



Research article

Non-resonance with one-sided superlinear growth for indefinite planar systems via rotation numbers

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Abstract: We consider the non-resonance with one-sided superlinear growth conditions for the indefinite planar system $z' = f(t, z)$ from a rotation number viewpoint, and obtain the existence of 2π -periodic solutions by applying a rotation number approach together with the Poincaré-Bohl theorem. We allow that the angular velocity of solutions of $z' = f(t, z)$ is controlled by the angular velocity of solutions of two positively homogeneous and oddly symmetric systems $z' = L_i(t, z)$, $i = 1, 2$ on the left half-plane, which have rotation numbers that satisfy $\rho(L_1) > n/2$ and $\rho(L_2) < (n + 1)/2$, and allow $f(t, z)$ to grow superlinearly on the right half-plane. In order to estimate the rotation angle difference of solutions, we develop a system methodology of “tracking” the angle difference of solutions of the system $z' = f(t, z)$ on each small interval on the given side under sign-varying conditions.

Keywords: periodic solutions; non-resonance; rotation numbers; one-sided superlinear; Poincaré-Bohl theorem

Mathematics Subject Classification: 34C25, 34B15, 34D15

1. Introduction

In this paper, we consider the existence of periodic solutions of the indefinite planar system

$$z' = f(t, z), \quad z = (x, y) \in \mathbb{R}^2, \tag{1.1}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2)$ is 2π -periodic with respect to the first variable and locally Lipschitz-continuous with respect to the second variable, guaranteeing uniqueness for the associated Cauchy problems.

Resonance and non-resonance problems are typical models in ordinary differential equations. There are many interesting results on these topics. Especially, many authors are interested in the study of these problems under asymmetric conditions, even one-sided superlinear growth conditions, which

can be traced back to the work of Dancer [1] and Fučík [2]. Dancer and Fučík studied the existence of 2π -periodic solutions of the equation

$$x'' + f(t, x) = 0 \quad (1.2)$$

when $f(t, x) = ax^+ - bx^- - p(t)$, where p is a 2π -periodic and continuous function, n is a positive integer, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$, satisfying

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{n}. \quad (1.3)$$

If the positive number pair (a, b) does not satisfy (1.3), then for any 2π -periodic continuous function p , Eq (1.2) has a 2π -periodic solution provided that $a \neq b$. For $n = 1, 2, \dots$, all of the number pairs (a, b) satisfying (1.3) are called the Fučík spectrum. When $a = b$, Eq (1.2) with f as above becomes

$$x'' + n^2x = p(t). \quad (1.4)$$

The linear Eq (1.4) has the classical linear resonance phenomenon, that is, periodic solutions and unbounded solutions cannot coexist; thus either the solutions of Eq (1.4) are all periodic solutions (cf. [3]), or the solutions of Eq (1.4) are all unbounded solutions.

Based on the study of the asymmetric nonlinearity f satisfying

$$A_+ \leq \liminf_{x \rightarrow +\infty} \frac{f(t, x)}{x} \leq \limsup_{x \rightarrow +\infty} \frac{f(t, x)}{x} \leq B_+, \quad (1.5)$$

$$A_- \leq \liminf_{x \rightarrow -\infty} \frac{f(t, x)}{x} \leq \limsup_{x \rightarrow -\infty} \frac{f(t, x)}{x} \leq B_-, \quad (1.6)$$

for some suitable constants $A_{\pm}, B_{\pm} \in [0, +\infty]$, the existence of solutions to Eq (1.2) is strictly related to the position of the rectangle $W = [A_+, B_+] \times [A_-, B_-]$ with respect to the set called the Fučík spectrum. According to the “position”, there are three kinds of problems, i.e., non-resonance (cf. [1, 2, 4–13]), simple resonance (cf. [14–16]) and double resonance (cf. [14, 17–20]). Especially, the book [21] is devoted to both resonance and non-resonance situations.

In particular, Drábek and Invernizzi [4] generalized Fučík’s work [2] to Eq (1.2), and assumed that (1.5) and (1.6) hold uniformly for t , and for some integer n , the relations

$$\frac{1}{\sqrt{A_+}} + \frac{1}{\sqrt{A_-}} < \frac{2}{n}, \quad \frac{1}{\sqrt{B_+}} + \frac{1}{\sqrt{B_-}} > \frac{2}{n+1} \quad (1.7)$$

hold. And a complementary result was obtained by Mawhin and Ward [5]; they considered a Liénard equation together with an assumption of the type $A_+ > 0$, $A_- > 0$, $B_- < 1/4$; a similar case refers to $A_+ > 0$, $A_- > 0$ and $B_+ < 1/4$.

Thereafter, Fabry and Habets [7] generalized the above work and considered a non-resonance situation with respect to the Dancer-Fučík spectrum, specifically, where f is allowed to have superlinear growth on one side as follows.

$$a_+(t) \leq \liminf_{x \rightarrow +\infty} \frac{f(t, x)}{x} \leq \limsup_{x \rightarrow +\infty} \frac{f(t, x)}{x} \leq b_+(t),$$

$$a_-(t) \leq \liminf_{x \rightarrow -\infty} \frac{f(t, x)}{x},$$

where $a_{\pm}(t)$, $b_{\pm}(t)$ satisfy the non-resonance conditions similar to that between two neighboring Fučík spectrum curves, and the following sign condition

$$\liminf_{x \rightarrow +\infty} \operatorname{sgn}(x)f(t, x) > 0.$$

It is noted that the particularity of the non-resonance conditions given in [7] is that f is allowed to grow superlinearly on the side of $x < 0$.

In these situations mentioned above the function f is possibly superlinear on one side. Concerning this problem, one need also notice the early works of Schmitt [22], Reissig [23], Fonda and Zanolin [6], Fernandes and Zanolin [24] and Fabry and Fonda [25].

In recent years, Qian [8] studied the existence of solutions of the Liénard equation by applying topological degree theory under asymmetric conditions. Such a work in [8] gave a contribution in the direction of considering the one-sided growth restrictions for the Liénard equation, as in [26, 27]. And then, Fonda and Sfecci [9] generalized the work of Fabry and Habets [7] and proposed a general method to prove the existence of periodic solutions for planar systems similar to system (1.1) that can be applied to many different circumstances involving the case of systems with superlinear growth in some direction; their work contributed in the direction of considering the one-sided growth restrictions for planar systems. Fonda and Sfecci [10] generalized the work in [9] to the case of higher dimensions and applied the corresponding results to the periodic problems associated with a system of coupled oscillators with one-sided superlinear nonlinearities and other types of retraction forces.

About the non-resonance case, one needs to also notice the works that yielded multiplicity results of periodic solutions done by Rebelo and Zanolin [28, 29], Liu [30], Zanini and Zanolin [31] and so on.

The cases of simple and double resonance seem to start from the work of Fabry [14], which considered the periodic boundary value problems under Landesman-Lazer conditions for Eq (1.2), where at least one of the inequalities in (1.7) is replaced by an equality; such cases mean that the rectangle $[A_+, B_+] \times [A_-, B_-]$ touches one or two of the Fučík curves. Sfecci [20] dealt with the problem of the existence of T -periodic solutions for Eq (1.2) with the asymmetric nonlinearity satisfying the following one-sided superlinear growth at infinity,

$$\lim_{x \rightarrow -\infty} \frac{f(t, x)}{x} = +\infty;$$

a double resonance condition, namely, there exists a constant $c > 0$ and an integer $N > 0$ such that

$$\mu_N x - c \leq f(t, x) \leq \mu_{N+1} x + c$$

for every $x > 0$ and every $t \in [0, T]$, where $\mu_j = (j\pi/T)^2$, and the Landesman-Lazer conditions (see the details in Theorem 1.1 of [20]).

Especially, there are a series of works on the existence of unbounded solutions (cf. [32–34] and the references therein) and coexistence of periodic solutions and unbounded solutions (cf. [32, 35, 36] and the references therein) under asymmetric conditions.

In this paper, we would like to generalize the results in [8, 9] to the case of non-resonance with one-sided superlinear growth for the indefinite planar system described by system (1.1) from a rotation

number viewpoint; we do not assume the sign conditions similar to that in [6–9]. Sign conditions were frequently needed in the previous works; as we know, without sign conditions, some classical theories or methods may not be used, and one has to overcome the difficulties stemming from the variable signs of nonlinearities and seek (or even develop) suitable methods. So the non-resonance in the sense of rotation number is different from the non-resonance in [6–9] (see Remark 1.2 in this paper, Example 1.1 in [37] or Remark 1.3 in [38]). The rotation number is an important index for dynamical systems on the torus or second order linear periodic equations with variable coefficients; it plays a very important role in the study, and some authors have used it to solve problems, see for instance [39–44].

