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*Research article*

# Anti-periodic synchronization of quaternion-valued high-order Hopfield neural networks with delays

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**Abstract:** This paper proposes a class of quaternion-valued high-order Hopfield neural networks with delays. By using the non-decomposition method, non-reduced order method, analytical techniques in uniform convergence functions sequence, and constructing Lyapunov function, we obtain several sufficient conditions for the existence and global exponential synchronization of anti-periodic solutions for delayed quaternion-valued high-order Hopfield neural networks. Finally, an example and its numerical simulations are given to support the proposed approach. Our results play an important role in designing inertial neural networks.

**Keywords:** inertial neural networks; quaternion; anti-periodic solutions; non-reduced order method; synchronization
**Mathematics Subject Classification:** 34D06, 34D23, 34K13, 34K24

## 1. Introduction

Hopfield neural networks model, which was first discussed by Babcock and Westervelt, has been widely applications such as signal processing, pattern recognition, associative memory, and so on. Since high-order Hopfield neural networks have stronger approximation properties, faster convergence speed, and larger storage capacity, numerous works have intensively analyzed high-order Hopfield neural networks in recent years. In particular, many good results are explored by some authors, especially the existence and stability of solutions for Hopfield neural networks [1–7].

As is well known, a new multidimensional neural networks model, which has been described quaternion-valued neural networks model, is 4-dimensional and represents a generalization of the real-valued and complex-valued neural networks. Quaternion-valued neural networks have received the interest of many scholars and there are some good research results about dynamic behaviors. For instance, in [8–12], some authors have discussed the stability of quaternion-valued neural networks with delays. In [13,14], some authors have discussed the existence and global exponential stability of pseudo almost periodic solutions and almost automorphic solutions for quaternion-valued neural networks with delays and leakage terms. In [15,16], some authors have discussed the existence and exponential stability of anti-periodic solutions for quaternion-valued neural networks.

Synchronization has played a key role in network control and system design. Synchronization has become a very hot topic across many research fields such as neural networks, information processing, biological systems, and so on. Recently, in [17–20], some authors have explored the synchronization of neural networks with time-varying delays. In [21–23], some authors have explored periodic synchronization, anti-periodic synchronization, and almost automorphic synchronization for delayed neural networks. Anti-periodic synchronization is one of the dynamic properties of neural networks. So far, there are many research results about synchronization, but there are few involved in counter-periodic synchronization. Therefore, it is of great significance to study the anti-periodic synchronization of neural networks.

As well known, neural network models are often described by first-order differential equations. However, inertial electronic neural networks, which were introduced by Babcock and Westervelt (see [24]), are modeled by second-order differential equations. Their dynamics can be more complex compared with first-order neural network models. Recently, in [25–30], many authors have explored the inertial neural networks, and there are many research results. In [31], some authors have explored the complex-valued inertial neural networks by applying the reduced-order and Lyapunov functional methods. In [32], some authors have explored the complex-valued inertial neural networks by using the non-reduced order method and establishing a novel Lyapunov function.

With the inspiration from the previous research, to fill the gap in the research field of quaternion-valued high-order Hopfield neural networks, the work of this article comes from three main motivations. (1) Recently, in [16], some authors have discussed the existence and exponential stability of anti-periodic solutions for inertial quaternion-valued high-order Hopfield neural networks by using the non-decomposition method and reduced-order method; in [22], some authors have discussed the anti-periodic synchronization of quaternion-valued generalized cellular neural networks by using decomposition method. However, using the reduced-order method or decomposition method will increase the complexity of the calculation. (2) In [25–30], some authors have discussed the stability of solutions for high-order real-valued neural networks by the non-reduced order method. Up to now, there has been no paper about high-order quaternion-valued neural networks yet by using the non-reduced order method. (3) In practical applications, synchronization is an interesting and significant dynamical property for differential equations, thus it is worth studying the anti-periodic synchronization of quaternion-valued neural network models. Therefore, in this paper, to overcome the complexity of the calculation, we will investigate the existence and exponential synchronization of anti-periodic solutions for quaternion-valued high-order Hopfield neural networks by using the non-decomposition method, non-reduced order method, analytical techniques in uniform convergence functions sequence, and Lyapunov function method.

Compared with the previous kinds of literature, the main contributions of this paper are listed as follows. (1) Firstly, to the best of our knowledge, without separating the quaternion-valued neural networks into real-valued neural networks. Therefore, the results are less conservative and more general. (2) Secondly, this paper, considering the model as a whole, applies the non-reduced order method to investigate differential equations. (3) Thirdly, our method in this paper can be used to study the anti-periodic synchronization for other types of quaternion-valued neural networks.

This paper is organized as follows: In Section 2, we introduce some definitions and preliminary lemmas. In Section 3, we establish some sufficient conditions for the existence anti-periodic solutions of system (2.1), global exponential synchronization for system (2.1) and system (3.7). In Section 4, one numerical example is provided to verify the effectiveness of the theoretical results. Finally, we draw a conclusion in Section 5.

**Notations**: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$ denotes the set of non-negative real numbers, $\mathbb{H}$ denotes the set of quaternion numbers, $\mathbb{H}^n$ denotes the $n$ dimensional quaternion numbers, $\|\cdot\|_{\mathbb{H}}$ represents the vector quaternion norm. For $x \in \mathbb{H}$, we define $\|x\|_{\mathbb{H}} = \sup\limits_{t\in[0,\omega]} \Big\{|x(t)|\Big\}$ and for $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{H}^n$, we define $\|x\|_{\mathbb{H}^n} = \sum\limits_{p=1}^{n} \|x_p\|_{\mathbb{H}}$. For $u \in \mathbb{H}^n$. Let

$$\mathbb{X} = \Big\{x \in C^1\Big([0, \frac{\omega}{2}], \mathbb{H}^n\Big) : x\Big(t + \frac{\omega}{2}\Big) = -x(t), t \in \mathbb{R}\Big\}$$

be one Banach space equipped with the norms

$$\|x\|_{\mathbb{X}} = \max\Big\{\|x\|_{\mathbb{H}^n}, \|\dot{x}\|_{\mathbb{H}^n}\Big\}.$$

## 2. Preliminaries

In this section, we shall first recall some fundamental definitions and lemmas.

The algebra of quaternion is defined as

$$\mathbb{H} := \Big\{x : x = x^R + ix^I + jx^J + kx^K\Big\},$$

where $x^R, x^I, x^J, x^K \in \mathbb{R}$ and the three imaginary units $i, j$ and $k$ obey the Hamolton's multiplication rules:

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,\ i^2 = j^2 = k^2 = -1.$$

For every $x \in \mathbb{H}$, the conjugate of quaternion $x$ is defined as $\bar{x} = x^R - ix^I - jx^J - kx^K$, its norm as $|x| = \sqrt{x\bar{x}} = \sqrt{(x^R)^2 + (x^I)^2 + (x^J)^2 + (x^K)^2}$.

In this paper, we will study the solutions of quaternion-valued high-order Hopfield neural networks with delays in the following model of differential equations:

$$\begin{aligned}\ddot{x}_p(t) \;=\; & -a_p(t)\dot{x}_p(t) - b_p(t)x_p(t) + \sum_{q=1}^{n} c_{pq}(t)f_q\big(x_q(t)\big) + \sum_{q=1}^{n} d_{pq}(t) \\ & \times g_q\big(x_q(t-\gamma_{pq}(t))\big) + \sum_{q=1}^{n}\sum_{\iota=1}^{n} e_{pq\iota}(t)h_q\big(x_q(t-\alpha_{pq\iota}(t))\big) \\ & \times h_\iota\big(x_\iota(t-\beta_{pq\iota}(t))\big) + U_p(t), \end{aligned} \tag{2.1}$$

where $p = 1, 2, \cdots, n$, $x_i(t) \in \mathbb{H}$ is the state vector of the $p$th unit at time $t$, $a_p(t) > 0$, $b_p(t) > 0$ represent the rate with which the $p$th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $c_{pq}, d_{pq}, e_{pq\iota} \in \mathbb{H}$ denote the strength of connectivity, the activation functions $f_q, g_q, h_q \in \mathbb{H}$ show how the $q$th neuron reacts to input, time-varying delay factors satisfy that $\gamma_{pq}(t), \alpha_{pq\iota}(t), \beta_{pq\iota}(t) \in \mathbb{R}_+$, $U_p \in \mathbb{H}$ denotes the $p$th component of an external input source introduced from outside the network to the unit $p$ at time $t$.

The initial value of system (2.1) is the following

$$x_p(s) = \varphi_p^x(s), \ \ \dot{x}_p(s) = \psi_p^x(s), \ \ s \in [-\tau, 0], \tag{2.2}$$

where $\varphi_p^x(s), \psi_p^x \in C\Big([-\tau, 0], \mathbb{H}\Big)$, $p = 1, 2, \cdots, n$, $\tau = \max\Big\{\gamma, \alpha, \beta\Big\}$, $\gamma = \max\limits_{1 \le p,q \le n} \Big\{ \sup\limits_{t \in [0,\omega]} \gamma_{pq}(t) \Big\}$, $\alpha = \max\limits_{1 \le p,q,\iota \le n} \Big\{ \sup\limits_{t \in [0,\omega]} \alpha_{pq\iota}(t) \Big\}$, $\beta = \max\limits_{1 \le p,q,\iota \le n} \Big\{ \sup\limits_{t \in [0,\omega]} \beta_{pq\iota}(t) \Big\}$.

