclic chromatic number of subdivision graphs of different special graphs are given. In literatures [21–24], the exact value or upper bound of acyclic chromatic number of different product graphs of special graphs and special graphs is obtained.

In this paper, we will give the upper bounds of the star and acyclic chromatic numbers of generalized lexicographic product of graphs. Let G be a connected graph of order n and $h_n = (H_x)_{x\in V(G)}$ be a disjoint graph sequence. The *generalized lexicographic product* of G and h_n is obtained by following two steps: (i) Replace each vertex x of G with H_x; (ii) Connect the vertex of H_x with the vertex of H_y if and only if $xy \in E(G)$. In particular, when every graph in h_n is isomorphic to H, $G[h_n]$ is abbreviated as $G[H]$, and it is called the lexicographic product of G and H. In literature [25], Szumny et al. gave the star chromatic number of the lexicographic product of path on order 4 with complete graph on order t, and the lexicographic product of cycle on order 5 with complete graph of order t respectively. We will generalize their results to the lexicographic product of path or cycle on order n with complete graph of order t.

In order to study the star and acyclic coloring of the generalized lexicographic product of graphs, the colorful neighbor star and acyclic coloring is introduced. Let $\sigma = (V_1, V_2, \ldots, V_k)$ be a $k-$star coloring of G, where $\chi_s(G) = k$. If there exists a color class V_{i_0} such that different neighbors of each vertex in V_{i_0} are colored differently, then σ is said to be a $k-colorful\ neighbor\ star\ coloring$ of G, and

V_{i_0} is said to be a colorful neighbor color class of σ. Let $\sigma' = (V_1, V_2, \ldots, V_k)$ be a $k-$acyclic coloring of G, where $a(G) = k$. If there exists a color class V_{i_0} such that different neighbors of each vertex in V_{i_0} are colored differently, then σ' is said to be a $k-colorful$ *neighbor acyclic coloring* of G, and V_{i_0} is said to be a colorful neighbor color class of σ'. According to the above definitions, it is easy to get that paths of order at least 3 have 3−colorful neighbor star coloring, and cycles of order n have 3−colorful neighbor star and acyclic coloring where $n \neq 5$. However, a graph with $k-$star (or $k-$acyclic) coloring does not necessarily have $k-$colorful neighbor star (or acyclic) coloring. For example, in Figure 1, it is obvious that $\chi_s(G) = 3$, but it can be proved that G does not have 3−colorful neighbor star coloring. The proof is as follows.

Let σ be any 3−star coloring of G, and the color set is $C = \{a, b, c\}$. Since $G[\{x_1, x_2, \cdots, x_6\}]$ is a cycle on order 6, the 3 colors in C must be represented at the vertices of this cycle. According to $G[\{x_1, x_2, \cdots, x_6\}]$, 3 pairs of relative vertices x_1 and x_4, x_2 and x_5, x_3 and x_6 can be divided into two different modes. Mode I is "exactly one pair of relative vertices in three pairs are the same color", mode II is "each pair of relative vertices in three pairs are the same color", whether it's mode I or mode II, $G[\{x_1, x_2, \cdots, x_6\}]$ must have three consecutive different top points with different colors, so assume that $\sigma(x_1) = a$, $\sigma(x_2) = b$, $\sigma(x_3) = c$. If mode I appears, $\sigma(x_4) = a$, $\sigma(x_5) = c$, $\sigma(x_6) = \sigma(x_7) = \sigma(x_8) = \sigma(x_9) = b$. Obviously, there is no colorful neighbor color class. If mode II appears, $\sigma(x_4) = \sigma(x_9) = a$, $\sigma(x_5) = \sigma(x_8) = b$, $\sigma(x_6) = \sigma(x_7) = c$. Obviously, there is no colorful neighbor color class. Therefore, σ is not 3−colorful neighbor star coloring of G. Since the selection of σ is arbitrary, G does not exist 3−colorful neighbor star coloring.