Now back to our generalization, the main difficulty of this problem is how to use the rotation numbers of positively homogeneous systems to estimate the rotation angle difference of solutions of system (1.1) under the one-sided superlinear growth conditions and the sign-varying condition. To this end, we develop a system methodology of “tracking” the angle difference of the solutions of system (1.1) on each small interval on the given side under sign-varying conditions, which is different from that in [13], because the the assumptions on the right half-plane are different, and so are the methods used to estimate the lower bound of the rotation angle difference. One can complete the estimation of the lower bound directly by exploiting the relationship between the rotation numbers and the 2π -rotation numbers (see the definition in Section 2 in [43]) of the solutions in [13].

Here, and throughout the paper, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix. $\rho(L_i), i = 1, 2$ represents the rotation numbers of the systems $z' = L_i(t, z), i = 1, 2$; see the definition in Section 2.

In this paper, we suppose that

(H_0) $f \in C(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2)$ is 2π -periodic with respect to t , the solution of the initial value problem of system (1.1) is unique and exists globally;

(H_1) $\rho(L_1) > \frac{n}{2}$, $L_1(t, z)$ is continuous, 2π -periodic with respect to t , positively homogeneous of degree 1 and odd with respect to z , and when $x \leq 0$ (x is the first component of z),

$$\liminf_{\lambda \rightarrow +\infty} \left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle \geq \langle JL_1(t, z), z \rangle$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z \in \mathbb{S}^1$;

(H_2) $\rho(L_2) < \frac{n+1}{2}$, $L_2(t, z)$ is continuous, 2π -periodic with respect to t , positively homogeneous of degree 1 and odd with respect to z , and when $x \leq 0$ (x is the first component of z),

$$\limsup_{\lambda \rightarrow +\infty} \left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle \leq \langle JL_2(t, z), z \rangle$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z \in \mathbb{S}^1$;

(H_3) When $x > 0$ (x is the first component of z),

$$\lim_{\lambda \rightarrow +\infty} \left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle = +\infty$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z \in \mathbb{S}^1$;

(H_4) $y \operatorname{sgn}(f_1(t, 0, y)) < 0$ (f_1 is the first component of f) and $y \operatorname{sgn}(L_{i1}(t, 0, y)) < 0$ (L_{i1} is the first component of L_i), $i = 1, 2$ when $|y| \gg 1$.

Then our main result is as follows.

Theorem 1.1. *Suppose that conditions $(H_0) - (H_4)$ hold, then system (1.1) has a 2π -periodic solution.*

Remark 1.1. *Regarding condition (H_0) , we directly assume the existence, uniqueness and global existence of solutions, because the application of the Poincaré-Bohl theorem is based on the assumption that the Poincaré map of the system is well defined, which requires the global existence of solutions of the corresponding initial value problem. In fact, there are many conditions guaranteeing the existence, uniqueness and global existence of solutions wherein the uniqueness of solutions of the initial value problem can be guaranteed by the local Lipschitz continuity of f with respect to z . And for global existence, for example, it can be guaranteed that, if $f(t, z)$ satisfies the quasi-linear growth condition, namely, there exists $a, b \in L^1(t_0, t_0 + 2\pi)$ such that*

$$|f(t, z)| \leq a(t)|z| + b(t), \quad \text{for a. e. } t \in [0, 2\pi] \text{ and } z \in \mathbb{R}.$$

In [9], Fonda and Sfecci introduced a rotating regular spiral assumption and an angular speed controlling assumption to guarantee the global existence of the solutions to system (1.1); see the assumptions (H_1) and (H_3) in Theorem 2.3 in [9].

Remark 1.2. *In Theorem 1.1, we do not assume the sign conditions. As we know, sign conditions were frequently needed in the previous works on periodic solutions problems, including the classic works of Jacobowitz and Hartman [45, 46], Ding, Iannacci and Zanolin [47], Fonda and Toader [48] and the references therein. Once there is a lack of sign conditions, the continuation method (see its applications in [49, 50]) may not be used. The basic idea of the continuation method is to reduce nonlinear systems similar to system (1.1) to simple systems by applying homotopy invariance of the topological degree, and the simple systems are usually autonomous ones. Then one can compute the topological degrees of some of the operators associated with the autonomous systems, and the degrees are not zero; when the indefinite weight appears, the reduced autonomous systems are not easy to find, but the indefinite systems still meet some non-resonant conditions, such as the non-resonant conditions in the assumptions (H_1) and (H_2) , (H_2) in [37] and (H_{PB}) in [38]. Consider the one-sided superlinear conditions for the system*

$$-y' = g_1(t, x), \quad x' = g_2(t, y)$$

in Corollary 3.11 of [9], which requires all constants in the formulas (36)–(39) to be positive. Compared with the asymmetric problems in Theorem 1.1 in [8], the sign conditions are needed (Because the oddness assumption for the positively homogeneous systems may not be satisfied in [8], the comparison lies only in the sign conditions). So the notion of non-resonance with one-sided superlinearity in the sense of rotation numbers is different from that in [8, 9] in the sense without sign conditions.

Remark 1.3. *Fabry and Fonda [12] studied periodically forced planar Hamiltonian systems, have obtained sharp non-resonance results, and have provided new insights into the non-resonance problems by using the energy and a phase as variables. Under the framework, some classical situations can be covered, like asymmetric oscillators, positively homogeneous Hamiltonians of degree 2 or Hamiltonians with separated variables. In the final section of [12], the authors considered the more general system $Jz' = F(t, z)$, which is similar to system (1.1) in the present paper, and obtained existence conditions for the periodic solutions via a comparison with Hamiltonian systems. However, such Hamiltonian systems may not be found in indefinite problems, so we obtain*

the existence of the periodic solutions via a comparison with the positively homogeneous planar systems $z' = L_i(t, z)$, $i = 1, 2$, which may not be isochronous.

Remark 1.4. In Theorem 1.1, conditions (H_1) – (H_3) show the control of the angular velocity by the homogeneous systems on the left half-plane and the superlinearity on the right half-plane, and (H_4) shows that the solution curve rotates counterclockwise. These conditions have equivalent forms, so Theorem 1.1 could be stated in a more general way. We will discuss this in Section 4.

Remark 1.5. In Theorem 1.1, n is an integer, and it can be equal to 0. In this case, we can obtain that the solution curve of system (1.1) rotates less than one turn in 2π time, so it also fits the framework of the Poincaré-Bohl theorem adding other conditions.

For the controls with half-integers, from the superlinear growth condition for the right half-plane, one can see that the solution curve for the system $z' = f(t, z)$ can rotate half a circle in a sufficiently short time when it is on the right half-plane, this means that the solution curve for the system $z' = L(t, z)$ has not rotated out of the left half-plane if their initial state is on the left half-plane (see the proof of Lemma 3.8). Thus, in this sense, half-integers for the rotation numbers of the systems $z' = L_i(t, z)$, $i = 1, 2$ are enough to estimate the rotation angle difference. And half-integers are also used for the control of the rotations in [41].

For the reader's convenience, we present the Poincaré-Bohl fixed point theorem as follows.

Theorem 1.2. (Poincaré-Bohl theorem) Let $\Omega \subset \mathbb{R}^m$ be an open bounded set containing the origin, and $\varphi : \overline{\Omega} \rightarrow \mathbb{R}^m$ be a continuous function such that

$$\varphi(u) \neq \lambda u, \quad \text{for every } u \in \partial\Omega \text{ and } \lambda > 1.$$

Then, φ has a fixed point in $\overline{\Omega}$.

This paper is organized as follows. We introduce the rotation numbers and rotation angle differences of the positive homogeneous systems in Section 2. We show the estimations of the rotation angle differences of the solutions of system (1.1) in Section 3. In Section 4, we give the proof, more general states and applications of Theorem 1.1. Finally, in Section 5, we give the proofs of some technical lemmas.

2. Rotation number and rotation angle difference

In this section, similar to that in Section 2 in [13], we consider the relationship between the rotation number of the positively homogeneous planar system

$$z' = L(t, z), \quad z = (x, y) \in \mathbb{R}^2, \quad (2.1)$$

which is oddly symmetric, and the rotation angle difference of the planar system (1.1). For the sake of presentation, we retell them as follows.

Let $L : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a L^1 -Carathéodory function, 2π -periodic with respect to the first variable, positively homogeneous of degree 1 with respect to the second variable, namely

$$L(t, \lambda z) = \lambda L(t, z), \quad \forall \lambda \geq 0, \quad \forall z \in \mathbb{R}^2,$$

and it is also odd with respect to the second variable. In the following, denote by \mathcal{Q} the set of all of the functions with the above properties.

Next, we define the rotation number of system (2.1) and the rotation angle difference of system (2.1) and system (1.1).