To study the existence of $\frac{\omega}{2}$-anti-periodic solution of system (2.1), we need the following assumptions:

$(H_1)$ For $p, q, \iota = 1, 2, \cdots, n$, $a_p, b_p, \gamma_{pq}, \alpha_{pq\iota}, \beta_{pq\iota} : \mathbb{R} \to \mathbb{R}_+$, $c_p, d_p, e_{pq\iota}, U_p : \mathbb{R} \to \mathbb{H}$, there exists $\omega > 0$ such that

$$a_p(t + \frac{\omega}{2}) = a_p(t), \ \ b_p(t + \frac{\omega}{2}) = b_p(t), \ \ U_p(t + \frac{\omega}{2}) = -U_p(t),$$

$$\gamma_{pq}(t + \frac{\omega}{2}) = \gamma_{pq}(t), \ \ \alpha_{pq\iota}(t + \frac{\omega}{2}) = \alpha_{pq\iota}(t), \ \ \beta_{pq\iota}(t + \frac{\omega}{2}) = \beta_{pq\iota}(t),$$

$$c_{pq}(t + \frac{\omega}{2}) f_q(u) = -c_{pq}(t) f_q(-u), \ \ d_{pq}(t + \frac{\omega}{2}) g_q(u) = -d_{pq}(t) g_q(-u),$$

$$e_{pq\iota}(t + \frac{\omega}{2}) h_q(u) h_\iota(v) = -e_{pq\iota}(t) h_q(-u) h_\iota(-v);$$

$(H_2)$ For $p, q, \iota = 1, 2, \cdots, n$, $t \in \mathbb{R}$, $\gamma_{pq}(t), \alpha_{pq\iota}(t), \beta_{pq\iota}(t)$ are continuously differentiable defined on $\mathbb{R}^+$, and $1 - \dot{\gamma}_{pq}(t) > 0$, $1 - \dot{\alpha}_{pq\iota}(t) > 0$, $1 - \dot{\beta}_{pq\iota}(t) > 0$, there exist positive constants $\dot{\gamma}_{pq}$, $\dot{\alpha}_{pq\iota}, \dot{\beta}_{pq\iota}$ such that

$$\dot{\gamma}_{pq} = \max_{1 \le p,q \le n} \Big\{ \sup_{t \in [0,\omega]} \dot{\gamma}_{pq}(t) \Big\},$$

$$\dot{\alpha}_{pq\iota} = \max_{1 \le p,q,\iota \le n} \Big\{ \sup_{t \in [0,\omega]} \dot{\alpha}_{pq\iota}(t) \Big\},$$

$$\dot{\beta}_{pq\iota} = \max_{1 \le p,q,\iota \le n} \Big\{ \sup_{t \in [0,\omega]} \dot{\beta}_{pq\iota}(t) \Big\};$$

$(H_3)$ For $q = 1, 2, \cdots, n$, there exist positive constants $L_f, L_g, L_h$ such that

$$|f_q(u) - f_q(v)| \le L_f |u - v|,$$

$$|g_q(u) - g_q(v)| \le L_g |u - v|,$$

$$|h_q(u) - h_q(v)| \le L_h |u - v|;$$

$(H_4)$ There exist positive constants $M_q^f$, $M_q^g$, $M_q^h$ such that $|f_q(u)| \le M_q^f$, $|g_q(u)| \le M_q^g$, $|h_q(u)| \le M_q^h$, $\forall u \in \mathbb{H}$, $q = 1, 2, \cdots, n$;

$(H_5)$ There exists a positive constant $\lambda$ such that

$$\begin{aligned}&2\lambda+4-2a_p^- -2b_p^- +\sum_{q=1}^{n}(c_{pq}^+)^2L_f^2(2\lambda+3)\\&+\sum_{q=1}^{n}\frac{(d_{pq}^+)^2(L_g)^2e^{2\lambda\gamma}}{1-\dot{\gamma}_{pq}}+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_\iota^h)^2e^{2\lambda\alpha}}{1-\dot{\alpha}_{pq\iota}}\\&+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_q^h)^2e^{2\lambda\beta}}{1-\dot{\beta}_{pq\iota}}<0,\end{aligned}$$

where

$$a_p^-=\inf_{[0,\omega]}a_p(t),\ a_p^+=\sup_{[0,\omega]}a_p(t),\ b_p^-=\inf_{[0,\omega]}b_p(t),\ b_p^+=\sup_{[0,\omega]}b_p(t),$$

$$c_{pq}^+=\max_{1\le p,q\le n}\|c_{pq}(t)\|_{\mathbb{H}},\ U_p=\max_{1\le p\le n}\|U_p(t)\|_{\mathbb{H}},$$

$$d_{pq}^+=\max_{1\le p,q\le n}\|d_{pq}(t)\|_{\mathbb{H}},\ e_{pq\iota}^+=\max_{1\le p,q,\iota\le n}\|e_{pq\iota}(t)\|_{\mathbb{H}}.$$

**Definition 2.1.** *A continuous function $x=(x_1,x_2\cdots,x_n)^T:[0,+\infty]\to\mathbb{H}^n$ is said to be a solution of system* (2.1)*, if*

(*i*) *$x(s)=\varphi^x(s),\dot{x}(s)=\psi^x(s)$ for $s\in[-\tau,0]$, where $\varphi^x=(\varphi_1^x,\varphi_2^x,\cdots,\varphi_n^x)^T$, $\varphi_p^x\in C([-\tau,0],\mathbb{H})$, $\psi^x=(\psi_1^x,\psi_2^x,\cdots,\psi_n^x)^T$, $\psi_p^x\in C([-\tau,0],\mathbb{H})$, $p=1,2,\cdots,n$;*

(*ii*) *$x(t)$ satisfies system* (2.1) *for $t\ge 0$.*

**Definition 2.2.** *A solution $x$ of system* (2.1) *is said to be $\frac{\omega}{2}$-anti-periodic solution of system* (2.1)*, if there exists $\omega>0$ such that*

$$x(t+\frac{\omega}{2})=-x(t).$$

**Definition 2.3.** *The drive system* (2.1) *and the response system* (3.7) *are said to be globally exponentially synchronized, if there exist constants $\epsilon>0$ and $N>0$ such that*

$$\|y(t)-x(t)\|_{\mathbb{X}}\le N\|\chi\|_{\mathbb{X}}e^{-\epsilon t},\forall t>0,$$

*where $x=(x_1,x_2,\cdots,x_n)^T$ is a solution of system* (2.1) *with the initial value $\varphi^x=(\varphi_1^x,\varphi_2^x,\cdots,\varphi_n^x)^T$ and $\psi^x=(\psi_1^x,\psi_2^x,\cdots,\psi_n^x)^T$, $y=(y_1,y_2,\cdots,y_n)^T$ is a solution of system* (3.4) *with the initial value $\varphi^y=(\varphi_1^y,\varphi_2^y,\cdots,\varphi_n^y)^T$ and $\psi^y=(\psi_1^y,\psi_2^y,\cdots,\psi_n^y)^T$,*

$$\|\chi\|_{\mathbb{X}}=\max\left\{\|\varphi^y-\varphi^x\|_{\mathbb{X}},\|\psi^y-\psi^x\|_{\mathbb{X}}\right\}.$$

**Lemma 2.1.** *[33] For all $u,v\in\mathbb{H}$, $\bar{u}v+\bar{v}u\le\bar{u}u+\bar{v}v$.*

**Lemma 2.2.** *Suppose that $(H_1)$–$(H_5)$ hold. Let $x=(x_1,x_2,\cdots,x_n)^T$ and $x^*=(x_1^*,x_2^*,\cdots,x_n^*)^T$ are two solutions of* (2.1) *with initial value*

$$\varphi^x=(\varphi_1^x,\varphi_2^x,\cdots,\varphi_n^x)^T,\psi^x=(\psi_1^x,\psi_2^x,\cdots,\psi_n^x)^T,$$

*and*

$$\varphi^{x^*} = (\varphi_1^{x^*}, \varphi_2^{x^*}, \cdots, \varphi_n^{x^*})^T, \psi^x = (\psi_1^{x^*}, \psi_2^{x^*}, \cdots, \psi_n^{x^*})^T.$$

*Then for* $\forall t > 0$, *there exist two constants* $M > 1$ *and* $\lambda > 0$ *such that*

$$\|x(t) - x^*(t)\|_{\mathbb{H}^n} \leq M\|\phi\|_{\mathbb{X}} e^{-\lambda t},$$

*and*

$$\|\dot{x}(t) - \dot{x}^*(t)\|_{\mathbb{H}^n} \leq M\|\phi\|_{\mathbb{X}} e^{-\lambda t},$$

*where*

$$\|\phi\|_{\mathbb{X}} = \max\Big\{\|\varphi^x - \varphi^{x^*}\|_{\mathbb{X}}, \|\psi^x - \psi^{x^*}\|_{\mathbb{X}}\Big\}.$$

*Proof.* Denote $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ and $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$ are two solutions of (2.1). Let $w_p(t) = x(t) - x_p^*(t)$, then

$$\begin{aligned} \ddot{w}_p(t) &= -a_p(t)\dot{w}_p(t) - b_p(t)w_p(t) + \sum_{q=1}^{n} c_{pq}(t)(f_q(x_q(t)) - f_q(x_q^*(t))) \\ &\quad + \sum_{q=1}^{n} d_{pq}(t)(g_q(x_q(t - \gamma_{pq}(t))) - g_q(x_q^*(t - \gamma_{pq}(t)))) \\ &\quad + \sum_{q=1}^{n}\sum_{\iota=1}^{n} e_{pq\iota}(t)(h_q(x_q(t - \alpha_{pq\iota}(t)))h_\iota(x_\iota(t - \beta_{pq\iota}(t))) \\ &\quad - h_q(x_q^*(t - \alpha_{pq\iota}(t)))h_\iota(x_\iota^*(t - \beta_{pq\iota}(t)))). \end{aligned} \tag{2.3}$$

We define a Lyapunov function as follows:

$$\begin{aligned} V(t) &= \sum_{p=1}^{n} e^{2\lambda t}\bar{\dot{w}}_p(t)\dot{w}_p(t) + \sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^+)^2 L_f^2 e^{2\lambda t}\bar{w}_q(t)w_q(t) \\ &\quad + \sum_{p=1}^{n}\sum_{q=1}^{n} \frac{(d_{pq}^+)^2(L_g)^2 e^{2\lambda\gamma}}{1 - \dot{\gamma}_{pq}} \int_{t-\gamma_{pq}(t)}^{t} e^{2\lambda s}|w_q(s)|^2 ds \\ &\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n} \frac{(e_{pq\iota}^+)^2 L_h^2 (M_\iota^h)^2 e^{2\lambda\alpha}}{1 - \dot{\alpha}_{pq\iota}} \int_{t-\alpha_{pq\iota}(t)}^{t} e^{2\lambda s}|w_q(s)|^2 ds \\ &\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n} \frac{(e_{pq\iota}^+)^2 L_h^2 (M_q^h)^2 e^{2\lambda\beta}}{1 - \dot{\beta}_{pq\iota}} \int_{t-\beta_{pq\iota}(t)}^{t} e^{2\lambda s}|w_\iota(s)|^2 ds. \end{aligned} \tag{2.4}$$

