Mathematics

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Novel operations of weighted hesitant fuzzy sets and their group decision making application

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Abstract: Weighted hesitant fuzzy set (WHFS) is an extension of hesitant fuzzy set (HFS), in which the weights indicate that the decision maker has different confidence in giving every possible assessment of the membership degree. In this paper, we redefine the union and intersection operations of weighted hesitant fuzzy elements (WHFEs), investigate their operation properties, and propose the variance function of the weighted hesitant fuzzy element (WHFE) to compare WHFEs. Furthermore, we develop two aggregation operators such as weighted hesitant fuzzy ordered weighted averaging (WHFOWA) and weighted hesitant fuzzy ordered weighted geometric (WHFOWG) operators to aggregate weighted hesitant fuzzy information, and present multiple-attribute group decision making algorithm under weighted hesitant fuzzy environment. Finally, four numerical examples are used to illustrate the effectiveness of our proposed aggregation operators.

1. Introduction

Fuzzy set introduced by Zadeh [1] has achieved a great success in many fields such as approximate reasoning, fuzzy control, fuzzy decision making and so on. With the complexity in the real decision making process, some researchers generalized the concept of fuzzy set. Considering the difficulty to establish the membership degree of an element to a fuzzy set, Torra [2, 3] introduced the concept of hesitant fuzzy set (HFS) which permitted the membership degree having a set of possible values, and could reflect the human's hesitancy more objectively than the other classical extensions of fuzzy

set. After then, many researchers have paid attention on this topic and obtained some meaningful conclusions. Generally speaking, most of these works can be mainly divided into three categories:

1) Aggregation operators. Aggregation operators are important tools to aggregate fuzzy information, but how to aggregate different fuzzy information usually relied on different domain knowledge. Hence, aiming at different scenarios, many researchers proposed different aggregation operators. For example, Bedregal et al. [4] investigated the classical aggregation operator for hesitant fuzzy element(HFE), Xia et al. [5, 6] proposed a series of aggregation operators for hesitant fuzzy information, Wei [7] and Zeng et al. [8] investigated hesitant fuzzy prioritized operators, respectively. Similar to exponentiation, Zhang [9] introduced hesitant fuzzy power aggregation operators, Zhu et al. [10] and Yu et al. [11] investigated hesitant fuzzy geometric Bonferroni means and generalized hesitant fuzzy Bonferroni means, respectively, Peng et al. [12] investigated the continuous hesitant fuzzy aggregation operators and applied in decision making.

2) Information measure. Information measure is the basis of all kinds of decision making methods, thus many scholars investigated the meaningful topic. For example, Xu et al. [13–15] investigated the distance, similarity measure and correlation measure of hesitant fuzzy sets (HFSs), Peng et al. [16] presented the generalized hesitant fuzzy weighted distance and applied it in multiple criteria decision making, Li et al. [17, 18] proposed several kinds of distance of HFSs based on the hesitancy degree of hesitant fuzzy element (HFE), Chen et al. [19] investigated correlation coefficient of HFSs and apply it into clustering analysis, Farhadinia [20] investigated the relationship among entropy, similarity measure and distance measure for HFSs and interval-valued hesitant fuzzy sets, Zeng et al. [21] investigated the relationship between the distance and the similarity measure of HFSs and applied in pattern recognition.

3) Extensions of HFSs. All kinds of extensions of HFSs are used to further describe the imprecise information in the real life. For example, Khan et al. [22–24] introduced several novel similarity measures for the q-rung orthopair fuzzy sets. Zeng et al. [25, 26] introduced weighted interval-valued hesitant fuzzy set and weighted hesitant fuzzy linguistic term set, and applied in group decision making, respectively, Zhu et al. [27] introduced dual hesitant fuzzy set, Chen et al. [28] and Wei et al. [29] developed interval-valued hesitant fuzzy sets, respectively, Rodríguez et al. [30, 31] investigated hesitant fuzzy linguistic term sets for decision making, Wei et al. [32] introduced some aggregation operators for hesitant fuzzy linguistic term sets and applied in multi-criteria decision making, Zhu et al. [33] introduced linguistic preference relation under hesitant fuzzy environment, Liao et al. [34, 35] investigated distance and similarity measures between hesitant fuzzy linguistic term sets and the consistency and consensus of hesitant fuzzy preference relation, and applied in group decision making, Onar et al. [36] and Xu et al. [37] utilized hesitant fuzzy Technique for Order Preference by Similarity to an Ideal Solution(TOPSIS) to obtain optimal strategy, respectively, Zhang et al. [38] investigated the extension of VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method based on hesitant fuzzy set(HFS), Qian et al. [39] proposed the generalized hesitant fuzzy set and applied in decision support system.

It needs to point out that each possible value involved in the HFS is distributed with the same importance or preference (weight). However, in some practical applications, especially in multi-attribute decision making, several possible values may sometimes have different importance, thus they are distributed different weights because decision makers hesitate among several possible values. Namely, the weights of the possible values are different. For instance, assuming that five

experts are authorized to evaluate one supplier anonymously according to a given criteria or attribute, and each assessment that can indicate the satisfaction degree of the expert to the supplier is provided to the decision maker. Suppose the five assessments provided by the five experts are 0.6, 0.7, 0.6, 0.8 and 0.8, respectively, and the experts cannot persuade each other, thus, while determining the satisfaction degree of the supplier, the decision maker hesitate among three values, 0.6, 0.7, 0.8. According to the approach of constructing HFS in general, the membership degree of an element to a set can be represented as the hesitant fuzzy element (HFE) $\{0.6, 0.7, 0.8\}$. But, in fact, 0.6 is provided by two experts, 0.8 is provided by two experts, but only one expert provides 0.7. Hence, the decision maker may prefer to 0.6 and 0.8 while determining the membership degree, it is obvious that the classical HFS or HFE cannot accurately represent these important fuzzy information. Based on this, Zhang and Wu [40] introduced the concept of weighted hesitant fuzzy set (WHFS) and presented some operations of weighted hesitant fuzzy elements (WHFEs) and some operations based on T-norm.

Recently, Zhu and Xu [41] introduced the concept of probabilistic hesitant fuzzy set (PHFS). It needs to point out that there exists some differences between WHFS and PHFS, the former assigns weights to each HFE, where the weights are from experts, and the latter allocates probabilities to every HFE, where the set of HFE is a countable set and probability can be determined by its frequency. Hence, these two variants express different application background. Inspired by the idea of WHFS in Zhang and Wu [40], considering the different background of group decision making in the real life, in this paper, we redefine some operations such as union and intersection of weighted hesitant fuzzy elements (WHFEs), propose the variance function of WHFE and present the ranking rule to compare WHFEs based on the score function and variance function. Furthermore, we investigate the characteristics of weighted hesitant fuzzy element and develop two kinds of aggregation operators such as weighted hesitant fuzzy ordered weighted averaging (WHFOWA) operator and weighted hesitant fuzzy ordered weighted geometric (WHFOWG) operator to aggregate weighted hesitant fuzzy information, and develop the mathematical model of multi-attribute group decision making. Finally, four numerical examples are used to illustrate the effectiveness and feasibility of our proposed method.

The organization of our study is as follows. In Section 2, we review some basic notions of HFS and HFE. In Section 3, we review the concept of WHFS, redefine the intersection and union operations of weighted hesitant fuzzy elements (WHFEs), investigate their operation properties, propose the variance function of weighted hesitant fuzzy element, and present the ranking rule to compare the WHFEs based on the score function and variance function. In Section 4, we develop two aggregation operators such as weighted hesitant fuzzy ordered weighted averaging (WHFOWA) and weighted hesitant fuzzy ordered weighted geometric (WHFOWG) operators to aggregate weighted hesitant fuzzy information, and propose multi-attribute group decision making algorithm. In Section 5, four numerical examples are used to illustrate the effectiveness of our techniques. The conclusion is given in the last section.

2. Preliminaries

In this paper, we use $X = \{x_1, x_2, \cdots, x_n\}$ to denote the discourse set, HFS and HFE stand for hesitant fuzzy set and hesitant fuzzy element, respectively, WHFS and WHFE stand for weighted hesitant fuzzy set and weighted hesitant fuzzy element, respectively, h and h^w stand for hesitant fuzzy element (HFE)

and weighted hesitant fuzzy element (WHFE), respectively.

Definition 1. [2] Given a fixed set X, then a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of [0, 1].

