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Research article

Spatiotemporal patterns induced by cross-diffusion on vegetation model

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Abstract: This paper considers the influence of cross-diffusion on semi-arid ecosystems based on simplified Hardenberg's reaction diffusion model. In the square region, we analyze the properties of this model and give the relaxation time correspond to the system to prejudge the approximate time of this system stabilization process. The numerical results are constant with the theory very well.

Keywords: semi-arid ecosystems; Turing bifurcation; steady-state bifurcation; cross-diffusion; relaxation time
Mathematics Subject Classification: 34K18, 35B32

1. Introduction

Turing pattern is a kind of pattern formation caused by local instability of the system. In 1952, Turing [1] pointed out in his paper that diffusion will make steady-state unstable and eventually form a Turing pattern. Some people [2, 3] have discovered many similarities between the dynamical instability of the uniform and the equilibrium phase transition when the system is far away from the thermodynamic equilibrium. In recent decades, the phenomenon of desertification has become more and more serious. Based on this, the exploration of the stability of the ecosystem has become a hot spot [4–6].

Patterns of vegetation, well known for one of the most attractive and interesting characters of aerial images, are ubiquitous in semi-arid regions, in which water is regarded as the limiting resource for plant growth. In fact, the existence of vegetation is affected by many factors, such as overgrazing, deforestation and so on [7]. Some models of vegetation consider the local effect between vegetation and water. It is clear that water is the source of life which is limited in arid and semi-arid areas and has a great impact on resulting in the formation of vegetation patterns. Hence, the study of relationship between the water and vegetation can play a certain guiding role in the stability of vegetation structure.

For the formation of vegetation patterns in semi-arid areas, many various models have been established. R. Lefever [8] established a univariate model (dimensionless model) that only included vegetation biomass:

$$\frac{\partial u}{\partial t} = (1-\mu)u + (A-1)u^2 - u^3 + \frac{1}{2}\left(L^2 - u\right)\nabla^2 u - \frac{1}{8}u\nabla^4 u, \tag{1.1}$$

where u represents the number of vegetation or vegetation density, μ represents the mortality of vegetation, A describes the sensitivity of mutual inhibition and promotion between vegetation, and L represents the ratio of the spatial distance between vegetation mutual promotion and inhibition, model (1.1) reveals the relationship between the short-distance mutual promotion of individual vegetation and vegetation communities and the long-distance competition for resource mechanisms.

In 1999, Klausmeier [9] established a mathematical model containing two variables (vegetation and water) for the formation of regular vegetation patterns in semi-arid areas:

$$\begin{cases} \frac{\partial N}{\partial T} = RJWN^2 - MN + D_1 \Delta N, \\ \frac{\partial W}{\partial T} = A - LW - RWN^2 + \frac{\partial W}{\partial X}, \end{cases} \tag{1.2}$$

among them, N, W respectively represent the number or density of vegetation and the density of water. The results showed that the regular vegetation pattern in the semi-arid area is caused by the traveling wave instability of the reaction-diffusion convection equation. The establishment of this model has provided much convenience for the study of vegetation patterns in semi-arid areas.

Later, some authors [10] introduced the soil water diffusion feedback existing between vegetation roots and soil water into the klausmeier model:

$$\begin{cases} \frac{\partial n}{\partial t} = \left(\frac{\gamma w}{1+\sigma w} - \nu\right)n - n^2 + \nabla^2 n, \\ \frac{\partial w}{\partial t} = p - (1-\rho n)w - w^2 n + \delta \nabla^2 (w - \beta n), \end{cases} \tag{1.3}$$

where the term $\frac{\gamma w}{(1+\sigma w)}n$ describes plant growth at a rate that grows linearly with w for dry soil. The spread of plants is modeled by the diffusion term ∇^2. Model (1.3) contains a source tern p representing precipitation and a loss term $-(1-\rho n)w$ representing evaporation. We ignore the evaporation reduced by vegetation evaporation ($\rho > 0$). Local uptake of water by plants (mostly transpiration) modeled by the tern $-w^2 n$. β describes positive feedback effects of water and biomass. n represents plant population density and w represents groundwater concentration. This nonlinear equation reflects the influence of mutual promotion and competition. The spatial term simulates the spread of vegetation and the transfer of water, and the cross-diffusion term explains the absorption of water by the roots of the vegetation.

In this paper, we ignore the evaporation reduced by vegetation and mainly study the properties of simplified Hardenberg's model with reaction diffusion when $\rho = 0$ as follows:

$$\begin{cases} \frac{\partial n}{\partial t} = \left(\frac{\gamma w}{1+\sigma w} - \nu\right)n - n^2 + d\nabla^2 n, \\ \frac{\partial w}{\partial t} = p - w - w^2 n + d\delta \nabla^2 (w - \beta n), \end{cases} \tag{1.4}$$

with Neumann boundary condition:

$$\begin{cases} n(x,y,0) = n_0, \ w(x,y,0) = w_0, \ (x,y) \in \Omega, \\ \frac{\partial n}{\partial \varrho} = \frac{\partial w}{\partial \varrho} = 0, \ (x,y) \in \partial\Omega, \end{cases}$$

where $\Omega = [0,l] \times [0,l]$, l is a positive bounded constant which gives out the size of the system in the directions of x and y, ϱ is the outward unit normal vector of the boundary $\partial\Omega$.

The rest of this paper is organized as follows: In Section 2, we analyze the stability of semi-arid ecosystems model and give the positive steady state existence conditions in order to explore the model's linear stability and Turing bifurcation conditions. Then in Section 3, we derive the amplitude equations of this model and consider the selection of Turing pattern close to the onset $\beta = \beta_T$ given later. Next in Section 4, based on the amplitude equations derived in Section 3, we analyze the conditions in different situations of Turing pattern steady-state. And in Section 5, the relaxation time describing the standard time of system to settle down is given to better the numerical simulations. At last, conclusions are shown from aspects of mathematics and biology, respectively, in Section 6.

2. Stability analysis of semi-arid ecosystems model

In this section, we analyze the stability of the corresponding ODE model at first,

$$\begin{cases} \frac{\partial n}{\partial t} = f(n,w), \\ \frac{\partial w}{\partial t} = g(n,w), \end{cases} \tag{2.1}$$

where

$$\begin{cases} f(n,w) = \left(\frac{\gamma w}{1+\sigma w} - v\right) n - n^2, \\ g(n,w) = p - w - w^2 n, \end{cases} \quad and \quad \begin{cases} \frac{\partial n}{\partial t} = f(n,w) + d\nabla^2 n, \\ \frac{\partial w}{\partial t} = g(n,w) + \delta d \nabla^2 (w - \beta n), \end{cases} \tag{2.2}$$

with Neumann boundary condition:

$$\begin{cases} n\,(x,y,0) = n_0, \ w\,(x,y,0) = w_0, \ \ (x,y) \in \Omega, \\ \frac{\partial n}{\partial \varrho} = \frac{\partial w}{\partial \varrho} = 0, \ \ (x,y) \in \partial\Omega, \end{cases}$$

where $\Omega = [0,l] \times [0,l]$, l is a positive bounded constant which gives out the size of the system in the directions of x and y, ϱ is the outward unit normal vector of the boundary $\partial\Omega$.

2.1. Positive steady state existence conditions

We first consider about the existence of the solutions. For model (2.1), when it doesn't change with time, the second of equations

$$\begin{cases} \frac{\partial n}{\partial t} = f(n,w) = 0, \\ \frac{\partial w}{\partial t} = g(n,w) = 0, \end{cases} \tag{2.3}$$

can be rewritten to the following:

$$w^3(-\gamma + \sigma v) + w^2(-\sigma + v) + w(\sigma p - 1) + p = 0. \tag{2.4}$$

Theorem 2.1. *(1) If* $(-\gamma + \sigma v) < 0$ *holds, then the solution* w_* *of Eq* (2.4) *must be positive.*
(2) If $(-\gamma + \sigma v) < 0$, $v - \sigma < 0$ *and* $\sigma p - 1 < 0$, *then Eq* (2.4) *has the unique positive real solution. Furthermore, if* $p(-\gamma + \sigma v) + v < 0$, *then the system* (2.3) *has the unique positive equilibrium* (n_*, w_*).

Proof. Denote

$$F(w) = w^3(-\gamma + \sigma v) + w^2(-\sigma + v) + w(\sigma p - 1) + p,$$

since the parameters are positive, so

$$F(0) = p > 0,$$

thus it is obvious that $F(+\infty) < 0$. So there exists w_* making $F(w_*) = 0$.

Denote

$$F(w) = w^3(-\gamma + \sigma v) + w^2(-\sigma + v) + w(\sigma p - 1) + p,$$

so

$$F'(w) = 3(-\gamma + \sigma v)w^2 + 2(-\sigma + v)w + (\sigma p - 1) < 0$$

when $w > 0$, since $F(0) = p > 0$ and $F(+\infty) < 0$, so there is only one positive solution w_* making $F(w_*) = 0$.

Since $F(p) = p^3(-\gamma + \sigma v) + p^2(-\sigma + v) + \sigma p^2$ and $p(-\gamma + \sigma v) + v < 0$, then $F(p) < 0$, and since $F'(w) < 0$ is proved in (2) when $w > 0$, so $p > w_*$, notice that $p - w_* - w_*^2 n_* = 0$ is equivalent to $n_* = \frac{p - w_*}{w_*^2}$, that is to say $n_* > 0$.

