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Research article

Some generalized fixed point results of Banach and ĆiriĆ type in extended fuzzy b-metric spaces with applications

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Abstract: In this paper, some generalized fixed point results of Banach and Ćirić type in the context of extended fuzzy b-metric spaces are established. For authenticity of the aforesaid results a nontrivial supporting example is also provided. Eventually, an application for the existence of a solution for an integral equation is established which shows corporeality of the obtained results. The presented work generalizes some well known fixed point results from the existing literature.

Keywords: fixed point; extended fuzzy b-metric spaces; Banach and Ćirić type contractions; integral equation
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1. Introduction

Zadeh [50] presented the concept of a fuzzy set and fuzzy logic. Unlike classical logic, which states whether an element belongs to a set or not, fuzzy logic expresses the bonding of an element to a set as a positive real value in unit interval [0,1]. With the introduction of fuzzy logic, fuzzy mathematics began to evolve. If the distance between points is not exact real number, then the factor of inaccuracy is incorporated in the metric, a distance measuring function. Kramosil and Michalek [24] generalized probabilistic metric space by introducing the concept of fuzzy metric space. Kramosil's notion of fuzzy metric was improved by George and Veeramani [11, 12], who defined Hausdorff topology on fuzzy metric space, which could not be defined on the Kramosil's fuzzy metric space.

One of the several spaces in which the theory of fixed point has been investigated is fuzzy metric space. The Banach's contraction principle is one of the most useful and important theorems in classical functional analysis. Its utility is not only to prove that, a contraction in a complete metric space has unique fixed point, but also to show that the Picard iteration converges to the fixed point. This powerful result in fuzzy metric space was first generalized by M. Grabiec [17].

Theorem 1.1. *([17] fuzzy Banach contraction theorem) Let* $(S, \mathrm{M}, *)$ *be a complete fuzzy metric space and* $\mathtt{f} : S \to S$ *be a mapping such that*

$$\mathrm{M}(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \xi) \geq \mathrm{M}\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right) \ \ \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \tag{1.1}$$

where $\delta \in (0, 1)$*. Then* $\mathtt{f}$ *has unique fixed point.*

Subsequently, many researchers investigated fixed point theory in fuzzy metric space, for example, [8, 9, 13, 15, 17, 18, 28, 36–40, 48, 49].

There are several metric space extensions, in addition to fuzzy metric space. With the goal of generalizing the Banach contraction principle [4], Bakhtin [3] presented b-metric space where the condition of triangular inequality is weakened. We recommend [1, 7, 25, 29, 32, 45–47] for more detail. Hassanzadeh [22] considered relationship between b-metric and fuzzy metric space. Nǎdǎban [33] introduced fuzzy b-metric space to generalise b-metric space. Mehmood *et al.* [27] presented the notion of extended fuzzy b-metric to generalize fuzzy b-metric. Rome *et al.* [43] generalized extended fuzzy b-metric by introducing μ-extended fuzzy b-metric space.

Many researchers have made attempts to relax the essential condition of continuity contraction in Banach contraction principle see for example [23, 42].

One of the most well-known results in generalizations of Banach's contraction principle where the Picard iteration still converges to the fixed point of map is the Ćirić's fixed point theorem [6]. A self mapping $\mathfrak{f}$ on a metric space (S, d) is said to be quasi-contraction iff there exists $\delta \in (0, 1)$ such that for all $\mathfrak{r}, \mathfrak{z} \in S$

$$d(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}) \leq \delta \max \{d(\mathfrak{r}, \mathfrak{z}), d(\mathfrak{r}, \mathfrak{f}\mathfrak{r}), d(\mathfrak{z}, \mathfrak{f}\mathfrak{z}), d(\mathfrak{r}, \mathfrak{f}\mathfrak{z}), d(\mathfrak{f}\mathfrak{z}, \mathfrak{r})\}.$$

According to Ćirić's fixed point theorem, every quasi-contraction on T-orbitally complete metric space has unique fixed point. D. Rakić *et al.* [35] generalized Ćirić's fixed point theorem in the setting of fuzzy b-metric space.

The aim of this paper is to generalize Banach and Ćirić's fixed point results in the context of extended fuzzy b-metric space ($\mathcal{EF}_b\mathcal{MS}$ for short). We prove a very useful lemma, which can be used in extended fuzzy b-metric to verify Cauchyness of a sequence. Finally, we investigate the applicability of the obtained results to integral equations. A nontrivial example is provided to confirm authenticity of our results.

2. Preliminaries

In this section some terms and definitions are provided which will be used in the main work of this manuscript. Throughout this paper the symbol $\mathbb{N}$ will stand for positive integers and the real numbers will be represented by $\mathtt{R}$, whereas S will signify an arbitrary non-empty set.

Definition 2.1. *[26] A binary operation* $*$: $[0,1]^2 \to [0,1]$ *is called continuous t-norm, if the following conditions hold:*
(01) $*$ *is commutative and associative,*
(02) $*$ *is continuous,*
(03) $*(\mathfrak{w},1)=\mathfrak{w},\ \ \forall \mathfrak{w} \in [0,1]$,
(04) $*(\mathfrak{w}_1,\mathfrak{w}_2) \leq *(\mathfrak{w}_3,\mathfrak{w}_4)$ *whenever* $\mathfrak{w}_1 \leq \mathfrak{w}_3$ *and* $\mathfrak{w}_2 \leq \mathfrak{w}_4, \forall \mathfrak{w}_1,\mathfrak{w}_2,\mathfrak{w}_3,\mathfrak{w}_4 \in [0,1]$.

Some frequently used example of continuous t-norm are $\mathfrak{w}_1 *_L \mathfrak{w}_2 = \max\{\mathfrak{w}_1+\mathfrak{w}_2-1,0\}$, $\mathfrak{w}_1 *_P \mathfrak{w}_2 = \mathfrak{w}_1\mathfrak{w}_2$, and $\mathfrak{w}_1 *_{\mathrm{M}} \mathfrak{w}_2 = \min\{\mathfrak{w}_1,\mathfrak{w}_2\}$. These are respectively called Lukasievicz t-norm, product t-norm and minimum t-norm.

Definition 2.2. *[19] Let* $*$ *be a t-norm and define* $*_\mathfrak{n} : [0,1]\times[0,1] \to [0,1]$, *as:*

$$*_1(\mathfrak{r}) = *(\mathfrak{r},\mathfrak{r}),\ \ *_{\mathfrak{n}+1}(\mathfrak{r}) = *(*_\mathfrak{n}(\mathfrak{r}),\mathfrak{r}),\ \ \forall \mathfrak{n}\in\mathbb{N}, \mathfrak{r}\in[0,1].$$

Then the t-norm $*$ is said to be *Hadžić*-type (H-type for short) if the family $\{*_\mathfrak{n}(\mathfrak{r})\}_{\mathfrak{n}\in\mathbb{N}}$ is equicontinuous at $\mathfrak{r}=1$, that is, for every $\beta\in(0,1)$, there exists $\gamma\in(0,1)$ such that $\mathfrak{r}\in(1-\gamma,1]$ implies that $*^{\mathfrak{n}}\mathfrak{x} > 1-\beta$, for all $\mathfrak{n}\in\mathbb{N}$. $*_{\min}$ is a trivial example of t-norm of H-type. From associativity of t-norm it follows that each t-norm $*$ can be extended in a unique manner to an $\mathfrak{n}$-array operation that takes for $(\mathfrak{r}_1,...,\mathfrak{r}_\mathfrak{n})\in[0,1]^\mathfrak{n}$ the values

$$*_{i=1}^{1}\mathfrak{r}_i = \mathfrak{r}_1,\ \ *_{i=1}^{\mathfrak{n}}\mathfrak{r}_i = *(*_{i=1}^{\mathfrak{n}-1}(\mathfrak{r}_i,\mathfrak{r}_\mathfrak{n})) = *(\mathfrak{r}_1,\mathfrak{r}_2,...,\mathfrak{r}_\mathfrak{n}).$$

Example 2.1. *The* $\mathfrak{n}$*-array extensions of t-norms* $*_M, *_L$, *and* $*_P$, *are as follows:*
$*_M(\mathfrak{r}_1,...,\mathfrak{r}_\mathfrak{n}) = \min(\mathfrak{r}_1,...,\mathfrak{r}_\mathfrak{n})$,
$*_L(\mathfrak{r}_1,...,\mathfrak{r}_\mathfrak{n}) = \max(\sum_{i=1}^{n}\mathfrak{r}_i - (\mathfrak{n}-1),0)$,
$*_P(\mathfrak{r}_1,...,\mathfrak{r}_\mathfrak{n}) = \Pi_{i=1}^{\mathfrak{n}}\mathfrak{r}_i$.