For conveniences, Xia and Xu [5] proposed a simple and convenient symbol to express the HFS

$$E = \{< x, h_E(x) > | x \in X\}$$

where $h_E(x)$ is a collection including multiple values in range [0,1], denoting the possible membership degree of the element $x \in X$ to the set E. $h = h_E(x)$ is called a hesitant fuzzy element (HFE).

Furthermore, Torra [2] and Xia and Xu [5] introduced the following operations for hesitant fuzzy elements (HFEs) h, h_1 and h_2, respectively.

(1) $h^- = \min h$, $h^+ = \max h$;

(2) $h^c = \bigcup_{\gamma \in h} \{1 - \gamma\}$;

(3) $h_1 \bigcup h_2 = \{\gamma \in h_1 \cup h_2 | \gamma \geq \max(h_1^-, h_2^-)\} = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$;

(4) $h_1 \bigcap h_2 = \{\gamma \in h_1 \cap h_2 | \gamma \leq \min(h_1^+, h_2^+)\} = \bigcap_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$;

(5) $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}, \lambda > 0$;

(6) $\lambda h = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}, \lambda > 0$;

(7) $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$;

(8) $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}$.

Furthermore, let $h_i, i = 1, 2, \cdots, n$ be a collection of HFEs, Liao et al. [34] introduced the following operations.

(9) $\oplus_{i=1}^n h_i = \bigcup_{\gamma_i \in h_i} \{1 - \prod_{i=1}^n (1 - \gamma_i)\}$;

(10) $\otimes_{i=1}^n h_i = \bigcup_{\gamma_i \in h_i} \{\prod_{i=1}^n \gamma_i\}$.

Meanwhile, Xia and Xu [5] introduced the score function of HFE h, $s(h) = \dfrac{1}{ł(h)} \sum_{\gamma \in h} \gamma$, where $ł(h)$ is the number of the elements in h. For any given two HFEs h_1 and h_2, Xia and Xu [5] give the ranking rule to compare the HFEs: If $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

After then, Farhadinia [42] and Rodríguez et al. [43] proposed the improved method to compare HFEs, respectively. Furthermore, Liao et al. [34] introduced the variance function as second ranking index to compare HFEs.

Definition 2. [34] For a given HFE h, $v(h) = \dfrac{1}{ł(h)} \sqrt{\sum_{\gamma_i, \gamma_j \in h} (\gamma_i - \gamma_j)^2}$ is called the variance function of HFE h, where $ł(h)$ is the number of the elements in h.

In the following, Liao et al. [34] proposed the ranking rule to compare HFEs based on the score function and variance function of HFE h.

(a) If $s(h_1) > s(h_2)$, then $h_1 > h_2$.

(b) If $s(h_1) = s(h_2)$, then

(1) If $v(h_1) > v(h_2)$, then $h_1 < h_2$;

(2) If $v(h_1) = v(h_2)$, then $h_1 = h_2$.

3. Weighted hesitant fuzzy set

Definition 3. [40] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a fixed set, then a weighted hesitant fuzzy set (WHFS) E^W on X is in terms of a function that when applied to X returns a subset with values as

$$E^w = \{< x, h_E^w(x) > | x \in X\}$$

where $h_E^w(x) = \{< \gamma_1, w_1 >, < \gamma_2, w_2 >, \cdots, < \gamma_m, w_m >\}$, $\gamma_j(j = 1, 2, \cdots, m)$ is a set of some values in [0, 1] which denotes the possible membership degree of the element $x \in X$ to the set E^W, and $w_j \in [0, 1](j = 1, 2, \cdots, m)$, $\sum_{j=1}^{m} w_j = 1$, is called the weight of γ_j, in which the weight w_j denotes the importance of γ_j being taken as the membership degree of x, or the preference value that the decision maker takes γ_j as the membership degree of element x, then $h^w = h_E^w(x)$ is called a weighted hesitant fuzzy element (WHFE).

Remark 1. Weighted hesitant fuzzy set (WHFS) is a variant of traditional HFS, and describes the human uncertainty more objectively and precisely.

Remark 2. The difference between HFS and WHFS is that the former assumes that the possible membership degrees have the equal importance, and the latter assumes that the possible membership degrees have different importance. Specially, If the weight $w_j = \dfrac{1}{m}$, for every $j = 1, 2, \cdots, m$ in WHFE h^w, then the WHFE

$$h^w = \{< \gamma_1, \frac{1}{m} >, < \gamma_2, \frac{1}{m} >, \cdots, < \gamma_m, \frac{1}{m} >\},$$

then the WHFE h^w will become the classical HFE $h = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$. Here, we suppose that the WHFE

$$h^w = \{< \gamma_1, \frac{1}{m} >, < \gamma_2, \frac{1}{m} >, \cdots, < \gamma_m, \frac{1}{m} >\}$$

is equivalent to the HFE $h = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$. Thus, in this paper, we will make no distinction and denote

$$h^w = \{< \gamma_1, \frac{1}{m} >, < \gamma_2, \frac{1}{m} >, \cdots, < \gamma_m, \frac{1}{m} >\}$$
$$\Longleftrightarrow h = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}.$$

Furthermore, Zhang and Wu [40] introduced some operations of WHFEs in the following.

Definition 4. [40] Given three WHFEs $h^w = \bigcup_{\gamma \in h^w}\{< \gamma, w_\gamma >\}$, $h_1^w = \bigcup_{\gamma_1 \in h_1^w}\{< \gamma_1, w_{\gamma_1} >\}$ and $h_2^w = \bigcup_{\gamma_2 \in h_2^w}\{< \gamma_2, w_{\gamma_2} >\}$, for $\lambda > 0$, then
  (1) $(h^w)^c = \bigcup_{\gamma \in h^w}\{< 1 - \gamma, w_\gamma >\}$;
  (2) $h_1^w \bigcup h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w}\{< \gamma_1 \vee \gamma_2, w_{\gamma_1} \cdot w_{\gamma_2\}} >\}$;
  (3) $h_1^w \bigcap h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w}\{< \gamma_1 \wedge \gamma_2, w_{\gamma_1} \cdot w_{\gamma_2\}} >\}$;
  (4) $(h^w)^\lambda = \bigcup_{\gamma \in h^w}\{< \gamma^\lambda, w_\gamma >\}$;
  (5) $\lambda h^w = \bigcup_{\gamma \in h^w}\{< 1 - (1 - \gamma)^\lambda, w_\gamma >\}$;
  (6) $h_1^w \oplus h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w}\{< \gamma_1 + \gamma_2 - \gamma_1\gamma_2, w_{\gamma_1} w_{\gamma_2} >\}$;
  (7) $h_1^w \otimes h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w}\{< \gamma_1\gamma_2, w_{\gamma_1} w_{\gamma_2} >\}$.
  Now we use an example to illustrate the shortcoming of the above operations such as "$\bigcup$" and "$\bigcap$".

Example 1. Given two WHFEs $h_1^w = \{<0.6, 0.5>, <0.7, 0.5>\}$ and $h_2^w = \{<0.3, 0.5>, <0.8, 0.5>\}$, because every possible value in WHFEs h_1^w and h_2^w has the same weight, then the WHFEs h_1^w and h_2^w can be thought of as the classic hesitant fuzzy elements (HFEs) or WHFEs h_1^w and h_2^w are equivalent to the HFEs h_1 and h_2. However, known by Definition 4, we have:

$$
\begin{aligned}
h_1^w \bigcup h_2^w \ &= \{<0.6 \vee 0.3, 0.5 \cdot 0.5>, <0.6 \vee 0.8, 0.5 \cdot 0.5>, <0.7 \vee 0.3, 0.5 \cdot 0.5>, <0.7 \vee 0.8, 0.5 \cdot 0.5>\} \\
&= \{<0.6, 0.25>, <0.8, 0.25>, <0.7, 0.25>, <0.8, 0.25>\} \\
&= \{<0.6, 0.25>, <0.7, 0.25>, <0.8, 0.5>\}.
\end{aligned}
$$

$$
\begin{aligned}
h_1^w \bigcap h_2^w \ &= \{<0.6 \wedge 0.3, 0.5 \cdot 0.5>, <0.6 \wedge 0.8, 0.5 \cdot 0.5>, <0.7 \wedge 0.3, 0.5 \cdot 0.5>, <0.7 \wedge 0.8, 0.5 \cdot 0.5>\} \\
&= \{<0.3, 0.25>, <0.6, 0.25>, <0.3, 0.25>, <0.7, 0.25>\} \\
&= \{<0.3, 0.5>, <0.6, 0.25>, <0.7, 0.25>\}.
\end{aligned}
$$

If we take WHFEs h_1^w and h_2^w as the HFEs h_1 and h_2, then known by the definitions of "$\bigcup$" and "$\bigcap$" for HFEs, we also have:

$$
h_1 \bigcup h_2 = \{0.6 \vee 0.3, 0.6 \vee 0.8, 0.7 \vee 0.3, 0.7 \vee 0.8\} = \{0.6, 0.7, 0.8\}.
$$

$$
h_1 \bigcap h_2 = \{0.6 \wedge 0.3, 0.6 \wedge 0.8, 0.7 \wedge 0.3, 0.7 \wedge 0.8\} = \{0.3, 0.6, 0.7\}.
$$

Because both $h_1 \bigcup h_2$ and $h_1 \bigcap h_2$ have three possible values and every possible value has the same weight, then we make use of the equivalent expression between HFE and WHFE, and use WHFE to denote them, namely,

$$
h_1^w \bigcup h_2^w = \{<0.6, \frac{1}{3}>, <0.7, \frac{1}{3}>, <0.8, \frac{1}{3}>\}
$$

and

$$
h_1^w \bigcap h_2^w = \{<0.3, \frac{1}{3}>, <0.6, \frac{1}{3}>, <0.7, \frac{1}{3}>\}.
$$

Obviously, we find that the above calculation results are inconsistent, and the later result seems more intuitive and logical than the former one.

