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Research article

Almost primes in generalized Piatetski-Shapiro sequences

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Abstract: We consider a generalization of Piatetski-Shapiro sequences in the sense of Beatty sequences, which is of the form $(\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}$ with real numbers $\alpha \geqslant 1, c > 1$ and β. In this paper, we prove that there are infinitely many R-almost primes in sequences $(\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}$ if $c \in (1, c_R)$ and c_R is an explicit constant depending on R.

Keywords: Piatetski-Shapiro sequences; almost primes; exponent pair
Mathematics Subject Classification: 11B83, 11L07

1. Introduction

For $1 < c \notin \mathbb{N}$, the Piatetski-Shapiro sequences are sequences of the form

$$\mathcal{N}^{(c)} := (\lfloor n^c \rfloor)_{n=1}^{\infty} \qquad (c > 1,\ c \notin \mathbb{N}).$$

Such sequences have been named in honor of Piatetski-Shapiro [5], who published the first paper in this problem. He showed that the counting function

$$\left|\{\text{prime } p \leqslant x : p \in \mathcal{N}^{(c)}\}\right| \sim \frac{x^{\frac{1}{c}}}{\log x} \qquad \text{as } x \to \infty,$$

holds for $1 < c < \frac{12}{11}$. The range for c of the asymptotic formula of $\pi^{(c)}(x)$ has been extended many times over the years and is currently known for all $c \in (1, \frac{243}{205})$ thanks to Rivat and Wu [6]. It is conjectured that there are infinitely many Piatetski-Shapiro primes for $c \in (1, 2)$. However, we can see that the best known bound for c in [6] is still far from 2 and it has not been improved for almost 20 years.

Several mathematicians approached this problem in a different direction. For every $R \geqslant 1$, we say that a natural number is an R-almost prime if it has at most R prime factors, counted with multiplicity. The study of almost primes is an intermediate step to the investigation of primes.

Baker, Banks, Guo and Yeager [1] proved that for any fixed $c \in (1, \frac{67}{66})$ there are infinitely many primes of the form $p = \lfloor n^c \rfloor$, where n is a natural number with at most eight prime factors. More precisely,

$$|\{n \leqslant x : n \text{ is an 8-almost prime and } \lfloor n^c \rfloor \text{ is prime}\}| \gg \frac{x}{\log^2 x}.$$

Provided that c_R is an explicit constant depending on R, for any fixed $c \in (1, c_R)$, Guo [4] proved that

$$\left|\{n \leqslant x : \lfloor n^c \rfloor \text{ is an } R\text{-almost prime}\}\right| \gg \frac{x}{\log x}$$

holds for all sufficiently large x.

For fixed real numbers α and β, the associated non-homogeneous Beatty sequence is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty},$$

where $\lfloor t \rfloor$ denotes the integral part of any $t \in \mathbb{R}$. Such sequences are also called generalized arithmetic progressions. If α is irrational, it follows from a classical exponential sum estimate of Vinogradov [9] that $\mathcal{B}_{\alpha,\beta}$ contains infinitely many prime numbers; in fact, one has the asymptotic relation

$$\#\{\text{prime } p \leqslant x : p \in \mathcal{B}_{\alpha,\beta}\} \sim \alpha^{-1}\pi(x) \qquad (x \to \infty),$$

where $\pi(x)$ is the prime counting function.

It is interesting to generalize the Piatetski-Shapiro sequences in the sense of Beatty sequences, since both Piatetski-Shapiro sequences and Beatty sequences produce infinitely many primes. Let $\alpha \geqslant 1$ and β be real numbers. We investigate the following generalized Piatetski-Shapiro sequences

$$\mathcal{N}_{\alpha,\beta}^{(c)} = (\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}.$$

Note that the special case $\mathcal{N}_{1,0}^{(c)}$ is the normal Piatetski-Shapiro sequences. In this paper, we prove that there are infinitely many almost primes in generalized Piatetski-Shapiro sequences.

Theorem 1.1. *For any fixed $c \in (1, c_R)$ we have*

$$\left|\{n \leqslant x : \lfloor \alpha n^c + \beta \rfloor \textit{ is an R-almost prime}\}\right| \gg \frac{x}{\log x}$$

holds for all sufficiently large x. In particular, we have

$$c_3 := \frac{329}{249} = 1.3319\ldots, \qquad c_4 := \frac{25882}{16071} = 1.6104\ldots,$$

and

$$c_R := 3 - \frac{128}{3(8R-1)} \qquad (R \geqslant 5).$$

2. Preliminaries

2.1. Notations

We denote by $\lfloor t \rfloor$ and $\{t\}$ the integer part and the fractional part of t, respectively. As is customary, we put $\mathbf{e}(t) := e^{2\pi i t}$. We make considerable use of the sawtooth function defined by

$$\psi(t) := t - \lfloor t \rfloor - \frac{1}{2} = \{t\} - \frac{1}{2} \qquad (t \in \mathbb{R}).$$

We use notation of the form $m \sim M$ as an abbreviation for $M < m \leqslant 2M$.