Klement et al. [26] extended t-norm $*$ to a countable infinite operation for any $(\mathfrak{r}_\mathfrak{n})_{\mathfrak{n}\in N}$ in $[0,1]$, where

$$*_{i=1}^{\infty}\mathfrak{r}_{\mathfrak{i}} = \lim_{\mathfrak{n}\to\infty} *_{i=1}^{\mathfrak{n}}\mathfrak{r}_{\mathfrak{i}}.$$

The sequence $(*_{i=1}^{\mathfrak{n}}\mathfrak{r}_{\mathfrak{i}})_{\mathfrak{n}\in N}$ being bounded from below and non increasing, is convergent. In the theory of fixed point [20,21], interesting families of t-norms $*$ and sequences $(\mathfrak{r}_\mathfrak{n})$ from $[0,1]$ are those which possess the properties $\lim\limits_{\mathfrak{n}\to\infty}\mathfrak{r}_\mathfrak{n} = 1$ and

$$\lim_{n\to\infty} *_{i=\mathfrak{n}}^{\infty}\mathfrak{r}_i = \lim_{\mathfrak{n}\to\infty} *_{i=1}^{\infty}\mathfrak{r}_{\mathfrak{n}+i} = 1.$$

The following lemma generates a large number of t-norms of H-type.

Lemma 2.1. *[41] Let* $*$ *be a t-norm and* $\epsilon\in(0,1]$. *If*

$$\mathfrak{r} *_\epsilon \mathfrak{y} = \begin{cases} \mathfrak{r} * \mathfrak{y} & ;\ \textit{if}\ max(\mathfrak{r},\mathfrak{y}) \leq 1-\epsilon, \\ min(\mathfrak{r},\mathfrak{y}) & ;\ \textit{if}\ max(\mathfrak{r},\mathfrak{y}) > 1-\epsilon. \end{cases}$$

Then $*_\epsilon$ *is a t-norm of H-type.*

Proposition 2.1. *[35] Let* $(\mathfrak{r}_\mathfrak{n})_{\mathfrak{n}\in N}$ *in* $[0,1]$ *be such that* $\lim_{\mathfrak{n}\to\infty}\mathfrak{r}_\mathfrak{n} = 1$ *and let* $*$ *of H-type. Then*

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty}\mathfrak{r}_i = \lim_{\mathfrak{n}\to\infty} *_{i=1}^{\infty}\mathfrak{r}_{\mathfrak{n}+i} = 1$$

.

Definition 2.3. *[11]* $(S, \mathrm{M}, *)$ *is called a fuzzy metric space, where* $*$ *is a continuous t-norm, and* M *is a fuzzy set on* $S^2 \times (0,\infty)$ *if for all* $\mathfrak{r}, \mathfrak{z}, \omega \in S$ *and* $\xi, \zeta > 0$:
(FM1) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) > 0$,
(FM2) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) = 1 \Leftrightarrow \mathfrak{r} = \mathfrak{z}$,
(FM3) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) = \mathrm{M}(\mathfrak{z}, \mathfrak{r}, \xi)$,
(FM4) $*(\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi), \mathrm{M}(\mathfrak{z}, \omega, \zeta)) \leq \mathrm{M}(\mathfrak{r}, \omega, \xi + \zeta)$,
(FM5) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \cdot) : (0,\infty) \to [0,1]$ *is continuous.*

Definition 2.4. *[33] For a continuous t-norm* $*$ *and and a fuzzy set* M *on* $S^2 \times (0,\infty)$, $(S, \mathrm{M}, *)$ *is called a fuzzy b-metric, if for all* $\mathfrak{r}, \mathfrak{z}, \omega \in S, \xi, \zeta > 0$ *and a given real numbers* $b \geq 1$:
(F_bM1) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) > 0$,
(F_bM2) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) = 1 \Leftrightarrow \mathfrak{r} = \mathfrak{z}$,
(F_bM3) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) = \mathrm{M}(\mathfrak{z}, \mathfrak{r}, \xi)$,
(F_bM4) $*(\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{b}), \mathrm{M}(\mathfrak{z}, \omega, \frac{\zeta}{b})) \leq \mathrm{M}(\mathfrak{r}, \omega, \xi + \zeta)$,
(F_bM5) $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \cdot) : (0,\infty) \to [0,1]$ *is continuous.*
When b = 1, fuzzy b-metric reduces to fuzzy metric. The following example demonstrates that the family of fuzzy b-metric is is effectively broader than that of fuzzy metric.

Example 2.2. *[8] Let* $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) = e^{\frac{-|\mathfrak{r}-\mathfrak{z}|^q}{\xi}}$, *where* $q > 1$ *is a real number. Obviously* M *is a fuzzy b-metric with* $b = 2^{q-1}$.

Notice that for $q = 2$ in the above example, it can be verified that $(S, \mathrm{M}, *)$ is not a fuzzy metric space.

Example 2.3. *[8] Let*(d, S) *be b-metric space and* $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi) = \frac{\xi}{\xi + \mathrm{d}(\mathfrak{r},\mathfrak{z})}$. *Then it can be verified that* $(S, \mathrm{M}, *_m)$*is fuzzy b-metric space.*

Definition 2.5. *[8] A function* $\mathtt{f} : \mathtt{R} \to \mathtt{R}$ *is said to be b-nondecreasing if* $\mathtt{f}(\mathfrak{r}) \geq \mathtt{f}(\mathfrak{z})$ *whenever* $\mathfrak{r} > b\mathfrak{z}$ *for all* $\mathfrak{r}, \mathfrak{z} \in \mathtt{R}$.

Lemma 2.2. *[8] Let* $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \cdot)$ *be a fuzzy b-metric. Then* $\mathrm{M}(\mathfrak{r}, \mathfrak{z}, \xi)$ *is b-nondecreasing w.r.t.* ξ *and for all* $\mathfrak{r}, \mathfrak{z} \in S$.

Remark 2.1. *A fuzzy b-metric space is not continuous in general.*

Definition 2.6. *[27] The 4-tuple* $(S, \mathrm{M}, *, \Omega)$ *is called* $\mathcal{EF}_b\mathcal{MS}$ *with function* $\Omega : S \times S \to [1,\infty)$, *where* $*$ *is continuous t-norm and* $\mathrm{M} : S \times S \to [0,\infty)$ *is fuzzy set such that for all* $\mathfrak{r}, \mathfrak{z}, \omega \in S$, *the following conditions are satisfied:*

($E\Omega_1$) $\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, 0) = 0$;
($E\Omega_2$) $\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) = 1, \forall \xi > 0 \Leftrightarrow \mathfrak{r} = \mathfrak{z}$;
($E\Omega_3$) $\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) = \mathrm{M}_\Omega(\mathfrak{z}, \mathfrak{r}, \xi)$;
($E\Omega_4$) $\mathrm{M}_\Omega(\mathfrak{r}, \omega, \Omega(\mathfrak{r}, \omega)(\xi + \zeta) \geq \mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) * \mathrm{M}_\Omega(\mathfrak{z}, \omega, \zeta)$, *for all* $\zeta, \xi > 0$;
($E\Omega_5$) $\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, .) : (0,\infty) \to [0,1]$ *is continuous and* $\lim_{t\to\infty}\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) = 1$.