Consequently, we think that it is necessary to improve and redefine the definitions of the union and intersection operations of WHFEs.

Definition 5. Given two WHFEs h_1^w, h_2^w, then

(1) $h_1^w \bigcup h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{< \max\{\gamma_1, \gamma_2\}, w'_{\max\{\gamma_1, \gamma_2\}} >\}$, for values in WHFEs, we choose maximum of two WHFEs, and for weight $w'_{\max\{\gamma_1, \gamma_2\}}$ can be determined through three steps:

1) If $\gamma_1 = \gamma_2$, then we choose average of two weights $w_{\max\{\gamma_1, \gamma_2\}} = \frac{w_{\gamma_1} + w_{\gamma_2}}{2}$ as new weight; if $\gamma_1 > \gamma_2$, then we choose maximum of two weights $w_{\max\{\gamma_1, \gamma_2\}} = w_{\gamma_1}$ as new weight, contrary, we choose minimum of two weights as new weight $w_{\max\{\gamma_1, \gamma_2\}} = w_{\gamma_2}$;

2) If there exists the several same values γ_i which belongs to the above calculation results $\bigcup\{< \gamma_i, w_{\gamma_i} >\}$, then the weight can be computed through arithmetic mean $w_{\max\{\gamma_1, \gamma_2, \cdots, \gamma_l\}} = \frac{1}{l} \sum_{k=1}^{l} w_{\gamma_k}$, where l is the number of γ_i in $\bigcup\{< \gamma_i, w_{\gamma_i} >\}$;

3) Owing to sum of multiple weights maybe beyond 1, we need to normalize $\{w_{\max\{\gamma_1, \gamma_2\}}\}$ to obtain new weight $\{w'_{\max\{\gamma_1, \gamma_2\}}\}$.

(2) $h_1^w \bigcap h_2^w = \bigcap_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{< \min\{\gamma_1, \gamma_2\}, w'_{\min\{\gamma_1,\gamma_2\}} >\}$, for values in WHFEs, we choose minimum of two WHFEs, and for weight $w'_{\min\{\gamma_1,\gamma_2\}}$ can be determined through three steps:

1) If $\gamma_1 = \gamma_2$, then we choose average of two weights $w_{\min\{\gamma_1,\gamma_2\}} = \frac{w_{\gamma_1} + w_{\gamma_2}}{2}$ as new weight; if $\gamma_1 > \gamma_2$, then we choose minimum of two weights $w_{\min\{\gamma_1,\gamma_2\}} = w_{\gamma_1}$ as new weight, contrary, we choose minimum of two weights as new weight $w_{\min\{\gamma_1,\gamma_2\}} = w_{\gamma_2}$;

2) If there exists the several same values γ_i which belongs to the above calculation results $\bigcap\{< \gamma_i, w_{\gamma_i} >\}$, then the weight can be computed through arithmetic mean $w_{\min\{\gamma_1, \gamma_2, \cdots, \gamma_l\}} = \frac{1}{l} \sum_{k=1}^{l} w_{\gamma_k}$, where l is the number of γ_i in $\bigcap\{< \gamma_i, w_{\gamma_i} >\}$;

3) Owing to sum of multiple weights maybe beyond 1, we need to normalize $\{w_{\min\{\gamma_1,\gamma_2\}}\}$ to obtain new weight $\{w'_{\min\{\gamma_1,\gamma_2\}}\}$.

Thus, according to Definition 5, we re-calculate Example 1 and have the following conclusions.

Example 2. Given two WHFEs $h_1^w = \{< 0.6, 0.5 >, < 0.7, 0.5 >\}$ and $h_2^w = \{< 0.3, 0.5 >, < 0.8, 0.5 >\}$, then known by Definition 5, we have:

$$
\begin{aligned}
& \{< 0.6 \vee 0.3, 0.5 >, < 0.6 \vee 0.8, 0.5 >, < 0.7 \vee 0.3, 0.5 >, < 0.7 \vee 0.8, 0.5 >\} \\
= & \{< 0.6, 0.5 >, < 0.8, 0.5 >, < 0.7, 0.5 >, < 0.8, 0.5 >\} \\
= & \{< 0.6, 0.5 >, < 0.7, 0.5 >, < 0.8, 0.5 >\}.
\end{aligned}
$$

Thus, we have:

$$
h_1^w \bigcup h_2^w = \{< 0.6, \frac{1}{3} >, < 0.7, \frac{1}{3} >, < 0.8, \frac{1}{3} >\}.
$$

Similarly,

$$
\begin{aligned}
& \{< 0.6 \wedge 0.3, 0.5 >, < 0.6 \wedge 0.8, 0.5 >, < 0.7 \wedge 0.3, 0.5 >, < 0.7 \wedge 0.8, 0.5 >\} \\
= & \{< 0.3, 0.5 >, < 0.6, 0.5 >, < 0.3, 0.5 >, < 0.7, 0.5 >\} \\
= & \{< 0.3, 0.5 >, < 0.6, 0.5 >, < 0.7, 0.5 >\}.
\end{aligned}
$$

Then we obtain

$$
h_1^w \bigcap h_2^w = \{< 0.3, \frac{1}{3} >, < 0.6, \frac{1}{3} >, < 0.7, \frac{1}{3} >\}.
$$

Remark 3. These calculation results are completely consistent with our intuition and logic, and show that Definition 5 is more reasonable.

In the following, we use an example to explain these operations of WHFEs.

Example 3. Let

$$
h_1^w = \{< 0.6, 0.3 >, < 0.7, 0.25 >, < 0.8, 0.45 >\}
$$

and

$$
h_2^w = \{< 0.65, 0.4 >, < 0.7, 0.35 >, < 0.9, 0.25 >\}
$$

be two WHFEs, and $\lambda > 0$, then we have:

(1)

$$
(h_1^w)^c = \{< 0.4, 0.3 >, < 0.3, 0.25 >, < 0.2, 0.45 >\},
$$

$$
(h_2^w)^c = \{< 0.35, 0.4 >, < 0.3, 0.35 >, < 0.1, 0.25 >\}.
$$

(2)

$$\{< 0.6 \vee 0.65, 0.4 >, < 0.6 \vee 0.7, 0.35 >, < 0.6 \vee 0.9, 0.25 >, < 0.7 \vee 0.65, 0.25 >, < 0.7 \vee 0.7, 0.3 >,$$
$$< 0.7 \vee 0.9, 0.25 >, < 0.8 \vee 0.65, 0.45 >, < 0.8 \vee 0.7, 0.45 >, < 0.8 \vee 0.9, 0.25 >\}$$
$$= \{< 0.65, 0.4 >, < 0.7, 0.35 >, < 0.9, 0.25 >, < 0.7, 0.25 >, < 0.7, 0.3 >,$$
$$< 0.9, 0.25 >, < 0.8, 0.45 >, < 0.8, 0.45 >, < 0.9, 0.25 >\}$$
$$= \{< 0.65, 0.4 >, < 0.7, 0.3 >, < 0.8, 0.45 >, < 0.9, 0.25 >\}.$$

It needs to point out that we compute the average of some weights as its final weight for the same values in the above calculation process.