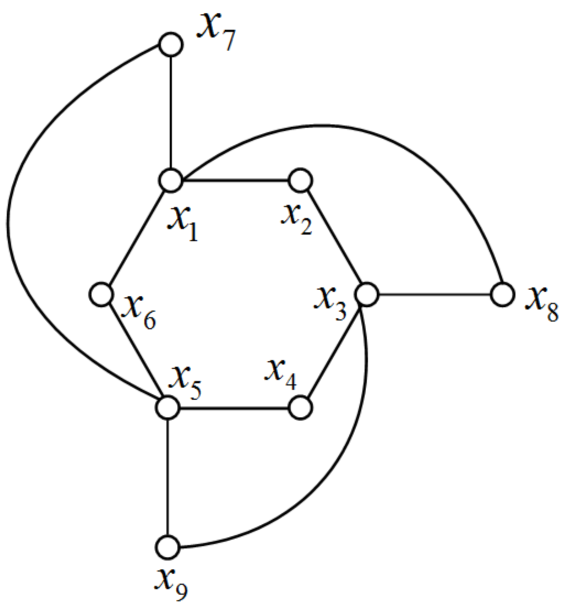

Figure 1. $\chi_s(G) = 3$.

2. The star coloring of generalized lexicographic product of graphs

Let G be a graph of order $n \geq 2$ and $h_n = (H_x)_{x\in V(G)}$ be a disjoint graph sequence, where each vertex in the H_k is expressed as (x, y). To facilitate the narrative, use $E(x, x')$ to denote the set of all edges of $G[h_n]$ between the vertices of H_x and the vertices of $H_{x'}$. According to the definition of generalized lexicographic product, the vertex set and the edge set of $G[h_n]$ are

$$V(G[h_n]) = \bigcup_{x\in V(G)} V(H_x), E(G[h_n]) = (\bigcup_{x\in V(G)} E(H_x))\bigcup(\bigcup_{xx'\in E(G)} E(x, x')). \tag{2.1}$$

In the following theorem, we give the upper bound of the star chromatic number of $G[h_n]$.

Theorem 2.1. Let $\sigma_G = (V_1, V_2, \ldots, V_k)$ be a colorful neighbor star coloring of G, where V_k is a colorful neighbor color class of σ_G, then

$$\chi_s(G[h_n]) \leq \max_{x\in V_k} \chi_s(H_x) + \sum_{i\in[k-1]} \max_{x\in V_i} |V(H_x)|. \tag{2.2}$$

Proof. Denote $\widetilde{G} = G[h_n]$, $p = \max_{x\in V_k} \chi_s(H_x)$, $q_i = \max_{x\in V_i} |V(H_x)|$, where $i \in [k-1]$. Let σ_1 be a $p-$star coloring of $\bigcup_{x\in V_k} H_x$, let σ_2 be a q_i-vertex coloring of $\bigcup_{x\in V_i} H_x$ where $i \in [k-1]$ and different vertices in each H_x are in different colors. Construct a $(p + \sum_{i\in[k-1]} q_i)-$vertex coloring σ of $\widetilde{G}$ as follows:

$$\sigma((x, y)) = \begin{cases} (\sigma_G(x), \sigma_1(y)) & \text{if } x \in V_k, \\ (\sigma_G(x), \sigma_2(y)) & \text{if } x \notin V_k. \end{cases} \tag{2.3}$$

Obviously, σ is a proper vertex coloring of $\widetilde{G}$. Then, it is shown that σ is a star coloring of $\widetilde{G}$. Take any path P of order 4 in $\widetilde{G}$, and the following five cases will be discussed:

Case 1. All four vertices of P are in $H_{x_{i_0}}$. Since σ is a star coloring limited to every H_x, P is not bicolored.

Case 2. Three vertices of P are in $H_{x_{i_0}}$ and another vertex is in $H_{x_{i_1}}$, where $x_{i_0}x_{i_1} \in E(G)$. Obviously, at least two of the three vertices of $H_{x_{i_0}}$ are adjacent to each other, then at least two colors are needed to color the three vertices. Since $x_{i_0}x_{i_1} \in E(G)$, the color of the vertices of $H_{x_{i_1}}$ must not belong to the color set of $H_{x_{i_0}}$. Therefore, P is not bicolored.

Case 3. Two vertices of P are in $H_{x_{i_0}}$ and the other two vertices are in $H_{x_{i_1}}$, where $x_{i_0}x_{i_1} \in E(G)$. Since $x_{i_0}x_{i_1} \in E(G)$, both x_{i_0} and x_{i_1} can't belong to V_k. Hence, P is not bicolored.