Assume that the solution of the initial value problem of system (2.1) exists and is unique. For a solution $z(t; z_0)$ with the initial value $z_0 \neq (0, 0)$, by the polar coordinates transformation, it can be written as

$$x(t) = r(t) \cos \theta(t), \quad y(t) = r(t) \sin \theta(t);$$

thus we have

$$\begin{aligned} \theta'(t) &= -\frac{\langle Jz', z \rangle}{|z|^2} = -\frac{\langle JL(t, z), z \rangle}{|z|^2} \\ &= -\langle JL(t, (\cos \theta, \sin \theta)), (\cos \theta, \sin \theta) \rangle \\ &=: \psi(t, \theta; L). \end{aligned} \tag{2.2}$$

Denote by $\theta(t; t_0, \theta_0)$ the unique solution of Eq (2.2) with $\theta(t_0; t_0, \theta_0) = \theta_0$. Since $\psi(t, \theta; L)$ is 2π -periodic for t and π -periodic for θ when $L(t, z)$ is 2π -periodic for t and odd for z , Eq (2.2) is a differential equation on a torus. Thus

$$\theta(t + 2\pi; t_0 + 2\pi, \theta_0) = \theta(t; t_0, \theta_0)$$

and

$$\theta(t; t_0, \theta_0 + \pi) = \theta(t; t_0, \theta_0) + \pi \tag{2.3}$$

hold for arbitrary $\theta_0, t_0 \in \mathbb{R}$. Therefore, we can define the rotation number of system (2.1) as follows;

$$\rho(L) := \lim_{t \rightarrow \infty} \frac{\theta(t_0 + t; t_0, \theta_0) - \theta_0}{t} = \lim_{k \rightarrow \infty} \frac{\theta(t_0 + 2k\pi; t_0, \theta_0) - \theta_0}{2k\pi},$$

and it is independent of t_0 and θ_0 ; see Theorem 2.1 in Chapter 2 of Hale [51]. By extension, we refer to $\rho(L)$ as the rotation number of the positively homogeneous planar system (2.1).

Then, we give the definitions of the rotation angle differences of system (2.1) and system (1.1). Denote the rotation angle difference of system (2.1) in the time segment $[t_0, t'_0]$ by $\Delta\theta(t_0, t'_0)$, $\Delta\theta(t_0, t'_0) := \theta(t'_0; t_0, \theta_0) - \theta_0$ and denote the rotation angle difference of system (1.1) in the time segment $[t_0, t'_0]$ by $\Delta\varphi(t_0, t'_0)$, $\Delta\varphi(t_0, t'_0) := \varphi(t'_0; t_0, r_0, \varphi_0) - \varphi_0$, where $\varphi(t; t_0, r_0, \varphi_0)$ denotes the angle component of the solutions of system (1.1) satisfying the initial condition $z(t_0; t_0, z_0) = z_0$, $t'_0 \leq t_0 + 2\pi$.

Based on the above definitions, we consider the comparison lemma about the relationship between the rotation angle differences of system (1.1) and system (2.1).

First, similar to that in [37, 41, 43, 52], we consider the relationship between the rotation number $\rho(L)$ of system (2.1) and the rotation angle difference $\Delta\theta(t_0, t'_0)$ of its solutions in the time segment $[t_0, t'_0]$. The proofs of Lemma 2.2 and Lemma 2.3 are also similar to that in [37]. For convenience, we will give the proofs of Lemmas 2.1–2.3 in Section 5.

Lemma 2.1. *Let $n \in \mathbb{Z}$; then*

- (i) $\rho(L) < n \Rightarrow \Delta\theta(t_0, t'_0) < 2n\pi, \forall (t_0, \theta_0) \in \mathbb{R}^2, t'_0 \leq t_0 + 2\pi;$
- (ii) $\rho(L) > n \Rightarrow \Delta\theta(t_0, t_0 + 2\pi) > 2n\pi, \forall (t_0, \theta_0) \in \mathbb{R}^2;$
- (iii) $\rho(L) = n$ if and only if there is at least one nontrivial 2π -periodic solution $\theta(t; t_0, \theta_0)$ of Eq (2.2) with $\theta(t_0 + 2\pi; t_0, \theta_0) - \theta_0 = 2n\pi$.

Next, we state the depending relationship lemma about the rotation angle difference $\Delta\theta(t_0, t'_0)$ of system (2.1) and the function $L(t, z)$.

Lemma 2.2. Assume $L_i(t, z) \in \mathcal{Q}$, $i = 1, 2$ and that for all $z \in \mathbb{R}^2$,

$$\langle JL_2(t, z) - JL_1(t, z), z \rangle \geq 0 \quad (2.4)$$

holds for a.e. $t \in [t_0, t'_0]$, $t'_0 \leq t_0 + 2\pi$. Then

$$\Delta\theta_2(t_0, t) \geq \Delta\theta_1(t_0, t), \quad \forall t \in [t_0, t'_0].$$

Second, we state the comparison result for the rotation angle difference of system (2.1) for different functions $L_i(t, z)$ ($i = 1, 2$).

Lemma 2.3. Suppose $L_i(t, z) \in \mathcal{Q}$, $i = 1, 2$. Assume also that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle JL_2(t, z), z \rangle \geq \langle JL_1(t, z), z \rangle - \delta|z|^2 \quad (2.5)$$

holds for all $z \in \mathbb{R}^2$ and a.e. $t \in [t_0, t'_0]$, $t'_0 \leq t_0 + 2\pi$. Then

$$\Delta\theta_2(t_0, t) \geq \Delta\theta_1(t_0, t) - \varepsilon, \quad \forall t \in [t_0, t'_0]. \quad (2.6)$$

Third, similar to the discussion in Section 2 in [13], we give the relationship of the rotation angle difference between system (2.1) and system (1.1). From (H_1) and (H_2) , we deduce the following two properties.

(H'_1) When $x \leq 0$, there exists $L_1(t, z) \in \mathcal{Q}$ satisfying $\forall \delta > 0, \exists l(t) \in L^1([t_0, t'_0], \mathbb{R}_+)$ such that

$$\left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle \geq \langle JL_1(t, z), z \rangle - \delta|z|^2 - l(t) \quad (2.7)$$

holds uniformly for a.e. $t \in [t_0, t'_0]$, $z \in S^1$ and $\lambda \geq 1$;

(H'_2) When $x \leq 0$, there exists $L_2(t, z) \in \mathcal{Q}$ satisfying $\forall \delta > 0, \exists l(t) \in L^1([t_0, t'_0], \mathbb{R}_+)$ such that

$$\left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle \leq \langle JL_2(t, z), z \rangle + \delta|z|^2 + l(t)$$

holds uniformly for a.e. $t \in [t_0, t'_0]$, $z \in S^1$ and $\lambda \geq 1$.

Then, applying the above two lemmas we have the following comparison lemma.

Lemma 2.4. (Comparison Lemma A) Suppose $L_i \in \mathcal{Q}$, $i = 1, 2$ and $f : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is 2π -periodic with respect to the first variable t and continuous with respect to the second variable z .

(i) If (H'_1) holds, then for each $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\Delta\varphi(t_0, t) \geq \Delta\theta_1(t_0, t) - \varepsilon, \quad \forall t \in [t_0, t'_0] \quad (2.8)$$

holds for every solution of system (1.1) satisfying $|z(t)| \geq R_\varepsilon, \forall t \in [t_0, t'_0]$.

(ii) If (H'_2) holds, then for each $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\Delta\varphi(t_0, t) \leq \Delta\theta_2(t_0, t) + \varepsilon, \quad \forall t \in [t_0, t'_0] \quad (2.9)$$

holds for every solution of system (1.1) satisfying $|z(t)| \geq R_\varepsilon, \forall t \in [t_0, t'_0]$.

Remark 2.1. By the assumption of oddness, the solutions of $z' = L(t, z)$ spend the same time in the left and right half-plane. In fact, using Eq (2.2) we can calculate the times spent by the solutions in the left half-plane and right half-plane using

$$\Delta t = \int_{\theta_1}^{\theta_2} \frac{1}{\theta'} d\theta = - \int_{\theta_1}^{\theta_2} \frac{1}{\langle JL(t, (\cos \theta, \sin \theta)), (\cos \theta, \sin \theta) \rangle} d\theta.$$

Then we can obtain that $\Delta t_L = \Delta t_R$ by a variable substitution. Δt_L and Δt_R denote the times spent by the solutions in the left and right half-planes, respectively.

3. Estimations of rotation angle differences for non-resonance with one-sided superlinear growth

In this section, we denote the solution of the equation $\theta' = \psi(t, \theta) - \varepsilon$ by $\theta_\varepsilon(t; t_0, \theta_0)$, where $\varepsilon > 0$, and the angle component of the solution of system (1.1) by $\varphi(t; t_0, r_0, \varphi_0)$. Because we only discuss the solutions for sufficiently large r_0 in the following parts and the estimation formulas hold for sufficiently large $r(t)$, we sometimes note $\varphi(t; t_0, r_0, \varphi_0)$ as $\varphi(t; t_0, \varphi_0)$ for short.

