



Research article

Bennett-Leindler nabla type inequalities via conformable fractional derivatives on time scales

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Abstract: In this work, we prove several new (γ, a) -nabla Bennett and Leindler dynamic inequalities on time scales. The results proved here generalize some known dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues. Our results will be proved by using integration by parts, chain rule and Hölder inequality for the (γ, a) -nabla-fractional derivative on time scales.

Keywords: Steffensen’s inequality; dynamic inequality; dynamic integral; time scales

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1. Introduction

In [13], Hardy presented the discrete version.

Theorem 1.1. If $\{r(n)\}_{n=0}^{\infty}$ is a nonnegative real sequence and $l > 1$, then

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{m=1}^n r(m) \right)^l \leq \left(\frac{l}{l-1} \right)^l \sum_{n=1}^{\infty} r^l(n). \tag{1.1}$$

Also, Hardy [14] gave the continuous version of (1.1).

Theorem 1.2. Let $r \geq 0$ be a continuous over $[0, \infty)$ and $l > 1$. Then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x r(s) ds \right)^l dx \leq \left(\frac{l}{l-1} \right)^l \int_0^\infty r^l(x) dx. \quad (1.2)$$

Copson [6] obtained a new version of inequality (1.1) by replacing the arithmetic mean of a sequence by a weighted arithmetic mean in the following manner: Let $b(m) \geq 0$, $\eta(m) \geq 0$ for all m . If $l > 1$, $c > 1$, then

$$\sum_{m=1}^\infty \frac{\eta(m)}{[\bar{\xi}(m)]^c} \left(\sum_{j=1}^m b(j)\eta(j) \right)^l \leq \left(\frac{l}{c-1} \right)^l \sum_{m=1}^\infty \eta(m) [\bar{\xi}(m)]^{l-c} b^l(m), \quad (1.3)$$

where $\bar{\xi}(m) = \sum_{j=1}^m \eta(j)$, and if $0 \leq c < 1 < l$, then

$$\sum_{m=1}^\infty \frac{\eta(m)}{[\bar{\xi}(m)]^c} \left(\sum_{j=1}^m b(j)\eta(j) \right)^l \leq \left(\frac{l}{1-c} \right)^l \sum_{m=1}^\infty \eta(m) [\bar{\xi}(m)]^{l-c} b^l(m). \quad (1.4)$$

The reverse versions of the inequalities (1.3) and (1.4), which have been derived by Bennett and Leindler [4, 17], can be deduced for $\bar{\xi}(m) \rightarrow \infty$, $b(m) \geq 0$ and $\eta(m) \geq 0$ for all m that if $0 < l < 1 < c$, then

$$\sum_{m=1}^\infty \frac{\eta(m)}{[\bar{\xi}(m)]^c} \left(\sum_{j=1}^m b(j)\eta(j) \right)^l \geq \left(\frac{lL}{c-1} \right)^l \sum_{m=1}^\infty \eta(m) [\bar{\xi}(m)]^{l-c} b^l(m), \quad (1.5)$$

where $L = \inf \frac{\bar{\xi}(m)}{\bar{\xi}(m+1)}$, and if $c \leq 0 < l < 1$, then

$$\sum_{m=1}^\infty \frac{\eta(m)}{[\bar{\xi}(m)]^c} \left(\sum_{j=1}^m b(j)\eta(j) \right)^l \geq \left(\frac{l}{1-c} \right)^l \sum_{m=1}^\infty \eta(m) [\bar{\xi}(m)]^{l-c} b^l(m), \quad (1.6)$$

respectively. Copson [7] gave the continuous version of the inequalities (1.5) and (1.6), respectively, as follows: Let η and f be nonnegative functions and $\bar{\xi}(\varphi) = \int_0^\varphi \eta(\varrho) d\varrho$, $B(\varphi) = \int_0^\varphi \eta(\varrho) f(\varrho) d\varrho$, $\bar{B}(\varphi) = \int_\varphi^\infty \eta(\varrho) f(\varrho) d\varrho$. If $0 < l \leq 1 < c$, $a > 0$ then

$$\int_a^\infty \frac{\eta(\varphi)}{[\bar{\xi}(\varphi)]^c} [B(\varphi)]^l dt \geq \left(\frac{p}{c-1} \right)^l \int_a^\infty \eta(\varphi) [\bar{\xi}(\varphi)]^{l-c} f^l(\varphi) dt, \quad (1.7)$$

and if $0 < l < 1$, $c < 1$ then

$$\int_a^\infty \frac{\eta(\varphi)}{[\bar{\xi}(\varphi)]^c} [\bar{B}(\varphi)]^l dt \geq \left(\frac{l}{1-c} \right)^l \int_a^\infty \eta(\varphi) [\bar{\xi}(\varphi)]^{l-c} f^l(\varphi) dt. \quad (1.8)$$

For further results on Hardy inequalities and other types see [1–3, 5, 8–10, 12, 15, 18–21, 23–27]. In [20] the author proved the time scales version of (1.1) and (1.2).

$$\int_a^\infty \left(\frac{\int_a^{\sigma(\varphi)} \eta(\varrho) \Delta\varrho}{\sigma(\varphi) - a} \right)^l \Delta\varphi < \left(\frac{l}{l-1} \right)^l \int_a^\infty \eta^l(\varphi) \Delta\varphi, \quad (1.9)$$

unless $\eta \equiv 0$.

In [11] El-Deeb et al. extended (1.9)

$$\int_a^\infty \frac{\tilde{\lambda}(\varsigma)\tilde{\Psi}^p(\varsigma)}{\tilde{\Lambda}^{\hat{\gamma}}(\varsigma)}\Delta\varsigma \geq \frac{p}{\hat{\gamma}-1} \int_a^\infty \tilde{\lambda}(\varsigma)\tilde{\Lambda}^{p-\hat{\gamma}}(\varsigma)\nu^p(\varsigma)\Delta\varsigma, \quad (1.10)$$

where

$$\tilde{\Psi}(\varsigma) = \int_a^\varsigma \tilde{\lambda}(\nu)\nu(\varrho)\Delta\nu, \quad \text{and} \quad \tilde{\Lambda}(\varsigma) = \int_a^\varsigma \tilde{\lambda}(\nu)\Delta\nu.$$

In 2021 Kayar et al. [16], established the time scale version unification of discrete and continuous Bennett-Leindler inequalities (1.5) and (1.7) as following theorem.

Theorem 1.3. Let λ, f be nonnegative, ld-continuous, ∇ -differentiable and ∇ -integrable functions on $[a, \infty)_{\mathbb{T}}$ where $a \in [0, \infty)_{\mathbb{T}}$. Define

$$\bar{\xi}(\wp) = \int_a^\wp \lambda(\varrho)\nabla\varrho \quad B(\wp) = \int_a^\wp \lambda(\varrho)f(\varrho)\nabla\varrho.$$

If $L = \inf_{\wp \in \mathbb{T}} \frac{\bar{\xi}^p(\wp)}{\xi(\wp)} > 0$, $0 < p < 1$ and $c \geq 1$, then

$$\int_a^\infty \frac{\lambda(\wp)}{[\bar{\xi}^c(\wp)]} [B(\wp)]^p \nabla\wp \geq \left(\frac{pL^c}{c-1}\right)^p \int_a^\infty \frac{\lambda(\wp)f^p(\wp)}{(\bar{\xi}(\wp))^{c-p}} \nabla\wp. \quad (1.11)$$

Lately, Zakarya et al. proved an α -conformable version of Hardy inequalities [22].