So, we have

$$h_1^w \bigcup h_2^w(x) = \{< 0.65, 0.2857 >, < 0.7, 0.2143 >, < 0.8, 0.3214 >, < 0.9, 0.1786 >\}.$$

(3) Similarly, we have

$$h_1^w \bigcap h_2^w = \{< 0.6, 0.2069 >, < 0.65, 0.2759 >, < 0.7, 0.2069 >, < 0.8, 0.3103 >\}.$$

(4)

$$(h_1^w)^\lambda = \{< 0.6^\lambda, 0.3 >, < 0.7^\lambda, 0.25 >, < 0.8^\lambda, 0.45 >\},$$
$$(h_2^w)^\lambda = \{< 0.65^\lambda, 0.4 >, < 0.7^\lambda, 0.35 >, < 0.9^\lambda, 0.25 >\}.$$

(5)

$$\lambda h_1^w = \{< 1 - 0.4^\lambda, 0.3 >, < 1 - 0.3^\lambda, 0.25 >, < 1 - 0.2^\lambda, 0.45 >\},$$
$$\lambda h_2^w(x) = \{< 1 - 0.35^\lambda, 0.4 >, < 1 - 0.3^\lambda, 0.35 >, < 1 - 0.1^\lambda, 0.25 >\}.$$

(6)

$$h_1^w \oplus h_2^w = \{< 0.86, 0.12 >, < 0.88, 0.105 >, < 0.96, 0.075 >, < 0.895, 0.1 >,$$
$$< 0.91, 0.0875 >, < 0.97, 0.0625 >, < 0.93, 0.18 >, < 0.94, 0.1575 >, < 0.98, 0.1125 >\}.$$

(7)

$$h_1^w \otimes h_2^w = \{< 0.39, 0.12 >, < 0.42, 0.105 >, < 0.54, 0.075 >, < 0.455, 0.1 >,$$
$$< 0.49, 0.0875 >, < 0.63, 0.0625 >, < 0.52, 0.18 >, < 0.56, 0.1575 >, < 0.72, 0.1125 >\}.$$

In the following we give some conclusions.

Theorem 1. For three WHFEs h^w, h_1^w and h_2^w, then $(h^w)^c, h_1^w \bigcup h_2^w, h_1^w \bigcap h_2^w, (h^w)^\lambda, \lambda h^w(x), h_1^w \oplus h_2^w, h_1^w \otimes h_2^w$ are also WHFEs.

Because all of these possible values belong to $[0, 1]$, and their weights are normalized or satisfy multiplication, thus all of the weights add up to 1, then the proof of Theorem 1 can be done.

Remark 4. Theorem 1 shows that the WHFEs are closed with respect to these operations.

Theorem 2. For three WHFEs h^w, h_1^w and h_2^w, and $\lambda > 0$, then we have
(1) $(h_1^w \bigcup h_2^w)^c = (h_1^w)^c \bigcap (h_2^w)^c$;
(2) $(h_1^w \bigcap h_2^w)^c = (h_1^w)^c \bigcup (h_2^w)^c$;
(3) $((h^w)^c)^\lambda = (\lambda h^w)^c$;
(4) $\lambda (h^w)^c = ((h^w)^\lambda)^c$;
(5) $(h_1^w \oplus h_2^w)^c = (h_1^w)^c \otimes (h_2^w)^c$;
(6) $(h_1^w \otimes h_2^w)^c = (h_1^w)^c \oplus (h_2^w)^c$.

Proof. Here, we only give the proof of (1), (3) and (5), the rest can be similarly proved.

(1)

$$
(h_1^w \bigcup h_2^w)^c = (\bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w, \gamma_1 \neq \gamma_2} \{< 1 - \max\{\gamma_1, \gamma_2\}, w'_{\max\{\gamma_1, \gamma_2\}} >\}) \bigcup
$$

$$
(\bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w, \gamma_1 = \gamma_2} \{< 1 - \max\{\gamma_1, \gamma_2\}, w'_{\max\{\gamma_1, \gamma_2\}} >\})
$$

$$
= (\bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w, \gamma_1 \neq \gamma_2} \{< \min\{1 - \gamma_1, 1 - \gamma_2\}, w'_{\max\{1-\gamma_1, 1-\gamma_2\}} >\}) \bigcup
$$

$$
(\bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w, \gamma_1 = \gamma_2} \{< \min\{1 - \gamma_1, 1 - \gamma_2\}, w'_{\max\{\gamma_1, \gamma_2\}} >\})
$$

$$
= (h_1^w)^c \bigcap (h_2^w)^c.
$$

(3)

$$
((h^w)^c)^\lambda = \bigcup_{\gamma \in h^w} \{< (1 - \gamma)^\lambda, w_\gamma >\},
$$

and

$$
(\lambda h^w)^c = \bigcup_{\gamma \in h^w} \{< 1 - (1 - (1 - \gamma)^\lambda), w_\gamma >\} = \bigcup_{\gamma \in h^w} \{< (1 - \gamma)^\lambda, w_\gamma >\} = ((h^w)^c)^\lambda.
$$

(5)

$$
(h_1^w \oplus h_2^w)^c = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{< 1 - (\gamma_1 + \gamma_2 - \gamma_1\gamma_2), w_{\gamma_1} w_{\gamma_2} >\}
$$

$$
= \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{< (1 - \gamma_1)(1 - \gamma_2), w_{\gamma_1} w_{\gamma_2} >\}
$$

$$
= (h_1^w)^c \otimes (h_2^w)^c.
$$

Here, we complete the proof of Theorem 2. □

Remark 5. Theorem 2 shows that both of $\bigcap$ and $\bigcup$ operations, and $\oplus$ and $\otimes$ operations satisfy De Morgan's law, respectively.

Definition 6. [40] For a given WHFE h^w, $s(h^w) = \sum_{\gamma \in h^w} w_\gamma \cdot \gamma$ was called as the score function of WHFE h^w.

Definition 7. For a given WHFE h^w,

$$
v(h^w) = \sqrt{\sum_{\gamma \in h^w} w_\gamma (\gamma - s(h^w))^2} \tag{1}
$$

is called as the variance function of WHFE h^w.

Remark 6. Known by the definitions of score function and variance function for HFS and WHFS, the main difference is that WHFS has weight adjustment. Hence, the calculations of score function and variance function are different.

In the following, we propose the ranking rule to compare any two WHFEs h_1^w and h_2^w.

(a) If $s(h_1^w) > s(h_2^w)$, then $h_1^w > h_2^w$.

(b) If $s(h_1^w) = s(h_2^w)$, then

   (1) If $v(h_1^w) > v(h_2^w)$, then $h_1^w < h_2^w$.

   (2) If $v(h_1^w) = v(h_2^w)$, then $h_1^w = h_2^w$.

4. Multi-attribute group decision making with weighted hesitant fuzzy information

In this section, we will propose two kinds of aggregation operators based on weighted hesitant fuzzy information, and investigate their related properties.

For given WHFEs collection, $h_1^w, h_2^w, \cdots, h_n^w$, Zhang and Wu [40] proposed two kinds of aggregation operators such as WHFWA and WHFWG operators as follows:

$$\text{WHFWA}(h_1^w, h_2^w, \cdots, h_n^w) = \bigoplus_{j=1}^{n}(\omega_j h_j^w) = \bigcup_{\gamma_1 \in h_1^w, \cdots, \gamma_n \in h_n^w} \left\{ < 1 - \prod_{j=1}^{n}(1 - \gamma_j)^{\omega_j}, w_{\gamma_1} w_{\gamma_2} \cdots w_{\gamma_n} > \right\},$$

$$\text{WHFWG}(h_1^w, h_2^w, \cdots, h_n^w) = \bigotimes_{j=1}^{n}(h_j^w)^{\omega_j} = \bigcup_{\gamma_1 \in h_1^w, \cdots, \gamma_n \in h_n^w} \left\{ < \prod_{j=1}^{n}\gamma_j^{\omega_j}, w_{\gamma_1} w_{\gamma_2} \cdots w_{\gamma_n} > \right\},$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weight vector of h_j^w, $j = 1, 2, \cdots, n$, and $\omega_j \geq 0$, $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 3. Let $h_1^w, h_2^w, \cdots, h_n^w$ be a collection of WHFEs, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be the weight vector with $\omega_j \geq 0$, $\sum_{j=1}^{n} \omega_j = 1$, then

$$\text{WHFWG}(h_1^w, h_2^w, \cdots, h_n^w) \leq \text{WHFWA}(h_1^w, h_2^w, \cdots, h_n^w).$$

Proof. Known by Wei et al. [32], we have

$$\prod_{j=1}^{n}\gamma_j^{\omega_j} \leq 1 - \prod_{j=1}^{n}(1 - \gamma_j)^{\omega_j}.$$

Since the weights corresponding to both $\prod_{j=1}^{n}\gamma_j^{\omega_j}$ and $1 - \prod_{j=1}^{n}(1 - \gamma_j)^{\omega_j}$ are the same as $w_{\gamma_1} w_{\gamma_2} \cdots w_{\gamma_n}$, hence we have

$$s(\text{WHFWG}(h_1^w, h_2^w, \cdots, h_n^w)) \leq s(\text{WHFWA}(h_1^w, h_2^w, \cdots, h_n^w)).$$

Hence, we complete the proof of Theorem 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Motivated by the idea of OWA operator introduced by Yager [44], we propose two kinds of aggregation operators to aggregate weighted hesitant fuzzy information.