Case 4. Two vertices of P are in $H_{x_{i_0}}$, the other two vertices are in $H_{x_{i_1}}$ and $H_{x_{i_2}}$ respectively, where the induced subgraph of $\{x_{i_0}, x_{i_1}, x_{i_2}\}$ in G is a path. If two vertices in $H_{x_{i_0}}$ are different colors, it is clear that P is not bicolored. If two vertices in $H_{x_{i_0}}$ are the same color, then $x_{i_0} \in V_k$. Since V_k is a colorful neighbor color class of σ_G, the other two vertices that are adjacent to the two vertices in $H_{x_{i_0}}$ must be different colors. Therefore, P is not bicolored.

Case 5. The four vertices of P are respectively in $H_{x_{i_0}}$, $H_{x_{i_1}}$, $H_{x_{i_2}}$, $H_{x_{i_3}}$, where $x_{i_0}x_{i_1}x_{i_2}x_{i_3} \in E(G)$. By definition of σ, P is not bicolored.

Therefore, σ is a $(p + \sum_{i\in[k-1]} q_i)-$star coloring of $\widetilde{G}$.

If the above $\sigma_G = (V_1, V_2, \ldots, V_k)$ is a 2−distance coloring of G (which every vertex must be colored in such a way that two vertices lying at distance less than or equal to 2 must be assigned different colors), each color class is colorful neighbor color class of σ_G. Theorem 2.1 can be applied to each color class, then

$$\chi_s(G[h_n]) \leq \min_{i\in[k]}\{\max_{x\in V_i}\chi_s(H_x) + \sum_{j\in[k],j\neq i}\max_{x\in V_j}|V(H_x)|\}. \tag{2.4}$$

Suppose G is a complete graph K_n of order $n \geq 2$, since any k−proper vertex coloring of K_n is a colorful neighbor star coloring of K_n, and each color class is colorful neighbor color class. Theorem 2.1 can be applied to each color class, so it can be obtained

$$\chi_s(K_n[h_n]) \leq \min_{i\in[n]}\{\chi_s(H_i) + \sum_{j\in[n],j\neq i}|V(H_j)|\}. \tag{2.5}$$

This upper bound is tight, as can be seen from Theorem 2.2.

According the definition of the joins of graphs and the generalized lexicographic product of graphs, the joins $G \vee H$ of G and H can be regarded as the generalized lexicographic product $K_2[h_2]$ of K_2 and the graph sequence h_2 of G and H. Lyons obtained the following results in reference [6] on the star coloring of the joins of any two simple graphs G and H.

$$\chi_s(G \vee H) = \min\{\chi_s(G) + |V(H)|, \chi_s(H) + |V(G)|\}. \tag{2.6}$$

By using the above formula and mathematical induction, more general results can be obtained. See Theorem 2.2 for details.

Theorem 2.2. For any integer $n \geq 2$,

$$\chi_s(K_n[h_n]) = \min_{i\in[n]}\{\chi_s(H_i) + \sum_{j\in[n],j\neq i}|V(H_j)|\}. \tag{2.7}$$

Proof. We argue by induction on the number of vertices of K_n, according to the formula 2.6, we have

$$\chi_s(K_2[h_2]) = \min\{\chi_s(H_1) + |V(H_2)|, \chi_s(H_2) + |V(H_1)\}. \tag{2.8}$$

That is, when $n = 2$, the theorem is valid.

Assume that $n = p$, the conclusion of the theorem holds, i.e.,

$$\chi_s(K_p[h_p]) = \min_{i\in[p]}\{\chi_s(H_i) + \sum_{j\in[p],j\neq i}|V(H_j)|\}. \tag{2.9}$$

We need to prove that the theorem is valid when $n = p + 1$. When $n = 2$, since $K_{p+1}[h_{p+1}] = K_p[h_p] \vee H_{p+1}$, according to the formula 2.6,

$$\begin{aligned}
\chi_s(K_{p+1}[h_{p+1}]) &= \chi_s(K_p[h_p] \vee H_{p+1}) \\
&= \min\{\chi_s(K_p[h_p]) + |V(H_{p+1})|, \chi_s(H_{p+1}) + \sum_{j\in[p]}|V(H_j)|\} \\
&= \min\{\min_{i\in[p]}\{\chi_s(H_i) + \sum_{j\in[p+1],j\neq i}|V(H_j)|\}, \chi_s(H_{p+1}) + \sum_{j\in[p]}|V(H_j)|\} \\
&= \min_{i\in[p+1]}\{\chi_s(H_i) + \sum_{j\in[p+1],j\neq i}|V(H_j)|\}.
\end{aligned}$$

By the induction hypothesis, the conclusion of the theorem holds.