First, similar to Lemma 4.3 in [37] and Lemmas 3.4 and Lemma 3.5 in [13], we give an orientation-preserving lemma and a sign lemma.

Lemma 3.1. (Orientation preserving lemma) Suppose that all of the solutions of the equation $\theta' = \psi(t, \theta)$ satisfying the initial value conditions exist uniquely; let $\theta(t; t_0, \theta_0)$ and $\theta(t; t_0, \theta_1)$ be two solutions of Eq (2.2) with the respective initial values $\theta(t_0; t_0, \theta_0) = \theta_0$ and $\theta(t_0; t_0, \theta_1) = \theta_1$, $t \in [t_0, t'_0]$. If $\theta_0 < \theta_1$ ($\theta_0 > \theta_1$), then

$$\theta(t; t_0, \theta_0) < \theta(t; t_0, \theta_1) \quad (\theta(t; t_0, \theta_0) > \theta(t; t_0, \theta_1)), \quad t \in [t_0, t'_0].$$

Lemma 3.2. (Sign lemma) Assume (H_4) ; if $\theta(t_*; t_0, \theta_0) \geq \frac{\pi}{2} + k\pi$, then $\forall t > t_*$, and we have $\theta(t; t_0, \theta_0) > \frac{\pi}{2} + k\pi$.

Second, we give the comparison lemma after the following two preliminary lemmas.

Lemma 3.3. Suppose that $\varepsilon > 0$, and $\psi(t, \theta)$ is a continuous function, 2π -periodic with respect to t , differentiable and π -periodic with respect to θ . Let $\theta(t; t_0, \theta_0)$ and $\theta_\varepsilon(t; t_0, \theta_0)$ be solutions of $\theta' = \psi(t, \theta)$ and $\theta' = \psi(t, \theta) - \varepsilon$, respectively. Then there exists $K_\varepsilon < 0$ such that $\theta_\varepsilon(t) \leq \theta(t) + K_\varepsilon(t - t_0)$, $t \geq t_0$.

Proof. Let $\bar{\theta}(t) = \theta_\varepsilon(t) - \theta(t)$; then $\bar{\theta}(t)$ satisfies

$$(\bar{\theta}(t))' = a(t)\bar{\theta}(t) - \varepsilon, \tag{3.1}$$

where $a(t) = \frac{\partial \psi}{\partial \theta}(t, \xi)$ and $\xi(t)$ is between $\theta(t)$ and $\theta_\varepsilon(t)$.

Integrating on both sides of Eq (3.1) from t_0 to t , we have

$$\bar{\theta}(t) = \int_{t_0}^t -\varepsilon \exp\left(\int_s^t a(\tau) d\tau\right) ds.$$

Take $K_\varepsilon = \max_{t_0 \leq t \leq t_0 + 2\pi} -\varepsilon \exp\left(\int_{t_0}^t a(s)ds\right)$; then

$$\bar{\theta}(t) \leq K_\varepsilon(t - t_0).$$

□

And by a direct application of the definition of the rotation number of the positively homogeneous system (2.1), we have the following lemma.

Lemma 3.4. *Suppose $\rho(L_1) > \frac{n}{2}$; then there exists $\varepsilon > 0$ such that $\rho(L_\varepsilon) > \frac{n}{2}$, where $L_\varepsilon(t, z) = L_1(t, z) - \varepsilon Jz$.*

We know that there exists L_ε such that $\rho(L_\varepsilon) > \frac{n}{2}$ under the conditions of Lemma 3.1. For this $\varepsilon > 0$, by the definition of rotation numbers there exists $\delta > 0$ such that

$$\theta_\varepsilon(t_0 + 2\pi; t_0, \theta_0) - \theta_0 > n\pi + 2\delta \quad (3.2)$$

holds for an arbitrary t_0 and θ_0 .

Lemma 3.5. (Comparison Lemma B) *Suppose that $\varepsilon > 0$, $\psi(t, \theta)$ is a continuous function, 2π -periodic with respect to t , differentiable and π -periodic with respect to θ . Then for the above $\varepsilon > 0$, there exists r_ε with $r_0 \geq r_\varepsilon$ such that*

$$\varphi(t; t'_0, \theta'_0) > \theta_\varepsilon(t; t'_0, \theta'_0) + \delta$$

holds for the angle variable of solutions in the left plane when $t \geq t'_0 + t_\varepsilon$, $\forall t'_0 \in [t_0, t_0 + 2\pi]$. It is given that $t_\varepsilon = \min\{t - t'_0 \mid t > t'_0, \theta_\varepsilon(t) - \theta_\varepsilon(t'_0) \geq \delta\} > 0$ and δ is sufficiently small such that $K_\varepsilon t_\varepsilon + \delta < 0$.

Proof. $\theta_\varepsilon(t; t_0, \theta_0)$ satisfies

$$\theta'_\varepsilon(t) = \psi_\varepsilon(t, \theta_\varepsilon(t)),$$

where $\psi_\varepsilon = -\langle JL(t, \cos \theta, \sin \theta), (\cos \theta, \sin \theta) \rangle - \varepsilon = \psi(t, \theta) - \varepsilon$. Take $\psi_M = \max_{(t, \theta) \in [0, 2\pi] \times [0, 2\pi]} |\psi_\varepsilon(t, \theta)|$; then

$t_\varepsilon \geq \frac{\delta}{\psi_M} > 0$. By (H_1) and Comparison Lemma A we can obtain that, for K_ε and t_ε , there exists $r_\varepsilon > 0$ such that

$$\varphi(t; t'_0, r'_0, \theta'_0) > \theta(t; t'_0, \theta'_0) - \varepsilon > \theta(t; t'_0, \theta'_0) + K_\varepsilon t_\varepsilon + \delta$$

holds for the angle variable of solutions in the left half-plane when $r'_0 \geq r_\varepsilon$, where $\varepsilon < -K_\varepsilon t_\varepsilon - \delta$.

Then by Lemma 5.1 and $K_\varepsilon < 0$,

$$\theta_\varepsilon(t; t'_0, \theta'_0) \leq \theta(t; t'_0, \theta'_0) + K_\varepsilon(t - t'_0) \leq \theta(t; t'_0, \theta'_0) + K_\varepsilon t_\varepsilon$$

holds when $t \geq t'_0 + t_\varepsilon$. Thus

$$\varphi(t; t'_0, r'_0, \theta'_0) > \theta_\varepsilon(t; t'_0, \theta'_0) + \delta,$$

that is

$$\varphi(t; t'_0, \theta'_0) > \theta_\varepsilon(t; t'_0, \theta'_0) + \delta.$$

□

Lemma 3.6. Assume (H_3) , then $\Delta_\varphi t \rightarrow 0$ when $r_0 \rightarrow +\infty$, where φ denotes the angle component of the solutions of system (1.1) and $\Delta_\varphi t$ denotes the time spent by the solutions of system (1.1) to rotate through the right half-plane.

Proof. Using the polar coordinate transformation $x = r \cos \varphi$, $y = r \sin \varphi$ we can obtain

$$\varphi' = -\frac{1}{r^2} \langle Jf(t, z), z \rangle,$$

then

$$\begin{aligned} \Delta_\varphi t &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2}{|\langle Jf(t, z), z \rangle|} d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{|\langle \frac{Jf(t, (r \cos \varphi, r \sin \varphi))}{r}, (\cos \varphi, \sin \varphi) \rangle|} d\varphi. \end{aligned} \quad (3.3)$$

Passing to the limit as $r_0 \rightarrow +\infty$, by the global existence of solutions, we can obtain $r(t) \rightarrow +\infty$. Then using (H_3) and the dominated convergence theorem, and by (3.3) we have

$$\lim_{r_0 \rightarrow +\infty} \Delta_\varphi t = 0.$$

□

Similar to the proof of Lemma 3.7 in [13], we have the following lemma.

Lemma 3.7. Assume that conditions (H_0) , (H_2) and (H_4) hold; and then we have that $\exists R_1 > 0$ for every solution of system (1.1) satisfying $|z(t)| \geq R_1$, $\forall t \in [t_0, t_0 + 2\pi]$,

$$\varphi(t_0 + 2\pi; t_0, \varphi_0) - \varphi_0 < 2(n + 1)\pi$$

holds for an arbitrary t_0 and φ_0 .

Lemma 3.8. Assume that conditions (H_0) , (H_1) , (H_3) and (H_4) hold; then we have that $\exists R_2 > 0$ for every solution of system (1.1) satisfying $|z(t)| \geq R_2$, $\forall t \in [t_0, t_0 + 2\pi]$,

$$\varphi(t_0 + 2\pi; t_0, \varphi_0) - \varphi_0 > 2n\pi$$

holds for an arbitrary t_0 and φ_0 .