2.2. Linear stability analysis and Turing bifurcation

Until now, we have already gotten the positive steady state (n_*, w_*) of (2.3). And now we discuss about its stability, then analyze the conditions for model (2.2) in 2-D space to admit the Turing bifurcation [11–14]. The Jacobian matrix corresponding to this equilibrium is

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} f_n & f_w \\ g_n & g_w \end{pmatrix}\Bigg|_{(n_*, w_*)} = \begin{pmatrix} -n_* & \gamma n_* \frac{1}{(1+\sigma w_*)^2} \\ -w_*^2 & -1 - 2w_* n_* \end{pmatrix}.$$

Denote

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} d & 0 \\ -\delta d\beta & \delta d \end{pmatrix},$$

then, we get the linear perturbations equation:

$$\begin{pmatrix} n_t \\ w_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} n \\ w \end{pmatrix} + \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \nabla^2 \begin{pmatrix} n \\ w \end{pmatrix}.$$

Therefore, the eigenvalues of linearized operator can be derived by discussing roots of following series of equations. Denote

$$D_k \triangleq \det\left(\lambda_k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left[\begin{pmatrix} f_n & f_w \\ g_n & g_w \end{pmatrix}\Bigg|_{(n_*, w_*)} + \begin{pmatrix} -dk^2 & 0 \\ d\delta\beta k^2 & -\delta dk^2 \end{pmatrix}\right]\right) = 0,$$

$$\begin{cases} Tr_k = -n_* - 1 - 2w_* n_* - dk^2(1 + \delta), \\ \mathrm{Det}_k = n_*(1 + 2w_* n_*) + w_*^2 \frac{\gamma n_*}{(1+\sigma w_*)^2} + k^2\left(d + 2w_* n_* d - \frac{d\delta\beta\gamma n_*}{(1+\sigma w_*)^2} + \delta d n_*\right) + k^4 \delta d^2, \end{cases}$$

where $k^2 = l^2 + m^2$, and $l = 1, 2, 3, \cdots, \ \ m = 1, 2, 3, \cdots$.

If there is no diffusion, k is equal to 0, and we get

$$\begin{cases} Tr_0 = -n_* - 1 - 2w_* n_* < 0, \\ Det_0 = n_*(1 + 2w_* n_*) + w_*^2 \frac{\gamma n_*}{(1+\sigma w_*)^2} > 0. \end{cases}$$

So (n_*, w_*) is stable.

Next we focus on the diffusion terms:

$$\begin{cases} Tr_k = Tr_0 - d(1+\delta)k^2 < 0, \\ Det_k = Det_0 + k^2 d\left(1 + 2w_*n_* - \frac{\delta\beta\gamma n_*}{(1+\sigma w_*)^2} + \delta n_*\right) + \delta d^2 k^4 \\ \qquad = \delta d^2\left(k^2 - k_T^2\right) + Det_0 - \delta d^2 k_T^4, \end{cases}$$

where

$$k_T^2 = \frac{\left(-1 - 2w_*n_* + \frac{\delta\beta\gamma n_*}{(1+\sigma w_*)^2} - \delta n_*\right)}{2\delta d}.$$

When the diffusion term is considered, the case is different. The sign of the Det_k is dominated by the cross diffusion parameter β. So we should discuss the cases following to make the Turing instability occur.

Denote

$$\beta_{00} = (n_*\delta + 1 + 2w_*n_*)\,\frac{(1+\sigma w_*)^2}{\delta\gamma n_*},$$

$$\beta_{01} = \beta_{00} + 2\sqrt{\delta\left[n_*\left(1 + 2w_*n_*\right) + \frac{w_*^2\gamma n_*}{(1+\sigma w_*)^2}\right]}\left(\frac{(1+\sigma w_*)^2}{\gamma\delta n_*}\right),$$

$$\beta_{02} = \frac{(\delta d + 2w_*n_* + 1 + \delta n_*)\,(1+\sigma w_*)^2}{\delta n_*\gamma}.$$

We assume that
(N0) $\beta > \beta_{00}$,
(N1) $\beta > \beta_{01}$,
(N2) $\beta > \beta_{02}$,
and the cases are as follows:
(1) If $\beta = 0$, then $Det_k > 0$, the positive equilibrium (n_*, w_*) is stable, the Turing instability will not take place;
(2) If $\beta > 0$, and then we will choose the conditions of β, furthermore:

(i) If (N0) holds, then the symmetry axis of equation $Det_k = 0$ is positive, that is to say, we can have k to get $(Det_k)_{\min}$.

(ii) If (N1) holds, then $(Det_k)_{\min} < 0$.

(iii) If (N2) holds, then $k_{\min}^2 > \frac{1}{2}$.

Condition (N1) guarantees that $(Det_k)_{\min} < 0$, where $k_{\min}^2 = k_T^2$. Moreover, condition (N2) guarantees that the minimal point $k_{min}^2 > 0.5$.

We know that $\beta = \beta_{02}(d)$ increases monotonically in d and intersects with $\beta = \beta_{01}$ at the point $d = d_0$, where $d_0 = 2\sqrt{\delta\left[n_*\left(1 + 2w_*n_*\right) + \frac{w_*^2}{(1+\sigma w_*)^2}\gamma n_*\right]}$.

We take

$$\beta_B = \begin{cases} \beta_{01}, & 0 < d < d_0, \\ \beta_{02}, & d \geq d_0, \end{cases}$$

then we get following conclusion:

Lemma 2.1. *Suppose (N0) holds, then (N1) and (N2) hold if and only if* $\beta > \beta_B$.

Denote

$$\beta_*(k,d) = \frac{n_*(1+2w_*n_*) + \gamma n_* \frac{w_*^2}{(1+\sigma w_*)^2} + k^2 d(1+2w_*n_* + \delta n_*) + k^4 \delta d^2}{\frac{dk^2 \delta \gamma n_*}{(1+\sigma w_*)^2}},$$

then $Det_k = 0$ *when* $\beta = \beta_*(k,d)$.

Lemma 2.2. *Assume that (N0) holds, function* $\beta = \beta_*(k,d)$ *has following properties:*
(a) It will reach the minimum $\beta = \beta_{01}$ *at* $d = d_m(k)$, *and* $d = d_m(k)$ *decreases monotonically as k increases, where*

$$d_m(k) = \sqrt{Det_0/\delta}/k^2.$$

(b) It increases monotonically in d and k when $d > d_m(k)$.

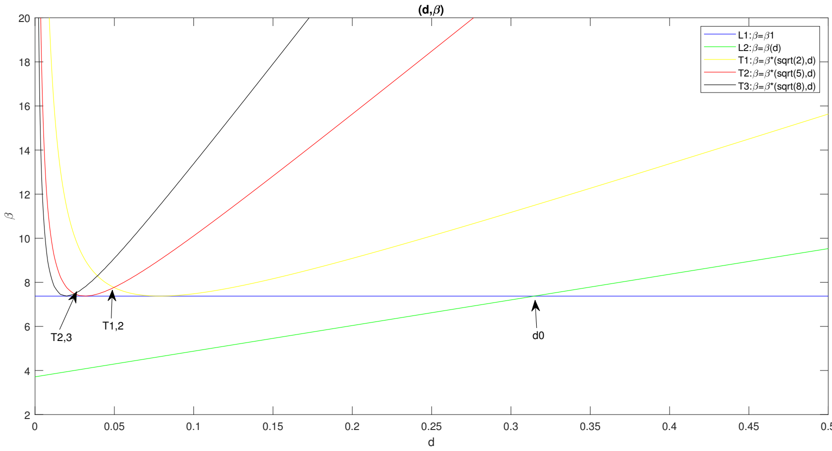

Figure 1. The graph of functions $\beta = \beta_{01}$, $\beta = \beta_{02}(d)$ and $\beta = \beta_*(k,d)$, $d > 0$, $k = \sqrt{2}, \sqrt{5}, \sqrt{8}, ...$, in the (d,β) plane, $\delta = 5.20$, $\gamma = 0.296$, $\sigma = 1.60$, $p = 1.60$.

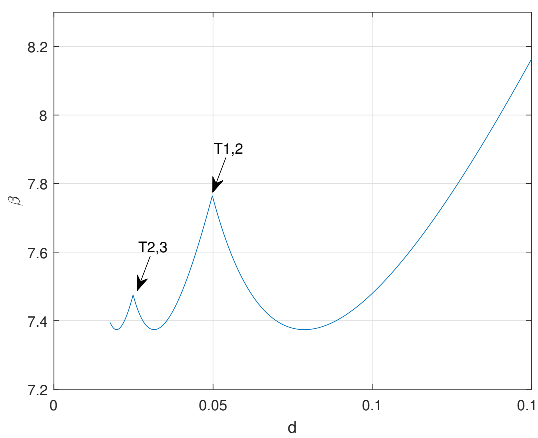

Figure 2. The first Turing bifurcation line T: $\beta = \beta_*(d)$, $d > 0$, and Turing-Turing bifurcation point $T_{k,k+1}$, $k = 1, 2...$, with $\delta = 5.20$, $\gamma = 0.296$, $\sigma = 1.60$, $p = 1.60$.

In Figure 1, a graph is present of functions $\beta = \beta_{01}$, $\beta = \beta_{02}(d)$ and $\beta = \beta_*(k,d)$, $d > 0$, $k = \sqrt{2}, \sqrt{5}, \sqrt{8}, ...$, and in Figure 2, we present a graph of Turing bifurcation line with different k-modes, which illustrates the results of Lemmas 2.1 and 2.2.

Theorem 2.2. *Assume that (N0) holds:*
(1) For any given k_i, where $k_i^2 = l^2 + m^2, l = 1,2,3,..., m = 1,2,3,...$, and k_i is ranked orderly according to the increasing of $l^2 + m^2$, like $k_0 = \sqrt{2}$, $k_1 = \sqrt{5}$, $k_2 = \sqrt{8}$, $k_3 = \sqrt{10}$, $k_4 = \sqrt{13}$, $k_5 = \sqrt{17}$, ... ,we have follows:

(i) When $\beta = \beta_* \triangleq \beta_*(k_1,d)$, $d \in (d_{k_1,k_1+1}, d_{k_1-1,k_1})$, if $k = k_1$, characteristic equation $D_k = 0$ has a simple real eigenvalue $\lambda = \lambda(k_1,\beta)$ with $\lambda(k_1,\beta_*) = 0$, $\frac{dD_{k_1}(\lambda,\beta)}{d\beta}\Big|_{\lambda=0,\beta=\beta_*} < 0$, and all other roots of $D_k(\lambda,\beta_*)$ are possession of negative real parts.

(ii) At $\beta = \beta_* \triangleq \beta_*(k_1,d)$, system (2.2) undergoes k_1-mode Turing bifurcation at (n_*, w_*).

(2) The relationship between β and β_* indicates the occurrence of Turing instability:

(i) $\beta < \beta_*(d)$, $d > 0$ is the critical curve to produce Turing instability.

(ii) When $\beta = \beta_*(d)$, $d > 0$, there is no Turing instability in system (2.2) with the asymptotically stable equilibrium (n_*, w_*).