In Theorem 2.1, if each H_i of $h_n = (H_x)_{x\in V(G)}$ isomorphic to H, we can get the upper bound of the star chromatic number of the lexicographic product of G and H, in Corollary 2.1.

Corollary 2.1. Let G and H be two graphs. If G exists a $k-$colorful neighbor star coloring, then

$$\chi_s(G[H]) \leq \chi_s(H) + (k-1)|V(H)|. \tag{2.10}$$

Suppose every H_i in $h_n = (H_x)_{x\in V(G)}$ is a graph of order m and the star chromatic number is r, we can get the star chromatic number of the generalized lexicographic product $P_n[h_n]$ or $C_n[h_n]$, see Theorem 2.3 and Theorem 2.4 for details.

Theorem 2.3. Let P_n be a path where $n \geq 4$, and $h_n = (H_x)_{x\in V(P_n)}$ be a disjoint graph sequence where every H_i is an $m-$order graph and the star chromatic number is r. Then, $\chi_s(P_n[h_n]) = 2m + r$.

Proof. Let $P_n = x_1x_2\cdots x_n$, and denote $\widetilde{G} = P_n[h_n]$. Since P_n exists a $3-$colorful neighbor star coloring, according to Theorem 2.1, $\chi_s(\widetilde{G}) \leq 2m + r$.

To prove $\chi_s(\widetilde{G}) \geq 2m + r$. Assume $\chi_s(\widetilde{G}) \leq 2m + r - 1$, and σ is a $(2m + r - 1)$-star coloring of $\widetilde{G}$. Obviously, for any $i = 1, 2, \cdots, n$, $r \leq |C_{V(H_{x_i})}| \leq m$. For any $j = 1, 2, \cdots, n-1$, $C_{V(H_{x_j})} \cap C_{V(H_{x_{j+1}})} = \emptyset$, and at least one of the two equations $|C_{V(H_{x_j})}| = m$ and $|C_{V(H_{x_{j+1}})}| = m$ is established. According to whether $|C_{V(H_{x_2})}| = m$ or $|C_{V(H_{x_3})}| = m$ is established, it can be divided into the following two cases:

Case 6. $|C_{V(H_{x_2})}| < m$ or $|C_{V(H_{x_3})}| < m$. Suppose $|C_{V(H_{x_2})}| < m$, then $|C_{V(H_{x_1})}| = |C_{V(H_{x_3})}| = m$ and $C_{V(H_{x_1})} \cap C_{V(H_{x_3})} = \emptyset$. Therefore, the three sets $C_{V(H_{x_1})}, C_{V(H_{x_2})}$ and $C_{V(H_{x_3})}$ are pairwise disjoint. Thus, the star coloring σ of $\widetilde{G}$ needs at least $2m + r$ colors, which contradicts the definition of σ.

Case 7. $|C_{V(H_{x_2})}| = |C_{V(H_{x_3})}| = m$. At this point, there will have $C_{V(H_{x_1})} \cap C_{V(H_{x_3})} \neq \emptyset$, otherwise, three sets $C_{V(H_{x_1})}, C_{V(H_{x_2})}$ and $C_{V(H_{x_3})}$ are pairwise disjoint. Same as the case 6, it can produce contradictory. By $C_{V(H_{x_1})} \cap C_{V(H_{x_3})} \neq \emptyset$, $C_{V(H_{x_2})} \cap C_{V(H_{x_4})} = \emptyset$. Hence, the three sets $C_{V(H_{x_1})}, C_{V(H_{x_2})}$ and $C_{V(H_{x_3})}$ are pairwise disjoint. So the star coloring σ of $\widetilde{G}$ needs at least $2m + r$ colors, which contradicts the definition of σ.

Therefore, $\chi_s(\widetilde{G}) \geq 2m + r$.

Theorem 2.4. Let C_n be a cycle where $n \geq 4$, and $h_n = (H_x)_{x\in V(P_n)}$ be a disjoint graph sequence where every H_i is an $m-$order graph and the star chromatic number is r. If $n \neq 5$, then $\chi_s(C_n[h_n]) = 2m + r$. Otherwise, $\chi_s(C_n[h_n]) = 3m + r$.