Before the proof, we want to emphasize that estimates in this lemma are uniform in the radius, and that these estimates are completed under the state that the radius is sufficiently large.

Proof. Case 1. Without loss of generality, we suppose $\theta_\varepsilon(t_0 + 2\pi; t_0, \theta_0) - \theta_0 < (n + 1)\pi$. First, suppose $\theta_0 \in [\frac{\pi}{2}, \frac{3\pi}{2} - \delta]$ and let $\theta_\varepsilon(t'_j) = \frac{\pi}{2} + j\pi$, $j = 1, \dots, n$ with $t_0 < t'_1 < \dots < t'_n < t_0 + 2\pi$. Then, by (3.2) we have

$$\frac{3\pi}{2} - \theta_0 + \theta_\varepsilon(t_0 + 2\pi; t_0, \theta_0) - \theta_\varepsilon(t'_n) > \pi. \quad (3.4)$$

It suffices to show that

$$\varphi(t_0 + 2\pi; t_0, \varphi_0, r_0) - \varphi_0 > 2n\pi$$

holds when r_0 is sufficiently large. Without loss of generality, we suppose $\varphi(t_0 + 2\pi; t_0, \varphi_0, r_0) - \varphi_0 < 2(n + 1)\pi$ and denote $\varphi_0 = \theta_0$ and $\varphi(t) = \varphi(t; t_0, \varphi_0, r_0)$ for short. Suppose that t_j and τ_j , $j = 1, \dots, n$ satisfy

$$\varphi(t_j) = \frac{\pi}{2} + (2j - 1)\pi, \quad \varphi(\tau_j) = \frac{\pi}{2} + 2j\pi, \quad j = 1, \dots, n,$$

respectively.

Next we compare τ_1 with t'_1 . Right now $\theta_\varepsilon(t)$ and $\varphi(t)$ have the same initial state (t_0, θ_0) , and both are in the left half-plane (see Figure 1, where $\Delta t_\theta = t'_1 - t_1$ denotes the time spent by the solution of system (2.1) rotating from t_1 to t'_1). By assumptions and Lemma 3.5 we obtain

$$\varphi(t; t_0, \theta_0) > \theta_\varepsilon(t; t_0, \theta_0) + \delta, \quad t \in (t_0, t'_1] \cap (t_0, \tau_1]$$

when $\theta_0 < \frac{3\pi}{2} - \delta$.

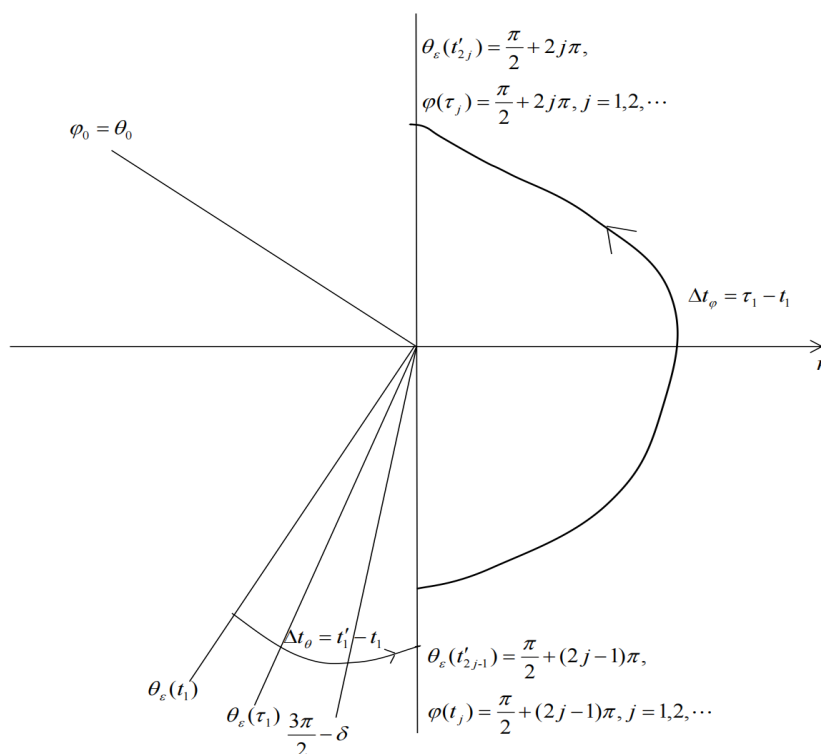


Figure 1. The present figure shows the tracking of the angle in Lemma 3.8, see the detailed explanation in Remark 3.1.

Then

$$\theta_\varepsilon(t_1; t_0, \theta_0) < \varphi(t_1; t_0, \theta_0) = \frac{\pi}{2} + \pi,$$

so, we have

$$\theta_\varepsilon(t_1; t_0, \theta_0) < \theta_\varepsilon(t'_1; t_0, \theta_0);$$

then by the orientation-preserving lemma (Lemma 3.1) we can obtain $t_1 < t'_1$. By the definition of $\varphi(t_j)$ and $\varphi(\tau_j)$, $j = 1, 2, \dots, n$, when the solution curve moves from t_1 to τ_1 , it is in the right half-plane (see

Figure 1, where $\Delta t_\varphi = \tau_1 - t_1$ denotes the time used by the solution of system (1.1) rotating from t_1 to τ_1). Then by (H_3) and Lemma 3.6 we know that the time $\tau_1 - t_1$ used by φ from t_1 to τ_1 is shorter than the time used by θ_ε from t_1 to t'_1 denoted by t_ε , that is $\tau_1 - t_1 < t_\varepsilon$; then we can obtain $\tau_1 < t'_1$.

So

$$\theta_\varepsilon(\tau_1; t_0, \theta_0) < \theta_\varepsilon(t'_1; t_0, \theta_0) = \theta_\varepsilon(t'_1; t'_1, \frac{\pi}{2} + \pi).$$

Take $\theta_1 = \theta_\varepsilon(\tau_1; t_0, \theta_0)$, then $\theta_1 < \frac{\pi}{2} + \pi$.

Therefore, by the orientation-reserving lemma (Lemma 3.1),

$$\theta_\varepsilon(t; \tau_1, \theta_1) < \theta_\varepsilon(t; \tau_1, \frac{\pi}{2} + \pi)$$

holds for $t > \tau_1$.

Next we compare τ_2 with t'_2 . φ starts from $(\tau_1, \frac{5\pi}{2})$ to $(t_2, \varphi(t_2)) = (t_2, \frac{\pi}{2} + 3\pi)$ right now; meanwhile θ_ε also starts from τ_1 to t_2 .

First, we compare t_2 and t'_2 . Suppose $t_2 \geq t'_2$, and apply Lemma 3.5 on the interval $[\tau_1, t_2]$. Because $L_1(t, z)$ is odd for z , (H_1) holds; then we can obtain

$$\varphi(t_2; \tau_1, \frac{\pi}{2} + 2\pi) > \theta_\varepsilon(t_2; \tau_1, \frac{\pi}{2} + 2\pi) > \theta_\varepsilon(t_2; t'_1, \frac{\pi}{2} + 2\pi), \quad (3.5)$$

$$\theta_\varepsilon(t'_2; t'_1, \frac{\pi}{2} + 2\pi) = \theta_\varepsilon(t'_2; t'_1, \frac{\pi}{2} + \pi) + \pi \leq \theta_\varepsilon(t_2; t'_1, \frac{\pi}{2} + 2\pi); \quad (3.6)$$

thus by (3.5) and (3.6) we have

$$\frac{\pi}{2} + 3\pi > \frac{\pi}{2} + 2\pi + \pi = \frac{\pi}{2} + 3\pi,$$

which is a contradiction. Therefore, $t_2 < t'_2$. And by (H_3) and Lemma 3.6 we know that the time used by φ from t_2 to τ_2 is shorter than the time used by θ_ε from t_2 to t'_2 denoted by t'_ε , that is $\tau_2 - t_2 < t'_\varepsilon$; so we know $\tau_2 < t'_2$.

So

$$\theta_\varepsilon(\tau_2; t_0, \theta_0) < \theta_\varepsilon(t'_2; t_0, \theta_0) = \theta_\varepsilon(t'_2; t'_2, \frac{\pi}{2} + 2\pi).$$

Take $\theta_2 = \theta_\varepsilon(\tau_2; t_0, \theta_0)$; then $\theta_2 < \frac{\pi}{2} + 2\pi$.

Therefore, by the orientation-reserving lemma (Lemma 3.1),

$$\theta_\varepsilon(t; \tau_2, \theta_2) < \theta_\varepsilon(t; \tau_2, \frac{\pi}{2} + 2\pi)$$

holds for $t > \tau_2$.