(iii) When $\beta > \beta_*(d)$, $d > 0$, the diffusion we add makes the equilibrium (n_*, w_*) unstable.

(3) On the critical curve $\beta = \beta_*(d)$, $d > 0$, k_i-mode Turing bifurcation occurs when $d \in (d_{k_i,k_i+1}, d_{k_i-1,k_i})$, and (k_i, k_{i+1})-mode Turing-Turing bifurcation occurs when $d = d_{k_i,k_i+1}$, $i \in N^*$, which means that one Turing bifurcation curves with wave numbers k_i and the other one with k_{i+1} are intersected with each other.

Proof. Firstly, we know directly that

$$Tr_k = Tr_0 - d(1+\delta)k^2 < 0, \ k \in N.$$

When $d \in (d_{k_i,k_{i+1}}, d_{k_{i-1},k_i})$, $\lambda = 0$ is a root of $D_{k_i}(\lambda,\beta_*)$. And $Det_{k_i} = 0$, $Det_k > 0$, where $k \neq k_i$, $k = \sqrt{2}, \sqrt{5}, \sqrt{8}, ...$. From a direct calculation, we obtain that

$$\frac{dD_{k_i}}{d\lambda}\Big|_{\lambda=0} = -Tr_{k_i} > 0,$$

thus $\lambda = 0$ is a simple root.

Next, we show that the transversal condition is valid. Let $\lambda = \lambda(k,\beta)$ be the root of $D_k(\lambda,\beta)$ satisfying $\lambda(k_i,\beta_*) = 0$, then

$$\frac{dD_k}{d\beta}\ |_{\lambda=0,\beta=\beta_*} = -Tr_k\frac{d\lambda}{d\beta} + \frac{dDet_k}{d\beta} = 0.$$

With the formulas above, we can obtain this:

$$\frac{d\lambda(k_i,\beta_*)}{d\beta} = \frac{k^2 d}{Tr_k}\left(-\frac{\delta\gamma n_*}{(1+\sigma w_*)^2}\right) < 0.$$

Remark 2.1. *Now we have three conclusions as follows:*
(1) $\beta < \beta_{01}$. The constant critical value β_{01} does not change with diffusion.
(2) $\beta < \beta_B(d)$, $d > 0$. The critical value $\beta_B(d)$ is determined by diffusion rather with mode number k.

(3) $\beta < \beta_*(d)$, $d > 0$. Both diffusion and mode number k have influence on the critical value $\beta_*(d)$. And equilibrium (n_*, w_*) is asymptotically stable.

We can see that both terms (1) and (2) in Remark 2.1 are sufficient conditions, which Turing instability does not take place under, and term (3) in Remark 2.1 is not only sufficient but also necessary.

Remark 2.2. *The critical curve of Turing instability $\beta = \beta_*(d)$, $d > 0$ is called the first Turing bifurcation curve, one with the corresponding characteristic equation having no root with positive real part. It is a piecewise smooth curve, with piecewise point called Turing-Turing bifurcation point $T_{k_i.k_{i+1}}$, see in Figure 2. Notice that wave number k and diffusion coefficient d affect the expression of the first Turing curve, so stable spatial pattern can be found with wave number k, where $k = \sqrt{2}, \sqrt{5}, \sqrt{8}\ldots$.*

Remark 2.3. *Focusing on Lemma 2.1 and Figure 2, It is clear that if diffusion ratio β is relatively constant, wave number k of spatial pattern is affected by the diffusion coefficient d. Smaller the diffusion coefficient d is, larger the wave number k is.*

3. Derivation of amplitude equation

We haven't determined the selection of Turing pattern based on the above discussion. In this section, we will take the selection of Turing pattern into consideration with free condition in 2-D space. β is close to the onset β_T, the eigenvalues are around zero corresponding to the critical modes which vary slowly, meanwhile the off-critical modes relax quickly, so only perturbations with k around k_T should be considered. Since amplitude equations dominate the dynamics of the system, we analyze the stability of different patterns near the onset using the amplitude equations. There are two methods to drive the coefficients of amplitude equations: One is symmetrical analysis and the other is standard multiple-scale analysis [15,16]. The critical value is

$$k_T^2 = k_{\min}^2 = \frac{1}{2\delta d}\left(\frac{\delta\beta\gamma n_*}{(1+\delta w_*)^2} - 2w_*n_* - 1 - n_*\delta\right).$$

The Taylor series expansion is

$$\begin{cases} f(n,w) = \frac{\partial f}{\partial n}\Big|_{(n_*,w_*)} n + \frac{\partial f}{\partial w}\Big|_{(n_*,w_*)} w + \frac{1}{2!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^2 f\Big|_{(n_*,w_*)} \\ \qquad + \frac{1}{3!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^3 f\Big|_{(n_*,w_*)} + o\left(\rho^3\right), \\ g(n,w) = n\frac{\partial g}{\partial n}\Big|_{(n_*,w_*)} + w\ \frac{\partial g}{\partial w}\Big|_{(n_*,w_*)} + \frac{1}{2!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^2 g\Big|_{(n_*,w_*)} \\ \qquad - \frac{1}{3!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^3 g\Big|_{(n_*,w_*)} + o\left(\rho^3\right), \end{cases}$$

and then

$$\begin{cases} f(n,w) = -n_*n + \gamma n_*\frac{1}{(1+\sigma w_*)^2}w + \frac{1}{2!}\Big(-1n^2 + 2wn\frac{\gamma}{(1+\sigma w_*)^2} \\ \qquad +w^2\gamma n_*(-2)\frac{\sigma}{(1+\sigma w_*)^3}\Big) + \frac{1}{3!}\Big(nw^2\cdot 3\gamma(-2)\left(1+\sigma w_*\right)^{-3}\sigma \\ \qquad +w^3\gamma(-2)(-3)\left(1+\sigma w_*\right)^{-4}\sigma^2\Big) + o\left(\rho^3\right), \\ g(n,w) = -w_*^2 n + (-1-2w_*n_*)w + \frac{1}{2!}\left(2nw\left(-2w_*\right) + w^2\left(-2n_*\right)\right) \\ \qquad +\frac{1}{3!}\left(3nw^2(-2)\right) + o\left(\rho^3\right) \end{cases} \tag{3.1}$$

is considered, and then substituted into system (2.2). The terms $O(\rho^4)$ are omitted for the sake of convenience, then we get a system which approaches system (2.2) as follows:

$$\begin{cases} n_t = \frac{\partial f}{\partial n}\Big|_{(n_*,w_*)} n + \frac{\partial f}{\partial w}\Big|_{(n_*,w_*)} w + \frac{1}{2!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^2 f\Big|_{(n_*,w_*)} \\ \qquad + \frac{1}{3!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^3 f\Big|_{(n_*,w_*)} + o\left(\rho^3\right) + d\nabla^2 n, \\ w_t = n\frac{\partial g}{\partial n}\Big|_{(n_*,w_*)} + w\,\frac{\partial g}{\partial w}\Big|_{(n_*,w_*)} + \frac{1}{2!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^2 g\Big|_{(n_*,w_*)} \\ \qquad - \frac{1}{3!}\left(n\frac{\partial}{\partial n} + w\frac{\partial}{\partial w}\right)^3 g\Big|_{(n_*,w_*)} + o(\rho)^3 + \delta d\nabla^2(w - \beta n), \end{cases} \tag{3.2}$$

and by substituting the specific expressions into Eq (3.2), we get

$$\begin{cases} n_t = -n_* n + \gamma n_* \frac{1}{(1+\sigma w_*)^2} w + \frac{1}{2!}\Big(-1n^2 + 2wn\frac{\gamma}{(1+\sigma w_*)^2} \\ \qquad + w^2\gamma n_*(-2)\frac{\sigma}{(1+\sigma w_*)^3}\Big) + \frac{1}{3!}\Big(nw^2 \cdot 3\gamma(-2)\left(1+\sigma w_*\right)^{-3}\sigma \\ \qquad + w^3\gamma(-2)(-3)\left(1+\sigma w_*\right)^{-4}\sigma^2\Big) + o\left(\rho^3\right) + d\nabla^2 n, \\ w_t = -w_*^2 n + (-1 - 2w_* n_*)w + \frac{1}{2!}\left(2nw\left(-2w_*\right) + w^2\left(-2n_*\right)\right) \\ \qquad + \frac{1}{3!}\left(3nw^2(-2)\right) + o\left(\rho^3\right) + \delta d\nabla^2(w - \beta n). \end{cases}$$

The solutions of model (3.2) can be expanded to

$$u \triangleq \begin{pmatrix} n \\ w \end{pmatrix} = \sum_{j=1}^{3} \begin{pmatrix} A_j^n \\ A_j^w \end{pmatrix} \exp\left(ik_j\gamma\right) + c.c, \tag{3.3}$$

where $c.c$ stands for the complex conjugate.

Denote

$$u = \begin{pmatrix} n \\ w \end{pmatrix} = \varepsilon \begin{pmatrix} \bar{n}_1 \\ \bar{w}_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \bar{n}_2 \\ \bar{w}_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} \bar{n}_3 \\ \bar{w}_3 \end{pmatrix} + \cdots \tag{3.4}$$

and

$$N_{12} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix},$$

where N_1, N_2 are given in Appendix. Then, model (3.2) can be changed to as the follows:

$$\frac{\partial u}{\partial t} = Lu + N, \tag{3.5}$$

where

$$L = \begin{pmatrix} -n_* + d\nabla^2 & \gamma n_* \frac{1}{(1+\sigma w_*)^2} \\ -w_*^2 - \delta d\beta\nabla^2 & \delta d\nabla^2 - (1 + 2w_* n_*) \end{pmatrix}. \tag{3.6}$$

For this system (3.5), the parameter β behavior near the onset β_T is analyzed. In this way, β is expanded in the following term:

$$\beta_T - \beta = \varepsilon\beta_1 + \varepsilon^2\beta_2 + \varepsilon^3\beta_3 + \cdots, \tag{3.7}$$

where ε is considered as a small parameter. The variable u and the nonlinear term N are expanded according to ε so that we obtain the follows:

$$N_{34} = \begin{pmatrix} N_3 \\ N_4 \end{pmatrix},$$

where N_3, N_4 are given in Appendix.