Theorem 1.4. Assume that \mathbb{T} is a time scale with $\omega \in (0, \infty)_{\mathbb{T}}$. Let λ and ξ be rd-continuous and α -fractional differentiable functions on $[\omega, \infty)_{\mathbb{T}}$. Define

$$\chi(\wp) = \int_\omega^\infty \lambda(\varrho)\Delta_\alpha\varrho \quad \text{and} \quad \Theta(\wp) = \int_\omega^\infty \lambda(\varrho)\xi(\varrho)\Delta_\alpha\varrho.$$

Then, for $k \leq 0 < h < 1$, and $\alpha \in (0, 1]$, we have that

$$\int_\omega^\infty \frac{\lambda(\wp)}{\chi^{k-\alpha+1}(\wp)} (\Theta^\sigma(\wp))^h \Delta_\alpha\wp \geq \left(\frac{h}{\alpha-k}\right)^h \int_\omega^\infty \lambda(\wp)\xi^h(\wp)\chi^{h-k+\alpha-1}(\wp)\Delta_\alpha\wp.$$

We will need the following chain rule for γ -nabla derivative, integration by parts for γ -nabla derivative [29] and generalized γ -nabla Hölder fractional inequality on timescales [28] respectively

$$\nabla_a^\gamma(\varpi \circ \xi)(\wp) = \varpi'(\xi(c))\nabla_a^\gamma(\xi(\wp)). \quad (1.12)$$

$$\int_d^b \varpi(\wp)[\nabla_a^\gamma\xi(\wp)]\nabla_a^\gamma\wp = [\varpi(\wp)\xi(\wp)]_d^b - \int_d^b [\nabla_a^\gamma\varpi(\wp)]\xi^p(\wp)\nabla_a^\gamma\wp. \quad (1.13)$$

$$\int_d^b |\varpi(\wp)\xi(\wp)|\nabla_a^\gamma\wp \leq \left(\int_d^b |\varpi(\wp)|^p\nabla_a^\gamma\wp\right)^{1/p} \left(\int_d^b |\xi(\wp)|^q\nabla_a^\gamma\wp\right)^{1/q}, \quad (1.14)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < \gamma \leq 1$. Now, we start to state our main results.

2. Main results

We focus in this section, on investigating corresponding results for γ -nabla conformable time scales.

Theorem 2.1. Let \mathbb{T} be a time scale with $r \in [0, \infty)_{\mathbb{T}}$, $\gamma \in (0, 1]$ and $\wp \geq a$. In addition, let \mathfrak{J} and λ be nonnegative ld-continuous and (γ, a) -nabla fractional differentiable functions on $[r, \infty)_{\mathbb{T}}$ where

$$\Omega(\wp) = \int_{\wp}^{\infty} \lambda(\varrho) \nabla_a^{\gamma} \varrho \quad \text{and} \quad \Psi(\wp) = \int_r^{\wp} \lambda(\varrho) \mathfrak{J}(\varrho) \nabla_a^{\gamma} \varrho, \quad \wp \in [r, \infty)_{\mathbb{T}}.$$

If $0 < p < \gamma$ and $c \leq \gamma - 1$, then

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^{\gamma} \wp \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \int_r^{\infty} \frac{\lambda(\wp) \mathfrak{J}^p(\wp) [\Psi(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}} \nabla_a^{\gamma} \wp. \quad (2.1)$$

Proof. Using (1.13), with

$$\nabla_a^{\gamma} \eta(\wp) = \lambda(\wp) / [\Omega^{\rho}(\wp)]^{c-\gamma+1}, \quad \xi(\wp) = [\Psi(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\alpha+1} \nabla_a^{\gamma} \wp = [\eta(\wp) \Psi^{p-\alpha+1}(\wp)]_r^{\infty} + \int_r^{\infty} (-\eta^{\rho}(\wp)) \nabla_a^{\gamma} (\Psi^{p-\gamma+1}(\wp)) \nabla_a^{\gamma} \wp,$$

where we assumed that

$$\eta(\wp) = - \int_{\wp}^{\infty} \lambda(\varrho) / \Omega^{c-\gamma+1}(\varrho) \nabla_a^{\gamma} \varrho. \quad (2.2)$$

Using $\Psi(r) = 0$ and $\eta(\infty) = 0$, we get

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^{\gamma} \wp = \int_r^{\infty} -\eta^{\rho}(\wp) \nabla_a^{\gamma} (\Psi^{p-\gamma+1}(\wp)) \nabla_a^{\gamma} \wp. \quad (2.3)$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$\nabla_a^{\gamma} (\Psi^{p-\gamma+1}(\wp)) = \frac{p-\gamma+1}{\Psi^{\gamma-p}(d)} \nabla_a^{\gamma} \Psi(\wp) \geq \frac{(p-\gamma+1)\lambda(\wp)\mathfrak{J}(\wp)}{\Psi^{\gamma-p}(\wp)}. \quad (2.4)$$

Next note $\nabla_a^{\gamma} \Omega(\wp) = -\lambda(\wp) \leq 0$. By chain rule, we see that

$$\begin{aligned} \nabla_a^{\gamma} (\Omega(\wp))^{\gamma-c} &= (\gamma-c) \int_0^1 \frac{\nabla_a^{\gamma} \Omega(\wp) dh}{[h\Omega(\wp) + (1-h)\Omega^{\rho}(\wp)]^{c-\alpha+1}} \\ &= -(\gamma-c) \int_0^1 \frac{\lambda(\wp) dh}{[h\Omega(\wp) + (1-h)\Omega^{\rho}(\wp)]^{c-\gamma+1}} \\ &\geq -(\gamma-c) \int_0^1 \frac{\lambda(\wp) dh}{[h\Omega^{\rho}(\wp) + (1-h)\Omega^{\rho}(\wp)]^{c-\gamma+1}} \\ &= -(\gamma-c) \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}. \end{aligned}$$

This implies that

$$\frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\gamma+1}} \geq \frac{-1}{\gamma-c} \nabla_a^\gamma (\Omega(\varphi))^{\gamma-c}, \quad (2.5)$$

and then, we have that

$$\begin{aligned} -\eta^p(\varphi) &= \int_{\rho(\varphi)}^\infty \frac{\lambda(\varrho)}{[\Omega^p(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho \geq \frac{-1}{\gamma-c} \int_{\rho(\varphi)}^\infty \nabla_a^\gamma (\Omega(\varrho))^{\gamma-c} \nabla_a^\gamma \varrho \\ &= \frac{1}{(\gamma-c)(\Omega^p(\varphi))^{c-\gamma}}. \end{aligned} \quad (2.6)$$

Using (2.4) and (2.6) in (2.3) yields

$$\left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\gamma+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^p \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\Omega^p(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p. \quad (2.7)$$