Example 2.4. *[27] Let* $S = \{1, 2, 3\}$ *and define* $\mathrm{d}_b : S \times S \to R$ *by* $\mathrm{d}(\mathfrak{r}, z) = |\mathfrak{r} - z|^2$. *Then it is simple to demonstrate that* (S, d_b) *is a b-metric space. Define the mapping*

$$\Omega : S \times S \to [1, \infty),\ \Omega(\mathfrak{r}, \mathfrak{z}) = 1 + \mathfrak{r} + \mathfrak{z}.$$

Let $\mathrm{M}_\Omega : S \times S \times [0, \infty) \to [0, 1]$ *be given by the rule:*

$$\mathrm{M}_\Omega(\mathfrak{r}, z, \xi) = \begin{cases} \frac{\xi}{\xi + \mathrm{d}_{b(\mathfrak{r},\mathfrak{z})}}, & if\ \xi > 0, \\ 0, & if\ \xi = 0, \end{cases}$$

and take the continuous t-norm $*_\mathrm{M}$, *that is,* $\xi_1 * \xi_2 = \xi_1 *_M \xi_2 = \min\{\xi_1, \xi_2\}$.
Then $(S, \mathrm{M}_\Omega, *, \Omega)$ *is an* $\mathcal{EF}_b\mathcal{MS}$.

Remark 2.2. *It is worth mentioning that fuzzy b-metric is special type of extended fuzzy b-metric when* $\Omega(\mathfrak{r}, \mathfrak{z}) = b \geq 1$.

Definition 2.7. *[5, 11, 27] Let* $(S, \mathrm{M}_\Omega, *, \Omega)$ *be an* $\mathcal{EF}_b\mathcal{MS}$. *A sequence* $\{\mathfrak{r}_\mathfrak{n}\}$:
(a) *converges to* $\mathfrak{r}$ *if* $n \to \infty$ *then* $\mathrm{M}_\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}, \xi) \to 1$ *for each* $\xi > 0$. *In this case, we write* $\lim\limits_{\mathfrak{n}\to\infty} \mathfrak{r}_\mathfrak{n} = \mathfrak{r}$.
(b) *is called M-Cauchy if for each* $\epsilon \in (0, 1)$ *and* $\xi > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $\mathrm{M}_\Omega(\mathfrak{r}_\mathfrak{m}, \mathfrak{r}_\mathfrak{n}, \xi) > 1 - \epsilon$, *for all* $\mathfrak{m}, \mathfrak{n} \geq n_0$. $\lim_{\mathfrak{n}\to\infty} \mathrm{M}_\Omega(\mathfrak{r}_{\mathfrak{n}+m}, \mathfrak{r}_\mathfrak{n}, \xi) = 1$, *for all* $\xi > 0$ *and each* $m, \mathfrak{n} \in \mathbb{N}$.
(c) *is called G-Cauchy if* $\lim_{\mathfrak{n}\to\infty} \mathrm{M}_\Omega(\mathfrak{r}_{\mathfrak{n}+m}, \mathfrak{r}_\mathfrak{n}, \xi) = 1$, *for all* $\xi > 0$ *and each* $m, \mathfrak{n} \in \mathbb{N}$.

Definition 2.8. *[27] An* $\mathcal{EF}_b\mathcal{MS}$, *is said to be M-complete(G-complete), provided every M-Cauchy (G-Cauchy) sequence converges in it.*

For more details on fuzzy topology, we refer the reader to [5].

3. Main results

Stimulated and inspired by the concept presented in [35, 43], we present several new fixed point results in $\mathcal{EF}_b\mathcal{MS}$. From now onward, $\Omega : S \times S \to [1, \infty)$ will represent a bounded function. First we prove the following lemmas, which will be used in our main work.

Lemma 3.1. *Let* $(S, \mathrm{M}_\Omega, *, \Omega)$ *be an* $\mathcal{EF}_b\mathcal{MS}$ *by* $\Omega : S \times S \to [1, \infty)$ *and* $\{\mathfrak{r}_\mathfrak{n}\}$ *is a sequence in it. Assume that for* $\delta \in (0, \frac{1}{\kappa})$ *and* $\mathfrak{n} \in \mathbb{N}$

$$\mathrm{M}_\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+1}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}-1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right), \xi > 0, \tag{3.1}$$

with $\kappa = \limsup\limits_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$ *and*

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} \mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1, \frac{\xi}{\nu^i}\right) = 1, t > 0. \tag{3.2}$$

where $\mathfrak{r}_0, \mathfrak{r}_1 \in S$ *and* $\nu \in (0, 1)$. *Then* $\{\mathfrak{r}_\mathfrak{n}\}$ *is Cauchy sequence.*

Proof. Clearly $\sum_{i=1}^{\infty}\varrho^i$ converges for $\varrho \in (\delta\kappa, 1) \subset (0,1)$ and therefore there exists $\mathfrak{n}_0 \in \mathbb{N}$ such that $\sum_{i=\mathfrak{n}}^{\infty}\varrho^i < 1$ for every $\mathfrak{n} > \mathfrak{n}_0$. Due to M_Ω being b-nondecreasing and by property $(E\Omega_4)$, for all $\xi > 0$ and $\mathfrak{n} > \mathfrak{m} > \mathfrak{n}_0$, we get the following

$$\begin{aligned}
&\mathrm{M}_\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m}, \xi)\\
&\geq \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m}, \frac{\xi\sum_{i=\mathfrak{n}}^{\mathfrak{n}+m+1}\varrho^i}{\kappa}\right)\\
&\geq *\left(\mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+1}, \frac{\xi\varrho^n}{\kappa\Omega(\mathfrak{r}_\mathfrak{n} + \mathfrak{r}_{\mathfrak{n}+m})}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+m}, \frac{\xi\sum_{i=\mathfrak{n}+1}^{\mathfrak{n}+m-1}\varrho^i}{\kappa\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m})}\right)\right)\\
&\geq *\Bigg(\mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+1}, \frac{\xi\varrho^\mathfrak{n}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n} + \mathfrak{r}_{\mathfrak{n}+m})}\right), *\Bigg(\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+2}, \frac{\xi\varrho^{\mathfrak{n}+1}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+m})}\right),\\
&* \Bigg(\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+2}, \mathfrak{r}_{\mathfrak{n}+3}, \frac{\xi\varrho^{\mathfrak{n}+2}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+2}, \mathfrak{r}_{\mathfrak{n}+m})}\right)\ldots,\\
&\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+m-1}, \mathfrak{r}_{\mathfrak{n}+m}, \frac{\xi\varrho^{\mathfrak{n}+m-1}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+2}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+3}, \mathfrak{r}_{\mathfrak{n}+m})}\right)\ldots\Bigg)\Bigg).
\end{aligned}$$

By (3.1), it turns out that

$$\mathrm{M}_\Omega(\mathfrak{r}_{\mathfrak{n},\mathfrak{n}+1}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1, \frac{\xi}{\delta^\mathfrak{n}}\right), \forall \mathfrak{n} \in \mathbb{N}, \xi > 0,$$

and since $\kappa \geq 1$ and $\mathfrak{n} > m$, we have

$$\begin{aligned}
&\mathrm{M}_\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m}, \xi)\\
&\geq *\Bigg(\mathrm{M}_\Omega\left(\mathfrak{r}_o, \mathfrak{r}_1, \frac{\xi\varrho^\mathfrak{n}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n} + \mathfrak{r}_{\mathfrak{n}+m})\delta^\mathfrak{n}}\right), *\Bigg(\mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1 \frac{\xi\varrho^{\mathfrak{n}+1}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+m})\delta^{\mathfrak{n}+1}}\right),\\
&* \Bigg(\mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1 \frac{\xi\varrho^{\mathfrak{n}+2}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+2}, \mathfrak{r}_{\mathfrak{n}+m})\delta^{\mathfrak{n}+2}}\right)\ldots\Bigg)\Bigg)\\
&\mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1, \frac{\xi\varrho^{\mathfrak{n}+m-1}}{\kappa\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{\mathfrak{n}+2}, \mathfrak{r}_{\mathfrak{n}+m})\Omega(\mathfrak{r}_{n+3}, \mathfrak{r}_{\mathfrak{n}+m})\delta^{\mathfrak{n}+m-1}}\ldots\right)\Bigg)\Bigg)\\
&= *_{i=\mathfrak{n}}^{\infty}\mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1, \frac{\xi\varrho^i}{\kappa\prod_{j=\mathfrak{n}}^{i}\Omega(\mathfrak{r}_j, \mathfrak{r}_{n+m})\delta^i}\right)\\
&\geq *_{i=\mathfrak{n}}^{\infty}\mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1, \frac{\xi\varrho^i}{\kappa^{(i+2-\mathfrak{n})}\delta^i}\right)\\
&\geq *_{i=\mathfrak{n}}^{\infty}\mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1, \frac{\xi\varrho^i}{\kappa^i\delta^i}\right)\\
&\geq *_{i=\mathfrak{n}}^{\infty}\mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathfrak{r}_1, \frac{\xi}{\nu^i}\right)
\end{aligned}$$

where $\nu = \frac{\kappa\delta}{\varrho}$. Since $\nu \in (0,1)$, by (3.2) it follows that $\{\mathfrak{r}_\mathfrak{n}\}$ is Cauchy sequence. □