Throughout the paper, implied constants in symbols O, $\ll$ and $\gg$ may depend (where obvious) on the parameters α, ε but are absolute otherwise. For given functions F and G, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leqslant C|G|$ holds with some constant $C > 0$. $F \asymp G$ means that $F \ll G \ll F$.

2.2. Technical lemmas

As we have mentioned the following notion plays a crucial role in our arguments. We specify it to the form that is suited to our applications; it is based on a result of Greaves [2] that relates level of distribution to R-almost primality. More precisely, we say that an N-element set of integers $\mathcal{A}$ has a level of distribution D if for a given multiplicative function $f(d)$ we have

$$\sum_{d \leqslant D} \max_{\gcd(s,d)=1} \left| \left|\{a \in \mathcal{A},\ a \equiv s \bmod d\}\right| - \frac{f(d)}{d} N \right| \leqslant \frac{N}{\log^2 N}.$$

As in [2, pp. 174–175] we define the constants

$$\delta_2 := 0.044560, \qquad \delta_3 := 0.074267, \qquad \delta_4 := 0.103974$$

and

$$\delta_R := 0.124820, \qquad R \geqslant 5.$$

We have the following result, which is [2, Chapter 5, Proposition 1].

Lemma 2.1. *Suppose $\mathcal{A}$ is an N-element set of positive integers with a level of distribution D and degree ρ in the sense that*

$$a < D^{\rho} \qquad (a \in \mathcal{A})$$

holds with some real number $\rho < R - \delta_R$. Then

$$\left|\{a \in \mathcal{A} : a \text{ is an } R\text{-almost prime}\}\right| \gg_{\rho} \frac{N}{\log N}.$$

Lemma 2.2. *Let $M \geqslant 1$ and λ be positive real numbers and let H be a positive integer. If $f : [1, M] \to \mathbb{R}$ is a real valued function with three continuous derivatives, which satisfies*

$$\lambda \leqslant |f^{(3)}(x)| \ll \lambda \qquad \text{for } 1 \leqslant x \leqslant M,$$

then for the sum

$$S = \frac{1}{H} \sum_{h=H+1}^{2H} \left| \sum_{m=1}^{M_h} \mathbf{e}\left(\frac{h}{H} f(m)\right) \right|,$$

where the integer M_h *satisfies* $1 \leqslant M_h \leqslant M$ *for each* $h \in [H+1, 2H]$*, we have*

$$S \ll M^{\varepsilon} \left(M\lambda^{1/6} H^{-1/9} + M\lambda^{1/5} + M^{3/4} \right) + \lambda^{-1/3}.$$

Proof. See [7, Theorem 1]. □

Lemma 2.3. *For any* $H \geqslant 1$ *there are numbers* a_h, b_h *such that*

$$\left| \psi(t) - \sum_{0<|h|\leqslant H} a_h \, \mathbf{e}(th) \right| \leqslant \sum_{|h|\leqslant H} b_h \, \mathbf{e}(th),$$

where

$$a_h \ll \frac{1}{|h|}, \qquad b_h \ll \frac{1}{H}.$$

Proof. See [8]. □

We also need the method of exponent pair. A detailed definition of exponent pair can be found in [3, p. 31]. For an exponent pair (k, l), we denote

$$A(k, l) := \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2} \right)$$

and

$$B(k, l) := \left(l - \frac{1}{2}, k + \frac{1}{2} \right),$$

the A-process and B-process of the exponent pair, respectively.