For $t \geq 0$, $p, q, \iota = 1, 2, \cdots, n$, by (2.3), (2.4) and Lemma 2.1, we can gain

$$\begin{aligned} \frac{d^+}{dt}V(t) &= \sum_{p=1}^{n} e^{2\lambda t}\Big\{2\lambda\bar{\dot{w}}_p(t)\dot{w}_p(t) + \bar{\ddot{w}}_p(t)\dot{w}_p(t) + \bar{\dot{w}}_p(t)\ddot{w}_p(t)\Big\} \\ &\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^+)^2 L_f^2 e^{2\lambda t}\Big\{2\lambda\bar{w}_q(t)w_q(t) + \bar{\dot{w}}_q(t)w_q(t) + \bar{w}_q(t)\dot{w}_q(t)\Big\} \end{aligned}$$

$$
\begin{aligned}
& +\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\lambda\gamma}}{1-\dot{\gamma}_{pq}}e^{2\epsilon t}|w_q(t)|^2-\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\lambda\gamma}}{1-\dot{\gamma}_{pq}} \\
& \times e^{2\lambda(t-\gamma_{pq}(t))}|w_q(t-\gamma_{pq}(t))|^2(1-\dot{\gamma}_{pq}(t))+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^{+})^2 \\
& \times\frac{L_h^2(M_\iota^h)^2e^{2\lambda\alpha}}{1-\dot{\alpha}_{pq\iota}}e^{2\lambda t}|w_q(t)|^2-\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2e^{2\lambda\alpha}}{1-\dot{\alpha}_{pq\iota}} \\
& \times e^{2\lambda(t-\alpha_{pq}(t))}|w_q(t-\alpha_{pq}(t))|^2(1-\dot{\alpha}_{pq}(t))+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^{+})^2 \\
& \times\frac{L_h^2(M_q^h)^2e^{2\lambda\beta}}{1-\dot{\beta}_{pq\iota}}e^{2\lambda t}|w_\iota(t)|^2-\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\lambda\beta}}{1-\dot{\beta}_{pq\iota}} \\
& \times e^{2\lambda(t-\beta_{pq}(t))}|w_\iota(t-\beta_{pq}(t))|^2(1-\dot{\beta}_{pq\iota}(t)) \\
\leq\ & \sum_{p=1}^{n}e^{2\lambda t}\Big\{2\lambda|\dot{w}_p(t)|^2-2a_p^{-}|\dot{w}_p(t)|^2-b_p^{-}(|w_p(t)|^2+|\dot{w}_p(t)|^2) \\
& +\sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2\Big(2\lambda|w_q(t)|^2+2|w_q(t)|^2+|\dot{w}_q(t)|^2\Big)+|\dot{w}_p(t)|^2 \\
& +\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\lambda\gamma}}{1-\dot{\gamma}_{pq}}|w_q(t)|^2+|\dot{w}_p(t)|^2+\sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^{+})^2 \\
& \times\frac{L_h^2(M_\iota^h)^2e^{2\lambda\alpha}}{1-\dot{\alpha}_{pq\iota}}|w_q(t)|^2+|\dot{w}_p(t)|^2+\sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^{+})^2 \\
& \times\frac{L_h^2(M_q^h)^2e^{2\lambda\beta}}{1-\dot{\beta}_{pq\iota}}|w_\iota(t)|^2+|\dot{w}_p(t)|^2\Big\} \\
\leq\ & \sum_{p=1}^{n}e^{2\lambda t}\Big\{2\lambda+4-2a_p^{-}-2b_p^{-}+\sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2(2\lambda+3) \\
& +\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\lambda\gamma}}{1-\dot{\gamma}_{pq}}+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2e^{2\lambda\alpha}}{1-\dot{\alpha}_{pq\iota}} \\
& +\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\lambda\beta}}{1-\dot{\beta}_{pq\iota}}\Big\}\cdot\max\Big\{|w_p(t)|^2,|\dot{w}_p(t)|^2\Big\} \\
<\ & 0,
\end{aligned}
$$

which implies that $V(t)\leq V(0)$ and we can get that

$$
\begin{aligned}
V(0) =\ & \sum_{p=1}^{n}\bar{w}_p(0)\dot{w}_p(0)+\sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2\bar{w}_q(0)w_q(0) \\
& +\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\lambda\gamma}}{1-\dot{\gamma}_{pq}}\int_{-\gamma_{pq}(0)}^{0}e^{2\lambda s}|w_q(s)|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^{2}L_{h}^{2}(M_{\iota}^{h})^{2}e^{2\lambda\alpha}}{1-\dot{\alpha}_{pq\iota}}\int_{-\alpha_{pq\iota}(0)}^{0}e^{2\lambda s}|w_{q}(s)|^{2}ds\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^{2}L_{h}^{2}(M_{q}^{h})^{2}e^{2\lambda\beta}}{1-\dot{\beta}_{pq\iota}}\int_{-\beta_{pq\iota}(0)}^{0}e^{2\lambda s}|w_{\iota}(s)|^{2}ds\\
\leq\ &\sum_{p=1}^{n}\|\dot{w}_{p}(s)\|_{\mathbb{H}}^{2}+\sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^{+})^{2}L_{f}^{2}\|w_{q}(s)\|_{\mathbb{H}}^{2}\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^{2}(L_{g})^{2}e^{2\lambda\gamma}}{2\lambda(1-\dot{\gamma}_{pq})}\|w_{q}(s)\|_{\mathbb{H}}^{2}\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^{2}L_{h}^{2}(M_{\iota}^{h})^{2}e^{2\lambda\alpha}}{2\lambda(1-\dot{\alpha}_{pq\iota})}\|w_{q}(s)\|_{\mathbb{H}}^{2}\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^{2}L_{h}^{2}(M_{q}^{h})^{2}e^{2\lambda\beta}}{2\lambda(1-\dot{\beta}_{pq\iota})}\|w_{\iota}(s)\|_{\mathbb{H}}^{2}\\
\leq\ &\sum_{p=1}^{n}M^{2}(\|\dot{w}_{p}(s)\|_{\mathbb{H}}^{2}+\|w_{p}(s)\|_{\mathbb{H}}^{2}),
\end{aligned}
$$

where $s\in[-\tau,0]$,

$$M^{2}:=\max\Big\{1,A\Big\},$$

and

$$
\begin{aligned}
A\ :=\ &\sum_{q=1}^{n}(c_{pq}^{+})^{2}L_{f}^{2}+\sum_{q=1}^{n}\frac{(d_{pq}^{+})^{2}(L_{g})^{2}e^{2\lambda\gamma}}{2\lambda(1-\dot{\gamma}_{pq})}+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^{2}L_{h}^{2}(M_{\iota}^{h})^{2}e^{2\lambda\alpha}}{2\lambda(1-\dot{\alpha}_{pq\iota})}\\
&+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^{2}L_{h}^{2}(M_{q}^{h})^{2}e^{2\lambda\beta}}{2\lambda(1-\dot{\beta}_{pq\iota})}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
e^{2\lambda t}|\dot{w}_{p}(t)|^{2}\ &\leq\ M^{2}(\|\dot{w}_{p}(s)\|_{\mathbb{H}}^{2}+\|w_{p}(s)\|_{\mathbb{H}}^{2})\\
&\leq\ M^{2}\Big(\|\dot{w}_{p}(s)\|_{\mathbb{H}}+\|w_{p}(s)\|_{\mathbb{H}}\Big)^{2}
\end{aligned}
$$

and

$$
\begin{aligned}
(c_{pq}^{+})^{2}L_{f}^{2}e^{2\lambda t}|w_{p}(t)|^{2}\ &\leq\ M^{2}(\|\dot{w}_{p}(s)\|_{\mathbb{H}}^{2}+\|w_{p}(s)\|_{\mathbb{H}}^{2})\\
&\leq\ M^{2}\Big(\|\dot{w}_{p}(s)\|_{\mathbb{H}}+\|w_{p}(s)\|_{\mathbb{H}}\Big)^{2},
\end{aligned}
$$

that is,

$$|\dot{w}_{p}(t)|\leq M(\|\dot{w}_{p}(s)\|_{\mathbb{H}}+\|w_{p}(s)\|_{\mathbb{H}})e^{-\lambda t},$$

and

$$|w_{p}(t)|\leq\frac{M}{c_{pq}^{+}L_{f}}(\|\dot{w}_{p}(s)\|_{\mathbb{H}}+\|w_{p}(s)\|_{\mathbb{H}})e^{-\lambda t}.$$

Hence, for $t > 0$,we have that

$$\|x(t) - x^*(t)\|_{\mathbb{H}^n} \leq 2M\|\phi\|_{\mathbb{X}} e^{-\lambda t},$$

and

$$\|\dot{x}(t) - \dot{x}^*(t)\|_{\mathbb{H}^n} \leq \frac{2M}{c_{pq}^+ L_f}\|\phi\|_{\mathbb{X}} e^{-\lambda t},$$

where

$$\|\phi\|_{\mathbb{X}} = \max\Big\{\|\varphi^x - \varphi^{x^*}\|_{\mathbb{X}}, \|\psi^x - \psi^{x^*}\|_{\mathbb{X}}\Big\}.$$

The proof is completed. □

## 3. Main results

In this section, we will investigate the existence and global exponential synchronization of anti-periodic solutions of quaternion-valued high-order Hopfield neural networks with delays (2.1).