And L can be expanded as follows:

$$L = L_T + (\beta_T - \beta)\begin{pmatrix} 0 & 0 \\ \delta d\nabla^2 & 0 \end{pmatrix}, \tag{3.8}$$

where

$$L_T = \begin{pmatrix} -n_* + d\nabla^2 & \gamma n_* \frac{1}{(1+\sigma w_*)^2} \\ -w_*^2 - \delta d\beta_T\nabla^2 & \delta d\nabla^2 - (1 + 2w_* n_*) \end{pmatrix}. \tag{3.9}$$

Separating the dynamical behavior of the system (3.5) according to different time scale is regarded as the core of the standard multiple-scale analysis scale. We just need to separate the time scale (i.e. $T_0 = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$). Each T_i corresponds to the dynamical behaviors of the variables, whose scales are ε^{-1}, an independent variable. So the derivative with respect to time converts to the following term:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \cdots . \tag{3.10}$$

For model (3.5), we consider the following result:

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial T_0} + \varepsilon\frac{\partial A}{\partial T_1} + \varepsilon^2\frac{\partial A}{\partial T_2} + \cdots . \tag{3.11}$$

Substituting Eqs (3.6)–(3.9) into Eq (3.5) and expanding it according to different orders of ε, we can obtain three equations as follows:

$$\varepsilon : L_T\begin{pmatrix} \bar{n}_1 \\ \bar{w}_1 \end{pmatrix} = 0, \tag{3.12}$$

$$\varepsilon^2 : L_T\begin{pmatrix} \bar{n}_2 \\ \bar{w}_2 \end{pmatrix} = \begin{pmatrix} F_1^n \\ F_1^w \end{pmatrix}, \tag{3.13}$$

where F_1^n, F_1^w are given in Appendix.

$$\varepsilon^2 : L_T\begin{pmatrix} \bar{n}_3 \\ \bar{w}_3 \end{pmatrix} = \begin{pmatrix} F_2^n \\ F_2^w \end{pmatrix}, \tag{3.14}$$

where F_2^n , F_2^w are given in Appendix.

For the first order of ε, as L_T is the linear operator of the system near the onset, $(n_1, w_1)^T$ is linear combination of the eigenvectors that corresponds to the eigenvalue zero. Solving the solution of the first order of ε, we can obtain

$$\begin{pmatrix} \bar{n}_1 \\ \bar{w}_1 \end{pmatrix} = \begin{pmatrix} \theta \\ 1 \end{pmatrix}\sum_{j=1}^{3} w_j \exp\left(ik_j\gamma\right) + c.c, \tag{3.15}$$

where

$$\theta = \frac{\gamma n_*}{(n_* + dk^2)(1 + \sigma w_*)^2},$$

$|k_j| = |k_T|$, and $|w_j|$ is the amplitude of the mode $exp(ik_j r)$ if the system is under the first-order perturbation. The perturbational term of the higher order determines its form. For the second order of ε, we obtain Eq (3.13) above.

First we directly solve the equation with the solutions of Eq (3.15) which we have already calculated and by the comparing with the corresponding coefficients. We will get the following results:

$$\begin{aligned} \begin{pmatrix} \bar{n}_2 \\ \\ \bar{w}_2 \end{pmatrix} &= \begin{pmatrix} \bar{u}_2^0 \\ \\ \bar{v}_2^0 \end{pmatrix} + \sum_{j=1}^3 \begin{pmatrix} \bar{u}_2^j \\ \\ \bar{v}_2^j \end{pmatrix} \exp\left(ik_j\gamma\right) + \sum_{j=1}^3 \begin{pmatrix} \bar{u}_2^{jj} \\ \\ \bar{v}_2^{jj} \end{pmatrix} \exp\left(i2k_j\gamma\right) \\ &+ \sum_{j=1}^3 \begin{pmatrix} \bar{u}_2^{j,j+1} \\ \\ \bar{v}_2^{j,j+1} \end{pmatrix} \exp\left(i\left(k_j - k_{j+1}\right)\gamma\right) + C.C. \end{aligned} \tag{3.16}$$

Secondly, according to the Fredholm condition, the vector function of the right-hand side of Eq (3.13) must be orthogonal with the zero eigenvectors of operator L_T^+, so that the existence of the nontrivial solution of this equation can be ensured. L_T^+ is the adjoint operator of L_T, it is as follow:

$$L_T^+ = \begin{pmatrix} -n_* + d\nabla^2 & -w_*^2 + \delta d\left(-\beta_T\right)\nabla^2 \\ \gamma n_* \frac{1}{(1+\sigma w_*)^2} & (-1 - 2w_* n_*) + \delta d\nabla^2 \end{pmatrix}.$$

In this system, the zero eigenvectors of operator L_T^+ are

$$\begin{pmatrix} 1 \\ \varphi \end{pmatrix} \exp\left(-ik_T r\right) + c.c\ .$$

The orthogonality condition is

$$(1, \varphi)\begin{pmatrix} F_1^n \\ F_1^w \end{pmatrix} = 0,$$

that is

$$(1, \varphi)\left[\frac{\partial}{\partial T_1}\begin{pmatrix} \bar{n}_1 \\ \bar{w}_1 \end{pmatrix} - R_1 - R_2\right] = 0, \tag{3.17}$$

where R_1, R_2 are given in Appendix.

For the third order of the ε in Eq (3.14), as with its second step of the second order, we have

$$(1, \varphi)\begin{pmatrix} F_2^n \\ F_2^w \end{pmatrix} = 0,$$

that is

$$B_1 + (B_2 + B_3 + B_4 + B_5 + B_6 + B_7) + \varphi(B_8 + B_9 + B_{10} + B_{11}) = 0,$$

where B_1–B_{11} are given in Appendix.

Expanding the above equation, we have

$$C_1 + (C_2 + C_3) + \varepsilon^3(C_4 + C_5 + C_6) = 0, \tag{3.18}$$

where C_1–C_6 are given in Appendix.

Denote $A_i = A_i^v$, and then, we expand amplitude A_1 in the following form:

$$A_1 = \varepsilon W_1 + \varepsilon^2 V_1 + \dots . \tag{3.19}$$

Substituting it into Eq (3.18), the amplitude equation corresponding to A_1 can be obtained as follows:

$$\tau_0 \frac{\partial A_1}{\partial T_1} = \mu A_1 + h\bar{A}_2\bar{A}_3 - \left(g_1 |A_1|^2 + g_2(|A_2|^2 + |A_3|^2)\right), \tag{3.20}$$

where C_1–C_6 are given in Appendix.

4. Analysis of Turing pattern steady-state

Eq (3.20) can be decomposed to mode $\rho_i = |A_i|$ with a corresponding phase angle φ_i. Then, four differential equations of the real variables are obtained as follows [15–17]:

$$\begin{cases} \tau_0 \frac{\partial \varphi}{\partial t} = -h\frac{\rho_1^2\rho_2^2+\rho_1^2\rho_3^2+\rho_2^2\rho_3^2}{\rho_1\rho_2\rho_3} \sin\varphi, \\ \tau_0 \frac{\partial \rho_1}{\partial t} = \mu\rho_1 + h\rho_2\rho_3\cos\varphi - g_1\rho_1^3 - g_2\left(\rho_2^2 + \rho_3^2\right)\rho_1, \\ \tau_0 \frac{\partial \rho_2}{\partial t} = \mu\rho_2 + h\rho_1\rho_3\cos\varphi - g_1\rho_2^3 - g_2\left(\rho_1^2 + \rho_3^2\right)\rho_2, \\ \tau_0 \frac{\partial \rho_3}{\partial t} = \mu\rho_3 + h\rho_1\rho_2\cos\varphi - g_1\rho_3^3 - g_2\left(\rho_1^2 + \rho_2^2\right)\rho_3, \end{cases} \tag{4.1}$$

where $\varphi = \varphi_1 + \varphi_2 + \varphi_3$.

The dynamical system (4.1) has four kinds of solutions:

(i) The stationary state $(0, 0, 0)$ is stable;

(ii) Stripe pattern, given by

$$\rho_1 = \sqrt{\frac{\mu}{g_1}}, \quad \rho_2 = \rho_3 = 0,$$

and exist only when μ is of the same sign as g_1;

(iii) Hexagonal pattern, given by

$$\rho_1 = \rho_2 = \rho_3 = \rho_\pm = \frac{|h| \pm \sqrt{h^2 + 4(g_1 + 2g_2)\mu}}{2(g_1 + 2g_2)}.$$

Denote

$$\mu_1 = \frac{-h^2}{4(g_1 + 2g_2)},$$

the existence condition of $\rho_1,\ \rho_2,\ \rho_3$ is $\mu > \mu_1$;

(iiii) Mixed structure pattern, given by

$$\rho_1 = \frac{|h|}{g_2 - g_1}, \quad \rho_2 = \rho_3 = \sqrt{\frac{\mu - g_1\rho_1^2}{g_1 + g_2}},$$

where $g_2 > g_1$.

Without discussing solution (i) which is trivial all time, we firstly take the stability of stripe pattern into consideration. Setting $\rho_1 = \rho_0 + \delta\rho_1$, $\rho_2 = \delta\rho_2$, $\rho_3 = \delta\rho_3$, substituting them into Eq (4.1), and then linearizing, we get

$$\frac{\partial}{\partial t}\begin{pmatrix} \delta\rho_1 \\ \delta\rho_2 \\ \delta\rho_3 \end{pmatrix} = \begin{pmatrix} \mu - 3g_1\rho_0^2 & 0 & 0 \\ 0 & \mu - g_2\rho_0^2 & |h|\rho_0 \\ 0 & |h|\rho_0 & \mu - g_2\rho_0^2 \end{pmatrix}\begin{pmatrix} \delta\rho_1 \\ \delta\rho_2 \\ \delta\rho_3 \end{pmatrix}.$$

The characteristic equation is

$$(-2\mu - s)\left(\left(\mu - \frac{g_2}{g_1}\mu - s\right)^2 - \frac{|h|^2}{g_1}\mu\right) = 0,$$

so the eigenvalues are

$$s_1 = -2\mu, \quad s_{2,3} = \mu\left(1 - \frac{g_2}{g_1}\right) \pm |h|\sqrt{\mu/g_1},$$

denote

$$\mu_2 = 0, \qquad \mu_3 = \frac{h^2 g_1}{(g_1 - g_2)^2},$$

then we get stable stripe pattern when $\mu > \mu_2$ and $\mu > \mu_3$.