Corollary 3.1. *Let $\{\mathfrak{r}_\mathfrak{n}\}$ is a sequence in $\mathcal{EF}_b\mathcal{MS}$ $(S, \mathrm{M}_\Omega, *, \Omega)$ with $\Omega : S \times S \to [1,\infty)$ and let $*$ is of H-type. If there is $\delta \in (0, \frac{1}{\kappa})$, where $\kappa = \limsup_{m,\mathfrak{n}\to\infty}\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$ and*

$$\mathrm{M}_\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+1}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}-1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right), \forall \mathfrak{n} \in \mathbb{N}, \xi > 0, \tag{3.3}$$

then $\{\mathfrak{r}_\mathfrak{n}\}$ *is Cauchy sequence.*

Lemma 3.2. *If for some* $\delta \in (0,1)$ *and* $\mathfrak{r}, \mathfrak{z} \in S$*,*

$$\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), \xi > 0 \tag{3.4}$$

then $\mathfrak{r} = \mathfrak{z}$.

Proof. Condition (3.4) implies that

$$\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta^\mathfrak{n}}\right), \forall \mathfrak{n} \in \mathbb{N}, \xi > 0.$$

Now

$$\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) \geq \lim_{\mathfrak{n}\to\infty} \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta^\mathfrak{n}}\right) = 1, \xi > 0,$$

and $(E\Omega_2)$ implies that $\mathfrak{r} = \mathfrak{z}$. □

Theorem 3.1. *Let* $(S, \mathrm{M}_\Omega, *, \Omega)$ *be a complete* $\mathcal{EF}_b\mathcal{MS}$ *and let* $\mathtt{f} : S \to S$*. Assume that there exist* $\delta \in (0, \frac{1}{\kappa})$*, with* $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$ *such that*

$$\mathrm{M}_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right) \ \ \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \tag{3.5}$$

and there are $\mathfrak{r}_0 \in S$ *and* $\nu \in (0,1)$ *such that*

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} \mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathtt{f}\mathfrak{r}_0, \frac{\xi}{\nu^i}\right) = 1, \xi > 0. \tag{3.6}$$

Where $\{\mathfrak{r}_n\} \subset S$*, is defined by* $\mathfrak{r}_{\mathfrak{n}+1} = \mathtt{f}\mathfrak{r}_\mathfrak{n}$*,* $\mathfrak{n} \in \mathbb{N} \cup \{0\}$*. Then* $\mathtt{f}$ *has unique fixed point .*

Proof. Putting $\mathfrak{r} = \mathfrak{r}_{n-1}$ and $\mathfrak{z} = \mathfrak{r}_n$ in (3.8), we have

$$\mathrm{M}_\Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+1}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}-1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right), \forall\ \mathfrak{n} \in \mathbb{N}, \xi > 0,$$

it follows that $\mathfrak{r}_\mathfrak{n}$ is a Cauchy sequence by Lemma 3.1. Since $(S, \mathrm{M}_\Omega, *, \Omega)$ is complete, there exist $\mathfrak{r} \in S$ such that $\lim_{\mathfrak{n}\to\infty} \mathfrak{r}_\mathfrak{n} = \mathfrak{r}$ and therefore

$$\lim_{\mathfrak{n}\to\infty} \mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{r}_\mathfrak{n}, \xi) = 1,\ \xi > 0. \tag{3.7}$$

Using (3.8) and $(E\Omega_4)$, it follows that

$$\begin{aligned} \mathrm{M}_\Omega(\mathtt{f}\mathfrak{r}, \mathfrak{r}, \xi) &\geq *\left(\mathrm{M}_\Omega\left(\mathtt{f}\mathfrak{r}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{2\Omega(\mathtt{f}\mathfrak{r}, \mathfrak{r})}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}, \frac{\xi}{2\Omega(\mathtt{f}\mathfrak{r}, \mathfrak{r})}\right)\right) \\ &\geq *\left(\mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{2\Omega(\mathtt{f}\mathfrak{r}, \mathfrak{r})\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}, \frac{\xi}{2\Omega(\mathtt{f}\mathfrak{r}, \mathfrak{r})}\right)\right). \end{aligned}$$

If $n \to \infty$, by (3.7), we have

$$\mathrm{M}_\Omega(\mathtt{f}\mathfrak{r}, \mathfrak{r}, \xi) \geq *(1,1) = 1.$$

Therefore $\mathfrak{r}$ is fixed point of $\mathtt{f}$. To show uniqueness, assume that $\mathfrak{z} \neq \mathfrak{r}$ is another fixed point of $\mathtt{f}$. That is $\mathfrak{z} = \mathtt{f}(\mathfrak{z}) \neq \mathtt{f}(\mathfrak{r}) = \mathfrak{r}$. Then using (3.8), it follows that

$$\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) = \mathrm{M}_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), \ \xi > 0.$$

Which by Lemma 3.2 gives a contradiction $\mathfrak{r} = \mathfrak{z}$. Hence $\mathtt{f}$ has unique fixed point. □

Corollary 3.2. *([17] fuzzy Banach contraction theorem) Let* $(S, \mathrm{M}, *)$ *be a complete fuzzy metric space and* $\mathtt{f} : S \to S$ *be a mapping such that*

$$\mathrm{M}(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \xi) \geq \mathrm{M}\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right) \ \ \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \tag{3.8}$$

where $\delta \in (0, 1)$*. Then* $\mathtt{f}$ *has unique fixed point .*

Proof. Proof follows directly from Theorem 3.1, by taking $\Omega : S \times S \to [1, \infty)$ to be the constant function $\Omega(\mathfrak{r}) = 1$ for all $\mathfrak{r} \in S$. □

Corollary 3.3. *([35] Theorem 2.4) Let* $(S, \mathrm{M}, *)$ *be a complete fuzzy b-metric space and* $\mathtt{f} : S \to S$ *be a mapping. Assume that there exist* $\delta \in (0, \frac{1}{b})$*, such that*

$$\mathrm{M}(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \xi) \geq \mathrm{M}\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right) \ \ \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \tag{3.9}$$

and there are $\mathfrak{r}_0 \in S$ *and* $\nu \in (0, 1)$ *such that*

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} \mathrm{M}\left(\mathfrak{r}_0, \mathtt{f}\mathfrak{r}_0, \frac{\xi}{\nu^i}\right) = 1, \xi > 0. \tag{3.10}$$

Then $\mathtt{f}$ *has unique fixed point .*

Proof. Take $\Omega : S \times S \to [1, \infty)$, in above Theorem, to be a constant function $\Omega(\mathfrak{r}) = b$ for all $\mathfrak{r} \in S$. Where $b \geq 1$. □

The following example elaborates that Theorem 3.1 is proper generalization of fuzzy Banach contraction theorem ([17] Theorem 5).