Definition 8. Let $h_1^w, h_2^w, \cdots, h_n^w$ be a collection of WHFEs, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be the weight vector with $\omega_j \geq 0$, $\sum_{j=1}^{n} \omega_j = 1$, and WHFOWA and WHFOWG operators are defined in the following, where $(h_{\sigma(1)}^w, h_{\sigma(2)}^w, \cdots, h_{\sigma(n)}^w)$ represents a permutation of $h_{\sigma(i)}^w \geq h_{\sigma(j)}^w$ for all $i < j$.

$$\text{WHFOWA}(h_1^w, h_2^w, \cdots, h_n^w) = \bigoplus_{j=1}^{n}(\omega_j h_{\sigma(j)}^w)$$

$$= \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}^w, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}^w} \left\{ < 1 - \prod_{j=1}^{n}(1 - \gamma_{\sigma(j)})^{\omega_j}, w_{\gamma_{\sigma(1)}} w_{\gamma_{\sigma(2)}} \cdots w_{\gamma_{\sigma(n)}} > \right\}.$$

$$\text{WHFOWG}(h_1^w, h_2^w, \cdots, h_n^w) = \bigotimes_{j=1}^{n}(h_{\sigma(j)}^w)^{\omega_j}$$

$$= \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}^w, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}^w} \left\{ < \prod_{j=1}^{n} \gamma_{\sigma(j)}^{\omega_j}, w_{\gamma_{\sigma(1)}} w_{\gamma_{\sigma(2)}} \cdots w_{\gamma_{\sigma(n)}} > \right\}.$$

Now we will give the detail algorithm steps in order to apply these two kinds of aggregation operators in multi-attribute group decision making based on weighted hesitant fuzzy information.

Let $A = \{A_1, A_2, \cdots, A_n\}$ and $C = \{C_1, C_2, \cdots, C_m\}$ be the collection of alternatives and the set of attributes, respectively. $E = \{E_1, E_2, \cdots, E_t\}$ and $r_{ij}^k \in [0, 1]$ represent the set of experts and the assessment from the expert E_k respect to the alternative A_i under every attribute C_j, respectively. Our goal is to choose the best alternative.

Hence, we utilize WHFEs as our decision making data to aggregate these fuzzy information and compare their score function and variance function in order to make the ranking relation.

Step 1. For every alternative A_i under each attribute C_j, considering two kinds of different cases, we construct the WHFE h_{ij}^w by incorporating the experts' assessments, respectively.

Case 1. The weights of the experts are unknown, then

$$h_{ij}^w(C_j) = \{< r_{ij}, w_{ij} > | w_{ij} = p/t\} \tag{2}$$

where $r_{ij} \in \cup_k \{r_{ij}^k\}$, p is the number of the experts who give the assessment r_{ij} and t is the number of experts.

Case 2. The weight vector of the experts, $v = (v_1, v_2, \cdots, v_t)^T$ with $v_k \geq 0$ and $\sum_{k=1}^{t} v_k = 1$, is given, then

$$h_{ij}^w(C_j) = \{< r_{ij}, w_{ij} > | w_{ij} = \sum_{k \in N(r_{ij})} v_k\} \tag{3}$$

where $r_{ij} \in \cup_k \{r_{ij}^k\}$ and $N(r_{ij})$ denotes the collection of the experts who give the assessment r_{ij}.

For convenience, we use an example to illustrate the establishing of weight for weighted hesitant fuzzy element $h^w(x)$.

Example 4. To compare three kinds of cars such as A, B and C under the attribute "*Comfort*" and ten experts($E_1, E_2, \cdots, E_{10}$) are authorized to provide their assessments. For A, the assessments provided by the experts are 0.6, 0.7, 0.8, 0.6, 0.75, 0.7, 0.7, 0.8, 0.6, 0.7, respectively. Now we merge the ten experts' assessments by incorporating the evaluated values.

If the weights of the experts are unknown, then according to Eq (2), we obtain the WHFE

$$h^w(x) = \{< 0.6, 0.3 >, < 0.7, 0.4 >, < 0.75, 0.1 >, < 0.8, 0.2 >\}.$$

If the weight vector of the experts is given as

$$v = (0.1, 0.12, 0, 08, 0, 09, 0.15, 0.08, 0.1, 0.08, 0.1, 0.1)^T,$$

then according to Eq (3), we get the WHFE

$$h^w(x) = \{< 0.6, 0.29 >, < 0.7, 0.4 >, < 0.75, 0.15 >, < 0.8, 0.16 >\}.$$

Here, we give the calculation process of $< 0.6, 0.29 >$. Since three experts, E_1, E_4 and E_9, give their assessment values 0.6, and the weights associated with them are $0.1, 0.09$ and 0.1, respectively, then $w = 0.1 + 0.09 + 0.1 = 0.29$, so we have $< 0.6, 0.29 >$.

Step 2. Assume that $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T$ with $\omega_j \geq 0$ and $\sum_{j=1}^m \omega_j = 1$ is the weight vector of the attributes. For every alternative A_i, we aggregate the WHFEs $h_{ij}^w(C_j), j = 1, 2, \cdots, m$, by applying the WHFWA, WHFWG, WHFOWA and WHFOWG operators to derive the overall aggregation value $h^w(A_i)$. Here we utilize WHFOWA operator, then

$$h^w(A_i) = \text{WHFOWA}(h_{i1}^w(C_1), h_{i2}^w(C_2), \cdots, h_{im}^w(C_m)).$$

Step 3. Calculate the score function $s(h^w(A_i))$ and variance function $v(h^w(A_i))$ of $h^w(A_i)$.

Step 4. Rank the alternatives according to $s(h^w(A_i))$ and $v(h^w(A_i)), i = 1, 2, \cdots, n$, and choose the best one.

5. Numerical examples

To illustrate the effectiveness of our proposed approach, we present four numerical examples in this section.

Example 5. There are four software packages A_1, A_2, A_3, A_4 to be selected. And these packages will be considered by three experts $E_k(k = 1, 2, 3)$ from four attributes:
(1) Investment in new software (C_1);
(2) Performance improvement (C_2);
(3) Cost of transferring systems (C_3);
(4) Reliability (C_4). Because of different attributes having different importance, thus we have the weight vector, $w = (0.3, 0.25, 0.25, 0.2)^T$.

After completion of decision making process, we obtain decision making matrix $A^{(k)} = (r_{ij}^{(k)})(k = 1, 2, 3)$ (see Tables 1–3) described by weighted hesitant fuzzy element. Every value in WHFE represents the membership degree. In the following, we give the ranking relation of four software packages by applying our proposed approach.

Table 1. The preference matrix provided by E_1.

	C_1	C_2	C_3	C_4
A_1	0.6	0.8	0.5	0.6
A_2	0.5	0.9	0.8	0.7
A_3	0.7	0.6	0.7	0.6
A_4	0.8	0.7	0.8	0.9

Table 2. The preference matrix provided by E_2.

	C_1	C_2	C_3	C_4
A_1	0.6	0.85	0.65	0.6
A_2	0.7	0.8	0.8	0.75
A_3	0.8	0.6	0.75	0.6
A_4	0.8	0.7	0.8	0.85

Table 3. The preference matrix provided by E_3.

	C_1	C_2	C_3	C_4
A_1	0.8	0.85	0.65	0.7
A_2	0.7	0.8	0.8	0.75
A_3	0.7	0.8	0.75	0.9
A_4	0.9	0.75	0.85	0.9

Case 1. Assume that the weights of the experts are unknown.