Applying Hölder inequality on the term

$$\left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\Omega^p(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p,$$

with indices $1/p$ and $1/1-p$ and

$$\begin{aligned} F(\varphi) &= \frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\Omega^p(\varphi))^{p(c-\gamma)}} \quad \text{and} \quad \mathfrak{I}(\varphi) = \left(\frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\gamma+1}} \right)^{1-p} [\Psi(\varphi)]^{(1-p)(p-\gamma+1)} \\ \left(\int_r^\infty F^{1/p}(\varphi) \nabla_a^\gamma \varphi \right)^p &= \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\Omega^p(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p \\ &\geq \frac{\int_r^\infty F(\varphi) \mathfrak{I}(\varphi) \nabla_a^\gamma \varphi}{\left(\int_r^\infty \mathfrak{I}^{1-p}(\varphi) \nabla_a^\gamma \varphi \right)^{p-1}} = \left[\int_r^\infty \frac{\lambda^p(\varphi) (\lambda(\varphi) [\Omega^p(\varphi)]^{\gamma-c-1})^{1-p} \mathfrak{I}^p(\varphi) [\Psi(\varphi)]^{(1-p)(p-\gamma+1)}}{(\Psi(\varphi))^{p(\alpha-p)} (\Omega^p(\varphi))^{p(c-\alpha)}} \nabla_a^\gamma \varphi \right] \\ &\quad \times \left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\alpha+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1} \\ &= \left(\int_a^\infty \frac{\lambda(\varphi) \mathfrak{I}^p(\varphi) [\Psi(\varphi)]^{1-\gamma}}{(\Omega^p(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi \right) \left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\gamma+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1}. \end{aligned} \quad (2.8)$$

From (2.8) and (2.7) yields

$$\begin{aligned} &\left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\gamma+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^p \\ &\geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \left(\int_r^\infty \frac{\lambda(\varphi) \mathfrak{I}^p(\varphi) [\Psi(\varphi)]^{1-\gamma}}{(\Omega^p(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi \right) \left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\gamma+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1}. \end{aligned} \quad (2.9)$$

Therefore,

$$\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^p(\varphi)]^{c-\gamma+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \int_r^\infty \frac{\lambda(\varphi) \mathfrak{I}^p(\varphi) [\Psi(\varphi)]^{1-\gamma}}{(\Omega^p(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi.$$

□

Remark 2.1. In Theorem 2, if we take $\gamma = 1$ then inequality (2.1) reduces to

$$\int_r^\infty \frac{\lambda(\wp)\Psi^p(\wp)}{(\Omega^p(\wp))^c} \nabla \wp \geq \left(\frac{p}{1-c}\right)^p \int_r^\infty \frac{\lambda(\wp)\mathfrak{I}^p(\wp)}{(\Omega^p(\wp))^c} \nabla \wp,$$

where

$$\Psi(\wp) = \int_r^\wp \lambda(\varrho)\mathfrak{I}(\varrho)\nabla \varrho \quad \text{and} \quad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)\nabla \varrho,$$

which is Theorem 3.1 in [16].

Corollary 2.1. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.1) obtains

$$\int_r^\infty \frac{\lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{(\Omega(\wp))^{c-\gamma+1}} (\wp - a)^{\gamma-1} d\wp \geq \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \int_r^\infty \frac{\lambda(\wp)\mathfrak{I}^p(\wp)\Psi^{1-\gamma}(\wp)}{(\Omega(\wp))^{c-\gamma+1}} (\wp - a)^{\gamma-1} dt,$$

where

$$\Psi(\wp) = \int_r^\wp \lambda(\varrho)\mathfrak{I}(\varrho)(\wp - a)^{\gamma-1} d\varrho \quad \text{and} \quad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)(\wp - a)^{\gamma-1} ds.$$

Corollary 2.2. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.1) get

$$\sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\Psi^{p-\gamma+1}(h\wp)}{\Omega^{c-\gamma+1}(h\wp-h)} (\rho^{\gamma-1}(h\wp) - a)_h^{(\gamma-1)} \geq \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\mathfrak{I}^p(h\wp)\Psi^{1-\gamma}(\wp)}{\Omega^{c-\gamma+1}(h\wp-h)} (\rho^{\gamma-1}(h\wp) - a)_h^{(\gamma-1)},$$

where

$$\Psi(\wp) = h \sum_{\varrho=\frac{\wp}{h}}^\wp \lambda(h\varrho)\mathfrak{I}(h\varrho)(\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)} \quad \text{and} \quad \Omega(\wp) = h \sum_{\varrho=\frac{\wp}{h}}^\infty \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)}.$$

Corollary 2.3. From Corollary 2, assume $\mathbb{T} = \mathbb{Z}$ and $h = 1$, then (2.1) obtains

$$\sum_{\wp=r}^\infty \frac{\lambda(\wp)\Psi^{p-\gamma+1}(h\wp)}{\Omega^{c-\gamma+1}(\wp-1)} (\rho^{\gamma-1}(\wp) - a)^{(\gamma-1)} \geq \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp=r}^\infty \frac{\lambda(\wp)\mathfrak{I}^p(\wp)\Psi^{1-\gamma}(\wp)}{\Omega^{c-\gamma+1}(\wp-1)} (\rho^{\gamma-1}(\wp) - a)^{(\gamma-1)},$$

where

$$\Psi(\wp) = \sum_{\varrho=r}^\wp \lambda(\varrho)\mathfrak{I}(\varrho)(\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)} \quad \text{and} \quad \Omega(\wp) = h \sum_{\varrho=\wp}^\infty \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)}.$$

Corollary 2.4. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.10) obtains

$$\sum_{\wp \in (r, \infty)} \frac{\wp(\rho^{\gamma-1}(\wp) - a)_{\tilde{q}}^{(\gamma-1)} \lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{\Omega^{c-\gamma+1}(\wp)} \geq \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp \in (r, \infty)} \frac{\wp(\rho^{\gamma-1}(\wp) - a)_{\tilde{q}}^{(\gamma-1)} \lambda(\wp)\mathfrak{I}^p(\wp)\Psi^{1-\gamma}(\wp)}{\Omega^{c-\gamma+1}(\wp)},$$

where

$$\Psi(\wp) = (\tilde{q}-1) \sum_{\varrho \in (r, \wp)} \varrho \lambda(\varrho)\mathfrak{I}(\varrho)(\rho^{\gamma-1}(\varrho) - a)_{\tilde{q}}^{(\gamma-1)} \quad \text{and} \quad \Omega(\wp) = (\tilde{q}-1) \sum_{\varrho \in (\wp, \infty)} \varrho \lambda(\varrho)(\rho^{\gamma-1}(\varrho) - a)_{\tilde{q}}^{(\gamma-1)}.$$

Theorem 2.2. Let \mathbb{T} be a time scale with $r \in [0, \infty)_{\mathbb{T}}$, $\gamma \in (0, 1]$ and $\wp \geq a$. In addition, let \mathfrak{J} and λ be nonnegative ld-continuous and (γ, a) -nabla fractional differentiable functions on $[r, \infty)_{\mathbb{T}}$ where

$$\Omega(\wp) = \int_{\wp}^{\infty} \lambda(\varrho) \nabla_a^{\gamma} \varrho \quad \text{and} \quad \bar{\Psi}(\wp) = \int_{\wp}^{\infty} \lambda(\varrho) \mathfrak{J}(\varrho) \nabla_a^{\gamma} \varrho, \quad \wp \in [r, \infty)_{\mathbb{T}}.$$

If $0 < p < \gamma$ and $c \geq \gamma$, then

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\bar{\Psi}^{\rho}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma} \wp \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \int_r^{\infty} \frac{\lambda(\wp) \mathfrak{J}^p(\wp) [\bar{\Psi}^{\rho}(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}} \nabla_a^{\gamma} \wp. \quad (2.10)$$