Proof. Since if $n \geq 4$ and $n \neq 5$, C_n exists $3-$colorful neighbor star coloring, it can be obtained according to Theorem 2.1, $\chi_s(C_n[h_n]) \leq 2m + r$. On the other hand, if $n \geq 4$, then $P_n[h_n] \subseteq C_n[h_n]$, it can be obtained according to Theorem 2.3, $\chi_s(C_n[h_n]) \geq 2m + r$. Hence, $\chi_s(C_n[h_n]) = 2m + r$.

If $n = 5$, let $C_5 = x_1x_2x_3x_4x_5x_1$, denote $\widetilde{G} = C_5[h_5]$. Since C_5 exists $4-$colorful neighbor star coloring, it can be obtained according to Theorem 2.1, $\chi_s(\widetilde{G}) \leq 3m + r$. To prove $\chi_s(\widetilde{G}) \geq 3m + r$. Assume $\chi_s(\widetilde{G}) \leq 3m + r - 1$ and σ is a $(3m + r - 1)$-star coloring of $\widetilde{G}$. Same as the proof process of Theorem 2.3, it can be known that, for any three successive vertices x_i, x_j and x_k in C_5, the sets $C_{V(H_{x_i})}, C_{V(H_{x_j})}$ and $C_{V(H_{x_k})}$ are pairwise disjoint, and there is at most one vertex set whose color number is less than m. Easy to prove, there are at most two vertex sets $V(H_{x_{i'}})$, $V(H_{x_{i''}})$ in $\widetilde{G}$ whose color number less than m. Otherwise, there will be a bicolored path of order 4, which contradicts the definition of σ. Suppose $|C_{V(H_1)}| < m$, $|C_{V(H_3)}| < m$. Obviously, the sets $C_{V(H_1)}, C_{V(H_2)}$ and $C_{V(H_5)}$ are pairwise disjoint, and $|C_{V(H_2)}| = |C_{V(H_5)}| = m$. The sets $C_{V(H_2)}, C_{V(H_3)}$ and $C_{V(H_4)}$ are pairwise disjoint, and $|C_{V(H_2)}| = |C_{V(H_4)}| = m$. Since $C_{V(H_4)} \cap C_{V(H_5)} \neq \emptyset$, the star coloring σ of $\widetilde{G}$ needs at least $3m + r$ colors, which contradicts the definition of σ.

In 2018, Karthick in literature [26] proved that $\chi_s(P_4[K_t]) = 3t$, $\chi_s(C_5[K_t]) = 4t$. By means of Theorem 2.3 and Theorem 2.4 in this paper, the above results can be extended to the lexicographic product of path or cycle of order n and complete graph of order t, where $n \geq 4$. See Corollary 2.2 for specific results.

Corollary 2.2. $\chi_s(P_n[K_t]) = 3t$, where $n \geq 4$. (2) $\chi_s(C_n[K_t]) = 3t$, where $n \geq 4$ and $n \neq 5$.

3. The acyclic coloring of generalized lexicographic product of graphs

In this section, we study the acyclic coloring of generalized lexicographic product $G[H_n]$.

In the following theorem, we establish an upper bound of the acyclic chromatic number of $G[H_n]$.

Theorem 3.1. Let $\sigma_G = (V_1, V_2, \ldots, V_k)$ be a colorful neighbor acyclic coloring of G, where V_k is a colorful neighbor color class of σ_G, then

$$a(G[h_n]) \leq \max_{x \in V_k} a(H_x) + \sum_{i \in [k-1]} \max_{x \in V_i} |V(H_x)|. \tag{3.1}$$

Proof. Denote $\widetilde{G} = G[h_n]$, $p = \max_{x \in V_k} a(H_x)$, $q_i = \max_{x \in V_i} |V(H_x)|$, where $i \in [k-1]$. And let σ_1 be a p-acyclic coloring of $\bigcup_{x \in V_k} H_x$, let σ_2 be a q_i−vertex coloring of $\bigcup_{x \in V_i} H_x$ where $i \in [k-1]$ and different vertices in each H_x are in different colors. Construct a $(p + \sum_{i \in [k-1]} q_i)$−vertex coloring σ of $\widetilde{G}$ as follows:

$$\sigma((x, y)) = \begin{cases} (\sigma_G(x), \sigma_1(y)) & if\ x \in V_k, \\ (\sigma_G(x), \sigma_2(y)) & if\ x \notin V_k. \end{cases} \tag{3.2}$$

Obviously, σ is a proper vertex coloring of $\widetilde{G}$. Then, it is shown that σ is an acyclic coloring of $\widetilde{G}$. Take any cycle C of order 4 in $\widetilde{G}$, and the following four cases will be discussed:

Case 8. All vertices of C are in $H_{x_{i_0}}$. Since σ is an acyclic coloring limited to every H_x, C is not bicolored.