**Theorem 3.1.** *Assume that assumptions* $(H_1)$–$(H_5)$ *hold. Then system* (2.1) *has at least an* $\frac{\omega}{2}$*-anti-periodic solution.*

*Proof.* Let $u(t) = (u_1(t), u_2(2), \cdots, u_n(t))^T \in \mathbb{H}^n$ be a solution of system (2.1) with initial conditions

$$u_p(s) = \varphi_p^u(s), \;\; \dot{u}_p(s) = \psi_p^u(s), \;\; s \in [-\tau, 0], \tag{3.1}$$

where $p = 1, 2, \cdots, n, \varphi_p^u(s), \psi_p^u(s) \in \mathbb{H}$.

For any nonnegative integer $m$, from $(H_1)$, we can get that

$$\begin{aligned}
& \Big((-1)^{m+1}u_p\Big(t + \frac{\omega}{2}(m+1)\Big)\Big)'' \\
= \;& -(-1)^{m+1}a_p\Big(t + \frac{\omega}{2}(m+1)\Big)\Big(u_p\Big(\Big(t + \frac{\omega}{2}(m+1)\Big)\Big)\Big)' \\
& -(-1)^{m+1}b_p\Big(t + \frac{\omega}{2}(m+1)\Big)u_p\Big(\Big(t + \frac{\omega}{2}(m+1)\Big)\Big) \\
& + \sum_{q=1}^{n}(-1)^{m+1}c_{pq}\Big(t + \frac{\omega}{2}(m+1)\Big)f_q\Big(u_q\Big(t + \frac{\omega}{2}(m+1)\Big)\Big) \\
& + \sum_{q=1}^{n}(-1)^{m+1}d_{pq}\Big(t + \frac{\omega}{2}(m+1)\Big)g_q\Big(u_q\Big(t + \frac{\omega}{2}(m+1) \\
& -\gamma_{pq}\Big(t + \frac{\omega}{2}(m+1)\Big)\Big)\Big) + \sum_{q=1}^{n}\sum_{\iota=1}^{n}(-1)^{m+1}e_{pq\iota}\Big(t + \frac{\omega}{2}(m+1)\Big) \\
& \times h_q\Big(u_q\Big(t + \frac{\omega}{2}(m+1) - \alpha_{pq\iota}\Big(t + \frac{\omega}{2}(m+1)\Big)\Big)\Big) \\
& \times h_\iota\Big(u_\iota\Big(t + \frac{\omega}{2}(m+1) - \beta_{pq\iota}\Big(t + \frac{\omega}{2}(m+1)\Big)\Big)\Big) \\
& +(-1)^{m+1}U_p\Big(t + \frac{\omega}{2}(m+1)\Big) \\
= \;& -a_p(t)\Big((-1)^{m+1}u_p\Big(\Big(t + \frac{\omega}{2}(m+1)\Big)\Big)\Big)'
\end{aligned}$$

$$
\begin{aligned}
&-b_p(t)\Big((-1)^{m+1}u_p\Big(\Big(t+\frac{\omega}{2}(m+1)\Big)\Big)\Big)\\
&+\sum_{q=1}^{n}c_{pq}(t)f_q\Big((-1)^{m+1}u_q\Big(t+\frac{\omega}{2}(m+1)\Big)\Big)\\
&+\sum_{q=1}^{n}d_{pq}(t)g_q\Big((-1)^{m+1}u_q\Big(t+\frac{\omega}{2}(m+1)-\gamma_{pq}(t)\Big)\Big)\\
&+\sum_{q=1}^{n}\sum_{\iota=1}^{n}e_{pq\iota}(t)h_q\Big((-1)^{m+1}u_q\Big(t+\frac{\omega}{2}(m+1)-\alpha_{pq\iota}(t)\Big)\Big)\\
&\times h_\iota\Big((-1)^{m+1}u_\iota\Big(t+\frac{\omega}{2}(m+1)-\beta_{pq\iota}(t)\Big)\Big)+U_p(t), \qquad (3.2)
\end{aligned}
$$

where $p = 1, 2, \cdots, n$, $t + \frac{\omega}{2}(m+1) \geq 0$.

Hence, For $t + \frac{\omega}{2}(m+1) \geq 0$, $(-1)^{m+1}u\Big(t + \frac{\omega}{2}(m+1)\Big)$ is a solution of system (2.1). Clearly, $v(t) = -u\Big(t + \frac{\omega}{2}\Big)$ is a solution of system (2.1) with initial conditions

$$
v_p(s) = \varphi_p^v(s),\ \ \dot{v}_p(s) = \psi_p^v(s),\ \ s \in [-\tau, 0],
$$

where $p = 1, 2, \cdots, n$, $\varphi_p^v(s), \psi_p^v(s) \in \mathbb{H}$.

By Lemma 2.2, we can choice a constant $M > 1$ such that

$$
\|u(t) - v(t)\|_{\mathbb{H}^n} \leq M\|\phi\|_{\mathbb{X}}e^{-\lambda t},
$$

and

$$
\|\dot{u}(t) - \dot{v}(t)\|_{\mathbb{H}^n} \leq M\|\phi\|_{\mathbb{X}}e^{-\lambda t},
$$

where

$$
\|\phi\|_{\mathbb{X}} = \max\Big\{\|\varphi^u - \varphi^v\|_{\mathbb{X}}, \|\psi^u - \psi^u\|_{\mathbb{X}}\Big\}.
$$

Therefore,

$$
\begin{aligned}
&\Big\|(-1)^m u\Big(t+\frac{\omega}{2}m\Big) - (-1)^{m+1}u\Big(t+\frac{\omega}{2}(m+1)\Big)\Big\|_{\mathbb{H}^n}\\
\leq\ & M\|\phi\|_{\mathbb{X}}e^{-\lambda(t+\frac{\omega}{2}m)} \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|\Big((-1)^m u\Big(t+\frac{\omega}{2}m\Big)\Big)' - \Big((-1)^{m+1}u\Big(t+\frac{\omega}{2}(m+1)\Big)\Big)'\Big\|_{\mathbb{H}^n}\\
\leq\ & M\|\phi\|_{\mathbb{X}}e^{-\lambda(t+\frac{\omega}{2}m)}. \qquad (3.4)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(-1)^{m+1}u_p\Big(t+\frac{\omega}{2}(m+1)\Big) &= u_p(t) + \sum_{\nu=0}^{m}\Big\{(-1)^{\nu+1}u_p\Big(t+\frac{\omega}{2}(\nu+1)\Big)\\
&\quad -(-1)^{\nu}u_p\Big(t+\frac{\omega}{2}\nu\Big)\Big\}, \qquad (3.5)
\end{aligned}
$$

and

$$\Big((-1)^{m+1}u_p\Big(t+\frac{\omega}{2}(m+1)\Big)\Big)' = u_p'(t)+\sum_{\nu=0}^{m}\Big\{\Big((-1)^{\nu+1}u_p\Big(t+\frac{\omega}{2}(\nu+1)\Big)\Big)' - \Big((-1)^{\nu}u_p\Big(t+\frac{\omega}{2}\nu\Big)\Big)'\Big\}. \tag{3.6}$$

Therefore, (3.3) and (3.4) suggest that $(-1)^{m+1}u_p(t+\frac{\omega}{2}(m+1))$ and $\Big((-1)^{m+1}u_p\Big(t+\frac{\omega}{2}(m+1)\Big)\Big)'$ are uniformly convergent to a differentiable function $u^*(t) = (u_1^*(t), u_2^*(t), \cdots, u_n^*(t))^T \in \mathbb{H}^n$ and its derivative $(u^*)'(t) = ((u_1^*)'(t), (u_2^*)'(t), \cdots, (u_n^*)'(t))^T \in \mathbb{H}^n$ on any compact set of $\mathbb{R}$.

Furthermore,

$$\begin{aligned} u_p^*\Big(t+\frac{\omega}{2}\Big) &= \lim_{m\to+\infty}(-1)^m u_p\Big(t+\frac{\omega}{2}+\frac{\omega}{2}m\Big) \\ &= -\lim_{m\to+\infty}(-1)^{m+1}u_p\Big(t+\frac{\omega}{2}(m+1)\Big) \\ &= -u_p^*(t), \end{aligned}$$

entails that $u^*(t)$ is $\frac{\omega}{2}$-anti-periodic on $\mathbb{R}$.

According to $(H_1)$-$(H_5)$ and the continuity of the right side of (3.2), we can easily show that $\Big((-1)^{m+1}u(t+\frac{\omega}{2}(m+1))\Big)''$ uniformly converges to a continuous function on any compact set of $\mathbb{R}$. Therefore, for any compact set of $\mathbb{R}$, letting $m\to+\infty$, we can obtain

$$\begin{aligned} (u_p^*)''(t) &= -a_p(t)(u_p^*)'(t) - b_p(t)u_p^*(t) + \sum_{q=1}^{n} c_{pq}(t)f_q(u_p^*(t)) \\ &\quad + \sum_{q=1}^{n} d_{pq}(t)g_q(u_p^*(t-\gamma_{pq}(t))) + \sum_{q=1}^{n}\sum_{\iota=1}^{n} e_{pq\iota}(t) \\ &\quad \times h_q(u_p^*(t-\alpha_{pq\iota}(t)))h_\iota(u_\iota^*(t-\beta_{pq\iota}(t))) + U_p(t), \end{aligned}$$

which implies that $u^*(t)$ is an $\frac{\omega}{2}$-anti-periodic solution of system (2.1).

By Lemma 2.2, it is easy to see that $u^*(t)$ is globally exponentially stable. This completes the proof. □

**Remark 3.1.** *Considering the object model as an entirety, unlike the method in literature [16], we show that system* (2.1) *has an* $\frac{\omega}{2}$*-anti-periodic solution by applying the same way as that in Theorem 3.1 of [25–30], namely, the non-reduced order method and analytical techniques in uniform convergence functions sequence. Therefore, this will bring us great convenience in the calculation.*

Next, to investigate drive-response synchronization, we will consider the neural network system (2.1) as the master system, and the slave system is given by

$$\begin{aligned} \ddot{y}_p(t) &= -a_p(t)\dot{y}_p(t) - b_p(t)y_p(t) + \sum_{q=1}^{n} c_{pq}(t)f_q(y_q(t)) + \sum_{q=1}^{n} d_{pq}(t) \\ &\quad \times g_q(y_q(t-\gamma_{pq}(t))) + \sum_{q=1}^{n}\sum_{\iota=1}^{n} e_{pq\iota}(t)h_q(y_q(t-\alpha_{pq\iota}(t))) \end{aligned}$$

$$\times h_\iota(y_\iota(t-\beta_{pq\iota}(t))) + U_p(t) + \varepsilon_p(t), \tag{3.7}$$

where $p = 1, 2, \cdots, n$, $y_p(t) : \mathbb{R} \to \mathbb{H}$ denotes the state of the response system, $\varepsilon_p(t) \in \mathbb{H}$ is a state-feedback controller, other notations are the same as those in system (2.1).