Proof. Using (1.13), with

$$\nabla_a^{\gamma} \eta(\wp) = \lambda(\wp) / [\Omega^{\rho}(\wp)]^{c-\gamma+1}, \quad \xi(\wp) = [\bar{\Psi}(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\bar{\Psi}^{\rho}(\wp)]^{p-\alpha+1} \nabla_a^{\gamma} \wp = [\eta(\wp) \bar{\Psi}^{p-\alpha+1}(\wp)]_r^{\infty} + \int_r^{\infty} (\eta(\wp)) \nabla_a^{\gamma} (-\bar{\Psi}^{p-\gamma+1}(\wp)) \nabla_a^{\gamma} \wp,$$

where we assumed that

$$\eta(\wp) = \int_r^{\wp} \lambda(\varrho) / [\Omega^{\rho}(\varrho)]^{c-\gamma+1} \nabla_a^{\gamma} \varrho.$$

Using $\bar{\Psi}(\infty) = 0$ and $\eta(r) = 0$, we get

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\bar{\Psi}^{\rho}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma} \wp = \int_r^{\infty} \eta(\wp) \nabla_a^{\gamma} (-\bar{\Psi}^{p-\gamma+1}(\wp)) \nabla_a^{\gamma} \wp. \quad (2.11)$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$\nabla_a^{\gamma} (-\bar{\Psi}^{p-\gamma+1}(\wp)) = -\frac{p-\gamma+1}{\Psi^{\gamma-p}(d)} \nabla_a^{\gamma} \Psi(\wp) \geq \frac{(p-\gamma+1)\lambda(\wp)\mathfrak{J}(\wp)}{\bar{\Psi}^{\gamma-p}(\wp)}. \quad (2.12)$$

Next note $\nabla_a^{\gamma} \Omega(\wp) = -\lambda(\wp) \leq 0$. By chain rule, we see that

$$\begin{aligned} \nabla_a^{\gamma} (\Omega(\wp))^{\gamma-c} &= (\gamma-c) \int_0^1 \frac{\nabla_a^{\gamma} \Omega(\wp) dh}{[h\Omega(\wp) + (1-h)\Omega^{\rho}(\wp)]^{c-\alpha+1}} \\ &= (c-\gamma) \int_0^1 \frac{\lambda(\wp) dh}{[h\Omega(\wp) + (1-h)\Omega^{\rho}(\wp)]^{c-\gamma+1}} \\ &\geq (c-\gamma) \int_0^1 \frac{\lambda(\wp) dh}{[h\Omega^{\rho}(\wp) + (1-h)\Omega^{\rho}(\wp)]^{c-\gamma+1}} \\ &= (c-\gamma) \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}. \end{aligned}$$

This implies that

$$\frac{-\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} \geq \frac{-1}{c-\gamma} \nabla_a^{\gamma} (\Omega(\wp))^{\gamma-c}, \quad (2.13)$$

and thus, we get

$$\begin{aligned}\eta(\varphi) &= \int_r^\varphi \frac{\lambda(\varrho)}{[\Omega^\rho(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho = \int_r^\infty \frac{\lambda(\varrho)}{[\Omega^\rho(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho - \int_\varphi^\infty \frac{\lambda(\varrho)}{[\Omega^\rho(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho \geq - \int_\varphi^\infty \frac{\lambda(\varrho)}{[\Omega^\rho(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho \\ &= \frac{-1}{c-\gamma} \int_\varphi^\infty \nabla_a^\gamma (\Omega(\varrho))^{\gamma-c} \nabla_a^\gamma \varrho = \frac{1}{(c-\gamma)(\Omega^\rho(\varphi))^{\gamma-c}}.\end{aligned}\quad (2.14)$$

Substituting (2.13), (2.14) into (2.11) obtains

$$\left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^\rho(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^p \geq \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\Omega^\rho(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p.\quad (2.15)$$

Applying Hölder inequality on the term

$$\left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\Omega^\rho(\varphi))^{p(\gamma-c)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p,$$

with indices $1/p$ and $1/1-p$ and

$$\begin{aligned}F(\varphi) &= \frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\Omega^\rho(\varphi))^{p(c-\gamma)}} \quad \text{and} \quad \mathfrak{I}(\varphi) = \left(\frac{\lambda(\varphi)}{[\Omega^\rho(\varphi)]^{c-\gamma+1}} \right)^{1-p} [\overline{\Psi}^\rho(\varphi)]^{(1-p)(p-\gamma+1)} \\ \left(\int_r^\infty F^{1/p}(\varphi) \nabla_a^\gamma \varphi \right)^p &= \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\Omega^\rho(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p \\ &\geq \frac{\int_r^\infty F(\varphi) \mathfrak{I}(\varphi) \nabla_a^\gamma \varphi}{\left(\int_r^\infty \mathfrak{I}^{\frac{1}{1-p}}(\varphi) \nabla_a^\gamma \varphi \right)^{p-1}} = \left[\int_r^\infty \frac{\lambda^p(\varphi) (\lambda(\varphi) [\Omega^\rho(\varphi)]^{\gamma-c-1})^{1-p} \mathfrak{I}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{(1-p)(p-\gamma+1)}}{(\overline{\Psi}^\rho(\varphi))^{p(\alpha-p)} (\Omega^\rho(\varphi))^{p(c-\alpha)}} \nabla_a^\gamma \varphi \right] \\ &\quad \times \left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^\rho(\varphi)]^{c-\alpha+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1} \\ &= \left(\int_r^\infty \frac{\lambda(\varphi) \mathfrak{I}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{1-\gamma}}{(\Omega^\rho(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi \right) \left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^\rho(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1}.\end{aligned}\quad (2.16)$$

Substituting (2.16) into (2.15) yields

$$\begin{aligned}\left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^\rho(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^p \\ \geq \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^\infty \frac{\lambda(\varphi) \mathfrak{I}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{1-\gamma}}{(\Omega^\rho(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi \right) \left(\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^\rho(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1}.\end{aligned}\quad (2.17)$$

Therefore,

$$\int_r^\infty \frac{\lambda(\varphi)}{[\Omega^\rho(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \geq \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \int_r^\infty \frac{\lambda(\varphi) \mathfrak{I}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{1-\gamma}}{(\Omega^\rho(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi.$$

□

Remark 2.2. In Theorem 2, if we take $\gamma = 1$ then inequality (2.10) reduces to

$$\int_r^\infty \frac{\lambda(\wp)\bar{\Psi}^p(\wp)}{(\Omega^p(\wp))^c} \nabla\wp \geq \left(\frac{p}{c-1}\right)^p \int_r^\infty \frac{\lambda(\wp)\mathfrak{I}^p(\wp)}{(\Omega^p(\wp))^{c-p}} \nabla\wp,$$

where

$$\bar{\Psi}(\wp) = \int_\wp^\infty \lambda(\varrho)\mathfrak{I}(\varrho)\nabla\varrho \quad \text{and} \quad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)\nabla\varrho,$$

which is Theorem 3.4 in [16].