Secondly, we focus on the stability of hexagonal pattern. Substituting $\rho_i = \rho_0 + \sigma_i$, where $i = 1, 2, 3$, into Eq (4.1), then linearizing, we get

$$\frac{\partial}{\partial t}\begin{pmatrix} \delta\rho_1 \\ \delta\rho_2 \\ \delta\rho_3 \end{pmatrix} = \begin{pmatrix} a & b & b \\ b & a & b \\ a & b & a \end{pmatrix}\begin{pmatrix} \delta\rho_1 \\ \delta\rho_2 \\ \delta\rho_3 \end{pmatrix},$$

where

$$a = \mu - (3g_1 + 2g_2)\rho_0^2, \ \ b = |h|\rho_0 - 2g_2\rho_0^2.$$

The character equation is

$$(a - s)^3 - 3b^2(a - s) + 2b^3 = 0,$$

so the eigenvalues are

$$s_1 = s_2 = -b + a, \quad s_3 = 2b + a,$$

denote

$$\mu_4 = \frac{2g_1 + g_2}{(g_2 - g_1)^2}h^2,$$

only when

$$\mu < \mu_4,$$

all the eigenvalues for ρ_+ are negative, which means that we have the stable hexagon pattern; while the eigenvalues s_1, s_2 for ρ_- are positive which means that we have the unstable hexagon pattern. And the solutions (iiii) are always unstable, so that we do not make a discussion about it.

In next section, we will simulate the following solutions:

(1)

$$S_p : \sqrt{\frac{\mu}{g_1}},$$

(2)

$$H_0 : \frac{h + \sqrt{h^2 + 4(g_1 + 2g_2)\mu}}{2(g_1 + 2g_2)},$$

(3)

$$H_\pi : \frac{h - \sqrt{h^2 + 4(g_1 + 2g_2)\mu}}{2(g_1 + 2g_2)},$$

where if the $\varphi = 0$ in system (4.1), the H_0 mode hexagon is stable when $\mu \in (\mu_1, \mu_4)$ and unstable when $\mu > \mu_4$ while the H_π mode hexagon is unstable when $\mu > \mu_2$, and if the $\varphi = \pi$ in system (4.1), the H_0 mode hexagon is unstable when $\mu \in (\mu_2, \mu_4)$ and stable when $\mu > \mu_4$ or $\mu \in (\mu_1, \mu_2)$ while the H_π mode hexagon is stable when $\mu > \mu_2$.

5. Numerical simulations

In this section, extensive numerical simulations of the spatially extended model (1.4) in two-dimensional space are performed, with the qualitative results shown here. The space and time of the problem need to be discretized in order to solve differential equations with the help of computer. A discrete domain with $M * N$ lattice sites whose space between the lattice points is Δh is built up to solve the continuous problem corresponding to the reaction-diffusion system in two-dimensional space. The time evolution is in a discrete process, with steps of Δt, and can be solved by the Euler method. Then, we calculate the Laplacian describing diffusion by using finite differences in the discrete system. The Turing pattern obtained from numerical simulations of Eq (2.2) satisfies initial and boundary conditions. We choose the time step $\Delta t = 0.0005$ and a system size of $50 * 50$ with the space step $\Delta x = 0.50$, $\Delta y = 0.50$. We keep $\gamma = 1.6$, $\delta = 5.2$, $p = 0.2958$, $\sigma = 1.6$, $v = 0.2$. By calculating, we analyze the Turing pattern of the system that we have already discussed about in following steps:

(1) First of all, we calculate the control parameters β and the important control parameters μ_1, μ_2, μ_3, μ_4 by the parameters given above and give a picture to show that result directly;

(2) Secondly, we calculate the relaxation time τ whose expression is given later, which means the time of the system to nearly settle down, and then we give the evolution pictures of μ in different intervals, such as $[\mu_1, \mu_2]$, $[\mu_2, \mu_3]$ and $[\mu_3, \mu_4]$ as well.

5.1. Parameters calculate

The values of control parameter β is corresponding to parameters $\mu_1 = -0.00467$, $\mu_2 = 0$, $\mu_3 = 0.07179$, $\mu_4 = 0.30022$, and $\beta_T = 7.3738$ is the critical value of Turing bifurcation. Then we present the figures below to show the different solutions' stability regarded to μ_1, μ_2, μ_3, μ_4.

In Figure 3, we take the amplitude as a function of the control parameter μ which is used to describe the bifurcation of the stripe pattern S_p and the spot pattern H_0 or H_π respectively.

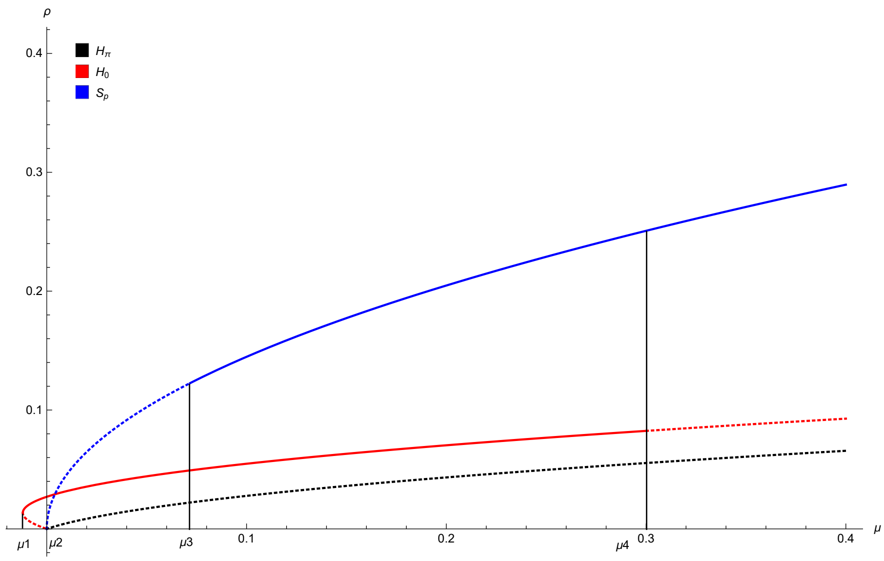

Figure 3. Turing bifurcation diagram of amplitude equation(3.17) with μ_1, μ_2, μ_3, μ_4.

5.2. *Numerical analysis of evolutions*

Firstly, by the analyze in Section 3, we obtain the relaxation time τ as follow:

$$\tau = \frac{\theta + \phi}{\beta_T \delta d \theta k_T^2 \phi},$$

then we get 4.15492 after substituting the parameters into the formula of τ. It is equivalent to 8310 iterations when we choose the time step as $\Delta t = 0.0005$.

Secondly, we give the evolution pictures with μ in different intervals, such as $[\mu_1, \mu_2]$, $[\mu_2, \mu_3]$ and $[\mu_3, \mu_4]$. And different types of patterns are observed in the process of numerical simulations. The distributions of water species w and vegetation species n are always in the same type. Here, we only presert the distribution of the population density pattern of vegetation species n.

5.2.1. Uniform state and hexagon pattern

We take $\mu = -1.08725 * 10^{-6} \in (\mu_1, \mu_2)$. The pattern of uniform stationary solution occupies the whole domain eventually at the end. Figure 4 shows the evolution process of the spatial pattern of species n after 0 iterations, 500 iterations, 8310 iterations, 20000 iterations, 300000 iterations, 400000 iterations and 500000 iterations.

5.2.2. Hexagon pattern

We take $\mu = 0.0717635 \in (\mu_2, \mu_3)$. At this time, the whole domain is occupied with H_0 hexagon pattern eventually. Figure 5 shows the evolution process of the spatial pattern of species n after the same numbers before. And the relation time is also 8310 iterations. From the Figure 5, we can find when the pattern settle down, the stripes decrease gradually into nonexistent while the spots are still in our sight.

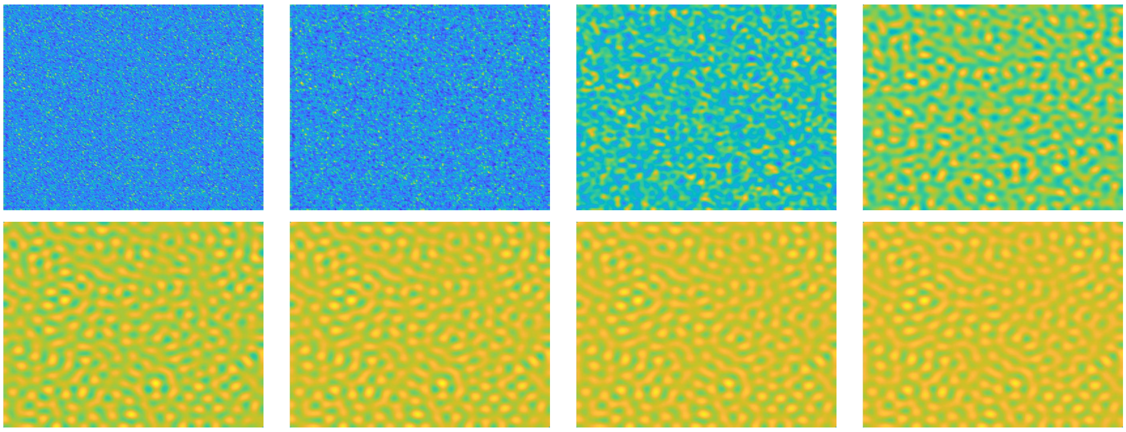

Figure 4. Square pattern under the condition of $\mu = -1.08725 * 10^{-6} \in (\mu_1, \mu_2)$ with the random initial state.