Example 3.1. *Let* $S = [0, 1]$ *and* $\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) = e^{\frac{-|\mathfrak{r}-\mathfrak{z}|}{\xi}}$, *for all* $\mathfrak{r}, \mathfrak{z} \in S$*. It can be verified that* $(S, \mathrm{M}_\Omega, *, \Omega)$ *is complete* $\mathcal{EF}_b\mathcal{MS}$ *with mapping* $\Omega : S \times S \to [1, \infty)$ *defined by* $\Omega(\mathfrak{r}, \mathfrak{z}) = 1 + \mathfrak{r}\mathfrak{z}$*, and continuous t-norm* $*$ *as usual product.*

Let $\mathtt{f} : S \to S$ *be such that* $\mathtt{f}(\mathfrak{r}) = 1 - \frac{1}{3}\mathfrak{r}$*. For all* $\xi > 0$*, we have*

$$\mathrm{M}_\Omega\left(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \frac{1}{3}\xi\right) = e^{\frac{-\frac{2}{3}|\mathfrak{r}-\mathfrak{z}|}{\xi}} > e^{\frac{-|\mathfrak{r}-\mathfrak{z}|}{\xi}} = \mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi) \ \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0,$$

for $0 < \delta = \frac{1}{3} < \kappa = \frac{1}{2}$*. that is conditions of Theorem 3.1 are satisfied. Therefore,* $\mathtt{f}$ *has unique fixed point* $\frac{3}{4} \in [0, 1] = S$*.*

Theorem 3.2. *Let* $\mathtt{f}$ *be a self mapping on a complete extended fuzzy b-metric space* $(S, M_\Omega, *, \Omega)$*. Assume that there exist* $\delta \in (0, \frac{1}{\kappa})$ *with* $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$ *such that*

$$M_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \xi) \geq \min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{f}\mathfrak{z}, \mathfrak{z}, \frac{\xi}{\delta}\right)\right\}, \tag{3.11}$$

for all $\mathfrak{r}, \mathfrak{z} \in S,\ \xi > 0$*, and there exist* $\mathfrak{r}_0 \in S$ *and* $v \in (0, 1)$

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} M_\Omega\left(\mathfrak{r}_0, \mathfrak{f}\mathfrak{r}_0, \frac{\xi}{v^i}\right) = 1, \tag{3.12}$$

for each $\xi > 0$*. Where* $\{\mathfrak{r}_n\} \subset S$*, is defined by* $\mathfrak{r}_{\mathfrak{n}+1} = \mathfrak{f}\mathfrak{r}_\mathfrak{n}$*,* $\mathfrak{n} \in \mathbb{N} \cup \{0\}$*.*

Then $\mathtt{f}$ *is unique fixed point.*

Proof. By (3.11) with $\mathfrak{r} = \mathfrak{r}_n$ and $\mathfrak{z} = \mathfrak{z}_{\mathfrak{n}-1}$, for every $\xi > 0$ and for all $\mathfrak{n} \in \mathbb{N}$, we have

$$\begin{aligned} M_\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi) &\geq \min\left\{M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right)\right\} \\ &\geq \min\left\{M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right)\right\}. \end{aligned}$$

If $M_\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi) \geq M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right)$, then Lemma 3.2, implies that $\mathfrak{r}_\mathfrak{n} = \mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{n} \in \mathbb{N}$. That is $\mathfrak{n}$ is fixed point of $\mathtt{f}$.

Therefore

$$M_\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi) \geq M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right),\ \mathfrak{n} \in \mathbb{N}, \xi > 0,$$

and we have that $\mathfrak{r}_\mathfrak{n}$ is Cauchy sequence by Lemma 3.1. As a result there is a $\mathfrak{r} \in S$ such that $\lim_{\mathfrak{n}\to\infty} \mathfrak{r} = \mathfrak{r}_\mathfrak{n}$ and therefore

$$\lim_{\mathfrak{n}\to\infty} M_\Omega(\mathfrak{r}, \mathfrak{r}_\mathfrak{n}\mathfrak{n}, \xi) = 1,\ \xi > 0. \tag{3.13}$$

Let's show that $\mathfrak{r}$ is fixed point of $\mathtt{f}$. Let $\varrho_1 \in (\delta\kappa, 1)$ an $\varrho_2 = 1 - \varrho_1$. By (3.11) we have

$$\begin{aligned} &M_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \xi) \\ &\geq *\left(M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{r}_\mathfrak{n}, \frac{\xi\varrho_1}{\Omega(f\mathfrak{r}, \mathfrak{r})}\right), M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})}\right)\right) \\ &\geq *\Bigg(\min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{r}_\mathfrak{n}, \frac{\xi\varrho_1}{\delta\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi\varrho_1}{\delta\Omega(f\mathfrak{r}, \mathfrak{r})}\right),\right. \\ &\left. M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+1}, \frac{\xi\varrho_1}{\delta\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})}\right)\right\}, M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})}\right)\Bigg). \end{aligned}$$

Letting $\mathfrak{n} \to \infty$ and using (3.13), we obtain

$$M_\Omega(\mathfrak{f}_\mathfrak{r}, \mathfrak{r}, \xi) \geq *\left(M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi\varrho_1}{\kappa\delta}\right), 1\right) = M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi}{v}\right).$$

Where $v = \frac{\kappa\delta}{\varrho_1} \in (0, 1)$, we have

$$M_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \xi) \geq M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{v}\right).$$

From Lemma 3.2 it follows that $\mathfrak{f}\mathfrak{r} = \mathfrak{r}$. To show uniqueness, suppose $\mathfrak{z} \neq \mathfrak{r}$ is another fixed point of $\mathfrak{f}$. By condition (3.11) we get

$$\begin{aligned}
&M_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \xi) \\
&\geq \min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{z}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\delta}\right)\right\} \\
&= \min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), 1, 1\right\} = M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right) = M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\delta}\right), \ \forall \xi > 0.
\end{aligned}$$

Lemma 3.2 gives $\mathfrak{r} = \mathfrak{z}$. Hence $\mathfrak{f}$ has unique fixed point. □

In the following we present fuzzy version of Ćirić quasicontraction in the setting of $\mathcal{EF}_b\mathcal{MS}$.

Theorem 3.3. *Let $\mathfrak{f}$ be self mapping on a complete $\mathcal{EF}_b\mathcal{MS}$ $(S, M_\Omega, *_m, \Omega)$. Suppose there exists $\delta \in (0, \frac{1}{k^2})$, with $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$, such that*

$$\begin{aligned}
M_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \xi) \geq \min\Bigg\{&M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{f}\mathfrak{z}, \mathfrak{z}, \frac{\xi}{\delta}\right), \\
&M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{z}, \frac{2\xi}{\delta}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\delta}\right)\Bigg\}, \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0.
\end{aligned} \tag{3.14}$$

Where $\{\mathfrak{r}_n\} \subset S$, is defined by $\mathfrak{r}_{\mathfrak{n}+1} = \mathfrak{f}\mathfrak{r}_\mathfrak{n}$, $\mathfrak{n} \in \mathbb{N} \cup \{0\}$, for some $\mathfrak{r}_0 \in S$. Then $\mathfrak{f}$ has a unique fixed point.