Corollary 2.5. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.10) gets

$$\int_r^\infty \frac{\lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{(\Omega(\wp))^{c-\gamma+1}} (\wp-a)^{\gamma-1} d\wp \geq \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \int_r^\infty \frac{\lambda(\wp)\mathfrak{I}^p(\wp)\Psi^{1-\gamma}(\wp)}{(\Omega(\wp))^{c-p-\gamma+1}} (\wp-a)^{\gamma-1} dt,$$

where

$$\bar{\Psi}(\wp) = \int_\wp^\infty \lambda(\varrho)\mathfrak{I}(\varrho)(\wp-a)^{\gamma-1} d\varrho \quad \text{and} \quad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)(\wp-a)^{\gamma-1} ds.$$

Corollary 2.6. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.10) gets

$$\sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\bar{\Psi}^{p-\gamma+1}(h\wp)}{\Omega^{c-\gamma+1}(h\wp-h)} (\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)} \geq \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\mathfrak{I}^p(h\wp)\bar{\Psi}^{1-\gamma}(\wp)}{\Omega^{c-p-\gamma+1}(h\wp-h)} (\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)},$$

where

$$\bar{\Psi}(\wp) = h \sum_{\varrho=\frac{\wp}{h}}^\infty \lambda(h\varrho)\mathfrak{I}(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)} \quad \text{and} \quad \Omega(\wp) = h \sum_{\varrho=\frac{\wp}{h}}^\infty \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)}.$$

Corollary 2.7. From Corollary 2, assume $\mathbb{T} = \mathbb{Z}$, and $h = 1$, then (2.10) obtains

$$\sum_{\wp=r}^\infty \frac{\lambda(\wp)\bar{\Psi}^{p-\gamma+1}(h\wp)}{\Omega^{c-p-\gamma+1}(\wp-1)} (\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)} \geq \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \sum_{\wp=r}^\infty \frac{\lambda(\wp)\mathfrak{I}^p(\wp)\bar{\Psi}^{1-\gamma}(\wp)}{\Omega^{c-p-\gamma+1}(\wp-1)} (\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)},$$

where

$$\bar{\Psi}(\wp) = \sum_{\varrho=\wp}^\infty \lambda(\varrho)\mathfrak{I}(\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)} \quad \text{and} \quad \Omega(\wp) = h \sum_{\varrho=\wp}^\infty \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)}.$$

Corollary 2.8. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.10) obtains

$$\sum_{\wp \in (r, \infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)} \lambda(\wp)\bar{\Psi}^{p-\gamma+1}(\wp)}{\Omega^{c-\gamma+1}(\rho(\wp))} \geq \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \sum_{\wp \in (r, \infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)} \lambda(\wp)\mathfrak{I}^p(\wp)\bar{\Psi}^{1-\gamma}(\wp)}{\Omega^{c-p-\gamma+1}(\rho(\wp))},$$

where

$$\bar{\Psi}(\wp) = (\tilde{q}-1) \sum_{\varrho \in (\wp, \infty)} \varrho \lambda(\varrho)\mathfrak{I}(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)} \quad \text{and} \quad \Omega(\wp) = (\tilde{q}-1) \sum_{\varrho \in (\wp, \infty)} \varrho \lambda(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)}.$$

Theorem 2.3. Let \mathbb{T} be a time scale with $r \in [0, \infty)_{\mathbb{T}}$, $\gamma \in (0, 1]$ and $\wp \geq a$. In addition, let \mathfrak{J} and λ be nonnegative ld-continuous and (γ, a) -nabla fractional differentiable functions on $[r, \infty)_{\mathbb{T}}$ where

$$\bar{\Omega}(\wp) = \int_r^{\wp} \lambda(\varrho) \nabla_a^\gamma \varrho \quad \text{and} \quad \bar{\Psi}(\wp) = \int_{\wp}^{\infty} \lambda(\varrho) \mathfrak{J}(\varrho) \nabla_a^\gamma \varrho, \quad \wp \in [r, \infty)_{\mathbb{T}}.$$

If $0 < p < \gamma$ and $c \leq \gamma - 1$, then

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}} [\bar{\Psi}^p(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \int_r^{\infty} \frac{\lambda(\wp) \mathfrak{J}^p(\wp) [\bar{\Psi}^p(\wp)]^{1-\gamma}}{(\bar{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp. \quad (2.18)$$

Proof. Using (1.13), with

$$\nabla_a^\gamma \eta(\wp) = \lambda(\wp) / [\bar{\Omega}(\wp)]^{c-\gamma+1}, \quad \xi(\wp) = [\bar{\Psi}(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}} [\bar{\Psi}^p(\wp)]^{p-\alpha+1} \nabla_a^\gamma \wp = [\eta(\wp) \bar{\Psi}^{p-\alpha+1}(\wp)]_r^{\infty} + \int_r^{\infty} (-\eta(\wp)) \nabla_a^\gamma (\bar{\Psi}^{p-\gamma+1}(\wp)) \nabla_a^\gamma \wp,$$

where we assumed that

$$\eta(\wp) = \int_r^{\wp} \lambda(\varrho) / [\bar{\Omega}(\varrho)]^{c-\gamma+1} \nabla_a^\gamma \varrho.$$

Using $\bar{\Psi}(\infty) = 0$ and $\eta(r) = 0$, we have that

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}} [\bar{\Psi}^p(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp = \int_r^{\infty} -\eta(\wp) \nabla_a^\gamma (\bar{\Psi}^{p-\gamma+1}(\wp)) \nabla_a^\gamma \wp. \quad (2.19)$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$-\nabla_a^\gamma (\bar{\Psi}^{p-\gamma+1}(\wp)) = -\frac{p-\gamma+1}{\bar{\Psi}^{\gamma-p}(d)} \nabla_a^\gamma \bar{\Psi}(\wp) \geq \frac{(p-\gamma+1)\lambda(\wp)\mathfrak{J}(\wp)}{\bar{\Psi}^{\gamma-p}(\wp)}. \quad (2.20)$$

Next note $\nabla_a^\gamma \bar{\Omega}(\wp) = \lambda(\wp) \leq 0$. By chain rule, we see that

$$\begin{aligned} \nabla_a^\gamma (\bar{\Omega}(\wp))^{\gamma-c} &= (\gamma-c) \int_0^1 \frac{\nabla_a^\gamma \bar{\Omega}(\wp) dh}{[h\bar{\Omega}(\wp) + (1-h)\bar{\Omega}^p(\wp)]^{c-\gamma+1}} \\ &= (\gamma-c) \int_0^1 \frac{\lambda(\wp) dh}{[h\bar{\Omega}(\wp) + (1-h)\bar{\Omega}^p(\wp)]^{c-\gamma+1}} \\ &\leq (\gamma-c) \int_0^1 \frac{\lambda(\wp) dh}{[h\bar{\Omega}(\wp) + (1-h)\bar{\Omega}(\wp)]^{c-\gamma+1}} \\ &= (\gamma-c) \frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}}. \end{aligned}$$

This implies that

$$\frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}} \geq \frac{-1}{\gamma-c} \nabla_a^\gamma (\bar{\Omega}(\wp))^{\gamma-c}, \quad (2.21)$$

and then, we have that

$$\begin{aligned}\eta(\varphi) &= \int_r^\varphi \frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho = \int_r^\infty \frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho - \int_\varphi^\infty \frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho \geq - \int_\varphi^\infty \frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho \\ &= \frac{1}{\gamma - c} \int_\varphi^\infty \nabla_a^\gamma (\overline{\Omega}(\varrho))^{\gamma-c} \nabla_a^\gamma \varrho = \frac{1}{(\gamma - c)(\overline{\Omega}(\varphi))^{c-\gamma}}.\end{aligned}\quad (2.22)$$

Substituting (2.21), (2.22) into (2.19) yields

$$\left(\int_r^\infty \frac{\lambda(\varphi)}{[\overline{\Omega}(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^p \geq \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{J}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\overline{\Omega}(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p.\quad (2.23)$$