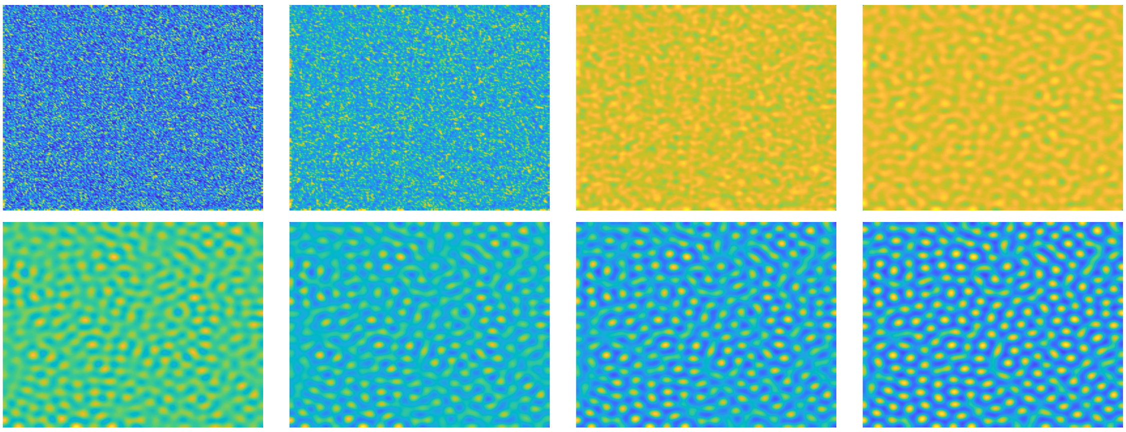

Figure 5. Square pattern under the condition of $\mu = 0.0717635 \in (\mu_2, \mu_3)$ with the random initial state.

5.2.3. Hexagon and strip pattern

We take $\mu = 0.261217 \in (\mu_3, \mu_4)$. At this time, H_0 hexagon pattern and strip are coexistence in the whole domain in the end. Figure 6 shows the evolution process of the spatial pattern of species n after the same iterations as above and with the same relaxation time 8310 iterations too, and we can find the coexistence of the H_0 hexagon pattern and the strips clearly as the theoretical analysis prejudged.

5.2.4. Strip pattern

We take $\mu = 0.300274 > \mu_4$. In this way, H_0 hexagon pattern is dismissing while the strips are substitute for the H_0 hexagon in the whole domain eventually. In Figure 7 forms squares consist of horizontal and vertical strips finally, and they are both shown with the same iteration numbers of evolution and the same relaxation time 8310 iterations as above. Comparing with the analysis, there should be only stripe pattern under this situation. Evidently the both series of the pictures are consistent with the theoretical analysis very well too.

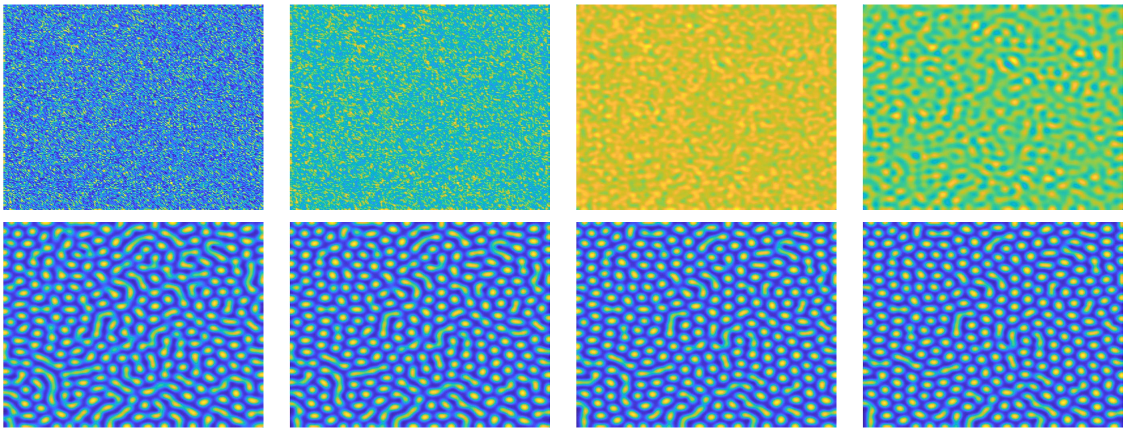

Figure 6. Square pattern under the condition of $\mu = 0.261217 \in (\mu_2, \mu_3)$ with the random initial state.

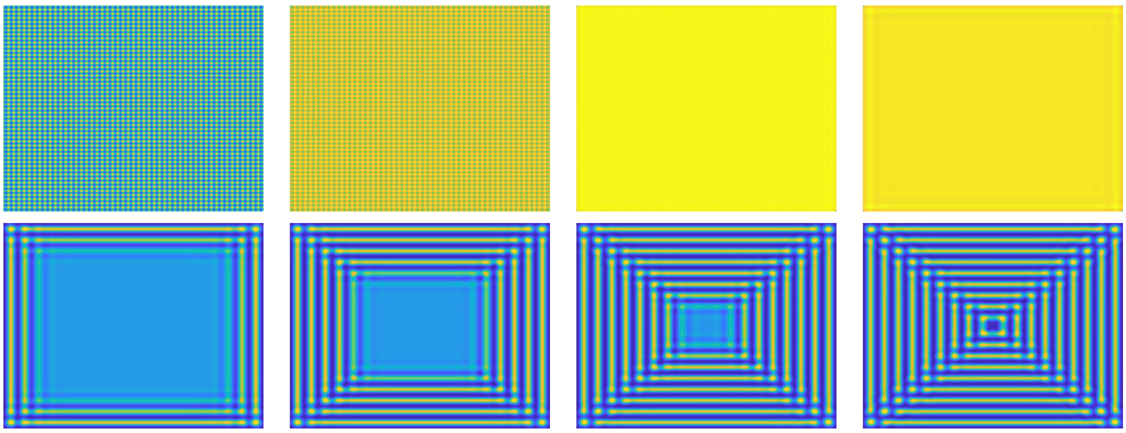

Figure 7. Square pattern under the condition of $\mu = 0.300274 > \mu_4$ and the uniform spots initial state.

6. Conclusions

For a semi-arid ecosystem model, a great amount of works discuss the stability and bifurcation based on Lyapunov method, but some of the studiers don't prefer to consider the diffusion effects, especially the cross diffusion. In this paper, we take it into consideration that a semi-arid ecosystems system is with cross diffusion effect in 2-D spatial domain. And some conditions of Turing instability to the model (1.4) is given with the help of Turing's bifurcation theory. We study both the amplitude equations and the stability of different patterns in detail. We also introduce the conception "relaxation time", with which we could find the standard to show the time when the system nearly settles down.

Furthermore, we also focus on the biology significance of the model (1.4). In the natural world, two species are in relationships of both coexistence and competition. The specie "water" may be recognized as a restrained survival source by the other specie "vegetation", meanwhile, the diffusion of the vegetation make a influence of the water since the water is absorbed by the vegetation root. This phenomenon are caused by cross-diffusion. For the sake of this, the effect of cross-diffusion on the pattern structure is considered and the results indicate that cross-diffusion is a critical term to the Turing spatial pattern formulation. Without cross diffusion, the instability will not take place in the system refer to Section 2 discussion. Besides, the numerical results are well consistent with the theory. When we take the stability of different patterns into consideration, such as spots, stripes, we find it is changing with the μ refer to Section 4 discussion. Biologically, it means that if we randomly plant some trees, it will occurs in four situations: (1) If $\mu < \mu_1$, the number of trees will probably decrease. So the region will change into a desert; (2) If $\mu_2 < \mu < \mu_3$, the pattern of trees is more like some spots called small bushes; (3) If $\mu_3 < \mu < \mu_4$, the pattern of the trees will change into the state of coexistence of both stripes and spots. So the gaps between two groups of the vegetation is either lines or labyrinths; (4) If $\mu > \mu_4$, the pattern of tree groups will be like some stripes. Based on the descriptions of the four situations, we would optimise the μ, dominated by parameters physically, larger than μ_2, so that the desert will probably not increase, especially when the pattern is strip, the μ should be lager than μ_3 in case it will degenerate into nothing. By these useful decisions, not only a constructive way to ensure the stationary pattern formation of the environment is presented, but also an interesting usage of Turing pattern is provided.

Conflict of interest

The authors declare that they have no conflicts of interest.

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Appendix

$$\begin{cases} N_1 = \frac{1}{2!}\left(-2n^2 + 2wn\frac{\gamma}{(1+\sigma w_*)^2} - 2w^2\gamma n_*\frac{\sigma}{(1+\sigma w_*)^3}\right) \\ \qquad +\frac{1}{3!}\left(-6nw^2\gamma\left(1+\sigma w_*\right)^{-3}\sigma + 6w^3\gamma n_*\sigma^2\left(1+\sigma w_*\right)^{-4}\right), \\ N_2 = \frac{1}{2!}\left(-4nww_* - 2w^2n_*\right) + \frac{1}{3!}\left(-6nw^2\right), \end{cases}$$

$$\begin{cases} N_3 = \varepsilon^2\left(-\bar{n}_1^2 + \frac{\gamma\bar{n}_1\bar{w}_1}{(1+\sigma w_*)^2} - \frac{\gamma\sigma\bar{w}_1^2\sigma n_*}{(1+w_*)^3}\right) \\ \qquad +\varepsilon^3(-2\bar{n}_1\bar{n}_2 - \frac{\gamma\bar{n}_1\sigma\bar{w}_1^2}{(1+\sigma w_*)^3} + \frac{\gamma\bar{n}_2\bar{w}_1}{(1+\sigma w_*)^2} \\ \qquad +\frac{\gamma\bar{n}_1\bar{w}_2}{(1+\sigma w_*)^2} + \frac{\gamma\sigma^2\bar{w}_1^3 n_*}{(1+\sigma w_*)^4} - \frac{2\gamma\sigma\bar{w}_1\bar{w}_2 n_*}{(1+\sigma w_*)^3}), \\ N_4 = \varepsilon^2\left(-2\bar{n}_1\bar{w}_1 w_* - \bar{w}_1^2 n_*\right). \end{cases}$$

$$\begin{cases} F_1^n = \frac{\partial}{\partial T_1}\bar{n}_1 - \left(-\bar{n}_1^2 + \frac{\gamma\bar{n}_1\bar{w}_1}{(1+\sigma w_*)^2} - \frac{\gamma\sigma\bar{w}_1^2 n_*}{(1+\sigma w_*)^3}\right), \\ F_1^w = \frac{\partial}{\partial T_1}\bar{w}_1 - \left(-2\bar{n}_1\bar{w}_1 w_* - \bar{w}_1^2 n_*\right) + \beta_1\delta dk_T^2\bar{n}_1. \end{cases}$$