Step 1. We utilize Eq (2) and obtain the WHFE $A = \{< r_{ij}, w_{ij} >\}$ under unknown experts weights environments as follows:

For A_1, we have

$$h^w_{A_1}(C_1) = \{< 0.6, 2/3 >, < 0.8, 1/3 >\}, h^w_{A_1}(C_2) = \{< 0.8, 1/3 >, < 0.85, 2/3 >\}.$$

$$h^w_{A_1}(C_3) = \{< 0.5, 1/3 >, < 0.65, 2/3 >\}, h^w_{A_1}(C_4) = \{< 0.6, 2/3 >, < 0.7, 1/3 >\}.$$

For A_2, we have

$$h^w_{A_2}(C_1) = \{< 0.5, 1/3 >, < 0.7, 2/3 >\}, h^w_{A_2}(C_2) = \{< 0.8, 2/3 >, < 0.9, 1/3 >\}.$$

$$h^w_{A_2}(C_3) = \{< 0.8, 1 >\}, h^w_{A_2}(C_4) = \{< 0.7, 1/3 >, < 0.75, 2/3 >\}.$$

For A_3, we have

$$h^w_{A_3}(C_1) = \{< 0.7, 2/3 >, < 0.8, 1/3 >\}, h^w_{A_3}(C_2) = \{< 0.6, 2/3 >, < 0.8, 1/3 >\}.$$

$$h^w_{A_3}(C_3) = \{< 0.7, 1/3 >, < 0.75, 2/3 >\}, h^w_{A_3}(C_4) = \{< 0.6, 2/3 >, < 0.9, 1/3 >\}.$$

For A_4, we have

$$h^w_{A_4}(C_1) = \{< 0.8, 2/3 >, < 0.9, 1/3 >\}, h^w_{A_4}(C_2) = \{< 0.7, 2/3 >, < 0.75, 1/3 >\}.$$

$$h^w_{A_4}(C_3) = \{< 0.8, 2/3 >, < 0.85, 1/3 >\}, h^w_{A_4}(C_4) = \{< 0.85, 1/3 >, < 0.9, 2/3 >\}.$$

Step 2. We apply WHFOWA operator to aggregate all of preference values $< r_{ij}, w_{ij} >$, $j = 1, 2, 3, 4$, and obtain $h^w(A_i)$, $i = 1, 2, 3, 4$ as follows:

$$h^w(A_1) = \{ \quad < 0.8632, 0.0494 >, < 0.8313, 0.0247 >, < 0.9144, 0.0988 >, < 0.8944, 0.0494 >,$$
$$< 0.9340, 0.0988 >, < 0.9186, 0.0494 >, < 0.9587, 0.1975 >, < 0.9490, 0.0988 >,$$
$$< 0.8526, 0.0247 >, < 0.8183, 0.0123 >, < 0.9078, 0.0494 >, < 0.8863, 0.0247 >,$$
$$< 0.9289, 0.0494 >, < 0.9123, 0.0247 >, < 0.9555, 0.0988 >, < 0.9451, 0.0494 >\}.$$

$$h^w(A_2) = \{ \quad < 0.9637, 0.0741 >, < 0.9785, 0.1481 >, < 0.9507, 0.037 >, < 0.9708, 0.0741 >,$$
$$< 0.9795, 0.1481 >, < 0.9878, 0.2963 >, < 0.9722, 0.0741 >, < 0.9835, 0.1481 >\}.$$

$$h^w(A_3) = \{ \quad < 0.9116, 0.0988 >, < 0.9244, 0.0494 >, < 0.9476, 0.1975 >, < 0.9552, 0.0988 >,$$
$$< 0.9048, 0.0494 >, < 0.9186, 0.0247 >, < 0.9435, 0.0988 >, < 0.9517, 0.0494 >,$$
$$< 0.8846, 0.0494 >, < 0.9014, 0.0247 >, < 0.9316, 0.0988 >, < 0.9415, 0.0494 >,$$
$$< 0.8757, 0.0247 >, < 0.8937, 0.0123 >, < 0.9263, 0.0494 >, < 0.937, 0.0247 >\}.$$

$$h^w(A_4) = \{ \quad < 0.9722, 0.0988 >, < 0.9887, 0.1975 >, < 0.9567, 0.0494 >, < 0.9825, 0.0988 >,$$
$$< 0.9609, 0.0494 >, < 0.9841, 0.0988 >, < 0.9392, 0.0247 >, < 0.9753, 0.0494 >,$$
$$< 0.9622, 0.0494 >, < 0.9847, 0.0988 >, < 0.9413, 0.0247 >, < 0.9762, 0.0494 >,$$
$$< 0.9469, 0.0247 >, < 0.9785, 0.0494 >, < 0.9175, 0.0123 >, < 0.9665, 0.0247 >\}.$$

Step 3. We calculate the score function and variance function of $h^w(A_i), i = 1, 2, 3, 4$, and have $s(h^w(A_1)) = 0.9256, v(h^w(A_1)) = 0.0345, s(h^w(A_2)) = 0.9790, v(h^w(A_2)) = 0.0090, s(h^w(A_3)) = 0.9307, v(h^w(A_3)) = 0.0213, s(h^w(A_4)) = 0.9747, v(h^w(A_4)) = 0.0148$.

Step 4. Rank the alternatives according to $s(h^w(A_i))$ and $v(h^w(A_i)), i = 1, 2, 3, 4$, therefore, we obtain the ranking relation of alternatives, $A_2 > A_4 > A_3 > A_1$, and the best alternative is A_2.

Case 2. Assume that the weights of the experts are $v = (0.4, 0.3, 0.3)^T$.

Step 1. We utilize Eq (3), and obtain the WHFSs $A = \{< r_{ij}, w_{ij} >\}$ as follows:
For A_1, we have

$$h^w_{A_1}(C_1) = \{< 0.6, 0.7 >, < 0.8, 0.3 >\}, h^w_{A_1}(C_2) = \{< 0.8, 0.4 >, < 0.85, 0.6 >\}.$$

$$h^w_{A_1}(C_3) = \{< 0.5, 0.4 >, < 0.65, 0.6 >\}, h^w_{A_1}(C_4) = \{< 0.6, 0.7 >, < 0.7, 0.3 >\}.$$

For A_2, we have

$$h^w_{A_2}(C_1) = \{< 0.5, 0.4 >, < 0.7, 0.6 >\}, h^w_{A_2}(C_2) = \{< 0.8, 0.6 >, < 0.9, 0.4 >\}.$$

$$h^w_{A_2}(C_3) = \{< 0.8, 1 >\}, h^w_{A_2}(C_4) = \{< 0.7, 0.4 >, < 0.75, 0.6 >\}.$$

For A_3, we have

$$h^w_{A_3}(C_1) = \{< 0.7, 0.7 >, < 0.8, 0.3 >\}, h^w_{A_3}(C_2) = \{< 0.6, 0.7 >, < 0.8, 0.3 >\}.$$

$$h_{A_3}^w(C_3) = \{< 0.7, 0.4 >, < 0.75, 0.6 >\}, h_{A_3}^w(C_4) = \{< 0.6, 0.7 >, < 0.9, 0.3 >\}.$$

For A_4, we have

$$h_{A_4}^w(C_1) = \{< 0.8, 0.7 >, < 0.9, 0.3 >\}, h_{A_4}^w(C_2) = \{< 0.7, 0.7 >, < 0.75, 0.3 >\}.$$

$$h_{A_4}^w(C_3) = \{< 0.8, 0.7 >, < 0.85, 0.3 >\}, h_{A_4}^w(C_4) = \{< 0.85, 0.3 >, < 0.9, 0.7 >\}.$$