Applying Hölder inequality on the term

$$\left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{J}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\overline{\Omega}(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p,$$

with indices $1/p$ and $1/1-p$ and

$$\begin{aligned}F(\varphi) &= \frac{\lambda^p(\varphi) \mathfrak{J}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\overline{\Omega}(\varphi))^{p(c-\gamma)}} \quad \text{and} \quad \mathfrak{J}(\varphi) = \left(\frac{\lambda(\varphi)}{[\overline{\Omega}(\varphi)]^{c-\gamma+1}} \right)^{1-p} [\overline{\Psi}^\rho(\varphi)]^{(1-p)(p-\gamma+1)} \\ \left(\int_r^\infty F^{1/p}(\varphi) \nabla_a^\gamma \varphi \right)^p &= \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{J}^p(\varphi)}{(\overline{\Psi}^\rho(\varphi))^{p(\gamma-p)} (\overline{\Omega}(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p \\ &\geq \frac{\int_r^\infty F(\varphi) \mathfrak{J}(\varphi) \nabla_a^\gamma \varphi}{\left(\int_r^\infty \mathfrak{J}^{1-p}(\varphi) \nabla_a^\gamma \varphi \right)^{p-1}} = \left[\int_r^\infty \frac{\lambda^p(\varphi) (\lambda(\varphi) [\overline{\Omega}(\varphi)]^{\gamma-c-1})^{1-p} \mathfrak{J}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{(1-p)(p-\gamma+1)}}{(\overline{\Psi}^\rho(\varphi))^{p(\alpha-p)} (\overline{\Omega}(\varphi))^{p(c-\alpha)}} \nabla_a^\gamma \varphi \right] \\ &\quad \times \left(\int_r^\infty \frac{\lambda(\varphi)}{[\overline{\Omega}(\varphi)]^{c-\alpha+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1} \\ &= \left(\int_r^\infty \frac{\lambda(\varphi) \mathfrak{J}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{1-\gamma}}{(\overline{\Omega}(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi \right) \left(\int_r^\infty \frac{\lambda(\varphi)}{[\overline{\Omega}(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1}.\end{aligned}\quad (2.24)$$

From (2.24) and (2.23) gets

$$\begin{aligned}&\left(\int_r^\infty \frac{\lambda(\varphi)}{[\overline{\Omega}(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^p \\ &\geq \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^\infty \frac{\lambda(\varphi) \mathfrak{J}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{1-\gamma}}{(\overline{\Omega}(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi \right) \left(\int_r^\infty \frac{\lambda(\varphi)}{[\overline{\Omega}(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1}.\end{aligned}\quad (2.25)$$

Therefore,

$$\int_r^\infty \frac{\lambda(\varphi)}{[\overline{\Omega}(\varphi)]^{c-\gamma+1}} [\overline{\Psi}^\rho(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \int_r^\infty \frac{\lambda(\varphi) \mathfrak{J}^p(\varphi) [\overline{\Psi}^\rho(\varphi)]^{1-\gamma}}{(\overline{\Omega}(\varphi))^{c-p-\gamma+1}} \nabla_a^\gamma \varphi.$$

□

Remark 2.3. In Theorem 2, if we take $\gamma = 1$ then inequality (2.18) reduces to

$$\int_r^\infty \frac{\lambda(\varphi) [\overline{\Psi}^p(\varphi)]^p}{(\overline{\Omega}(\varphi))^c} \nabla \varphi \geq \left(\frac{p}{1-c} \right)^p \int_r^\infty \frac{\lambda(\varphi) \mathfrak{J}^p(\varphi)}{(\overline{\Omega}(\varphi))^{c-p}} \nabla \varphi,$$

where

$$\overline{\Psi}(\varphi) = \int_\varphi^\infty \lambda(\varrho) \mathfrak{J}(\varrho) \nabla \varrho \quad \text{and} \quad \overline{\Omega}(\varphi) = \int_r^\varphi \lambda(\varrho) \nabla \varrho,$$

which is Theorem 3.9 in [16].

Corollary 2.9. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.18) gets

$$\int_r^\infty \frac{\lambda(\varphi) \overline{\Psi}^{p-\gamma+1}(\varphi)}{(\overline{\Omega}(\varphi))^{c-\gamma+1}} (\varphi - a)^{\gamma-1} d\varphi \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \int_r^\infty \frac{\lambda(\varphi) \mathfrak{J}^p(\varphi) \overline{\Psi}^{1-\gamma}(\varphi)}{(\overline{\Omega}(\varphi))^{c-p-\gamma+1}} (\varphi - a)^{\gamma-1} dt,$$

where

$$\overline{\Psi}(\varphi) = \int_\varphi^\infty \lambda(\varrho) \mathfrak{J}(\varrho) (\varphi - a)^{\gamma-1} d\varrho \quad \text{and} \quad \overline{\Omega}(\varphi) = \int_r^\varphi \lambda(\varrho) (\varphi - a)^{\gamma-1} ds.$$

Remark 2.4. In Corollary 2, if we take $\gamma = 1$ yields discrete Bennett-Leindler type inequality (1.8).

Corollary 2.10. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.18) gets

$$\sum_{\varphi=\frac{r}{h}}^\infty \frac{\lambda(h\varphi) \overline{\Psi}^{p-\gamma+1}(h\varphi - h)}{\overline{\Omega}^{c-\gamma+1}(h\varphi)} (\rho^{\gamma-1}(h\varphi) - a)_h^{(\gamma-1)} \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \sum_{\varphi=\frac{r}{h}}^\infty \frac{\lambda(h\varphi) \mathfrak{J}^p(h\varphi) \overline{\Psi}^{1-\gamma}(\varphi)}{\overline{\Omega}^{c-p-\gamma+1}(h\varphi)} (\rho^{\gamma-1}(h\varphi) - a)_h^{(\gamma-1)},$$

where

$$\overline{\Psi}(\varphi) = h \sum_{\varrho=\frac{\varphi}{h}}^\infty \lambda(h\varrho) \mathfrak{J}(h\varrho) (\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)} \quad \text{and} \quad \overline{\Omega}(\varphi) = h \sum_{\varrho=\frac{r}{h}}^\varphi \lambda(h\varrho) (\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)}.$$

Corollary 2.11. For $\mathbb{T} = \mathbb{Z}$, we take $h = 1$ in Corollary 2. In this case, inequality (2.18) reduces to

$$\sum_{\varphi=r}^\infty \frac{\lambda(\varphi) \overline{\Psi}^{p-\gamma+1}(\varphi - 1)}{\overline{\Omega}^{c-\gamma+1}(\varphi)} (\rho^{\gamma-1}(\varphi) - a)^{(\gamma-1)} \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \sum_{\varphi=r}^\infty \frac{\lambda(\varphi) \mathfrak{J}^p(\varphi) \overline{\Psi}^{1-\gamma}(\varphi)}{\overline{\Omega}^{c-p-\gamma+1}(\varphi)} (\rho^{\gamma-1}(\varphi) - a)^{(\gamma-1)},$$

where

$$\overline{\Psi}(\varphi) = \sum_{\varrho=r}^\infty \lambda(\varrho) \mathfrak{J}(\varrho) (\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)} \quad \text{and} \quad \overline{\Omega}(\varphi) = h \sum_{\varrho=r}^\varphi \lambda(h\varrho) (\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)}.$$

Remark 2.5. In Corollary 2, if we take $\gamma = 1$ and $r = 1$, yields discrete Bennett-Leindler type inequality (1.6), which is the converse of Copson inequality (1.4).