Proof. Take $\mathfrak{r} = \mathfrak{r}_\mathfrak{n}$ *and* $\mathfrak{z} = \mathfrak{r}_{\mathfrak{n}-1}$ in (3.14). By $(E\Omega_4)$ along with the assumption $* = *_m$, and Lemma 3.1, we have

$$\begin{aligned}
&M_\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi) \\
&\geq \min\Bigg\{M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), \\
&\min\left\{M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}-1})\delta}\right), M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}-1})\delta}\right)\right\}, \\
&M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right)\Bigg\} \\
&\geq \min\left\{M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}-1})\delta}\right), M_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}-1})\delta}\right)\right\}.
\end{aligned}$$

Using same arguments as used in the proof of Theorem 3.2, it turns out that

$$M_\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi) \geq M_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\kappa\delta}\right), \ \forall \mathfrak{n} \in \mathbb{N}, \xi > 0,$$

and $\{\mathfrak{r}_\mathfrak{n}\}$ is Cauchy . So there is $\mathfrak{r} \in S$ such that $\lim_{\mathfrak{n}\to\infty} \mathfrak{r} = \mathfrak{r}_\mathfrak{n}$ and therefore

$$\lim_{\mathfrak{n}\to\infty} M_\Omega(\mathfrak{r}, \mathfrak{r}_\mathfrak{n}, \xi) = 1, \ \xi > 0. \tag{3.15}$$

Let $\varrho_1 \in (\kappa^2\delta, 1)$ and $\varrho_2 = 1 - \varrho_1$. Using (3.14) and $(E\Omega_4)$ for $* = *_m$, we have

$$\begin{aligned}
&\mathrm{M}_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \xi)\\
&\geq \min\left\{\mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{r}_\mathfrak{n}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}}\right), \mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}_\mathfrak{n}, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})}\right)\right\}\\
&\geq \min\left\{\min\left\{\mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{r}_\mathfrak{n}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})\delta}\right),\right.\right.\\
&\mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}+1}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})\delta}\right), \min\left\{\mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}_\mathfrak{n})\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})\delta}\right),\right.\\
&\left.\left.\mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{r}_\mathfrak{n}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}_\mathfrak{n})\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})\delta}\right)\right\}, \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{r}_{\mathfrak{n}+1}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})\delta}\right)\right\},\\
&\left.\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r})}\right)\right\},\ \forall\ \mathfrak{n} \in \mathbb{N},\ \xi > 0.
\end{aligned}$$

Letting $\mathfrak{n} \to \infty$ and using (3.15), we obtain

$$\begin{aligned}
\mathrm{M}_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \xi) &\geq \min\left\{\min\left\{1, \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi\varrho_1}{\kappa\delta}\right),\quad 1, \min\left\{\mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi\varrho_1}{\kappa^2\delta}\right), 1\right\}, 1\right\}, 1\right\}\\
&= \mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi\varrho_1}{\kappa^2\delta}\right),\ \xi > 0
\end{aligned}$$

and by Lemma 3.2 with $\nu = \frac{\kappa^2\delta}{\varrho_1} \in (0, 1)$ it follows that $\mathfrak{f}\mathfrak{r} = \mathfrak{r}$. To show uniqueness, suppose $\mathfrak{z} = \mathfrak{f}(\mathfrak{z}) \neq \mathfrak{f}(\mathfrak{r}) = \mathfrak{r}$. By Condition (3.14) we get

$$\begin{aligned}
&\mathrm{M}_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \xi)\\
&\geq \min\left\{\mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{z}, \mathfrak{z}, \frac{\xi}{\delta}\right),\right.\\
&\left.\min\left\{\mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{\kappa\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\kappa\delta}\right)\right\}, \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\delta}\right)\right\}\\
&= \min\left\{\mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), 1, 1, \min\left\{1, \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\kappa\delta}\right)\right\}, \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right)\right\}\\
&= \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\kappa\delta}\right) = \mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\kappa\delta}\right)\ \forall\xi > 0.
\end{aligned}$$

Which gives contradiction $\mathfrak{r} = \mathfrak{z}$ in the view of Lemma 3.2. Hence $\mathfrak{f}$ has unique fixed point. □

Remark 3.1. *In the above theorem, the quasicontractive condition involves the strongest t-norm, that is minimum t-norm $*_m$, therefore is of least interest. In the next theorem, we relax this condition by using a t-norm weaker than $*_m$. This new contractive condition therefore ensures the existence of fixed point for a relatively broader class of t-norms.*

Theorem 3.4. *Let $(S, \mathrm{M}_\Omega, *, \Omega)$ with $* \geq *_p$ be a complete $\mathcal{EF}_b\mathcal{MS}$, and let $\mathfrak{f} : S \to S$. Assume that for some $\delta \in (0, \frac{1}{\kappa^2})$,* *with* $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$*,*

$$\begin{aligned} \mathrm{M}_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \xi) \geq \min\Bigg\{ & \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{z}, \mathfrak{z}, \frac{\xi}{\delta}\right), \\ & \sqrt{\mathrm{M}_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{z}, \frac{2\xi}{\delta}\right)}, \mathrm{M}_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\delta}\right)\Bigg\},\ \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \end{aligned} \tag{3.16}$$

and there is $\mathfrak{r}_0 \in S$ *and* $v \in (0,1)$ *such that*

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} \mathrm{M}_\Omega\left(\mathfrak{r}_0, f\mathfrak{r}_0, \frac{\xi}{v^i}\right) = 1,\ \xi > 0. \tag{3.17}$$

Where $\{\mathfrak{r}_n\} \subset S$*, is defined by* $\mathfrak{r}_{\mathfrak{n}+1} = \mathfrak{f}\mathfrak{r}_\mathfrak{n}$*,* $\mathfrak{n} \in \mathbb{N} \cup \{0\}$*. Then* $\mathfrak{f}$ *is unique fixed point.*

Proof. Taking $\mathfrak{r} = \mathfrak{r}_\mathfrak{n}$ *and* $\mathfrak{z} = \mathfrak{r}_{\mathfrak{n}-1}$ in Condition (3.16), by $(E\Omega_4)$ and $* \geq *_p$, we have

$$\begin{aligned} & \mathrm{M}_\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi) \\ & \geq \min\Bigg\{\mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), \\ & \sqrt{\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\Omega(\mathfrak{r}_{n+1}, \mathfrak{r}_{\mathfrak{n}-1})\delta}\right) \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_{\mathfrak{n}-1})\delta}\right)}, \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right)\Bigg\} \\ & \geq \min\Bigg\{\mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\delta}\right), \\ & \sqrt{\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\kappa\delta}\right) \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\kappa\delta}\right)}, \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\delta}\right)\Bigg\}. \end{aligned}$$

since $\mathrm{M}_\Omega(\mathfrak{r}, \mathfrak{z}, \xi)$ is b-nondecreasing in ξ and for $\mathfrak{w}_g, \mathfrak{w}_2 \in [0,1]$, $\min\{\mathfrak{w}_g, \mathfrak{w}_y\} \leq \sqrt{\mathfrak{w}_g.\mathfrak{w}_2}$, therefore

$$\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi\right) \geq \min\left\{\mathrm{M}_\Omega\left(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \frac{\xi}{\kappa\delta}\right), \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\kappa\delta}\right)\right\}.$$

Which implies

$$\mathrm{M}_\Omega(\mathfrak{r}_{\mathfrak{n}+1}, \mathfrak{r}_\mathfrak{n}, \xi) \geq \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_{\mathfrak{n}-1}, \frac{\xi}{\kappa\delta}\right) \text{ for } \mathfrak{n} \in \mathbb{N}, \xi > 0.$$

Otherwise from Lemma 3.2, it follows that $\mathfrak{r}_\mathfrak{n}$ is fixed point of $\mathfrak{f}$.