System (3.7) is supplemented with initial values given by

$$y_p(s) = \varphi_p^y(s), \ \ \dot{y}_p(s) = \psi_p^y(s), \ \ s \in [-\tau, 0],$$

where $\varphi_p^y(s), \psi_p^y(s) \in C\Big([-\tau, 0], \mathbb{H}\Big), p = 1, 2, \cdots, n.$

We are now in a position to discuss the problem of systems (2.1) and (3.7). Let $z_p = y_p - x_p,\ \ p = 1, 2, \ldots, n$, then from (2.1) and (3.7), the error system is given by

$$\begin{aligned} \ddot{z}_p(t) \ \ =\ \ & -a_p(t)\dot{z}_p(t) - b_p(t)z_p(t) + \sum_{q=1}^{n} c_{pq}(t)(f_q(y_q(t)) - f_q(x_q(t))) \\ & + \sum_{q=1}^{n} d_{pq}(t)(g_q(y_q(t-\gamma_{pq}(t))) - g_q(x_q(t-\gamma_{pq}(t)))) \\ & + \sum_{q=1}^{n}\sum_{\iota=1}^{n} e_{pq\iota}(t)(h_q(y_q(t-\alpha_{pq\iota}(t)))h_\iota(y_\iota(t-\beta_{pq\iota}(t))) \\ & - h_q(x_q(t-\alpha_{pq\iota}(t)))h_\iota(x_\iota(t-\beta_{pq\iota}(t)))) + \varepsilon_p(t). \end{aligned} \tag{3.8}$$

In order to realize synchronization between (2.1) and (3.7), the controller $\varepsilon_p$ is designed as

$$\varepsilon_p(t) = -\sigma_p(t)z_p(t) + \sum_{q=1}^{n} \vartheta_{pq}(t)z_q(t) + \sum_{q=1}^{n} \mu_{pq}(t)\varrho_q(z_q(t-\theta_{pq}(t))), \tag{3.9}$$

where $p = 1, 2, \cdots, n, \sigma_p, \theta_{pq} : \mathbb{R} \longrightarrow \mathbb{R}_+, \vartheta_{pq}, \mu_{pq}, \varrho_q \in \mathbb{H}.$

System (3.5) is supplemented with initial values given by

$$z_p(s) = \varphi_p^y(s) - \varphi_p^x(s), \ \ \dot{z}_p(s) = \psi_p^y(s) - \psi_p^x(s), \ \ s \in [-\tau, 0]. \tag{3.10}$$

**Theorem 3.2.** *Assume that* $(H_1)$–$(H_5)$ *hold. If the following conditions are satisfied:*

$(H_6)$ *For* $p = 1, 2, \cdots, n$, $\sigma_p, \theta_{pq} : \mathbb{R} \longrightarrow \mathbb{R}_+$, $\vartheta_{pq}, \mu_{pq}, \varrho_q \in \mathbb{H}$, *there exists a positive constant* $\omega > 0$ *such that*

$$\sigma_p(t+\frac{\omega}{2}) = \sigma_p(t), \ \ \vartheta_{pq}(t+\frac{\omega}{2}) = \vartheta_{pq}(t),$$

$$\theta_{pq}(t+\frac{\omega}{2}) = \theta_{pq}(t), \ \ \mu_{pq}(t+\frac{\omega}{2})\varrho_q(u) = -\mu_{pq}(t)\varrho_q(-u);$$

$(H_7)$ *For* $p = 1, 2, \cdots, n$, $\varrho_q(0) = 0$, *there exists a positive constant* $L_\varrho$ *such that*

$$|\varrho_q(u) - \varrho_q(v)| \le L_\varrho|u - v|;$$

$(H_8)$ *There exists a positive constant* $\epsilon$ *such that*

$$2\epsilon + 6 - 2a_p^- - 2(b_p^- + \sigma_p^-) + \sum_{q=1}^{n}(c_{pq}^+)^2L_f^2(3 + 2\epsilon) + \sum_{q=1}^{n}\frac{(d_{pq}^+)^2L_g^2e^{2\epsilon\gamma}}{1 - \dot{\gamma}_{pq}}$$

$$+\sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2\frac{1}{1-\dot{\alpha}_{pq\iota}}e^{2\epsilon\alpha}+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}}$$
$$+\sum_{q=1}^{n}(\vartheta_{pq}^{+})^2+\sum_{q=1}^{n}(\mu_{pq}^{+})^2L_\varrho^2e^{2\epsilon\theta}<0,$$

*where*

$$\sigma_p^-=\inf_{t\in[0,\omega]}\sigma_p(t),\ \ \sigma_p^+=\sup_{t\in[0,\omega]}\sigma_p(t),$$
$$\vartheta_{pq}^{+}=\max_{1\le q\le n}\|\vartheta_{pq}(t)\|_{\mathbb{H}},\ \ \mu_{pq}^{+}=\max_{1\le q\le n}\|\mu_{pq}(t)\|_{\mathbb{H}},\ \ \theta=\sup_{t\in[0,\omega]}\theta_{pq}(t).$$

*Then the drive system* (2.1) *and the response system* (3.7) *are globally exponentially synchronized.*

*Proof.* We define a Lyapunov function as follows:

$$\begin{aligned}V(t) &= \sum_{p=1}^{n}e^{2\epsilon t}\bar{\dot{z}}_p(t)\dot{z}_p(t)+\sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2e^{2\epsilon t}\bar{z}_q(t)z_q(t)\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\epsilon\gamma}}{1-\dot{\gamma}_{pq}}\int_{t-\gamma_{pq}(t)}^{t}e^{2\epsilon s}|z_q(s)|^2ds\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{1-\dot{\alpha}_{pq\iota}}\int_{t-\alpha_{pq\iota}(t)}^{t}e^{2\epsilon s}|z_q(s)|^2ds\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}}\int_{t-\beta_{pq\iota}(t)}^{t}e^{2\epsilon s}|z_\iota(s)|^2ds.\end{aligned}\tag{3.11}$$

For $t\ge 0$, $p,q,\iota=1,2,\cdots,n$, from (3.11) and by Lemma 2.1, we can gain

$$\begin{aligned}\frac{d^+}{dt}V(t) &= \sum_{p=1}^{n}e^{2\epsilon t}\Big\{2\epsilon\bar{\dot{z}}_p(t)\dot{z}_p(t)+\bar{\ddot{z}}_p(t)\dot{z}_p(t)+\bar{\dot{z}}_p(t)\ddot{z}_p(t)\Big\}\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2e^{2\epsilon t}\Big\{2\epsilon\bar{z}_q(t)z_q(t)+\bar{\dot{z}}_q(t)z_q(t)+\bar{z}_q(t)\dot{z}_q(t)\Big\}\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\epsilon\gamma}}{1-\dot{\gamma}_{pq}}e^{2\epsilon t}|z_q(t)|^2-\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\epsilon\gamma}}{1-\dot{\gamma}_{pq}}\\
&\times e^{2\epsilon(t-\gamma_{pq}(t))}|z_q(t-\gamma_{pq}(t))|^2(1-\dot{\gamma}_{pq}(t))+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^{+})^2\\
&\times\frac{L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{1-\dot{\alpha}_{pq\iota}}e^{2\epsilon t}|z_q(t)|^2-\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{1-\dot{\alpha}_{pq\iota}}\\
&\times e^{2\epsilon(t-\alpha_{pq}(t))}|z_q(t-\alpha_{pq}(t))|^2(1-\dot{\alpha}_{pq}(t))+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^{+})^2\\
&\times\frac{L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}}e^{2\epsilon t}|z_\iota(t)|^2-\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}}\end{aligned}$$

$$
\begin{aligned}
& \times e^{2\epsilon(t-\beta_{pq}(t))}|z_\iota(t-\beta_{pq}(t))|^2(1-\dot{\beta}_{pq\iota}(t)) \\
\leq\ & \sum_{p=1}^{n} e^{2\epsilon t}\Big\{2\epsilon|\dot{z}_p(t)|^2 - 2a_p^-|\dot{z}_p(t)|^2 - (b_p^- + \sigma_p^-)(|\dot{z}_p(t)|^2 \\
& +|z_p(t)|) + \sum_{q=1}^{n}(c_{pq}^+)^2L_f^2|z_q(t)|^2 + |\dot{z}_p(t)|^2 + \sum_{q=1}^{n}\frac{(d_{pq}^+)^2L_g^2}{1-\dot{\gamma}_{pq}} \\
& \times e^{2\epsilon\gamma}|z_q(t)|^2 + |\dot{z}_p(t)|^2 + \sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_\iota^h)^2}{1-\dot{\alpha}_{pq\iota}} \\
& \times e^{2\epsilon\alpha)}|z_q(t)|^2 + |\dot{z}_p(t)|^2 + \sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}} \\
& \times|z_\iota(t)|^2 + |\dot{z}_p(t)|^2 + \sum_{q=1}^{n}(c_{pq}^+)^2L_f^2\Big(2\epsilon|z_q(t)|^2 + |\dot{z}_q(t)|^2 \\
& +|z_q(t|^2)\Big) + \sum_{q=1}^{n}(\vartheta_{pq}^+)^2|z_q(t)|^2 + |\dot{z}_p(t)|^2 + \sum_{q=1}^{n}(\mu_{pq}^+)^2L_\varrho^2 \\
& e^{\epsilon\theta}|z_q(t)|^2 + |\dot{z}_p(t)|^2\Big\} \\
\leq\ & \sum_{p=1}^{n} e^{2\epsilon t}\Big\{2\epsilon + 6 - 2a_p^- - 2(b_p^- + \sigma_p^-) + \sum_{q=1}^{n}(c_{pq}^+)^2L_f^2(3+2\epsilon) \\
& +\sum_{q=1}^{n}\frac{(d_{pq}^+)^2L_g^2e^{2\epsilon\gamma}}{1-\dot{\gamma}_{pq}} + \sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{1-\dot{\alpha}_{pq\iota}} \\
& +\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}} + \sum_{q=1}^{n}(\vartheta_{pq}^+)^2 + \sum_{q=1}^{n}(\mu_{pq}^+)^2L_\varrho^2e^{2\epsilon\theta}\Big\} \\
& \times\max\Big\{|z_p(t)|^2, |\dot{z}_p(t)|^2\Big\} \\
<\ & 0,
\end{aligned}
$$