Then we can obtain $\tau_n < t'_n$ by a similar discussion as above. Notice that $\theta_\varepsilon(t'_n) = \frac{\pi}{2} + n\pi$, so

$$\theta_\varepsilon(\tau_n; t_0, \theta_0) < \theta_\varepsilon(t'_n; t_0, \theta_0) = \theta_\varepsilon(t'_n; t'_n, \frac{\pi}{2} + n\pi) = \frac{\pi}{2} + n\pi,$$

that is

$$\theta_n = \theta_\varepsilon(\tau_n; t_0, \theta_0) < \frac{\pi}{2} + n\pi.$$

By the orientation-reserving lemma (Lemma 3.1) we have

$$\theta_\varepsilon(t; \tau_n, \theta_n) < \theta_\varepsilon(t; \tau_n, \frac{\pi}{2} + n\pi),$$

when $t > \tau_n$. By Lemma 3.5, the translation property and the above inequality, and as compared on $[\tau_n, t_0 + 2\pi]$, we have

$$\begin{aligned} & \varphi(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) \\ & > \theta_\varepsilon(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) = \theta_\varepsilon(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + n\pi) + n\pi \\ & > \theta_\varepsilon(t_0 + 2\pi; \tau_n, \theta_n) + n\pi = \theta_\varepsilon(t_0 + 2\pi; t_0, \theta_0) + n\pi; \end{aligned}$$

combined with (3.4), we have

$$\varphi(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) > \frac{\pi}{2} + n\pi + \theta_0 - \frac{3\pi}{2} + \pi + n\pi = \theta_0 + 2n\pi,$$

that is

$$\varphi(t_0 + 2\pi; t_0, \theta_0) - \theta_0 > 2n\pi.$$

Case 2. Suppose $\theta_0 \in [\frac{3\pi}{2} - \delta, \frac{3\pi}{2}]$, consider the solution $\theta_\varepsilon(t; t_0, \frac{3\pi}{2} - \delta)$ of Eq (2.2) starting from $(t_0, \frac{3\pi}{2} - \delta)$ and denote $\theta_\varepsilon(t'_j) = \frac{\pi}{2} + j\pi$, $\varphi(t_j) = \frac{\pi}{2} + (2j - 1)\pi$, $\varphi(\tau_j) = \frac{\pi}{2} + 2j\pi$, $j = 1, \dots, n$.

Next we compare τ_1 and t'_1 . Right now, the initial state of $\theta_\varepsilon(t)$ is $(t_0, \frac{3\pi}{2} - \delta)$, which means that $\theta_0 = \frac{3\pi}{2} - \delta$, the initial state of $\varphi(t)$ is (t_0, φ_0) , with $\varphi_0 \geq \theta_0$, and that both are in the left half-plane. Then by Lemma 3.5,

$$\varphi(t; t_0, \varphi_0) > \theta_\varepsilon(t; t_0, \theta_0), \quad t > t_0$$

holds when $\varphi_0 \in [\frac{3\pi}{2} - \delta, \frac{3\pi}{2}]$; then $\theta_\varepsilon(t_1; t_0, \theta_0) < \varphi(t_1; t_0, \varphi_0) = \frac{\pi}{2} + \pi$; similar to that in Case 1 we have $t_1 < t'_1$. And by the superlinearity on the right half-plane we know that the time used by φ from t_1 to τ_1 is shorter than the time used by θ_ε from t_1 to t'_1 denoted by t'_ε , that is $\tau_1 - t_1 < t'_\varepsilon$; then we can obtain $\tau_1 < t'_1$.

Similarly, we can obtain

$$\tau_n < t'_n,$$

and by (3.2) we have

$$\frac{3\pi}{2} - \theta_0 + \theta_\varepsilon(t_0 + 2\pi; t_0, \frac{3\pi}{2} - \delta) - \theta_\varepsilon(t'_n) > \pi + \delta,$$

namely

$$\theta_\varepsilon(t_0 + 2\pi; t_0, \frac{3\pi}{2} - \delta) - \theta_\varepsilon(t'_n) > \pi.$$

Then

$$\begin{aligned} \varphi(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) & > \theta_\varepsilon(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) \\ & > \theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + 2n\pi) = \theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + n\pi) + n\pi \end{aligned}$$

$$\begin{aligned}
&> \theta_\varepsilon(t'_n) + \pi + n\pi \\
&= 2n\pi + \frac{3\pi}{2}.
\end{aligned}$$

Since $\varphi_0 \leq \frac{3\pi}{2}$, we have

$$\varphi(t_0 + 2\pi; t_0, \varphi_0) - \varphi_0 > 2n\pi.$$

Case 3. Suppose $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}]$; we discuss it in two cases.

First, when $\theta_0 = \frac{\pi}{2}$, take $\varphi_0 = \theta_0$. Right now $\theta_\varepsilon(t)$ and $\varphi(t)$ have the same initial state; denote $\theta_\varepsilon(t'_j) = \frac{\pi}{2} + j\pi$ and $\varphi(t_j) = \frac{\pi}{2} + (2j-1)\pi$, $\varphi(\tau_j) = \frac{\pi}{2} + 2j\pi$, $j = 1, \dots, n$. According to the discussion in Case 1 we can prove

$$\tau_2 < t'_2, \dots, \tau_n < t'_n.$$

And by Lemma 3.5, the translation property and (3.2), we have

$$\begin{aligned}
\varphi(t_0 + 2\pi; t_0, \theta_0) &= \varphi(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) > \theta_\varepsilon(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) \\
&> \theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + 2n\pi) = \theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + n\pi) + n\pi \\
&> \theta_0 + n\pi + n\pi,
\end{aligned}$$

that is

$$\varphi(t_0 + 2\pi; t_0, \theta_0) - \theta_0 > 2n\pi.$$

Second, when $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we consider the solution $\theta_\varepsilon(t; t_0, -\frac{\pi}{2})$ of Eq (2.2) starting from $(t_0, -\frac{\pi}{2})$, and denote $\theta_\varepsilon(t'_j) = \frac{\pi}{2} + j\pi$ and $\varphi(t_j) = \frac{\pi}{2} + (2j-1)\pi$, $\varphi(\tau_j) = \frac{\pi}{2} + 2j\pi$, $j = 0, \dots, n$; suppose that $\varphi(t)$ and $\theta_\varepsilon(t)$ have the same initial state $(t_0, -\frac{\pi}{2})$. By the superlinearity on the right half-plane we know that the time used by φ from t_0 to τ_0 is shorter than the time used by θ_ε from t_0 to t'_0 denoted by t'_ε ; thus we have $\tau_0 < t'_0$; then we can prove

$$\tau_1 < t'_1, \dots, \tau_n < t'_n,$$

similar to the discussion in Case 1.

Thus

$$\begin{aligned}
\varphi(t_0 + 2\pi; t_0, \theta_0) &= \varphi(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) > \theta_\varepsilon(t_0 + 2\pi; \tau_n, \frac{\pi}{2} + 2n\pi) \\
&> \theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + 2n\pi) = \theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + n\pi) + n\pi,
\end{aligned}$$

and by (3.2) we have

$$\frac{\pi}{2} - \theta_0 + \theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + n\pi) - \theta_\varepsilon(t'_n) > 2\delta,$$

namely

$$\theta_\varepsilon(t_0 + 2\pi; t'_n, \frac{\pi}{2} + n\pi) > \theta_0 + n\pi + 2\delta;$$

thus

$$\varphi(t_0 + 2\pi; t_0, \theta_0) - \theta_0 > 2n\pi.$$

□

Remark 3.1. In the proof of Lemma 3.8, an explanatory figure is used to show where the counting of the rotations starts and the estimates of the rotation angle differences in each region. Figure 1 corresponding to Case 1, shows the tracking at the beginning. The statement on the other small time periods can be seen similarly.

Remark 3.2. The proof of Lemma 3.8 is based on the system methodology of “tracking” the angle difference of the solutions of system (1.1) on each small interval on the given side; it gives an estimation of the lower bound of the rotation angle difference of the solutions of system (1.1). This is different from the proof of Lemma 3.6 in [13] and the proof of Lemma 4.2 in [37]. In fact, the proof idea of the lower bound of the rotation angle difference is similar to that in [13, 37], namely, one can describe the 2π -rotation numbers of the solutions (from which one can calculate the rotation angle differences) of the positive homogeneous systems by the rotation numbers of themselves first; then through the discussion of the relationship of the 2π -rotation numbers between the planar systems and the positive homogeneous systems, one can finally realize the estimations of the 2π -rotation number of the planar system by using the rotation numbers of the positive homogeneous systems. Moreover, the “tracking” is different from that in the proof of Lemma 3.7 in [13] because of the superlinearity on the right half-plane.

4. Proof, more general states and applications of Theorem 1.1

In this section, we consider the proof, more general states and applications of our main result.