Case 9. All vertices of C are in $H_{x_{i_0}}$ and $H_{x_{i_1}}$, where $x_{i_0}x_{i_1} \in E(G)$. In fact, there is at least one $x_{i_0}, x_{i_1} \notin V_k$. Therefore, C is not bicolored.

Case 10. All vertices of C are in $H_{x_{i_0}}$, $H_{x_{i_1}}$ and $H_{x_{i_2}}$, where $G[\{x_{i_0}, x_{i_1}, x_{i_2}\}]$ contains a path of length 2. Suppose without loss of generality that $x_{i_0}x_{i_1}x_{i_2}$ is a path in G, obviously, there are at least two vertices of C are in $H_{x_{i_1}}$. If these two vertices are in different color, C is not bicolored. If these two vertices are in same color, then $x_{i_1} \in V_k$. Since V_k is a colorful neighbor color class of σ_G, the color sets $H_{x_{i_0}}$ is disjoint with $H_{x_{i_2}}$. Therefore, C is not bicolored.

Case 11. The vertices of C are distributed in at least four different $H_{x_{i_0}}$, $H_{x_{i_1}}$, $H_{x_{i_2}}$, and $H_{x_{i_3}}$, where $G[\{x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}\}]$ contains a path of length 3. Suppose without loss of generality that $x_{i_0}x_{i_1}x_{i_2}x_{i_3}$ is a path in G, obviously, there are at least two vertices of C are in $H_{x_{i_1}}$ and at least two vertices of C are in $H_{x_{i_2}}$. According to the proof process of case 10, the vertices of C in $H_{x_{i_1}}$ and $H_{x_{i_2}}$ are at least in three colors.

Therefore, σ is a $(p + \sum_{i\in[k-1]} q_i)-$acyclic coloring of $\widetilde{G}$.

If the above $\sigma_G = (V_1, V_2, \ldots, V_k)$ is a $2-$distance coloring of G, each color class is colorful neighbor color class of σ_G. Theorem 3.1 can be applied to each color class, then

$$a(G[h_n]) \leq \min_{i\in[k]}\{\max_{x\in V_i} a(H_x) + \sum_{j\in[k],j\neq i} \max_{x\in V_j} |V(H_x)|\}. \tag{3.3}$$

When G is a complete graph K_n of order $n \geq 2$, since any $k-$proper vertex coloring of K_n is a colorful neighbor acyclic coloring of K_n, and each color class is colorful neighbor color class. Theorem 3.1 can be applied to each color class, so it can be obtained

$$a(K_n[h_n]) \leq \min_{i\in[n]}\{a(H_i) + \sum_{j\in[n],j\neq i} |V(H_j)|\}. \tag{3.4}$$

This upper bound is tight, see the Theorem 3.2.

About the acyclic coloring of the joins of any two simple graphs G and H, we have

$$a(G \vee H) = \min\{a(G) + |V(H)|, a(H) + |V(G)|\}. \tag{3.5}$$

By using the above formula and mathematical induction, more general results can be obtained. See Theorem 3.2 for details.

Theorem 3.2. For any integer $n \geq 2$,

$$a(K_n[h_n]) = \min_{i\in[n]}\{a(H_i) + \sum_{j\in[n],j\neq i} |V(H_j)|\}. \tag{3.6}$$

Proof. We argue by induction on the number of vertices of K_n. First of all, the result is valid for the case when n=2.

$$a(K_2[h_2]) = \min\{a(H_1) + |V(H_2)|, a(H_2) + |V(H_1)\}. \tag{3.7}$$

That is, when $n = 2$, the theorem is valid.