Lemma 3.1 implies that $\{\mathfrak{r}_\mathfrak{n}\}$ is Cauchy. Completeness of $(S, \mathrm{M}_\Omega, *, \Omega)$ implies that there is $\mathfrak{r} \in S$ such that $\lim_{\mathfrak{n}\to\infty} \mathfrak{r} = \mathfrak{r}_\mathfrak{n}$ and therefore

$$\lim_{\mathfrak{n}\to\infty} \mathrm{M}_\Omega\left(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}, \xi\right) = 1, \xi > 0. \tag{3.18}$$

Let $\varrho_1 \in (\kappa^2\delta, 1)$ and $\varrho_2 = 1 - \varrho_1$. By (3.16) and $(E\Omega_4)$ for $* \geq *_p$, we have

$$\mathrm{M}_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \xi)$$

$$\geq *\left(M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{r}_n, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})}\right), M_\Omega\left(\mathfrak{f}\mathfrak{r}_n, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})}\right)\right)$$
$$\geq *\left(\min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{r}_n, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right), M_\Omega\left(\mathfrak{r}_n, \mathfrak{r}_{n+1}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right),\right.\right.$$
$$\sqrt{M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r}_n)\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{r}_n, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r}_n)\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right)},$$
$$\left.\left.M_\Omega\left(\mathfrak{r}, \mathfrak{r}_{n+1}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right)\right\}, M_\Omega\left(\mathfrak{r}_{n+1}, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r}))}\right)\right)$$
$$\geq *\left(M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{r}_n, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})}\right), M_\Omega\left(\mathfrak{f}\mathfrak{r}_n, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})}\right)\right)$$
$$\geq *\left(\min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{r}_n, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right), M_\Omega\left(\mathfrak{r}_n, \mathfrak{r}_{n+1}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right),\right.\right.$$
$$\min\left\{M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r}_n)\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{r}_n, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r}_n)\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right)\right\},$$
$$\left.\left.M_\Omega\left(\mathfrak{r}, \mathfrak{r}_{n+1}, \frac{\xi\varrho_1}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r})\delta}\right)\right\}, M_\Omega\left(\mathfrak{r}_{n+1}, \mathfrak{r}, \frac{\xi\varrho_2}{\Omega(\mathfrak{f}\mathfrak{r},\mathfrak{r}))}\right)\right)$$
$$\forall n \in \mathbb{N}, \xi > 0.$$

Letting $n \to \infty$ and by (3.18), we get

$$M_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \xi) \geq *\left(\min\left\{1, M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{r}, \frac{\xi\varrho_1}{\kappa\delta}\right), 1, \min\left\{M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi\varrho_1}{\kappa^2\delta}\right), 1\right\}, 1\right\}, 1\right)$$
$$= M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi\varrho_1}{\kappa^2\delta}\right)\ \xi > 0,$$

and by Lemma 3.2 with $v = \frac{\kappa^2\delta}{\varrho_1} \in (0, 1)$ it follows that $\mathfrak{f}\mathfrak{r} = \mathfrak{r}$. To show uniqueness, suppose that $\mathfrak{z} \neq \mathfrak{r}$ is another fixed point of f. By Condition (3.16) we get

$$M_\Omega(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \xi) \geq \min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{\delta}\right) M_\Omega\left(\mathfrak{f}\mathfrak{z}, \mathfrak{z}, \frac{\xi}{\delta}\right)\right.$$
$$\left.\sqrt{M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{r}, \frac{\xi}{\kappa\delta}\right), M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\kappa\delta}\right)}, M_\Omega\left(\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\delta}\right)\right\}$$
$$\geq \min\left\{M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right), 1, 1, \min\left\{1, M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\kappa\delta}\right)\right\}, M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\delta}\right)\right\}$$
$$= M_\Omega\left(\mathfrak{r}, \mathfrak{z}, \frac{\xi}{\kappa\delta}\right) = M_\Omega\left(\mathfrak{f}\mathfrak{r}, \mathfrak{f}\mathfrak{z}, \frac{\xi}{\kappa\delta}\right),\ \forall \xi > 0.$$

Lemma 3.2 gives contradiction $\mathfrak{r} = \mathfrak{z}$. Hence $\mathfrak{f}$ has unique fixed point. □

4. Consequences

This section is about the construction of some fixed point results involving integral inequalities as consequences of our results. Define a function $T : [0, \infty) \to [0, \infty)$ as

$$T(t) = \int_0^t \mho(t)dt \ \ \forall t > 0, \tag{4.1}$$

where $T(t)$ is non-decreasing and continuous function. Also $\mho(t) > 0$ and $\mho(t) = 0$ iff $t = 0$.

Theorem 4.1. *Let* $(S, M_\Omega, *_m, \Omega)$ *be a complete extended fuzzy b-metric space and* $\mathtt{f}$ *is self mapping on* S. *Assume there exist* $\mathfrak{r}_0 \in S$, $\nu \in (0,1)$ *and* $\delta \in (0, \frac{1}{\kappa})$, *with* $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$ *such that*

$$\int_0^{M_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \delta\xi)} \mho(\xi)d\xi \geq \int_0^{M_\Omega(\mathfrak{r}, \mathfrak{z}, \xi)} \mho(\xi)d\xi \ \ \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \tag{4.2}$$

and

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} M_\Omega\left(\mathfrak{r}_0, \mathtt{f}\mathfrak{r}_0, \frac{\xi}{\nu^i}\right) = 1, \xi > 0.$$

Where $\{\mathfrak{r}_n\} \subset S$, *is defined by* $\mathfrak{r}_{\mathfrak{n}+1} = \mathtt{f}\mathfrak{r}_\mathfrak{n}$, $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. *Then* $\mathtt{f}$ *has unique fixed point* .

Proof. (4.1) along with (4.2) implies that

$$T\left(M_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \delta\xi)\right) \geq T\left(M_\Omega(\mathfrak{r}, \mathfrak{z}, \xi)\right).$$

As T is non-decreasing and continuous therefore

$$M_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \delta\xi) \geq M_\Omega(\mathfrak{r}, \mathfrak{z}, \xi).$$

Rest of the proof follows from Theorem 3.1. □

In the following, we present a more general form of Theorem 4.1 as a consequence of Theorem 3.2.

Theorem 4.2. *Let* $(S, M_\Omega, *_m, \Omega)$ *be a complete extended fuzzy b-metric space and* $\mathtt{f}$ *is self mapping on* S. *Assume there exist* $\mathfrak{r}_0 \in S$, $\nu \in (0,1)$ *and* $\delta \in (0, \frac{1}{\kappa})$, *with* $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$ *such that*

$$\int_0^{M_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \delta\xi)} \mho(\xi)d\xi \geq \int_0^{\mathcal{B}(\mathfrak{r}, \mathfrak{z}, \xi)} \mho(\xi)d\xi \ \ \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \tag{4.3}$$

and

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} M_\Omega\left(\mathfrak{r}_0, \mathtt{f}\mathfrak{r}_0, \frac{\xi}{\nu^i}\right) = 1, \xi > 0.$$

Where $\mathcal{B}(\mathfrak{r}, \mathfrak{z}, \xi) = \min\{M_\Omega(\mathfrak{r}, \mathfrak{z}, \xi), M_\Omega(\mathtt{f}\mathfrak{r}, \mathfrak{r}, \xi), M_\Omega(\mathtt{f}\mathfrak{z}, \mathfrak{z}, \xi)\}$, *and* $\{\mathfrak{r}_n\} \subset S$, *is defined by* $\mathfrak{r}_{\mathfrak{n}+1} = \mathtt{f}\mathfrak{r}_\mathfrak{n}$, $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. *Then* $\mathtt{f}$ *has unique fixed point* .

Proof. (4.1) along with (4.3) implies that

$$T\left(M_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \delta\xi)\right) \geq T\left(\mathcal{B}(\mathfrak{r}, \mathfrak{z}, \xi)\right).$$

As T is non-decreasing and continuous therefore

$$M_\Omega(\mathtt{f}\mathfrak{r}, \mathtt{f}\mathfrak{z}, \delta\xi) \geq \mathcal{B}(\mathfrak{r}, \mathfrak{z}, \xi).$$

Rest of the proof follows from Theorem 3.2. □

Theorem 4.3. *Let* $(S, \mathrm{M}_\Omega, *_m, \Omega)$ *be a complete extended fuzzy b-metric space and* $\mathtt{f}$ *is self mapping on* S. *Assume there exist* $\mathfrak{r}_0 \in S$, $\nu \in (0,1)$ *and* $\delta \in (0, \frac{1}{\kappa})$, *with* $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(\mathfrak{r}_\mathfrak{n}, \mathfrak{r}_m)$ *such that*

$$\int_0^{\mathrm{M}_\Omega(\mathtt{f}\mathfrak{r},\mathtt{f}\mathfrak{z},\delta\xi)} \mho(\xi)d\xi \geq \int_0^{\mathcal{Y}(\mathfrak{r},\mathfrak{z},\xi)} \mho(\xi)d\xi \;\; \forall \mathfrak{r}, \mathfrak{z} \in S, \xi > 0, \tag{4.4}$$

and

$$\lim_{\mathfrak{n}\to\infty} *_{i=\mathfrak{n}}^{\infty} \mathrm{M}_\Omega\left(\mathfrak{r}_0, \mathtt{f}\mathfrak{r}_0, \frac{\xi}{\nu^i}\right) = 1, \xi > 0.$$

Where $\mathcal{Y}(\mathfrak{r},\mathfrak{z},\xi) = \min\{\mathrm{M}_\Omega(\mathfrak{r},\mathfrak{z},\xi), \mathrm{M}_\Omega(\mathtt{f}\mathfrak{r},\mathfrak{r},\xi), \mathrm{M}_\Omega(\mathtt{f}\mathfrak{z},\mathfrak{z},\xi), \mathrm{M}_\Omega(\mathtt{f}\mathfrak{r},\mathfrak{z},2\xi), \mathrm{M}_\Omega(\mathfrak{r},\mathtt{f}\mathfrak{z},\xi)\}$, *and* $\{\mathfrak{r}_n\} \subset S$, *is defined by* $\mathfrak{r}_{\mathfrak{n}+1} = \mathtt{f}\mathfrak{r}_\mathfrak{n}$, $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. *Then* $\mathtt{f}$ *has unique fixed point* .