First, the proof idea of Theorem 1.1 is similar to that in [37], but there are some differences in details, so we state the proof as follows.

Proof of Theorem 1.1. Suppose that $z(t; z_0) = (x(t; z_0), y(t; z_0))$ is a solution of system (1.1) satisfying the initial value condition $z(t_0; z_0) = z_0 = (x_0, y_0)$. Take $\Omega_M = \{z \in \mathbb{R}^2 : |z| < M\}$, and define the Poincaré map

$$\begin{aligned} \mathcal{P} : \quad \bar{\Omega}_M &\rightarrow \mathbb{R}^2, \\ (x_0, y_0) &\mapsto (x(t_0 + 2\pi; z_0), y(t_0 + 2\pi; z_0)). \end{aligned}$$

By (H_0) we have the global existence of solutions; thus \mathcal{P} is well defined. Moreover, the uniqueness of the solution implies that the solution depends continuously on the initial value. Therefore \mathcal{P} is continuous. In addition, \mathcal{P} has a fixed point $z_0 \in \mathbb{R}^2$ if and only if system (1.1) has a 2π periodic solution. Then our aim is to prove that, for $M > 0$ sufficiently large, \mathcal{P} has a fixed point by the Poincaré-Bohl theorem.

Applying the global existence of the solutions of system (1.1) again, as well as the continuous dependence of the solutions with respect to the initial values, we can prove the elastic properties of the solutions of system (1.1). That is, for R_1 in Lemma 3.7 and R_2 in Lemma 3.8, there exists $M_R > 0$ such that $|z(t; z_0)| > R_i$, $i = 1, 2$ for $t \in [t_0, t_0 + 2\pi]$ when $|z_0| \geq M_R$.

Let $M = M_R$. For any $z_0 \in \partial\Omega_M$, we have $|z(t; z_0)| > R_i$, $i = 1, 2$ for $t \in [t_0, t_0 + 2\pi]$. Then by conditions (H_1) – (H_4) and Lemma 3.7 and Lemma 3.8 we have

$$2n\pi < \varphi(t_0 + 2\pi; t_0, \varphi_0) - \varphi_0 < 2(n + 1)\pi,$$

where $n \in \mathbb{N}$, which implies that $z(t; z_0)$ is not able to rotate any integer number of turns in 2π time for $t \in [t_0, t_0 + 2\pi]$. Hence $\mathcal{P}(z_0) \neq \lambda z_0$ for $z_0 \in \partial\Omega_M$ and $\lambda > 1$. Using the Poincaré-Bohl theorem, we have that \mathcal{P} has a fixed point. The proof is thus completed. \square

Second, we consider a more general state of Theorem 1.1. We assume the following conditions.

(H'_4) When $|y| \gg 1$, $y \operatorname{sgn}(f_1(t, 0, y)) > 0$ (f_1 is the first component of f) and $y \operatorname{sgn}(L_{i1}(t, 0, y)) > 0$ (L_{i1} is the first component of L_i), $i = 1, 2$.

(H_5) $\rho(L_1) > \frac{n}{2}$, $L_1(t, z)$ is continuous, 2π -periodic with respect to t , positively homogeneous of degree 1 and odd with respect to z ; and when $x \geq 0$ (x is the first component of z),

$$\liminf_{\lambda \rightarrow +\infty} \left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle \geq \langle JL_1(t, z), z \rangle$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z \in \mathbb{S}^1$.

(H_6) $\rho(L_2) < \frac{n+1}{2}$, $L_2(t, z)$ is continuous, 2π -periodic with respect to t , positively homogeneous of degree 1 and odd with respect to z ; and when $x \geq 0$ (x is the first component of z),

$$\limsup_{\lambda \rightarrow +\infty} \left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle \leq \langle JL_2(t, z), z \rangle$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z \in \mathbb{S}^1$.

(H_7) When $x < 0$ (x is the first component of z),

$$\lim_{\lambda \rightarrow +\infty} \left\langle \frac{Jf(t, \lambda z)}{\lambda}, z \right\rangle = +\infty$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z \in \mathbb{S}^1$.

Then by the similar proof idea of Theorem 1.1, we have the following more general state of Theorem 1.1.

Theorem 4.1. Assume (H_0) , and that $(H_1) - (H_2) - (H_3) - (H_4)$, $(H_1) - (H_2) - (H_3) - (H'_4)$, $(H_5) - (H_6) - (H_7) - (H_4)$ or $(H_5) - (H_6) - (H_7) - (H'_4)$ hold; then system (1.1) has a 2π -periodic solution.

More generally, $\{x \leq 0\}$ and $\{x > 0\}$ in conditions $(H_1) - (H_3)$ can be replaced by $\{y \leq 0\}$ and $\{y > 0\}$, and $\{x \geq 0\}$ and $\{x < 0\}$ in conditions $(H_5) - (H_7)$ can be replaced by $\{y \geq 0\}$ and $\{y < 0\}$; Assumption (H_4) and (H'_4) need to be replaced by the following conditions.

(H_8) When $|x| \gg 1$, $x \operatorname{sgn}(f_2(t, x, 0)) < 0$ (f_2 is the second component of f) and $x \operatorname{sgn}(L_{i2}(t, x, 0)) < 0$ (L_{i2} is the second component of L_i), $i = 1, 2$.

(H'_8) When $|x| \gg 1$, $x \operatorname{sgn}(f_2(t, x, 0)) > 0$ (f_2 is the second component of f) and $x \operatorname{sgn}(L_{i2}(t, x, 0)) > 0$ (L_{i2} is the second component of L_i), $i = 1, 2$.

Then we can obtain similar results to that in Theorem 4.1. Since the expression is similar, the result and some conditions are not stated here.

Finally, we give an application of Theorem 1.1 (or Theorem 4.1).

Example 4.1. Consider the following equations

$$x'' + a_i(t)x = 0, \quad x'' + b_i(t)x = 0, \quad i = 1, 2,$$

where $a_i(t), b_i(t) \in L^1([0, 2\pi])$ and $a_i(t) < b_i(t)$, $i = 1, 2$, are 2π -periodic functions with definitions are as follows.

$$a_1(t) = \begin{cases} (n + 1/2)^2, & t \in [0, \pi], \\ c(t), & t \in [\pi, 2\pi], \end{cases} \quad a_2(t) = \begin{cases} (n + \delta)^2, & t \in [0, \pi], \\ -\lambda^2, & t \in [\pi, 2\pi], \end{cases}$$

$$b_1(t) = \begin{cases} (n + 3/4)^2, & t \in [0, \pi], \\ -\mu^2, & t \in [\pi, 2\pi], \end{cases} \quad b_2(t) = \begin{cases} (n + 1/2)^2, & t \in [0, \pi], \\ d(t), & t \in [\pi, 2\pi], \end{cases}$$

where $n \in \mathbb{N}$, $c(t), d(t) \in L^1([\pi, 2\pi])$, $d(t) \geq 0$, $\delta \geq 2(n + \delta) \arctan |\lambda|/\pi > 0$ and $\arctan \mu \geq \pi/2 - \pi/(4(n + 3/4))$. For $i = 1$ or $i = 2$, we suppose that

$$a_i(t) \leq \liminf_{x \rightarrow -\infty} \frac{g(t, x)}{x} \leq \limsup_{x \rightarrow -\infty} \frac{g(t, x)}{x} \leq b_i(t) \quad (4.1)$$

holds uniformly for a.e. $t \in [0, 2\pi]$, and that

$$\lim_{x \rightarrow +\infty} \frac{g(t, x)}{x} = +\infty,$$

where $g(t, x)$ is continuous, 2π -periodic with respect to t and locally Lipschitz continuous with respect to x . Then we can prove (the proof is similar to that in [38])

$$n/2 < \rho(a_1), \rho(a_2), \rho(b_1), \rho(b_2) < (n + 1)/2.$$

Through verification, we can see that the conditions in Theorem 1.1 all hold; then $x'' + g(t, x) = 0$ has a 2π -periodic solution.

Remark 4.1. In this paper, we mainly consider the indefinite systems, where “indefinite” means that the nonlinearity in the research model is sign-varying. Specifically, for the typical second-order equation $x'' + a(t)h(x) = 0$, “indefinite” means that $a(t)$ is sign-varying. And for $x'' + g(t, x) = 0$, it means that $g(t, x)$ is sign-varying. In Example 4.1, $x'' + g(t, x) = 0$ is an indefinite equation, because by (4.1), we have that $g(t, x)$ is sign-varying when $-x \gg 1$. And the equivalent system of $x'' + g(t, x) = 0$ is $x' = -y$, $y' = g(t, x)$, which is an indefinite planar system. Besides, there are many indefinite systems (equations), such as $x'' + a_i(t)x = 0$, $i = 1, 2$ and $x'' + b_1(t)x = 0$, where $a_i(t)$, $i = 1, 2$ and $b_1(t)$ are defined as in Example 4.1.