$$\begin{cases} F_2^n = \frac{\partial\bar{n}_2}{\partial T_1} + \frac{\partial\bar{n}_1}{\partial T_2} - (-2\bar{n}_1\bar{n}_2 - \frac{\gamma\bar{n}_1\sigma\bar{w}_1^2}{(1+\sigma w_*)^3} + \frac{\gamma\bar{n}_2\bar{w}_1}{(1+\sigma w_*)^2} + \frac{\gamma\bar{n}_1\bar{w}_2}{(1+\sigma w_*)^2} \\ \qquad +\frac{\gamma\sigma^2\bar{w}_1^3 n_*}{(1+\sigma w_*)^4} - \frac{2\gamma\sigma\bar{w}_1\bar{w}_2 n_*}{(1+\sigma w_*)^3}), \\ F_2^w = \frac{\partial\bar{w}_2}{\partial T_1} + \frac{\partial\bar{w}_1}{\partial T_2} - \beta_2\delta dk_T^2\bar{n}_1 + \beta_1\delta dk_T^2\bar{n}_2. \end{cases}$$

$$\bar{U}_2^0 = f_0 * \sum_{j=1}^{3}|\bar{w}_i|^2, \quad \bar{V}_2^0 = q_0 * \sum_{j=1}^{3}|\bar{w}_i|^2, \quad \bar{U}_2^1 = \theta * \bar{V}_2^1,$$

$$\bar{U}_2^{11} = f_1 * \bar{w}_1^2, \quad \bar{V}_2^{11} = q_1 * \bar{w}_1^2, \quad \bar{U}_2^{12} = f_2 * \bar{w}_1 * \bar{W}_2', \quad \bar{V}_2^{11} = q_2 * \bar{w}_1 * \bar{W}_2',$$

$$f_0 = -\frac{f_{01}}{f_{02}},$$

$$\begin{aligned} f_{01} &= n_*^2(\gamma + 3w_*\gamma\sigma) + n_*(2w_*\theta^2 + \gamma\sigma + 6w_*^2\theta^2\sigma + 6w_*^3\theta^2\sigma^2 + 2w_*^4\theta^2\sigma^3) \\ &\quad +\theta(1+w_*\sigma)[-\gamma + \theta(1+w_*\sigma)^2], \end{aligned}$$

$$f_{02} = n_*(1+w_*\sigma)[1 + 2w_*\sigma + 2n_*w_*(1+w_*\sigma)^2 + w_*^2\left(\gamma + \sigma^2\right)].$$

$$q_0 = -\frac{q_{01}}{q_{02}},$$

$$\begin{aligned} q_{01} &= d^2k_Tk_T^2(1+w_*\sigma)[-w_*^2\gamma\sigma + n_*(1+w_*\sigma)^3] \\ &\quad +dk_Tk_T[w_*^2\gamma^2 + 2n_*w_*\gamma(1+w_*\sigma) + 2n_*^2(1+w_*\sigma)^4] \\ &\quad +n_*^2(1+w_*\sigma)[n_*(1+w_*\sigma)^3 + w_*\gamma(2+w_*\sigma)], \end{aligned}$$

$$q_{02} = (dk_Tk_T + n_*)^2(1+w_*\sigma)^2[1 + 2w_*\sigma + 2n_*w_*(1+w_*\sigma)^2 + w_*^2\left(\gamma + \sigma^2\right)].$$

$$f_1 = -n_*\gamma \frac{f_{111} + f_{112}}{(dk_T k_T + n_*)^2(1 + w_*\sigma)^4 f_{121}},$$

$$f_{111} = (1 + 2n_*w_* + 4dk_T k_T\delta)[n_*\gamma - (dk_T k_T + n_*)\gamma + (dk_T k_T + n_*)^2\sigma(1 + w_*\sigma)],$$

$$f_{112} = n_*(dk_T k_T + n_*)[2w_*\gamma + (dk_T k_T + n_*)(1 + w_*\sigma)^2],$$

$$f_{121} = (4\ dk_T k_T + n_*)(1 + 2n_*w_* + 4\ dk_T k_T\delta) + \frac{n_*\gamma(w_*^2 - 4dk_T k_T\beta_T\delta)}{(1 + w_*\sigma)^2},$$

$$q_1 = -\frac{n_*(q_{111} + q_{112} + q_{113})}{q_{121}},$$

$$\begin{aligned} q_{111} = {} & 4d^3k_T k_T^3(1 + w_*\sigma)(1 + 3w_*\sigma + \beta_T\gamma\delta\sigma + 3w_*^2\sigma^2 + w_*^3\sigma^3) \\ & + n_*^2(1 + w_*\sigma)[n_*(1 + w_*\sigma)^3 + w_*\gamma(2 + w_*\sigma)], \end{aligned}$$

$$\begin{aligned} q_{112} = {} & dk_T k_T[w_*^2\gamma^2 + 2n_*w_*\gamma(5 + 9w_*\sigma + 4w_*^2\sigma^2) \\ & + 2n_*^2(1 + w_*\sigma)\ (3 + 9w_*\sigma + 2\beta_T\gamma\delta\sigma + 9w_*^2\sigma^2 + 3w_*^3\sigma^3\ )], \end{aligned}$$

$$\begin{aligned} q_{113} = {} & d^2k_T k_T^2[\gamma\left(8w_* - 4\beta_T\gamma\delta + 15w_*^2\sigma + 7w_*^3\sigma^2\right) + n_*(1 + w_*\sigma)\ (9 + 27w_*\sigma \\ & + 8\beta_T\gamma\delta\sigma + 27w_*^2\sigma^2 + 9w_*^3\sigma^3\Big)], \end{aligned}$$

$$\begin{aligned} q_{121} = {} & (dk_T k_T + n_*)^2(1 + w_*\sigma)^2[16\ d^2\delta(k_T k_T + k_T k_T w_*\sigma)^2 \\ & + n_*(1 + 2w_*\sigma + 2n_*w_*(1 + w_*\sigma)^2 + w_*^2(\gamma + \sigma^2)) + 4dk_T k_T((1 + w_*\sigma)^2 \\ & + n_*(\delta - \beta_T\gamma\delta + 2w_*^3\sigma^2 + 2w_*(1 + \delta\sigma) + w_*^2\sigma(4 + \delta\sigma)))], \end{aligned}$$

$$f_2 = -\frac{n_*\gamma(f_{211} + f_{212})}{f_{221}},$$

$$f_{211} = (1 + 2n_*w_* + 3dk_T k_T\delta)[n_*\gamma - (dk_T k_T + n_*)\gamma + (dk_T k_T + n_*)^2\sigma(1 + w_*\sigma)],$$

$$f_{212} = n_*(dk_T k_T + n_*)[2w_*\gamma + (dk_T k_T + n_*)(1 + w_*\sigma)^2],$$

$$\begin{aligned} f_{221} = {} & (dk_T k_T + n_*)^2(1 + w_*\sigma)^4[(3\ dk_T k_T + n_*)(1 + 2n_*w_* + 3\ dk_T k_T\delta) \\ & + \tfrac{n_*\gamma\left(w_*^2 - 3dk_T k_T\beta_T\delta\right)}{(1+w_*\sigma)^2}]. \end{aligned}$$

$$q_2 = -\frac{n_*(q_{211} + q_{212} + q_{213})}{q_{221}(q_{222} + q_{223})},$$

$$\begin{aligned} q_{211} = {} & 3d^3k_T k_T^3(1 + w_*\sigma)(1 + 3w_*\sigma + \beta_T\gamma\delta\sigma + 3w_*^2\sigma^2 + w_3^3\sigma^3) \\ & + n_*^2(1 + w_*\sigma)\left[n_*(1 + w_*\sigma)^3 + w_*\gamma(2 + w_*\sigma)\right], \end{aligned}$$

$$\begin{aligned} q_{212} = {} & dk_T k_T[w_*^2\gamma^2 + 2n_*w_*\gamma\left(4 + 7w_*\sigma + 3w_*^2\sigma^2\right) \\ & + n_*^2(1 + w_*\sigma)(5 + 15w_*\sigma + 3\beta_T\gamma\delta\sigma + 15w_*^2\sigma^2 + 5w_*^3\sigma^3)], \end{aligned}$$

$$q_{213} = d^2 k_T k_T^2 [\gamma \left(6w_* - 3\beta_T \gamma\delta + 11 w_*^2 \sigma + 5 w_*^3 \sigma^2\right)$$
$$+n_*(1 + w_*\sigma)(7 + 21 w_*\sigma + 6\beta_T \gamma\delta\sigma + 21 w_*^2 \sigma^2 + 7 w_*^3 \sigma^3)],$$

$$q_{221} = (dk_T k_T + n_*)^2 (1 + w_*\sigma)^2,$$

$$q_{222} = 9\, d^2 \delta(k_T k_T + k_T k_T w_*\sigma)^2 + n_*[1 + 2w_*\sigma + 2n_* w_*(1 + w_*\sigma)^2 + w_*^2(\gamma + \sigma^2)],$$

$$q_{223} = 3dk_T k_T[(1 + w_*\sigma)^2 + n_*(\delta - \beta_T \gamma\delta + 2w_*^3 \sigma^2 + 2w_*(1 + \delta\sigma) + w_*^2 \sigma(4 + \delta\sigma))],$$

where

$$j + 1 = mod(j, 3) + 1.$$

$$R_1 = \begin{pmatrix} -\bar{n}_1^2 + \frac{\gamma \bar{n}_1 \bar{w}_1}{(1+\sigma w_*)^2} - \frac{\gamma\sigma \bar{w}_1^2 n_*}{(1+\sigma w_*)^2} \\ -2\bar{n}_1 \bar{w}_1 w_* - \bar{w}_1^2 n_*; \end{pmatrix},$$

$$R_2 = \beta_1 \begin{pmatrix} 0 & 0 \\ -\delta dk_T^2 & 0 \end{pmatrix} \begin{pmatrix} \bar{n}_1 \\ \bar{w}_1 \end{pmatrix},$$

$$B_1 = \frac{\partial \bar{V_2}'}{\partial T_1}(\theta + \varphi) + (\theta + \varphi)\frac{\partial \bar{w}_1}{\partial T_2} - \varphi\beta_2 \left(-\theta\delta dk_T^2\right) \bar{w}_1 + \beta_1 \varphi\delta dk_T^2 \bar{V_2}'\theta,$$