3. Proof of Theorem 1.1

Now we prove our main results. The set we sieve is

$$\mathcal{A} := \{ m \leqslant x^c : m = \lfloor \alpha n^c + \beta \rfloor \text{ for integer } n \}.$$

For any $d \leqslant D$, where D is a fixed power of x, we estimate

$$\mathcal{A}_d := \{ m \in \mathcal{A} : d \,|\, m \}.$$

We know that $rd \in \mathcal{A}$ if and only if

$$rd \leqslant \alpha n^c + \beta < rd + 1 \quad \text{and} \quad rd \leqslant x^c.$$

Within $O(1)$ the cardinality of $\mathcal{A}_d$ is equal to the number of integers $n \leqslant x$ for which the interval $((\alpha n^c + \beta - 1)d^{-1}, (\alpha n^c + \beta)d^{-1}]$ contains a natural number. Hence

$$|\mathcal{A}_d| = \sum_{n \leqslant x} \left(\left\lfloor (\alpha n^c + \beta) d^{-1} \right\rfloor - \left\lfloor (\alpha n^c + \beta - 1) d^{-1} \right\rfloor \right) + O(1)$$

$$= \mathcal{X}d^{-1} + \sum_{n\leqslant x}\Big(\psi((\alpha n^c+\beta-1)d^{-1})-\psi((\alpha n^c+\beta)d^{-1})\Big)+O(1),$$

where

$$\mathcal{X} := \sum_{n\leqslant x} 1 = x.$$

By Lemma 2.1 we need to show that for any sufficiently small $\varepsilon > 0$,

$$\sum_{d\leqslant D}\Big||\mathcal{A}_d| - \mathcal{X}d^{-1}\Big| \leqslant \mathcal{X}x^{-\frac{\varepsilon}{3}} = x^{1-\frac{\varepsilon}{3}},$$

for sufficiently large x. Splitting the range of d into dyadic subintervals, it is sufficient to prove that

$$\sum_{d\sim D_1}\Big|\sum_{N<n\leqslant N_1}\Big(\psi((\alpha n^c+\beta-1)d^{-1})-\psi((\alpha n^c+\beta)d^{-1})\Big)\Big| \ll x^{1-\frac{\varepsilon}{2}}, \tag{3.1}$$

holds uniformly for $D_1 \leqslant D, N \leqslant x, N_1 \sim N$. Our aim is to establish (3.1) with D as large as possible. We define

$$S := \sum_{N<n\leqslant N_1}\Big(\psi((\alpha n^c+\beta-1)d^{-1})-\psi((\alpha n^c+\beta)d^{-1})\Big). \tag{3.2}$$

By Lemma 2.3 and taking $H = Dx^{\varepsilon}$, we have

$$S = S_1 + O(S_2),$$

where

$$S_1 := \sum_{N<n\leqslant N_1}\sum_{0<|h|\leqslant H} a_h\Big(\mathbf{e}(h(\alpha n^c+\beta-1)d^{-1})-\mathbf{e}(h(\alpha n^c+\beta)d^{-1})\Big)$$

and

$$S_2 := \sum_{N<n\leqslant N_1}\sum_{|h|\leqslant H} b_h\Big(\mathbf{e}(h(\alpha n^c+\beta-1)d^{-1})+\mathbf{e}(h(\alpha n^c+\beta)d^{-1})\Big).$$

We split S_1 into two parts

$$S_1 = S_1^{(1)} + S_1^{(2)}, \tag{3.3}$$

where

$$S_1^{(1)} := \sum_{N<n\leqslant N_1}\sum_{0<|h|\leqslant H} a_h\Big(\mathbf{e}(h(\alpha n^c+\beta-1)d^{-1})-\mathbf{e}(h\alpha n^c d^{-1})\Big),$$

and

$$S_1^{(2)} := \sum_{N<n\leqslant N_1}\sum_{0<|h|\leqslant H} a_h\Big(\mathbf{e}(h\alpha n^c d^{-1})-\mathbf{e}(h(\alpha n^c+\beta)d^{-1})\Big).$$

We consider $S_1^{(1)}$. Writing that

$$\phi_h := \mathbf{e}(h(\beta-1)d^{-1}) - 1 \ll 1.$$

Using the exponent pair (k, l), we obtain that

$$S_1^{(1)} = \sum_{N<n\leqslant N_1}\sum_{0<|h|\leqslant H} a_h\phi_h\mathbf{e}(h\alpha n^c d^{-1})$$

$$
\begin{aligned}
&\ll \sum_{0<h\leqslant H} h^{-1}|\sum_{N<n\leqslant N_1} \mathbf{e}(h\alpha n^c d^{-1})| \\
&\ll \sum_{0<h\leqslant H} h^{-1}\left((hd^{-1}N^{c-1})^k N^l + (hd^{-1})^{-1}N^{1-c}\right) \\
&\ll H^k d^{-k} N^{kc-k+l} + H^{-1}dN^{1-c}. \qquad (3.4)
\end{aligned}
$$