which implies that $V(t) \leq V(0)$ and we can get that

$$
\begin{aligned}
V(0) =\ & \sum_{p=1}^{n}\bar{\dot{z}}_p(0)\dot{z}_p(0) + \sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^+)^2L_f^2\bar{z}_q(0)z_q(0) \\
& +\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^+)^2(L_g)^2e^{2\epsilon\gamma}}{1-\dot{\gamma}_{pq}}\int_{-\gamma_{pq}(0)}^{0}e^{2\epsilon s}|z_q(s)|^2ds \\
& +\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{1-\dot{\alpha}_{pq\iota}}\int_{-\alpha_{pq\iota}(0)}^{0}e^{2\epsilon s}|z_q(s)|^2ds \\
& +\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}}\int_{-\beta_{pq\iota}(0)}^{0}e^{2\epsilon s}|z_\iota(s)|^2ds
\end{aligned}
$$

$$\begin{aligned}
&\le \sum_{p=1}^{n}\bar{\dot{z}}_p(0)\dot{z}_p(0)+\sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2\bar{z}_q(0)z_q(0)\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\epsilon\gamma}}{1-\dot{\gamma}_{pq}}\int_{-\gamma}^{0}e^{2\epsilon s}|z_q(s)|^2ds\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{1-\dot{\alpha}_{pq\iota}}\int_{-\alpha}^{0}e^{2\epsilon s}|z_q(s)|^2ds\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{1-\dot{\beta}_{pq\iota}}\int_{-\beta}^{0}e^{2\epsilon s}|z_\iota(s)|^2ds\\
&\le \sum_{p=1}^{n}\|\dot{z}_p(s)\|_{\mathbb{H}}^2+\sum_{p=1}^{n}\sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2\|z_q(s)\|_{\mathbb{H}}^2\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\epsilon\gamma}}{2\epsilon(1-\dot{\gamma}_{pq})}\|z_q(s)\|_{\mathbb{H}}^2\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{2\epsilon(1-\dot{\alpha}_{pq\iota})}\|z_q(s)\|_{\mathbb{H}}^2\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{2\epsilon(1-\dot{\beta}_{pq\iota})}\|z_\iota(s)\|_{\mathbb{H}}^2\\
&\le \sum_{p=1}^{n}M^2(\|\dot{z}_p(s)\|_{\mathbb{H}}^2+\|z_p(s)\|_{\mathbb{H}}^2),
\end{aligned}$$

where $s\in[-\tau,0]$,

$$N^2 := \max\Big\{1, B\Big\},$$

and

$$\begin{aligned}
B &:= \sum_{q=1}^{n}(c_{pq}^{+})^2L_f^2+\sum_{q=1}^{n}\frac{(d_{pq}^{+})^2(L_g)^2e^{2\epsilon\gamma}}{2\epsilon(1-\dot{\gamma}_{pq})}+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_\iota^h)^2e^{2\epsilon\alpha}}{2\epsilon(1-\dot{\alpha}_{pq\iota})}\\
&\quad+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^{+})^2L_h^2(M_q^h)^2e^{2\epsilon\beta}}{2\epsilon(1-\dot{\beta}_{pq\iota})}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
e^{2\epsilon t}|\dot{z}_p(t)|^2 &\le N^2(\|\dot{z}_p(s)\|_{\mathbb{H}}^2+\|z_p(s)\|_{\mathbb{H}}^2)\\
&\le N^2\Big(\|\dot{z}_p(s)\|_{\mathbb{H}}+\|z_p(s)\|_{\mathbb{H}}\Big)^2
\end{aligned}$$

and

$$\begin{aligned}
(c_{pq}^{+})^2L_f^2e^{2\epsilon t}|z_p(t)|^2 &\le N^2(\|\dot{z}_p(s)\|_{\mathbb{H}}^2+\|z_p(s)\|_{\mathbb{H}}^2)\\
&\le N^2\Big(\|\dot{z}_p(s)\|_{\mathbb{H}}+\|z_p(s)\|_{\mathbb{H}}\Big)^2,
\end{aligned}$$

that is,

$$|\dot{z}_p(t)| \le N(\|\dot{z}_p(s)\|_{\mathbb{H}} + \|z_p(s)\|_{\mathbb{H}})e^{-\epsilon t},$$

and

$$|z_p(t)| \le \frac{N}{c_{pq}^{+}L_f}(\|\dot{z}_p(s)\|_{\mathbb{H}} + \|z_p(s)\|_{\mathbb{H}})e^{-\epsilon t}.$$

Hence, for $t \ge 0$,we have that

$$\left\|y - x\right\|_{\mathbb{X}} \le 2\max\left\{N, \frac{N}{c_{pq}^{+}L_f}\right\}\|\chi\|_{\mathbb{X}}e^{-\epsilon t},$$

where

$$\|\chi\|_{\mathbb{X}} = \max\left\{\|\varphi^{y} - \varphi^{x}\|_{\mathbb{X}}, \|\psi^{y} - \psi^{x}\|_{\mathbb{X}}\right\}.$$

Therefore, the drive system (2.1) and the response system (3.7) are globally exponentially synchronized. The proof is complete. □

**Remark 3.2.** *Compared with real-valued and complex-valued neural networks, the multiplication of quaternion-valued neural networks does not satisfy the commutative rule. The dynamical research of quaternion-valued neural networks is more difficult than real-valued neural networks. Even so, to overcome the complexity of calculation, we do not need to separate the quaternion-valued system into 4 real-valued systems.*

**Remark 3.3.** *In this paper, like the method in literature [34–36], we gain our main results by constructing an appropriate Lyapunov function. However, compared with [35,36], we gain anti-periodic synchronization of quaternion-valued inertial neural networks by using the non-decomposition, non-reduced order, and Lyapunov function method. Therefore, the obtained results in this paper are more general.*

**Remark 3.4.** *In [37–41], some authors have obtained stability and synchronization of neural networks by using the Lyapunov function method and inequality technique. In [42], some authors have obtained finite-time stability of CNNs with neutral networks by using the proof by contradiction and inequality technique. However, unlike the method of the above papers paper, we obtain the anti-periodic synchronization of quaternion-valued inertial neural networks by using the non-decomposition method, non-reduced order method, and analytical techniques in uniform convergence functions sequence, and constructing the Lyapunov function. Therefore, this paper aims on investigating the synchronization of quaternion-valued inertial neural networks.*

## 4. Illustrative example

In this section, we give one example to illustrate the feasibility and effectiveness of the main results.

**Example 4.1.** *Consider the following delayed quaternion-valued neural networks with two neurons as the drive system:*

$$\begin{aligned}\ddot{x}_p(t) &= -a_p(t)\dot{x}_p(t) - b_p(t)x_p(t) + \sum_{q=1}^{2} c_{pq}(t)f_q(x_q(t)) \\ &\quad + \sum_{q=1}^{2} d_{pq}(t)g_q(x_q(t-\gamma_{pq}(t))) + \sum_{q=1}^{2}\sum_{\iota=1}^{2} e_{pq\iota}(t) \\ &\quad \times h_q(x_q(t-\alpha_{pq\iota}(t)))h_\iota(x_\iota(t-\beta_{pq\iota}(t))) + U_p(t). \qquad (4.1)\end{aligned}$$

*The corresponding response system is given by*

$$\begin{aligned}\ddot{y}_p(t) &= -a_p(t)\dot{y}_p(t) - b_p(t)y_p(t) + \sum_{q=1}^{2} c_{pq}(t)f_q(y_q(t)) \\ &\quad + \sum_{q=1}^{2} d_{pq}(t)g_q(y_q(t-\gamma_{pq}(t))) + \sum_{q=1}^{2}\sum_{\iota=1}^{2} e_{pq\iota}(t) \\ &\quad \times h_q(y_q(t-\alpha_{pq\iota}(t)))h_\iota(y_\iota(t-\beta_{pq\iota}(t))) + U_p(t) + \varepsilon_p(t) \qquad (4.2)\end{aligned}$$

*and the controller is as follows:*

$$\varepsilon_p(t) = -\sigma_p(t)z_p(t) + \sum_{q=1}^{2}\vartheta_{pq}(t)z_q(t) + \sum_{q=1}^{2}\mu_{pq}(t)\varrho_q(z_q(t-\theta_{pq}(t))), \qquad (4.3)$$