Step 2. We apply WHFOWA operator to aggregate all of preference values $< r_{ij}, w_{ij} >$, $j = 1, 2, 3, 4$, and obtain $h^w(A_i)$, $i = 1, 2, 3, 4$ as follows:

$$
\begin{aligned}
h^w(A_1) = \{ \quad &< 0.8896, 0.0784 >, < 0.8539, 0.0336 >, < 0.9224, 0.1176 >, < 0.8973, 0.0504 >, \\
&< 0.9327, 0.1176 >, < 0.9109, 0.0504 >, < 0.9527, 0.1764 >, < 0.9374, 0.0756 >, \\
&< 0.8707, 0.0336 >, < 0.8288, 0.0144 >, < 0.9091, 0.0504 >, < 0.8797, 0.0216 >, \\
&< 0.9211, 0.0504 >, < 0.8956, 0.0216 >, < 0.9446, 0.0756 >, < 0.9266, 0.0324 >\}.
\end{aligned}
$$

$$
\begin{aligned}
h^w(A_2) = \{ \quad &< 0.9643, 0.0960 >, < 0.9749, 0.1440 >, < 0.9627, 0.0640 >, < 0.9737, 0.0960 >, \\
&< 0.9772, 0.1440 >, < 0.9839, 0.2160 >, < 0.9761, 0.096 >, < 0.9832, 0.1440 >\}.
\end{aligned}
$$

$$
\begin{aligned}
h^w(A_3) = \{ \quad &< 0.9263, 0.1372 >, < 0.9298, 0.0588 >, < 0.9480, 0.2058 >, < 0.9505, 0.0882 >, \\
&< 0.9136, 0.0588 >, < 0.9177, 0.0252 >, < 0.9391, 0.0882 >, < 0.9420, 0.0378 >, \\
&< 0.8943, 0.0588 >, < 0.8994, 0.0252 >, < 0.9255, 0.0882 >, < 0.9291, 0.0378 >, \\
&< 0.8761, 0.0252 >, < 0.8821, 0.0108 >, < 0.9127, 0.0378 >, < 0.9169, 0.0162 >\}.
\end{aligned}
$$

$$
\begin{aligned}
h^w(A_4) = \{ \quad &< 0.9744, 0.1029 >, < 0.9910, 0.2401 >, < 0.9553, 0.0441 >, < 0.9842, 0.1029 >, \\
&< 0.9608, 0.0441 >, < 0.9862, 0.1029 >, < 0.9315, 0.0189 >, < 0.9758, 0.0441 >, \\
&< 0.9604, 0.0441 >, < 0.9860, 0.1029 >, < 0.9309, 0.0189 >, < 0.9756, 0.0441 >, \\
&< 0.9393, 0.0189 >, < 0.9786, 0.0441 >, < 0.8941, 0.0081 >, < 0.9627, 0.0189 >\}.
\end{aligned}
$$

Step 3. We calculate the score function and variance function of $h^w(A_i)$, $i = 1, 2, 3, 4$, and have $s(h^w(A_1)) = 0.9199$, $v(h^w(A_1)) = 0.0280$, $s(h^w(A_2)) = 0.9766$, $v(h^w(A_2)) = 0.0068$, $s(h^w(A_3)) = 0.9290$, $v(h^w(A_3)) = 0.0187$, $s(h^w(A_4)) = 0.9769$, $v(h^w(A_4)) = 0.0168$.

Step 4. Rank the alternatives according to $s(h^w(A_i))$ and $v(h^w(A_i))$, $i = 1, 2, 3, 4$, therefore, we obtain the ranking of alternatives, $A_4 > A_2 > A_3 > A_1$, and the best alternative is A_4.

Similarly, we can use WHFOWG operator to aggregate weighted hesitant fuzzy information and have the following conclusions.

Case 1. Assume that the weights of the experts are unknown.

We calculate the score function and variance function of $h^w(A_i)$, $i = 1, 2, 3, 4$, and have $s(h^w(A_1)) = 0.4190$, $v(h^w(A_1)) = 0.0725$, $s(h^w(A_2)) = 0.4795$, $v(h^w(A_2)) = 0.0311$, $s(h^w(A_3)) = 0.4409$, $v(h^w(A_3)) = 0.0964$, $s(h^w(A_4)) = 0.6196$, $v(h^w(A_4)) = 0.0613$. and obtain the ranking of alternatives, $A_4 > A_2 > A_3 > A_1$, and the best alternative is A_4.

Case 2. Assume that the weights of the experts are $v = (0.4, 0.3, 0.3)^T$.

We calculate the score function and variance function of $h^w(A_i), i = 1, 2, 3, 4$, and have $s(h^w(A_1)) = 0.4078$, $v(h^w(A_1)) = 0.0758$, $s(h^w(A_2)) = 0.4874$, $v(h^w(A_2)) = 0.0277$, $s(h^w(A_3)) = 0.4227$, $v(h^w(A_3)) = 0.0976$, $s(h^w(A_4)) = 0.6042$, $v(h^w(A_4)) = 0.0664$, and obtain the ranking of alternatives, $A_4 > A_2 > A_3 > A_1$, and the best alternative is A_4.

By combining the calculation results of both WHFOWA and WHFOWG operators, we find that the best alternative is A_4.

In addition, we will make comparison analysis between our algorithm and the one of classical hesitant fuzzy set. Aimed at the data in Example 5, we retain the possible membership degrees and cancel the weight information, thus, these HFEs are listed in the following:

For A_1, we have

$$h_{A_1}(C_1) = \{0.6, 0.8\}, h_{A_1}(C_2) = \{0.8, 0.85\}, h_{A_1}(C_3) = \{0.5, 0.65\}, h_{A_1}(C_4) = \{0.6, 0.7\}.$$

For A_2, we have

$$h_{A_2}(C_1) = \{0.5, 0.7\}, h_{A_2}(C_2) = \{0.8, 0.9\}, h_{A_2}(C_3) = \{0.8\}, h_{A_2}(C_4) = \{0.7, 0.75\}.$$

For A_3, we have

$$h_{A_3}(C_1) = \{0.7, 0.8\}, h_{A_3}(C_2) = \{0.6, 0.8\}, h_{A_3}(C_3) = \{0.7, 0.75\}, h_{A_3}(C_4) = \{0.6, 0.9\}.$$

For A_4, we have

$$h_{A_4}(C_1) = \{0.8, 0.9\}, h_{A_4}(C_2) = \{0.7, 0.75\}, h_{A_4}(C_3) = \{0.8, 0.85\}, h_{A_4}(C_4) = \{0.85, 0.9\}.$$

We apply HFOWA operator to aggregate origin values $< \gamma_{ij} >, j = 1, 2, 3, 4$ and results $h(A_i), i = 1, 2, 3, 4$ are as follows:

$$h_{A_1} = \{0.984, 0.988, 0.9888, 0.9916, 0.988, 0.991, 0.9916, 0.9937,$$
$$0.992, 0.994, 0.9944, 0.9958, 0.994, 0.9955, 0.9958, 0.9969\}.$$

$$h_{A_2} = \{0.994, 0.995, 0.997, 0.9975, 0.9964, 0.997, 0.9982, 0.9985\}.$$

$$h_{A_3} = \{0.9856, 0.9964, 0.988, 0.997, 0.9928, 0.9982, 0.994, 0.9985,$$
$$0.9904, 0.9976, 0.992, 0.998, 0.9952, 0.9988, 0.996, 0.999\}.$$

$$h_{A_4} = \{0.9982, 0.9988, 0.9987, 0.9991, 0.9985, 0.999, 0.9989, 0.9992,$$
$$0.9991, 0.9994, 0.9993, 0.9996, 0.9992, 0.9995, 0.9994, 0.9996\}.$$

Then we calculate the score function and the variance function of $h(A_i), i = 1, 2, 3, 4$, and the results are listed as follows:

$$s(h(A_1)) = 0.9922, v(h(A_1)) = 0.0137, s(h(A_2)) = 0.9967, v(h(A_2)) = 0.0041.$$

$$s(h(A_3)) = 0.9948, v(h(A_3)) = 0.0158, s(h(A_4)) = 0.9990, v(h(A_4)) = 0.0016.$$

Therefore, we obtain the ranking relation of alternatives, $A_4 > A_2 > A_3 > A_1$, and the best alternative is A_4.

Remark 7. By analyzing the results of WHFE and HFE in Case 1 and Case 2, we find that the ranking result of WHFE is the same as the one of HFE though the weights of experts and the WHFEs are different, thus it shows that the weighted hesitant fuzzy element has higher sensitivity.

Example 6. [40] Assume that a factory intends to choose a new site for new buildings. Now there exist three possible alternatives $Y_i(i = 1, 2, 3)$ to be considered, three attributes include (1) G_1(Price); (2) G_2(Location); and (3) G_3 Environment, and the weight vector of three attributes is $w = (0.3, 0.2, 0.5)^T$. The related data is described as weighted hesitant fuzzy element, $h_{ij}^w = \bigcup_{\gamma_{ij} \in h_{ij}^w} \{< \gamma_{ij}, w_{\gamma_{ij}} >\}$, where γ_{ij} represents the possible membership degree which is the satisfaction degree of the alternative Y_i respect to the attribute G_j, and $w_{\gamma_{ij}}$ is the weight of γ_{ij}. The origin data is listed in Table 4.

Table 4. Weighted hesitant fuzzy decision matrix.

	G_1	G_2	G_3
Y_1	{(0.6,0.3),(0.5,0.3),(0.4,0.4)}	{(0.6,0.8),(0.4,0.2)}	{(0.5,0.3),(0.3,0.7)}
Y_2	{(0.4,0.6),(0.3,0.4)}	{(0.8,1)}	{(0.4,0.2),(0.3,0.3),(0.2,0.5)}
Y_3	{(0.8,1)}	{(0.7,0.1),(0.6,0.3),(0.5,0.6)}	{(0.2,0.5),(0.1,0.5)}

Then according to the related steps, we have the following calculation results.