Corollary 2.12. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.18) gets

$$\sum_{\wp \in (r, \infty)} \frac{\wp(\rho^{\gamma-1}(\wp) - a)^{\frac{(\gamma-1)}{\tilde{q}}} \lambda(\wp) \bar{\Psi}^{p-\gamma+1}(\rho(\wp))}{\bar{\Omega}^{c-\gamma+1}(\wp)} \geq \left(\frac{p-\gamma+1}{\gamma-c} \right)^p \sum_{\wp \in (r, \infty)} \frac{\wp(\rho^{\gamma-1}(\wp) - a)^{\frac{(\gamma-1)}{\tilde{q}}} \lambda(\wp) \mathfrak{J}^p(\wp) \bar{\Psi}^{1-\gamma}(\wp)}{\bar{\Omega}^{c-p-\gamma+1}(\wp)},$$

where

$$\bar{\Psi}(\wp) = (\tilde{q}-1) \sum_{\varrho \in (\wp, \infty)} \varrho \lambda(\varrho) \mathfrak{J}(\varrho) (\rho^{\gamma-1}(\varrho) - a)^{\frac{(\gamma-1)}{\tilde{q}}} \quad \text{and} \quad \bar{\Omega}(\wp) = (\tilde{q}-1) \sum_{\varrho \in (r, \wp)} \varrho \lambda(\varrho) (\rho^{\gamma-1}(\varrho) - a)^{\frac{(\gamma-1)}{\tilde{q}}}.$$

Theorem 2.4. Let \mathbb{T} be a time scale with $r \in [0, \infty)_{\mathbb{T}}$, $\gamma \in (0, 1]$ and $\wp \geq a$. In addition, let \mathfrak{J} and λ be nonnegative Id-continuous and (γ, a) -nabla fractional differentiable functions on $[r, \infty)_{\mathbb{T}}$ where

$$\bar{\Omega}(\wp) = \int_r^{\wp} \lambda(\varrho) \nabla_a^\gamma \varrho, \quad \bar{\Omega}(\infty) = \infty, \quad \Psi(\wp) = \int_r^{\wp} \lambda(\varrho) \mathfrak{J}(\varrho) \nabla_a^\gamma \varrho, \quad \wp \in [r, \infty)_{\mathbb{T}}.$$

If $L = \inf_{\wp \in \mathbb{T}} \frac{\bar{\Omega}^p(\wp)}{\bar{\Omega}(\wp)} > 0$, $0 < p < \gamma$ and $c \geq \gamma$, then

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \geq \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \int_r^{\infty} \frac{\lambda(\wp) \mathfrak{J}^p(\wp) [\Psi(\wp)]^{1-\gamma}}{(\bar{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp. \quad (2.26)$$

Proof. Using (1.13), with

$$\nabla_a^\gamma \eta(\wp) = \lambda(\wp) / [\bar{\Omega}(\wp)]^{c-\gamma+1}, \quad \xi(\wp) = [\Psi(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\alpha+1} \nabla_a^\gamma \wp = [\eta(\wp) \Psi^{p-\alpha+1}(\wp)]_r^{\infty} + \int_r^{\infty} (-\eta^\rho(\wp)) \nabla_a^\gamma (\Psi^{p-\gamma+1}(\wp)) \nabla_a^\gamma \wp,$$

where we assumed that

$$\eta(\wp) = - \int_{\wp}^{\infty} \lambda(\varrho) / [\bar{\Omega}(\varrho)]^{c-\gamma+1} \nabla_a^\gamma \varrho.$$

Using $\Psi(r) = 0$ and $\eta(\infty) = 0$, we have that

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\bar{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp = \int_r^{\infty} -\eta^\rho(\wp) \nabla_a^\gamma (\Psi^{p-\gamma+1}(\wp)) \nabla_a^\gamma \wp. \quad (2.27)$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$\nabla_a^\gamma (\Psi^{p-\gamma+1}(\wp)) = \frac{p-\gamma+1}{\Psi^{\gamma-p}(d)} \nabla_a^\gamma \Psi(\wp) \geq \frac{(p-\gamma+1)\lambda(\wp)\mathfrak{J}(\wp)}{\Psi^{\gamma-p}(\wp)}. \quad (2.28)$$

Next note $\nabla_a^\gamma \bar{\Omega}(\wp) = \lambda(\wp) \leq 0$. By chain rule, we see that

$$\nabla_a^\gamma (\bar{\Omega}(\wp))^{\gamma-c} = (\gamma-c) \int_0^1 \frac{\nabla_a^\gamma \bar{\Omega}(\wp) dh}{[h\bar{\Omega}(\wp) + (1-h)\bar{\Omega}^\rho(\wp)]^{c-\gamma+1}}$$

$$\begin{aligned}
&= -(c - \gamma) \int_0^1 \frac{\lambda(\varphi) dh}{[h\bar{\Omega}(\varphi) + (1 - h)\bar{\Omega}^p(\varphi)]^{c-\gamma+1}} \\
&\geq -(c - \gamma) \int_0^1 \frac{\lambda(\varphi) dh}{[h\bar{\Omega}^p(\varphi) + (1 - h)\bar{\Omega}(\varphi)]^{c-\gamma+1}} \\
&= -(c - \gamma) \frac{\lambda(\varphi)}{[\bar{\Omega}^p(\varphi)]^{c-\gamma+1}} = -(\gamma - c) \frac{\lambda(\varphi)}{[\bar{\Omega}^p(\varphi)]^{c-\gamma+1}} \frac{[\bar{\Omega}(\varphi)]^{c-\gamma+1}}{[\bar{\Omega}(\varphi)]^{c-\gamma+1}} \\
&\geq -(c - \gamma) \frac{\lambda(\varphi)}{L^{c-\gamma+1} [\bar{\Omega}(\varphi)]^{c-\gamma+1}}.
\end{aligned}$$

This implies that

$$\frac{\lambda(\varphi)}{[\bar{\Omega}(\varphi)]^{c-\gamma+1}} \geq \frac{-L^{c-\gamma+1}}{c - \gamma} \nabla_a^\gamma (\bar{\Omega}(\varphi))^{\gamma-c} \quad (2.29)$$

and then, we have that

$$\begin{aligned}
-\eta^p(\varphi) &= \int_{\rho(\varphi)}^\infty \frac{\lambda(\varrho)}{[\bar{\Omega}(\varrho)]^{c-\gamma+1}} \nabla_a^\gamma \varrho \geq - \int_{\rho(\varphi)}^\infty \frac{L^{c-\gamma+1}}{c - \gamma} \nabla_a^\gamma (\bar{\Omega}(\varphi))^{\gamma-c} \nabla_a^\gamma \varrho = \frac{L^{c-\gamma+1}}{c - \gamma} \left\{ (\bar{\Omega}^p(\varphi))^{\gamma-c} - (\bar{\Omega}(\infty))^{\gamma-c} \right\} \\
&= \frac{L^{c-\gamma+1}}{c - \gamma} (\bar{\Omega}^p(\varphi))^{\gamma-c} \geq \frac{L^{c-\gamma+1}}{c - \gamma} (\bar{\Omega}(\varphi))^{\gamma-c}.
\end{aligned} \quad (2.30)$$

Substituting (2.30), (2.28) into (2.27) yields

$$\left(\int_r^\infty \frac{\lambda(\varphi)}{[\bar{\Omega}(\varphi)]^{c-\gamma+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^p \geq \left(\frac{p - \gamma + 1}{c - \gamma} L^{c-\gamma+1} \right)^p \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\bar{\Omega}(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p. \quad (2.31)$$