$$B_2 = 2\left[\Psi_{11} |\bar{w}_1|^2 \bar{w}_1 + \Psi_{12}(|\bar{w}_2|^2 + |\bar{w}_3|^2)\bar{w}_1 + \theta(\bar{w}_2' \bar{u}_2^3 + \bar{w}_3' \bar{u}_2^2)\right],$$

$$B_3 = \frac{\gamma\sigma}{(1 + \sigma w_*)^3}[3\theta \bar{w}_1(|\bar{w}_1|^2 + 2(|\bar{w}_2|^2 + |\bar{w}_3|^2)],$$

$$B_4 = -\frac{\gamma}{(1 + \sigma w_*)^2}\left[\frac{\Psi_{11}}{\theta} |\bar{w}_1|^2 \bar{w}_1 + \frac{\Psi_{12}}{\theta}(|\bar{w}_2|^2 + |\bar{w}_3|^2)\bar{w}_1 + (\bar{w}_2' \bar{u}_2^3 + \bar{w}_3' \bar{u}_2^2)\right],$$

$$B_5 = -\frac{\gamma}{(1 + \sigma w_*)^2}\left[\Psi_{21} |\bar{w}_1|^2 \bar{w}_1 + \Psi_{22}(|\bar{w}_2|^2 + |\bar{w}_3|^2)\bar{w}_1 + \theta(\bar{w}_2' \bar{v}_2^3 + \bar{w}_3' \bar{v}_2^2)\right],$$

$$B_6 = -\frac{\gamma\sigma^2 n_*}{(1 + \sigma w_*)^4}\left[3\bar{w}_1 \left(|\bar{w}_1|^2 + 2(|\bar{w}_2|^2 + |\bar{w}_3|^2)\right)\right],$$

$$B_7 = \frac{2\gamma\sigma n_*}{(1 + \sigma w_*)^3}\left[\frac{\Psi_{21}}{\theta} |\bar{w}_1|^2 \bar{w}_1 + \frac{\Psi_{22}}{\theta}(|\bar{w}_2|^2 + |\bar{w}_3|^2)\bar{w}_1 + (\bar{w}_2' \bar{v}_2^3 + \bar{w}_3' \bar{v}_2^2)\right],$$

$$B_8 = 3\theta \bar{w}_1 \left[|\bar{w}_1|^2 + 2(|\bar{w}_2|^2 + |\bar{w}_3|^2)\right],$$

$$B_9 = 2w_* \left[\frac{\Psi_{11}}{\theta} |\bar{w}_1|^2 \bar{w}_1 + \frac{\Psi_{12}}{\theta}(|w_2|^2 + |w_3|^2)w_1 + (w_2' \bar{u}_2^3 + w_3' \bar{u}_2^2)\right],$$

$$B_{10} = 2w_* \left[\Psi_{21} |\bar{w}_1|^2 \bar{w}_1 + \Psi_{22}(|\bar{w}_2|^2 + |\bar{w}_3|^2)\bar{w}_1 + \theta(\bar{w}_2' \bar{v}_2^3 + \bar{w}_3' \bar{v}_2^2)\right],$$

$$B_{11} = 2n_*\left[\frac{\Psi_{21}}{\theta}\left|\bar{w}_1\right|^2\bar{w}_1 + \frac{\Psi_{22}}{\theta}(|\bar{w}_2|^2 + |\bar{w}_3|^2)\bar{w}_1 + (\bar{w}_2'\bar{v}_2^3 + \bar{w}_3'\bar{v}_2^2)\right].$$

$$\Psi_{11} = \theta(f_0 + f_1), \qquad \Psi_{12} = \theta(f_0 + 2 * f_2),$$
$$\Psi_{21} = \theta(q_0 + q_1), \qquad \Psi_{22} = \theta(q_0 + 2 * q_2).$$

$$\begin{aligned}
C_1 &= (\theta+\varphi)\left[\varepsilon\tfrac{\partial\bar{w}_1\varepsilon}{\partial T_1} + (\varepsilon\tfrac{\partial\varepsilon^2\bar{v}_1'}{\partial T_1} + \varepsilon^2\tfrac{\partial\varepsilon\bar{w}_1}{\partial T_2})\right] \\
&\quad +\varphi\delta dk_T^2\theta\left[\beta_1\varepsilon\bar{w}_1\varepsilon + \beta_2\varepsilon^2\bar{w}_1\varepsilon + \beta_1\varepsilon\bar{V}_2'\varepsilon^2\right], \\
C_2 &= 2\bar{w}_2'\varepsilon\bar{w}_3'\varepsilon\left[(\theta^2 - \tfrac{\gamma\theta}{(1+\sigma w_*)^2} + \tfrac{\gamma\sigma n_*}{(1+\sigma w_*)^3} + 2w_*\varphi\theta + \varphi n_*)\right] \\
&\quad +(\varepsilon\bar{w}_2'\bar{v}_2^3\varepsilon^2 + \bar{w}_3'\varepsilon\bar{v}_2^2\varepsilon^2)\left[2\theta^2 - \tfrac{\gamma\theta}{(1+\sigma w_*)^2} + 2w_*\varphi\theta\right], \\
C_3 &= (\varepsilon\bar{w}_2'\varepsilon^2\bar{v}_2^3 + \varepsilon\bar{w}_3'\varepsilon^2\bar{v}_2^2)\left[(-\frac{\gamma}{(1+\sigma w_*)^2})\theta + \frac{2\gamma\sigma n_*}{(1+\sigma w_*)^3} + 2w_*\theta\varphi + 2n_*\varphi\right], \\
C_4 &= \left|\bar{w}_1\right|^2\bar{w}_1\left[2\Psi_{11} + \tfrac{3\theta\gamma\sigma}{(1+\sigma w_*)^3} - \tfrac{\gamma}{(1+\sigma w_*)^2}\tfrac{\Psi_{11}}{\theta} - \tfrac{\gamma}{(1+\sigma w_*)^2}\Psi_{21} - \tfrac{3\gamma\sigma^2 n_*}{(1+\sigma w_*)^4}\right. \\
&\quad \left. +\tfrac{2\gamma\sigma n_*}{(1+\sigma w_*)^3}\tfrac{\Psi_{21}}{\theta} + (3\theta + 2w_*\tfrac{\Psi_{11}}{\theta} + 2w_*\Psi_{21} + 2n\tfrac{\Psi_{21}}{\theta})\varphi\right], \\
C_5 &= (|\bar{w}_2|^2 + |\bar{w}_3|^2)w_1\left[2\Psi_{12} + \tfrac{6\theta\gamma\sigma}{(1+\sigma w_*)^3} - \tfrac{\gamma}{(1+\sigma w_*)^2}\tfrac{\Psi_{12}}{\theta} - \tfrac{\gamma}{(1+\sigma w_*)^2}\Psi_{22}\right. \\
&\quad \left. -\tfrac{6\gamma\sigma^2 n_*}{(1+\sigma w_*)^4} + \tfrac{2\gamma\sigma n_*}{(1+\sigma w_*)^3}\tfrac{\Psi_{22}}{\theta} + \left(6\theta + 2w_*\tfrac{\Psi_{12}}{\theta} + 2w_*\Psi_{22} + 2n_*\tfrac{\Psi_{22}}{\theta}\right)\varphi\right], \\
C_6 &= (\bar{w}_2'\bar{u}_2^3 + \bar{w}_3'\bar{u}_2^2)\left[2\theta - \tfrac{\gamma}{(1+\sigma w_*)^2} + 2w_*\varphi\right] \\
&\quad +(\bar{w}_2'\bar{v}_2^3 + \bar{w}_3'\bar{v}_2^2)\left[-\tfrac{\gamma}{(1+\sigma w_*)^2}\theta + \tfrac{2\gamma\sigma n_*}{(1+\sigma w_*)^3} + 2w_*\theta\varphi + 2n_*\varphi\right].
\end{aligned}$$

$$\tau_0 = \frac{\frac{\theta+\varphi}{\beta_T}}{\theta\varphi\delta dk_T^2}, \qquad \mu = \frac{\beta - \beta_T}{\beta_T}.$$

$$h = \frac{2\left[-\theta + \frac{\gamma\theta}{(1+\sigma w_*)^2} - \frac{\gamma\sigma n_*}{(1+\sigma w_*)^3} - 2w_*\varphi\theta - \varphi n_*\right]}{\beta_T(\theta\varphi\delta dk_T^2)}.$$

$$g_1 = \frac{J}{\beta_T(\theta\varphi\delta dk_T^2)}, \quad g_2 = \frac{PA_1}{\beta_T(\theta\varphi\delta dk_T^2)}.$$

$$\begin{aligned}
J &= 2\Psi_{11} + \tfrac{3\theta\gamma\sigma}{(1+\sigma w_*)^3} - \tfrac{\gamma}{(1+\sigma w_*)^2}\tfrac{\Psi_{11}}{\theta} - \tfrac{\gamma}{(1+\sigma w_*)^2}\Psi_{21} - \tfrac{3\gamma\sigma^2 n_*}{(1+\sigma w_*)^4} \\
&\quad +\tfrac{2\gamma\sigma n_*}{(1+\sigma w_*)^3}\tfrac{\Psi_{21}}{\theta} + \left(3\theta + 2w_*\tfrac{\Psi_{11}}{\theta} + 2w_*\Psi_{21} + 2n_*\tfrac{\Psi_{21}}{\theta}\right)\varphi. \\
P &= 2\Psi_{12} + \tfrac{6\theta\gamma\sigma}{(1+\sigma w_*)^3} - \tfrac{\gamma}{(1+\sigma w_*)^2}\tfrac{\Psi_{12}}{\theta} - \tfrac{\gamma}{(1+\sigma w_*)^2}\Psi_{22} - \tfrac{6\gamma\sigma^2 n_*}{(1+\sigma w_*)^9} \\
&\quad +\tfrac{2\gamma\sigma n_*}{(1+\sigma w_*)^3}\tfrac{\Psi_{22}}{\theta} + \left(6\theta + 2w_*\tfrac{\Psi_{12}}{\theta} + 2w_*\Psi_{22} + 2n_*\tfrac{\Psi_{22}}{\theta}\right)\varphi.
\end{aligned}$$