Assume that $n = p$, the conclusion of the theorem holds, i.e.

$$a(K_p[h_p]) = \min_{i\in[p]}\{a(H_i) + \sum_{j\in[p],j\neq i} |V(H_j)|\}. \tag{3.8}$$

We need to prove that the theorem is valid when $n = p + 1$. When $n = 2$, since $K_{p+1}[h_{p+1}] = K_p[h_p] \vee H_{p+1}$, according to the above formula,

$$\begin{aligned} a(K_{p+1}[h_{p+1}]) &= a(K_p[h_p] \vee H_{p+1}) \\ &= \min\{a(K_p[h_p]) + |V(H_{p+1})|, a(H_{p+1}) + \sum_{j\in[p]} |V(H_j)|\} \\ &= \min\{\min_{i\in[p]}\{a(H_i) + \sum_{j\in[p+1],j\neq i} |V(H_j)|\}, a(H_{p+1}) + \sum_{j\in[p]} |V(H_j)|\} \\ &= \min_{i\in[p+1]}\{a(H_i) + \sum_{j\in[p+1],j\neq i} |V(H_j)|\}. \end{aligned}$$

According to the principle of mathematical induction, the conclusion of the theorem holds.

In Theorem 3.1, if each H_i of $h_n = (H_x)_{x\in V(G)}$ isomorphic to H, we can get the upper bound of the acyclic chromatic number of the lexicographic product of G and H, in Corollary 3.1.

Corollary 3.1. Let G and H be two graphs. If G exists $k-$colorful neighbor acyclic coloring, then

$$a(G[H]) \le a(H) + (k-1)|V(H)|. \tag{3.9}$$

When every H_i in $h_n = (H_x)_{x\in V(G)}$ is an $m-$order graph and the acyclic chromatic number is r, we can get the acyclic chromatic number of the generalized lexicographic product $C_n[h_n]$, see Theorem 3.3 for details.

Theorem 3.3. Let C_n be a cycle where $n \ge 4$, and $h_n = (H_x)_{x\in V(P_n)}$ be a disjoint graph sequence where every H_i is an $m-$order graph and the acyclic chromatic number is r. If n is even and $n \ge 2m+2$, then $a(C_n[h_n]) \le 2m + 1$. If n is odd, then $a(C_n[h_n]) \le 2m + r$.

Proof. Let $C_n = x_1x_2\cdots x_nx_1$, and denote $\widetilde{G} = C_5[h_5]$. Since if $n \ge 4$, C_n exists 3−colorful neighbor acyclic coloring, it can be obtained according to Theorem 3.1, $a(\widetilde{G}) \le 2m + r$. Thus, if n is odd, we have $a(C_n[h_n]) \le 2m + r$.

If n is even and $n \ge 2m + 2$. Let $\sigma_G = (V_1, V_2)$ be a acyclic coloring of C_n where $V_1 = \{x_i|(i)_2 = 0\}$, $V_2 = \{x_i|(i)_2 = 1\}$. To prove $a(\widetilde{G}) \le 2m + 1$. Construct a coloring σ of $\widetilde{G}$ as follows: Let $C_1 = \{0, 1, \dots, m\}$, $C_2 = \{m + 1, m + 2, \dots, 2m\}$. If $x_i \in V_1$, use the color set $C_1/\{(i/2)_{m+1}\}$ to color every H_{x_i}. If $x_i \in V_2$, use the color set C_2 to color every H_{x_i}. In particular, different vertices of every H_{x_i} are in different colors. It is easy to prove that σ is a $(2m + 1)$-acyclic coloring of $\widetilde{G}$. Thus, if n is even and $n \ge 2m + 2$, we have $a(C_n[h_n]) \le 2m + 1$.

For the upper bound of acyclic chromatic number of generalized lexicographic product $G[h_n]$, where the acyclic chromatic number of G is k but G does not exist $k-$colorful neighbor acyclic coloring, see Theorem 3.4.

Theorem 3.4. Let $\sigma_G = (V_1, V_2, \dots, V_k)$ be a acyclic coloring of G, then

$$a(G[h_n]) \le \sum_{i\in[k]} \max_{x\in V_i} |V(H_x)|. \tag{3.10}$$

Proof. Denote $\widetilde{G} = G[h_n]$, $p_i = \max\limits_{x\in V_i} |V(H_x)|$, where $i \in [k]$. Let σ_1 be a q_i-vertex coloring of $\bigcup\limits_{x\in V_i} H_x$, and different vertices of every H_x are in different colors, where $i \in [k]$. Construct a $(\sum\limits_{i\in[k]} p_i)-$coloring σ of $\widetilde{G}$ as follows:

$$\sigma((x, y)) = (\sigma_G(x), \sigma_1(y)). \tag{3.11}$$

It is easy to prove that σ is a proper vertex coloring $\widetilde{G}$. As a matter of fact, since different vertices of every H_x are in different color, and for any $x_{i_0}x_{i_1} \in E(G)$, we have the color set $H_{x_{i_0}}$ is disjoint with $H_{x_{i_1}}$. Any cycle of $\widetilde{G}$ is not bicolored. Therefore, σ is a $(\sum\limits_{i\in[k]} p_i)-$acyclic coloring of $\widetilde{G}$.