Applying Hölder inequality on the term

$$\left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\bar{\Omega}(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p,$$

with indices $1/p$ and $1/1 - p$ and

$$F(\varphi) = \frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\bar{\Omega}(\varphi))^{p(c-\gamma)}} \quad \text{and} \quad \mathfrak{I}(\varphi) = \left(\frac{\lambda(\varphi)}{[\bar{\Omega}(\varphi)]^{c-\gamma+1}} \right)^{1-p} [\Psi(\varphi)]^{(1-p)(p-\gamma+1)}$$

$$\begin{aligned}
\left(\int_r^\infty F^{1/p}(\varphi) \nabla_a^\gamma \varphi \right)^p &= \left(\int_r^\infty \left[\frac{\lambda^p(\varphi) \mathfrak{I}^p(\varphi)}{(\Psi(\varphi))^{p(\gamma-p)} (\bar{\Omega}(\varphi))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^\gamma \varphi \right)^p \\
&\geq \frac{\int_r^\infty F(\varphi) \mathfrak{I}(\varphi) \nabla_a^\gamma \varphi}{\left(\int_r^\infty \mathfrak{I}^{\frac{1}{1-p}}(\varphi) \nabla_a^\gamma \varphi \right)^{p-1}} = \left[\int_r^\infty \frac{\lambda^p(\varphi) (\lambda(\varphi) [\bar{\Omega}(\varphi)]^{\gamma-c-1})^{1-p} \mathfrak{I}^p(\varphi) [\Psi(\varphi)]^{(1-p)(p-\gamma+1)}}{(\Psi(\varphi))^{p(\alpha-p)} (\bar{\Omega}(\varphi))^{p(c-\alpha)}} \nabla_a^\gamma \varphi \right] \\
&\quad \times \left(\int_r^\infty \frac{\lambda(\varphi)}{[\bar{\Omega}(\varphi)]^{c-\alpha+1}} [\Psi(\varphi)]^{p-\gamma+1} \nabla_a^\gamma \varphi \right)^{p-1}
\end{aligned}$$

$$= \left(\int_r^\infty \frac{\lambda(\wp) \mathfrak{J}^p(\wp) [\overline{\Psi}^p(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp \right) \left(\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \right)^{p-1}. \quad (2.32)$$

From (2.32) and (2.31) gets

$$\begin{aligned} & \left(\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \right)^p \\ & \geq \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^\infty \frac{\lambda(\wp) \mathfrak{J}^p(\wp) [\overline{\Psi}^p(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp \right) \left(\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \right)^{p-1}. \end{aligned} \quad (2.33)$$

Therefore,

$$\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \geq \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \int_r^\infty \frac{\lambda(\wp) \mathfrak{J}^p(\wp) [\Psi(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp.$$

□

Remark 2.6. In Theorem 2, if we take $\gamma = 1$ then we get Theorem 1.

Corollary 2.13. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.26) gets

$$\int_r^\infty \frac{\lambda(\wp) \Psi^{p-\gamma+1}(\wp)}{(\overline{\Omega}(\wp))^{c-\gamma+1}} (\wp - a)^{\gamma-1} d\wp \geq \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \int_r^\infty \frac{\lambda(\wp) \mathfrak{J}^p(\wp) \Psi^{1-\gamma}(\wp)}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} (\wp - a)^{\gamma-1} dt,$$

where

$$\Psi(\wp) = \int_r^\wp \lambda(\varrho) \mathfrak{J}(\varrho) (\wp - a)^{\gamma-1} d\varrho \quad \text{and} \quad \overline{\Omega}(\wp) = \int_r^\wp \lambda(\varrho) (\wp - a)^{\gamma-1} ds.$$

Remark 2.7. In Corollary 2, if we take $L = \gamma = 1$ yields continuous variant of Bennett-Leindler type inequality (1.7).

Corollary 2.14. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.26) gets

$$\sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp) \Psi^{p-\gamma+1}(h\wp)}{\overline{\Omega}^{c-\gamma+1}(h\wp)} (\rho^{\gamma-1}(h\wp) - a)_h^{(\gamma-1)} \geq \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp) \mathfrak{J}^p(h\wp) \Psi^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(h\wp)} (\rho^{\gamma-1}(h\wp) - a)_h^{(\gamma-1)},$$

where

$$\Psi(\wp) = h \sum_{\varrho=\frac{r}{h}}^\wp \lambda(h\varrho) \mathfrak{J}(h\varrho) (\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)} \quad \text{and} \quad \overline{\Omega}(\wp) = h \sum_{\varrho=\frac{r}{h}}^\wp \lambda(h\varrho) (\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)}.$$

Corollary 2.15. For $\mathbb{T} = \mathbb{Z}$, we take $h = 1$ in Corollary 2. In this case, inequality (2.26) reduces to

$$\sum_{\wp=r}^\infty \frac{\lambda(\wp) \Psi^{p-\gamma+1}(h\wp)}{\overline{\Omega}^{c-p-\gamma+1}(\wp)} (\rho^{\gamma-1}(\wp) - a)^{(\gamma-1)} \geq \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \sum_{\wp=r}^\infty \frac{\lambda(\wp) \mathfrak{J}^p(\wp) \Psi^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(\wp)} (\rho^{\gamma-1}(\wp) - a)^{(\gamma-1)},$$

where

$$\Psi(\wp) = \sum_{\varrho=r}^{\wp} \lambda(\varrho) \mathfrak{J}(\varrho) (\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)} \quad \text{and} \quad \bar{\Omega}(\wp) = h \sum_{\varrho=r}^{\wp} \lambda(h\varrho) (\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)}.$$

Remark 2.8. In Corollary 2, if we take $\gamma = 1$ and $r = 1$, yields discrete Bennett-Leindler type inequality (1.5), which is the converse of Copson inequality (1.3).

Corollary 2.16. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.26) gets

$$\sum_{\wp \in (r, \infty)} \frac{\wp (\rho^{\gamma-1}(\wp) - a)_{\bar{q}}^{(\gamma-1)} \lambda(\wp) \Psi^{p-\gamma+1}(\wp)}{\bar{\Omega}^{-c-\gamma+1}(\rho(\wp))} \geq \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \sum_{\wp \in (r, \infty)} \frac{\wp (\rho^{\gamma-1}(\wp) - a)_{\bar{q}}^{(\gamma-1)} \lambda(\wp) \mathfrak{J}^p(\wp) \Psi^{1-\gamma}(\wp)}{\bar{\Omega}^{-c-p-\gamma+1}(\wp)},$$

where

$$\Psi(\wp) = (\bar{q} - 1) \sum_{\varrho \in (r, \wp)} \varrho \lambda(\varrho) \mathfrak{J}(\varrho) (\rho^{\gamma-1}(\varrho) - a)_{\bar{q}}^{(\gamma-1)} \quad \text{and} \quad \bar{\Omega}(\wp) = (\bar{q} - 1) \sum_{\varrho \in (r, \wp)} \varrho \lambda(\varrho) (\rho^{\gamma-1}(\varrho) - a)_{\bar{q}}^{(\gamma-1)}.$$

3. Conclusions

In this paper, with the help of a simple consequence of Keller's chain rule and Hölder inequality for the (γ, a) -nabla-fractional derivative on time scales, we generalized a number of Bennett and Leindler Hardy-type inequalities to a general time scale. Besides that, in order to obtain some new inequalities as special cases, we also extended our inequalities to discrete and continuous calculus. In order to illustrate the theorems for each type of inequality applied to various time scales such as \mathbb{R} , $h\mathbb{Z}$, $\bar{q}^{\mathbb{Z}}$ and \mathbb{Z} as a sub case of $h\mathbb{Z}$. For future studies researchers may obtain some different generalizations for dynamic Hardy inequality and its companion inequalities by using the results presented in this paper.

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Conflict of interest

The authors declare that there is no competing interest.

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