Step 1. Obtain origin data under weighted hesitant fuzzy environment, see Table 4.

Step 2. Apply WHFOWA operator to aggregate all WHFEs, and obtain $h^w(Y_i), i = 1, 2, 3$ in the following:

$$
\begin{aligned}
h^w(Y_1) = \{ \quad &< 0.7035, 0.072 >, < 0.7157, 0.1680 >, < 0.4429, 0.018 >, < 0.4657, 0.0420 >, \\
&< 0.6830, 0.072 >, < 0.6960, 0.1680 >, < 0.4043, 0.018 >, < 0.4287, 0.0420 >, \\
&< 0.6819, 0.096 >, < 0.6949, 0.224 >, < 0.4022, 0.024 >, < 0.4266, 0.056 >\}.
\end{aligned}
$$

$$
\begin{aligned}
h^w(Y_2) = \{ \quad &< 0.8671, 0.12 >, < 0.8677, 0.18 >, < 0.8683, 0.3 >, < 0.8434, 0.08 >, \\
&< 0.8442, 0.12 >, < 0.8449, 0.2 >\}.
\end{aligned}
$$

$$
\begin{aligned}
h^w(Y_3) = \{ \quad &< 0.8414, 0.05 >, < 0.8318, 0.05 >, < 0.8641, 0.15 >, < 0.8559, 0.15 >, \\
&< 0.8820, 0.3 >, < 0.8748, 0.3 >\}.
\end{aligned}
$$

Step 3. Calculate the score function and variance function, then we have:

$$s(h^w(Y_1)) = 0.6445, v(h^w(Y_1)) = 0.1072, s(h^w(Y_2)) = 0.8585.$$

$$v(h_A^w(Y_2)) = 0.0115, s(h^w(Y_3)) = 0.8687, v(h^w(Y_3)) = 0.0140.$$

Step 4. Rank the alternatives based on the score function and the variance function of every alternative, then we have the ranking relation: $A_3 > A_2 > A_1$.

Remark 8. The ranking result based on the WHFOWA operator is the same as that of WHFHWA operator in Zhang and Wu [40].

Remark 9. Numerical examples show that our proposed algorithm has good order preservation.

Example 7. In this example, we adopt the data from [5] by adding the same weights for HFEs data to construct the weighted hesitant fuzzy elements, list these data in Table 5, and make comparison.

After we compute the score function of our operator aggregation result, we have:

$$s(h^w(Y_1)) = 0.574, s(h^w(Y_2)) = 0.5978, s(h^w(Y_3)) = 0.5321, s(h^w(Y_4)) = 0.8116.$$

And the ranking of these alternatives are that: $Y_4 > Y_2 > Y_1 > Y_3$. By comparing with the result in [5], we find that there exists the same ranking order, but the consequence values give us more accurate information for alternatives.

Table 5. Weighted hesitant fuzzy decision matrix.

	G_1	G_2
Y_1	{(0.2,1/3),(0.4,1/3),(0.7,1/3)}	{(0.2,1/3),(0.6,1/3),(0.8,1/3)}
Y_2	{(0.2,1/4),(0.4,1/4),(0.7,1/4),(0.9,1/4)}	{(0.1,1/4),(0.2,1/4),(0.4,1/4),(0.5,1/4)}
Y_3	{(0.3,1/4),(0.5,1/4),(0.6,1/4),(0.7,1/4)}	{(0.2,1/4),(0.4,1/4),(0.5,1/4),(0.6,1/4)}
Y_4	{(0.3,1/3),(0.5,1/3),(0.6,1/3)}	{(0.2,1/2),(0.4,1/2))}
	G_3	G_4
Y_1	{(0.2,1/5),(0.3,1/5),(0.6,1/5),(0.7,1/5),(0.9,1/5)}	{(0.3,1/5),(0.4,1/5),(0.5,1/5),(0.7,1/5),(0.8,1/5)}
Y_2	{(0.3,1/4),(0.4,1/4),(0.6,1/4),(0.9,1/4)}	{(0.5,1/4),(0.6,1/4),(0.5,1/4),(0.9,1/4)}
Y_3	{(0.3,1/4),(0.5,1/4),(0.7,1/4),(0.8,1/4)}	{(0.2,1/4),(0.5,1/4),(0.6,1/4),(0.7,1/4)}
Y_4	{(0.5,1/3),(0.6,1/3),(0.7,1/3)}	{(0.8,1/2),(0.9,1/2))}

Example 8. In this example, we choose the data from [14] by adding the same weights for HFEs data to construct the weighted hesitant fuzzy elements, list these data in Table 6, and make comparison.

Suppose that the ideal alternative is $A^* = \{1\}$, then we can obtain the score function of operator aggregation result, and have:

$$s(h^w(A_1)) = 0.5875, s(h^w(A_2)) = 0.5871, s(h^w(A_3)) = 0.8456,$$

$$s(h^w(A_4)) = 0.7292, s(h^w(A_5)) = 0.7232.$$

Hence, we have the ranking order of alternatives: $A_3 > A_4 > A_5 > A_1 > A_2$.

Comparing with result from generalized hybrid hesitant weighted distance in [14], $A_3 > A_5 > A_4 > A_1 > A_2(\lambda = 1)$, we find that there exists the difference between the samples A_4 and A_5. In fact, it also reflects the difference between the two approaches.

Table 6. Weighted hesitant fuzzy decision matrix.

	P_1	P_2
A_1	{(0.5,1/3),(0.4,1/3),(0.3,1/3)}	{(0.9,1/4),(0.8,1/4),(0.7,1/4),(0.1,1/4)}
A_2	{(0.5,1/2),(0.3,1/2)}	{(0.9,1/4),(0.7,1/4),(0.6,1/4),(0.2,1/4)}
A_3	{(0.7,1/2),(0.6,1/2)}	{(0.9,1/2),(0.6,1/2)}
A_4	{(0.8,1/4),(0.7,1/4),(0.4,1/4),(0.3,1/4)}	{(0.7,1/3),(0.4,1/3),(0.2,1/3)}
A_5	{(0.9,1/5),(0.7,1/5),(0.6,1/5),(0.3,1/5),(0.1,1/5)}	{(0.8,1/4),(0.7,1/4),(0.6,1/4),(0.4,1/4)}
	P_3	P_4
A_1	{(0.5,1/3),(0.4,1/3),(0.2,1/3)}	{(0.9,1/4),(0.6,1/4),(0.5,1/4),(0.3,1/4)}
A_2	{(0.8,1/4),(0.6,1/4),(0.5,1/4),(0.1,1/4)}	{(0.7,1/3),(0.3,1/3),(0.4,1/3)}
A_3	{(0.7,1/3),(0.5,1/3),(0.3,1/3)}	{(0.6,1/2),(0.4,1/2)}
A_4	{(0.8,1/2),(0.1,1/2)}	{(0.9,1/3),(0.8,1/3),(0.6,1/3)}
A_5	{(0.9,1/3),(0.8,1/3),(0.7,1/3)}	{(0.9,1/4),(0.7,1/4),(0.6,1/4),(0.3,1/4)}

6. Conclusions

Considering that the weighted hesitant fuzzy set (WHFS) introduced by Zhang and Wu [40] is a powerful tool to describe hesitant fuzzy information, where its membership degrees are given with some different weights. In this paper, we redefine the union and intersection operations of WHFEs which are more intuitive and logical, investigate its operation properties, and propose the variance function of WHFE and the ranking rule to compare WHFEs based on the score function and variance function. Furthermore, motivated by the idea of OWA operator introduced by Yager [44], we propose two kinds of aggregation operators such as WHFOWA and WHFOWG operators, and apply in multiple-attribute group decision making. Finally, four numerical examples are used to illustrate that our proposed algorithm has good effectiveness.

In the future, we will focus on developing some novel operators under weighted hesitant fuzzy environment including Frank operator, Bonferroni means, OWA operator and variable weighted aggregation operator to enrich weighted hesitant fuzzy set theory and method. Furthermore, inspired by the idea of research in [22–24], we will extend our idea to q-rung orthopair fuzzy sets and propose some novel generalization of fuzzy sets. Finally, we hope that our research will be able to provide more new idea and new methods for multi-criteria and multiple-attribute group decision making based on weighted hesitant fuzzy information.

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Conflict of interest

We declare that we have no conflicts of interest.

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