The upper bound of Theorem 3.4 is tight, such as when every H_i of $h_n = (H_x)_{x\in V(G)}$ has m vertices, the acyclic chromatic number of $P_n[h_n]$ is $2m$, see Theorem 3.5 for details.

Theorem 3.5. Let P_n be a path where $n \geq 4$, and $h_n = (H_x)_{x \in V(P_n)}$ be a disjoint graph sequence where every H_i is an $m-$order graph. Then, $a(P_n[h_n]) = 2m$.

Proof. Let $P_n = x_1 x_2 \cdots x_n$, and denote $\widetilde{G} = P_n[h_n]$. Since P_n exists $2-$acyclic coloring, according to Theorem 3.4, $a(\widetilde{G}) \leq 2m$. To prove $a(\widetilde{G}) \geq 2m$. Assume $a(\widetilde{G}) \leq 2m-1$, and σ_0 is a $(2m-1)-$acyclic coloring of $\widetilde{G}$. Obviously, for any $i = 1, 2, \cdots, n$, $a(H_{x_i}) \leq |C_{V(H_{x_i})}| \leq m$. For any $j = 1, 2, \cdots, n-1$, $C_{V(H_{x_j})} \cap C_{V(H_{x_{j+1}})} = \emptyset$, and only one of the two equations $|C_{V(H_{x_j})}| = m$ and $|C_{V(H_{x_{j+1}})}| = m$ is true. According to whether $|C_{V(H_{x_2})}| = m$ is true or not, it can be divided into the following two cases:

Case 12. $|C_{V(H_{x_2})}| < m$. It is easy to prove that $|C_{V(H_{x_1})}| = |C_{V(H_{x_3})}| = m$ and $C_{V(H_{x_1})} \cap C_{V(H_{x_3})} = \emptyset$. Therefore, the three sets $C_{V(H_{x_1})}, C_{V(H_{x_2})}$ and $C_{V(H_{x_3})}$ are pairwise disjoint. Thus, the acyclic coloring σ_0 of $\widetilde{G}$ needs at least $2m + a(H_{x_2})$ colors, which contradicts the definition of σ_0.

Case 13. $|C_{V(H_{x_2})}| = m$. At this time, $|C_{V(H_{x_3})}| \geq a(H_{x_3})$. Then, $C_{V(H_{x_3})} \cap C_{V(H_{x_4})} = \emptyset$ and $C_{V(H_{x_2})} \cap C_{V(H_{x_4})} = \emptyset$. Thus, three sets $C_{V(H_{x_2})}, C_{V(H_{x_3})}$ and $C_{V(H_{x_4})}$ are pairwise disjoint. Therefore, the acyclic coloring σ_0 of $\widetilde{G}$ needs at least $2m + a(H_{x_3})$ colors, which contradicts the definition of σ_0.

Thus, $a(\widetilde{G}) \geq 2m$. Then, $a(\widetilde{G}) = 2m$.

4. Conclusions

In this paper, we mainly study the star and acyclic coloring of generalized lexicographic product of graphs. we present upper bound for the star and acyclic chromatic numbers of the generalized lexicographic product $G[h_n]$ of graph G and disjoint graph sequence h_n, where G exists a $k-$colorful neighbor star coloring or $k-$colorful neighbor acyclic coloring.

In addition, we obtain the exact value of the star chromatic numbers of the generalized lexicographic product of complete graph, path, cycle and special disjoint graph sequences. And we also obtain the exact value of the acyclic chromatic numbers of the generalized lexicographic product of complete graph, path and special disjoint graph sequence. These exact values can prove that the upper bounds we get are tight.

According to Theorem 2.2 and Theorem 2.4, the star chromatic number of the generalized lexicographic product of complete graph and disjoint graph sequence of the same order with star chromatic number and acyclic chromatic number is equal to its acyclic chromatic number. On this basis, we put forward the following problem to be solved:

What condition does the graph G satisfy if $\chi_s(G[h_n]) = a(G[h_n])$?

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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