For $S_1^{(2)}$, by a similar argument with ϕ_h replaced by φ_h defined by

$$\varphi_h := 1 - \mathbf{e}(h\beta d^{-1}) \ll 1.$$

One can derive that

$$S_1^{(2)} \ll H^k d^{-k} N^{kc-k+l} + H^{-1}dN^{1-c}. \qquad (3.5)$$

Now we consider S_2. The contribution of S_2 from $h = 0$ is

$$\sum_{N<n\leqslant N_1} b_h \ll NH^{-1}. \qquad (3.6)$$

By a simialar arguments of S_1 with a shift of n, the contribution of S_2 from $h \neq 0$ is

$$
\begin{aligned}
&= \sum_{N<n\leqslant N_1} \sum_{0<|h|\leqslant H} b_h \left(\mathbf{e}(h(\alpha n^c + \beta - 1)d^{-1}) + \mathbf{e}(h(\alpha n^c + \beta)d^{-1})\right) \\
&\ll \sum_{N<n\leqslant N_1} \sum_{0<|h|\leqslant H} b_h\phi_h \mathbf{e}(h\alpha n^c d^{-1}) + \sum_{N<n\leqslant N_1} \sum_{0<|h|\leqslant H} b_h\varphi_h \mathbf{e}(h\alpha n^c d^{-1}) \\
&\ll \sum_{0<h\leqslant H} H^{-1}\Big|\sum_{N<n\leqslant N_1} \mathbf{e}(h\alpha n^c d^{-1})\Big| \\
&\ll \sum_{0<h\leqslant H} H^{-1}(h^k d^{-k} N^{kc-k+l} + h^{-1}dN^{1-c}) \\
&\ll H^k d^{-k} N^{kc-k+l} + H^{-1}dN^{1-c}\log H. \qquad (3.7)
\end{aligned}
$$

Substituting (3.4) and (3.5) to (3.3), and combining (3.6) and (3.7), the left hand side of (3.1) is

$$
\begin{aligned}
&\ll \sum_{d\sim D_1} (H^k d^{-k} N^{kc-k+l} + H^{-1}dN^{1-c} \\
&\qquad + H^{-1}dN^{1-c}\log H + NH^{-1}) \ll Dx^{kc-k+l+k\varepsilon} + Dx^{1-c+\varepsilon}.
\end{aligned}
$$

Therefore, to make (3.1) to be true, we need that

$$Dx^{kc-k+l+k\varepsilon} \ll x^{1-\frac{\varepsilon}{2}}, \qquad (3.8)$$

and

$$Dx^{1-c+\varepsilon} \ll x^{1-\frac{\varepsilon}{2}}. \qquad (3.9)$$

Combining (3.8) and (3.9), we obtain that

$$D \ll \min\left(x^{c-\frac{3\varepsilon}{2}}, x^{1-kc+k-l-\varepsilon}\right). \qquad (3.10)$$

3.1. *Exponent pair estimation for $R = 3$*

By Lemma 2.1, $\mathcal{A}$ contains $\gg x/\log x$ R-almost primes. We apply the weighted sieve with the choice

$$R = 3, \qquad \delta_3 = 0.074267$$

and choose

$$\Lambda_R = 3 - \frac{3}{40} = \frac{117}{40} < R - \delta_R.$$

By (3.10) we require that

$$1 - kc + k - l > \frac{40}{117} \qquad \text{and} \qquad c > \frac{40}{117}, \tag{3.11}$$

then

$$c < \frac{77 - 117l}{117k} + 1.$$

Taking the exponent pair

$$BAAAAAB(0,1) = \left(\frac{19}{42}, \frac{32}{63}\right),$$

we have

$$c < \frac{329}{247} = 1.3319\ldots.$$

3.2. *Exponent pair estimation for $R = 4$*

Similarly, we apply the weighted sieve with the choice

$$R = 4, \qquad \delta_4 = 0.103974$$

and choose

$$\Lambda_R = 4 - \frac{13}{125} = \frac{487}{125} < R - \delta_R.$$

By taking the exponent pair

$$BABABAABAAB(0,1) = \left(\frac{33}{128}, \frac{75}{128}\right),$$

we can get

$$c < \frac{362 - 487l}{487k} + 1 = \frac{25882}{16071} = 1.6104\ldots.$$

3.3. *The bound of c for $R \geqslant 5$*

For $R \geqslant 5$, we estimate (3.2) by Lemma 2.2. From (3.4) we have

$$S_1^{(1)} \ll \log H \max_{1 \ll T \ll H} S(T, N),$$

where

$$S(T, N) := \frac{1}{T} \sum_{h \sim T} \sum_{n \sim N} \mathbf{e}(hd^{-1}n^c).$$