*where* $p = 1, 2$, $a_1(t) = 2.5 + 0.3\sin 2t$, $a_2(t) = 3 + \sin 2t$, $b_1(t) = 3 + 0.5\sin 2t$, $b_2(t) = 2.5 + 0.5\sin 2t$, $\sigma_1(t) = 1.8 + 0.3\sin 2t$, $\sigma_2(t) = 2 + 0.2\sin 2t$, $\gamma_{pq} = \frac{1}{2} + \frac{1}{6}\sin 2t$, $\alpha_{pql} = \frac{2}{5} + \frac{1}{5}\sin 2t$, $\beta_{pql} = \frac{1}{3} + \frac{1}{12}\sin 2t$, $\theta_{pq} = \frac{2}{3} + \frac{1}{3}\sin 2t$, *and*

$$(c_{pq})_{2\times 2} = \begin{pmatrix} 0.1\sin t + 0.5i\sin t + 0.3j\sin t & 0.2\sin t + 0.3i\sin t + 0.1k\sin t \\ 0.3\sin t + 0.1j\sin t + 0.2k\sin t & 0.2i\sin t + 0.3j\sin t + 0.2k\sin t \end{pmatrix},$$

$$(d_{pq})_{2\times 2} = \begin{pmatrix} 0.4\sin t + 0.2i\sin t + 0.3k\sin t & 0.3\sin t + 0.1i\sin t + 0.1j\sin t \\ 0.1i\sin t + 0.3j\sin t + 0.2k\sin t & 0.4\sin t + 0.2j\sin t + 0.2k\sin t \end{pmatrix},$$

$$(e_{pq1})_{2\times 2} = \begin{pmatrix} 0.3i\sin t + 0.1j\sin t + 0.1k\sin t & 0.5\sin t + 0.2j\sin t + 0.3k\sin t \\ 0.3\sin t + 0.2i\sin t + 0.1j\sin t & 0.2\sin t + 0.1i\sin t + 0.1k\sin t \end{pmatrix},$$

$$(e_{pq2})_{2\times 2} = \begin{pmatrix} 0.3\sin t + 0.1i\sin t + 0.2j\sin t & 0.1i\sin t + 0.2j\sin t + 0.3k\sin t \\ 0.1i\sin t + 0.2j\sin t + 0.2k\sin t & 0.2\sin t + 0.1i\sin t + 0.2j\sin t \end{pmatrix},$$

$$(\vartheta_{pq})_{2\times 2} = \begin{pmatrix} 0.5i\sin t + 0.3j\sin t + 0.2k\sin t & 0.3\sin t + 0.2j\sin t + 0.2k\sin t \\ 0.2\sin t + 0.1i\sin t + 0.2j\sin t & 0.4\sin t + 0.2i\sin t + 0.2k\sin t \end{pmatrix},$$

$$(\mu_{pq})_{2\times 2} = \begin{pmatrix} 0.3\sin t + 0.2j\sin t + 0.1k\sin t & 0.5i\sin t + 0.2j\sin t + 0.3k\sin t \\ 0.4\sin t + 0.2i\sin t + 0.2j\sin t & 0.3\sin t + 0.2i\sin t + 0.1j\sin t \end{pmatrix},$$

$$U_p = 0.5\sin t + 0.4i\sin t + 0.6j\sin t + k\sin t,$$

$$f_q = \frac{1}{4}\sin x_q^R + \frac{1}{4} i \sin x_q^I + \frac{1}{4} j \sin x_q^J + \frac{1}{4} k \sin x_q^K,$$
$$g_q = \frac{1}{3}\sin x_q^R + \frac{1}{3} i \sin x_q^I + \frac{1}{3} j \sin x_q^J + \frac{1}{3} k \sin x_q^K,$$
$$h_q = \frac{1}{5}\sin x_q^R + \frac{1}{5} i \sin x_q^I + \frac{1}{5} j \sin x_q^J + \frac{1}{5} k \sin x_q^K,$$
$$\varrho_q = \frac{1}{5}\sin z_q^R + \frac{1}{5} i \sin z_q^I + \frac{1}{5} j \sin z_q^J + \frac{1}{5} k \sin z_q^K.$$

*Let $\lambda = \epsilon = 0.5$, and by calculating, we have*

$$a_p^- = 2,\ b_p^- = 2,\ \sigma_p^- = 1.5,\ c_{pq}^+ = \frac{\sqrt{35}}{10},$$
$$d_{pq}^+ = \frac{\sqrt{29}}{10},\ e_{pql}^+ = \frac{\sqrt{38}}{10},\ \vartheta_{pq}^+ = \frac{\sqrt{38}}{10},\ \mu_{pq}^+ = \frac{\sqrt{38}}{10},$$
$$L_f = \frac{1}{4},\ L_g = \frac{1}{3}, L_h = \frac{1}{5},\ L_\varrho = \frac{1}{5},\ \omega = 2\pi,$$
$$M_q^f = 1,\ M_q^g = \frac{4}{3},\ M_q^h = \frac{4}{5},\ M_q^\varrho = \frac{4}{5},$$

$$\begin{aligned}&2\lambda + 4 - 2a_p^- - 2b_p^- + \sum_{q=1}^{n}(c_{pq}^+)^2 L_f^2 (2\lambda + 3)\\&+\sum_{q=1}^{n}\frac{(d_{pq}^+)^2 (L_g)^2 e^{2\lambda\gamma}}{1-\dot{\gamma}_{pq}} + \sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2 L_h^2 (M_\iota^h)^2 e^{2\lambda\alpha}}{1-\dot{\alpha}_{pq\iota}}\\&+\sum_{q=1}^{n}\sum_{\iota=1}^{n}\frac{(e_{pq\iota}^+)^2 L_h^2 (M_q^h)^2 e^{2\lambda\beta}}{1-\dot{\beta}_{pq\iota}} \approx -1.8125 < 0,\end{aligned}$$

$$\begin{aligned}&2\epsilon + 6 - 2a_p^- - 2(b_p^- + \sigma_p^-) + \sum_{q=1}^{n}(c_{pq}^+)^2 L_f^2 (3 + 2\epsilon)\\&+\sum_{q=1}^{n}(d_{pq}^+)^2 L_g^2 \frac{1}{1-\dot{\gamma}_{pq}} e^{2\epsilon\gamma} + \sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^+)^2\\&\times L_h^2 (M_\iota^h)^2 \frac{1}{1-\dot{\alpha}_{pq\iota}} e^{2\epsilon\alpha} + \sum_{q=1}^{n}\sum_{\iota=1}^{n}(e_{pq\iota}^+)^2\\&\times L_h^2 (M_q^h)^2 \frac{1}{1-\dot{\beta}_{pq\iota}} e^{2\epsilon\beta} + \sum_{q=1}^{n}(\vartheta_{pq}^+)^2\\&+\sum_{q=1}^{n}(\mu_{pq}^+)^2 L_\varrho^2 e^{2\epsilon\theta} \approx -1.2920 < 0.\end{aligned}$$

*It is not difficult to verify that all conditions ($H_1$)–($H_8$) are satisfied. Therefore, by Theorem* 3.1 *and Theorem* 3.2*, we have that system* (4.1) *has a unique $\pi$-anti-periodic solution, and the systems* (4.1) *and* (4.2) *are globally exponentially synchronized, which is shown in Figures 1–5.*