5. Proof of Lemmas 2.1–2.3

First, we recall the definition of the rotation number of an orientation-preserving homeomorphism. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be an orientation-preserving homeomorphism such that

$$h(\theta + 2n\pi) = h(\theta) + 2n\pi, \quad \text{for all } \theta \in \mathbb{R}, n \in \mathbb{Z}. \quad (5.1)$$

Then, we can define the rotation number of h as follows

$$\rho(h) = \lim_{k \rightarrow \infty} \frac{h^k(\theta) - \theta}{2k\pi},$$

and it is independent of the choice of θ ; where h^k is a map meaning k times iterations of h .

Before the proof of Lemma 2.1, similar to Proposition 2.1 in [41] or Lemma 2.1 in [52], we give the following lemma.

Lemma 5.1. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be an orientation-preserving homeomorphism satisfying (5.1) and n be an integer number. Then

- (i) $\rho(h) \leq n$ if and only if $\min_{\theta \in \mathbb{R}}(h(\theta) - \theta) \leq 2n\pi$,
(ii) $\rho(h) \geq n$ if and only if $\max_{\theta \in \mathbb{R}}(h(\theta) - \theta) \geq 2n\pi$.

Proof of Lemma 2.1. First, let $h : \mathbb{R} \rightarrow \mathbb{R}$ be the Poincaré map of Eq (2.2), that is

$$\begin{aligned} h : \quad \mathbb{R} &\rightarrow \mathbb{R}, \\ \theta_0 &\mapsto \theta(t_0 + 2\pi; t_0, \theta_0). \end{aligned}$$

By the uniqueness of the solutions of Eq (2.2) with respect to the initial values, the orientation-reserving lemma (Lemma 3.1) holds. Thus, h is an orientation-preserving homeomorphism. Then by (2.3), we have

$$h(\theta_0 + 2n\pi) = h(\theta_0 + n\pi) + n\pi = h(\theta_0) + 2n\pi, \text{ for all } \theta_0 \in \mathbb{R}, n \in \mathbb{Z}.$$

Then, by (ii) of Lemma 5.1, we have

$$\rho(L) < n \Rightarrow \max_{\theta_0 \in \mathbb{R}}(h(\theta_0) - \theta_0) < 2n\pi;$$

thus, $\Delta\theta(t_0, t'_0) \leq \max_{\theta_0 \in \mathbb{R}}(\theta(t_0 + 2\pi; t_0, \theta_0) - \theta_0) < 2n\pi$. Therefore (i) holds.

In a similar way, by (i) of Lemma 5.1, we have

$$\rho(L) > n \Rightarrow \min_{\theta_0 \in \mathbb{R}}(h(\theta_0) - \theta_0) > 2n\pi;$$

thus, $\Delta\theta(t_0, t_0 + 2\pi) \geq \min_{\theta_0 \in \mathbb{R}}(\theta(t_0 + 2\pi; t_0, \theta_0) - \theta_0) > 2n\pi$. Therefore (ii) holds.

Consequently, (iii) can be proved. \square

Next, similar to the proofs of Lemma 3.2 and Lemma 3.3 in [37], we prove Lemma 2.2 and Lemma 2.3.

Proof of Lemma 2.2. For each $\theta_0 \in \mathbb{R}$, let $\theta(t; \theta_0, L)$ be the unique solution of Eq (2.2) satisfying $\theta(t_0; \theta_0, L) = \theta_0$. Denote $\vartheta(t) =: \theta_2(t) - \theta_1(t)$, where $\theta_i(t) = \theta(t; \theta_0, L_i)$ for $i = 1, 2$. Note that $\vartheta(t_0) = 0$. We have

$$\begin{aligned} \frac{d\vartheta(t)}{dt} &= \psi(t, \theta_2; L_2) - \psi(t, \theta_1; L_1) \\ &= (\psi(t, \theta_2; L_2) - \psi(t, \theta_1; L_2)) + \psi(t, \theta_1; L_2) - (\psi(t, \theta_1; L_1)) \\ &=: a(t)\vartheta(t) + b(t), \end{aligned}$$

where $a(t) = \frac{\partial\psi}{\partial\theta}(t, \xi; L_2)$, $\xi = \xi(t)$ is in a closed interval with $\theta_1(t)$ and $\theta_2(t)$ being its boundary points; also $b(t) = \langle J(L_1(t, (\cos \theta_1, \sin \theta_1))) - L_2(t, (\cos \theta_1, \sin \theta_1)), (\cos \theta_1, \sin \theta_1) \rangle$. Then,

$$\vartheta(t) = \int_{t_0}^t b(s) \exp\left(\int_s^t a(\tau) d\tau\right) ds.$$

It follows from (2.4) that $b(t) \leq 0$ for a.e. $t \in [t_0, t'_0]$. Thus $\vartheta(t) \leq 0$ for all $t \in [t_0, t'_0]$; then we have

$$\theta_0 - \theta_2(t) \geq \theta_0 - \theta_1(t), \quad \forall t \in [t_0, t'_0],$$

in the clockwise sense. Or in the counterclockwise sense,

$$-\theta_2(t) - (-\theta_0) \geq -\theta_1(t) - (-\theta_0), \quad \forall t \in [t_0, t'_0].$$

Thus we have

$$\Delta\theta_2(t_0, t) \geq \Delta\theta_1(t_0, t), \quad \forall t \in [t_0, t'_0],$$

which completes the proof of Lemma 2.2. \square

Proof of Lemma 2.3. Let $\varepsilon > 0$ be fixed. For $\delta > 0$ small enough, suppose that (2.5) holds for all $z \in \mathbb{R}^2$ and a.e. $t \in [t_0, t'_0]$. According to Lemma 2.2, we have

$$\Delta\theta_2(t_0, t) \geq \Delta\theta_\delta(t_0, t), \quad \forall t \in [t_0, t'_0].$$

Here, $\Delta\theta_\delta(t_0, t)$ denotes the rotation angle difference of the solution $z(t)$ of the planar system $z' = L_1(t, z) + \delta Jz$. Hence, to prove the result, it suffices to prove that there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\Delta\theta_\delta(t_0, t) \geq \Delta\theta_1(t_0, t) - \varepsilon, \quad \forall t \in [t_0, t'_0]. \quad (5.2)$$

If not, for each $n \in \mathbb{N}$, there would be $t_n \in [t_0, t'_0]$ such that

$$\Delta\theta_{\frac{1}{n}}(t_0, t_n) < \Delta\theta_1(t_0, t_n) - \varepsilon. \quad (5.3)$$

Without loss of generality, we suppose $t_n \rightarrow \tau \in [t_0, t'_0]$. Since $\Delta\theta_1(\cdot, \cdot)$ is continuous on $[t_0, t'_0] \times \mathbb{R}$, passing to the upper limit on both sides of (5.3) gives

$$\limsup_{n \rightarrow \infty} \Delta\theta_{\frac{1}{n}}(t_0, t_n) \leq \Delta\theta_1(t_0, \tau) - \varepsilon. \quad (5.4)$$

Let $z_n(t)$ and $z(t)$ be, respectively, solutions of

$$z' = L_1(t, z) + \frac{1}{n}Jz \quad \text{and} \quad z' = L_1(t, z),$$

with $z_n(t_0) = v_n$ and $z(t_0) = v$. According to the theorem of continuous dependence of the solutions, it follows that $z_n(t) \rightarrow z(t)$ uniformly on $t \in [t_0, t'_0]$. Hence,

$$\begin{aligned} \Delta\theta_{\frac{1}{n}}(t_0, t_n) &= \int_{t_0}^{t_n} -\frac{\langle Jz'_n, z_n \rangle}{|z_n|^2} dt \\ &= \int_{t_0}^{t_n} -\frac{\langle JL_1(t, z_n), z_n \rangle - \frac{1}{n}\langle z_n, z_n \rangle}{|z_n|^2} dt \\ &\rightarrow \int_{t_0}^{\tau} -\frac{\langle JL_1(t, z), z \rangle}{|z|^2} dt \\ &= \Delta\theta_1(t_0, \tau), \end{aligned}$$

which is a contradiction with (5.4). Therefore, (2.6) is proved, and this completes the proof of Lemma 2.3. \square

6. Conclusions

In this paper, we have obtained the existence of 2π -periodic solutions for system (1.1) under non-resonance conditions with one-sided superlinear growth, it is formulated in an original way, with sufficiently general assumptions and overcoming the difficulties from the sign-varying conditions.

On the other hand, we have generalized the rotation number approach and developed a system methodology of “tracking” the angle difference of solutions of the system (1.1) on each small interval on the given side under sign-varying conditions.

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Conflict of interest

The author declares that there is no conflict of interest.

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