By Lemma 2.2 with $f(n) = Td^{-1}(n+N)^c$ and

$$\lambda = c(c-1)(c-2)Td^{-1}N^{c-3},$$

it follows that

$$\begin{aligned} S(T,N) &\ll N^{1+\varepsilon}(Td^{-1}N^{c-3})^{\frac{1}{6}}T^{-\frac{1}{9}} + N^{1+\varepsilon}(Td^{-1}N^{c-3})^{\frac{1}{5}} \\ &\quad + N^{\frac{3}{4}+\varepsilon} + (Td^{-1}N^{c-3})^{-\frac{1}{3}} \\ &\ll T^{\frac{1}{18}}d^{-\frac{1}{6}}N^{\frac{c}{6}+\frac{1}{2}+\varepsilon} + T^{\frac{1}{5}}d^{-\frac{1}{5}}N^{\frac{c}{5}+\frac{2}{5}+\varepsilon} \\ &\quad + N^{\frac{3}{4}+\varepsilon} + T^{-\frac{1}{3}}d^{\frac{1}{3}}N^{1-\frac{c}{3}}. \end{aligned}$$

Hence

$$S_1^{(1)} \ll H^{\frac{1}{18}}d^{-\frac{1}{6}}N^{\frac{c}{6}+\frac{1}{2}+\varepsilon} + H^{\frac{1}{5}}d^{-\frac{1}{5}}N^{\frac{c}{5}+\frac{2}{5}+\varepsilon} + N^{\frac{3}{4}+\varepsilon} + d^{\frac{1}{3}}N^{1-\frac{c}{3}}.$$

Similarly, we can get the estimation of $S_1^{(2)}$. The contribution of S_2 from $h \neq 0$ can be estimated by the same method and achieve the same upper bound. Together with the contribution of S_2 from $h = 0$, by (3.6) we obtain that the left-hand side of (3.1) is

$$\begin{aligned} \sum_{d \sim D_1} |S| &\ll \sum_{d \sim D_1} \left| H^{\frac{1}{18}}d^{-\frac{1}{6}}N^{\frac{c}{6}+\frac{1}{2}+\varepsilon} + H^{\frac{1}{5}}d^{-\frac{1}{5}}N^{\frac{c}{5}+\frac{2}{5}+\varepsilon} + N^{\frac{3}{4}+\varepsilon} + d^{\frac{1}{3}}N^{1-\frac{c}{3}} \right| \\ &\ll H^{\frac{1}{18}}D^{\frac{5}{6}}N^{\frac{c}{6}+\frac{1}{2}+\varepsilon} + H^{\frac{1}{5}}D^{\frac{4}{5}}N^{\frac{c}{5}+\frac{2}{5}+\varepsilon} + DN^{\frac{3}{4}}\log H + D^{\frac{4}{3}}N^{1-\frac{c}{3}} \\ &\ll D^{\frac{8}{9}}x^{\frac{c}{6}+\frac{1}{2}+\frac{19\varepsilon}{18}} + Dx^{\frac{c}{5}+\frac{2}{5}+\frac{6\varepsilon}{5}} + Dx^{\frac{3}{4}+\varepsilon} + D^{\frac{4}{3}}x^{1-\frac{c}{3}}. \end{aligned}$$

To ensure the left-hand side of (3.1) is $\ll x^{1-\varepsilon/2}$, we require that

$$D \ll \min\left(x^{\frac{9}{16}-\frac{3c}{16}-\varepsilon}, x^{\frac{3}{5}-\frac{c}{5}-\varepsilon}, x^{\frac{1}{4}-\varepsilon}, x^{\frac{c}{4}-\varepsilon}\right). \tag{3.12}$$

We apply the weighted sieve with the choice

$$\delta_R = 0.124820 \qquad (R \geqslant 5)$$

and choose

$$\Lambda_R = R - \frac{1}{8} < R - \delta_R.$$

To apply Lemma 2.1, by (3.12) we need that

$$\min\left(\frac{9}{16} - \frac{3c}{16}, \frac{3}{5} - \frac{c}{5}, \frac{1}{4}, \frac{c}{4}\right) > \frac{1}{R - \frac{1}{8}},$$

which gives that

$$c < 3 - \frac{128}{3(8R-1)}.$$

4. Conclusions

In this paper, we investigate the following generalized Piatetski-Shapiro sequences

$$\mathcal{N}_{\alpha,\beta}^{(c)} = (\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}.$$

We prove that there are infinitely many R-almost primes in generalized Piatetski-Shapiro sequences by the Van der Corput's method of exponential sums and exponent pairs.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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