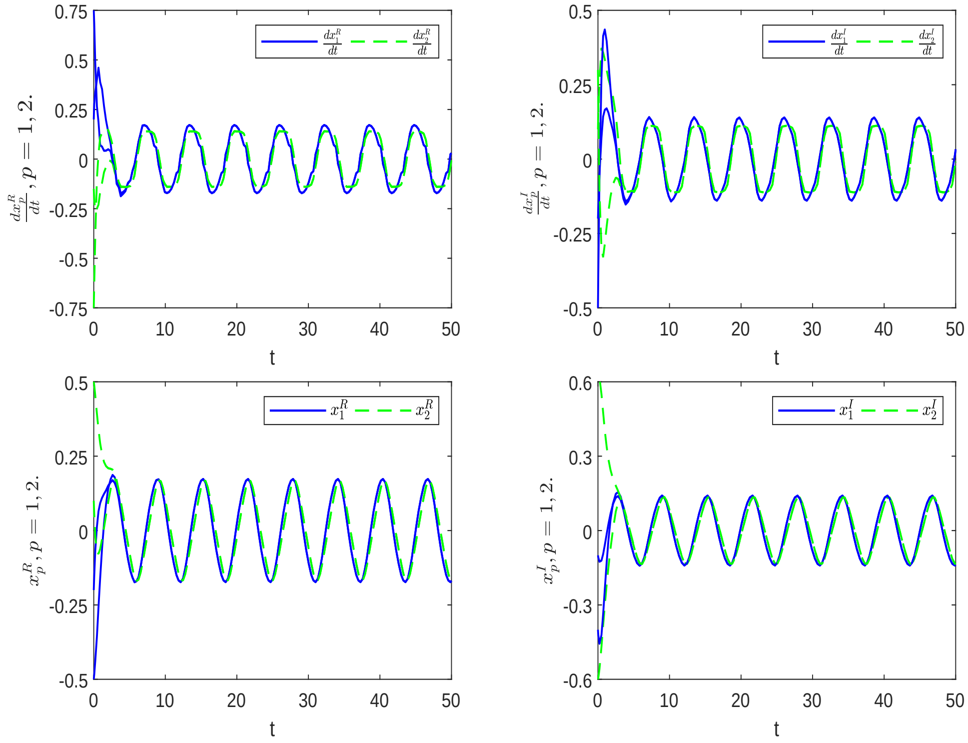


**Figure 1.** The states of $dx_p^R(t)/dt$, $dx_p^I(t)/dt$, $x_p^R(t)$ and $x_p^I(t)$, $p = 1, 2$.

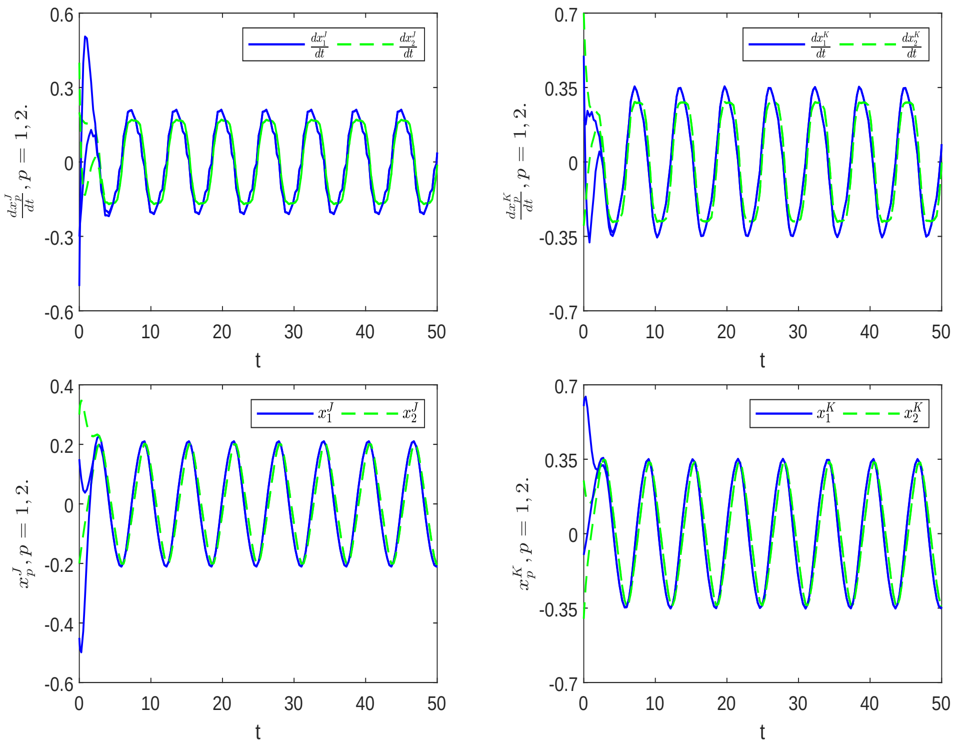


**Figure 2.** The states of $dx_p^J(t)/dt$, $dx_p^K(t)/dt$, $x_p^J(t)$ and $x_p^K(t)$, $p = 1, 2$.

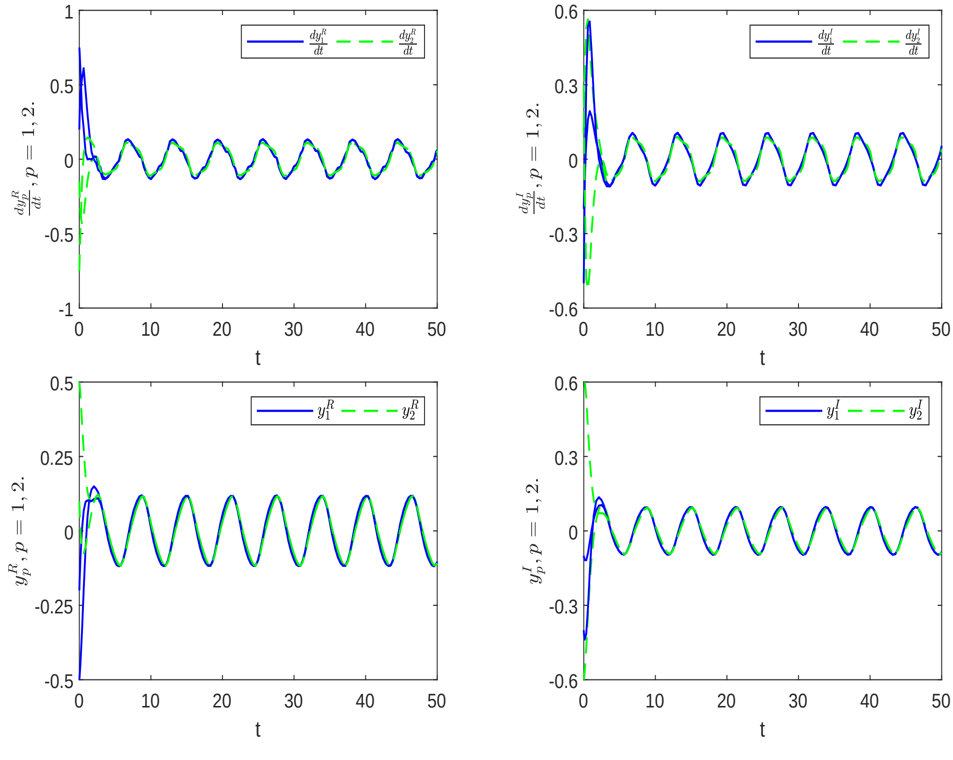


**Figure 3.** The states of $dy_p^R(t)/dt$, $dy_p^I(t)/dt$, $y_p^R(t)$ and $y_p^I(t)$, $p = 1, 2$.

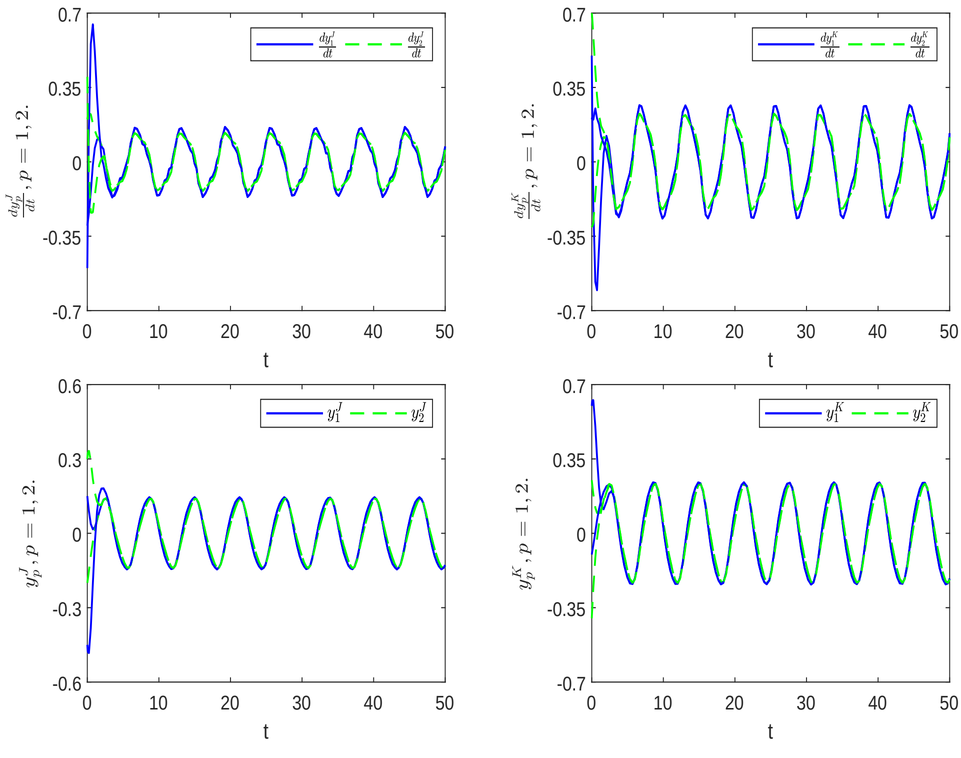


**Figure 4.** The states of $dy_p^J(t)/dt$, $dy_p^K(t)/dt$, $y_p^J(t)$ and $y_p^K(t)$, $p = 1, 2$.

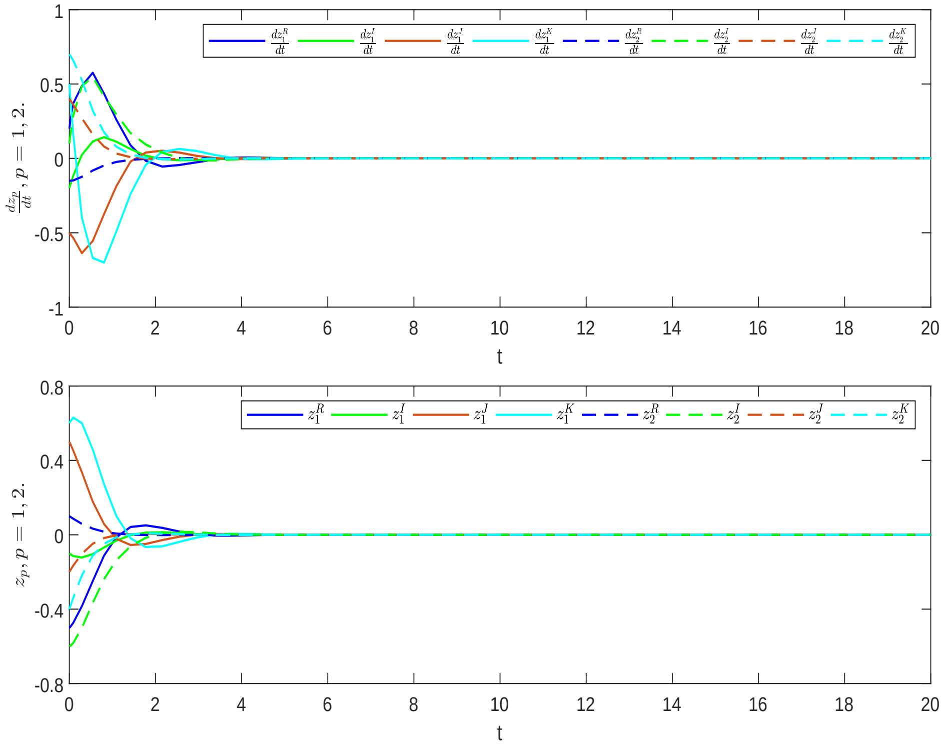

**Figure 5.** Synchronization errors $dz(t)/dt$ and $z(t)$.

**Remark 4.1.** *By Theorem* 3.1*, from Figures 1 and 2, we can see system* (4.1) *has a unique* $\pi$*-anti-periodic solution that is globally exponentially stable. From Figures 3 and 4, we can see system* (4.2) *has a unique* $\pi$*-anti-periodic solution that is globally exponentially stable.*

**Remark 4.2.** *When applying the controller* (4.3)*, from Figure 5, we can see the master and slave system can reach globally exponentially synchronized.*

## 5. Conclusions

This paper considers a class of delayed quaternion-valued high-order Hopfield neural networks. Although the multiplication of quaternion algebra does not satisfy the commutativity, without decomposing the quaternion-valued neural networks into real-valued neural networks. By using the non-reduce order method, analytical techniques in uniform convergence functions sequence, and the Lyapunov function method, we obtain several sufficient conditions for the existence of anti-periodic solutions for quaternion-valued high-order Hopfield neural networks, and by Lyapunov function, we establish the global exponential synchronization of anti-periodic solutions for quaternion-valued high-order Hopfield neural networks, one example is given. And in future research, I will study the almost periodicity of inertial quaternion-valued neural networks models by using the non-reduced order method.

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## Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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