Proof. (4.1) along with (4.4) implies that

$$T\left(\mathrm{M}_\Omega(\mathtt{f}\mathfrak{r},\mathtt{f}\mathfrak{z},\delta\xi)\right) \geq T\left(\mathcal{Y}(\mathfrak{r},\mathfrak{z},\xi)\right).$$

As T is non-decreasing and continuous therefore

$$\mathrm{M}_\Omega(\mathtt{f}\mathfrak{r},\mathtt{f}\mathfrak{z},\delta\xi) \geq \mathcal{Y}(\mathfrak{r},\mathfrak{z},\xi).$$

Rest of the proof follows from Theorem 3.3. □

In the same manner, results on integral inequalities can be obtained as a consequence of Theorem 3.4.

5. Application to integral equations

Integral equations find applications in a variety of scientific fields, such as biology, chemistry, physics, or engineering. It is a rapidly growing field in abstract space. Furthermore, fuzzy integral equations constitute one of the important branches of fuzzy analysis theory and play a vital role in numerical analysis. One of the important approaches used for the studying integral equations is to apply fixed point theory directly to the mapping defined by the right-hand side of the equation, or to develop homotopy methods, which are largely considered in fixed point theory. In particular, for its connection with the study of fuzzy integral problems, we highlight a very recent paper [14], in which the author proposes a homotopy analysis method to find an approximate solution of the two-dimensional non-linear fuzzy Volterra integral equation. We also refer the reader to [15, 16, 27, 31, 51] for other related works.

We apply our theory of fixed point to ensure the existence of solutions to the following type of integral equations:

$$u(t) = f(t) + \int_0^t H(t,s,u(s))ds, \; t \in [0,b], \tag{5.1}$$

where $b > 0$. The Banach space $C \equiv C([0,b],\mathbb{R})$ of all real continuous functions defined on $[0,b]$, with norm $\|u\| := \sup_{s\in[0,b]} |u(s)|$ for every $u \in C$, can be considered as a fuzzy Banach space [30] (for more details concerning the relation between Banach spaces and fuzzy Banach spaces, see [44]). Consider the fuzzy metric on C given by

$$M(u,v,\delta) = e^{-\frac{\sup_{s\in[0,b]} |u(s)-v(s)|^2}{\delta}},$$

for all $u, v \in C$ and $\delta > 0$, furnished with the t-norm $*_p$ defined as $x *_p y = xy$ for all $x, y \in [0, 1]$. Then $(C, M, *_p, \Omega)$ is a complete $\mathcal{EF}_b\mathcal{MS}$ for a bounded function $\Omega : C \times C \to [1, \infty)$.

In the following, we discuss the existence of solutions for the integral equations of the form (5.1).

Theorem 5.1. *Let $P : C \to C$ be an integral operator given by*

$$[P(u)](t) = f(t) + \int_0^t H(t, s, u(s))ds,\ u \in C,\ t \in [0, b].$$

Let $\{f_n\} \subset C$, be defined by $f_{\mathfrak{n}+1} = P(f_{\mathfrak{n}})$, $\mathfrak{n} \in \mathbb{N}\cup\{0\}$, for $f \in C$. Suppose there exists $\delta \in (0, \frac{1}{k^2})$, with $\kappa = \limsup_{m,\mathfrak{n}\to\infty} \Omega(f_{\mathfrak{n}}, f_m)$, where $\Omega : C\times C \to [1, \infty)$ is a bounded function and let $H \in C([0, b]\times[0, b]\times\mathbb{R}, \mathbb{R})$ satisfies the following condition:

(i) There exists a continuous and non-decreasing mapping $\psi : [0, 1] \to [0, 1]$ with $\psi(t) > t$ for all $t \in (0, 1)$, such that, for all $u, v \in C$, and every $\delta > 0$,

$$\sup_{s\in[0,b]} \left(\int_0^s |H(s, r, u(r)) - H(s, r, v(r))|dr\right)^2 \leq -\ln\left(\psi\left(e^{-\frac{\sup_{s\in[0,b]} |u(s)-v(s)|^2}{\kappa\delta}}\right)\right).$$

Then, the integral Eq (5.1) has a solution $u^* \in C$.

Proof. For all $u, v \in C$, and $\delta > 0$, we have

$$\begin{aligned} M(P(u), P(v), \kappa\delta) &= e^{-\frac{\sup_{s\in[0,b]} |[P(u)](s)-[P(v)](s)|^2}{\kappa\delta}} \\ &\geq e^{-\frac{\sup_{s\in[0,b]} \left(\int_0^s |H(s,r,u(r))-H(s,r,v(r))|dr\right)^2}{\kappa\delta}} \\ &\geq e^{-\frac{\sup_{s\in[0,b]} \left(\int_0^s |H(s,r,u(r))-H(s,r,v(r))|dr\right)^2}{\delta}} \\ &\geq \psi\left(e^{-\frac{\sup_{s\in[0,b]} |u(s)-v(s)|^2}{\delta}}\right) \\ &= M(u, v, \delta). \end{aligned}$$

Hence, using Theorem 3.1, P has a fixed point $u^* \in C$, which is a solution to the integral Eq (5.1). □

Remark 5.1. *With slight modification, Theorems 3.2–3.4 can also be applied to the solution of integral equation of type* (5.1).

6. Application to nonlinear fractional differential equations

The important objective in this section is to study the existence and uniqueness of solutions to a nonlinear fractional differential equation (NFDE)by applying Theorem 3.1. Considering the Banach spacer $\mathcal{S} \equiv C([0, 1], \mathbb{R})$ of all continuous real valued functions defined on $[0, 1]$ with with norm

$||u|| := \sup_{s\in[0,1]} |u(s)|$ for every $u \in \mathcal{S}$, we study the existence of unique solutions to a non-linear fractional differential equation

$$D^{\sigma}_{0+}(\mathtt{u(t)}) = \mathtt{g(t,u(t))}, \quad t \in (0,1) \tag{6.1}$$

with boundary conditions

$$\mathtt{u}(0) + \acute{u}(0) = 0, \quad \mathtt{u}(1) + \acute{u}(1) = 0,$$

where $\mathtt{u} \in \mathcal{S}, \sigma \in (1,2]$ and $\mathtt{f} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Notice that, $\mathtt{u} \in \mathcal{S}$ is a solution of (6.1) whenever $\mathtt{u} \in \mathcal{S}$ solves the following integral equation

$$\begin{aligned}\mathtt{u(s)} =& \frac{1}{\Gamma(\sigma)}\int_0^1 (1-\tau)^{\sigma-1}(1-s)\mathtt{f}(\tau,\mathtt{x}(\tau))\mathrm{d}\tau + \frac{1}{\Gamma(\sigma-1)}\int_0^1 (1-\tau)^{\sigma-2}(1-s)\mathtt{f}(\tau,\mathtt{x}(\tau))\mathrm{d}\tau \\ &+ \frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathtt{f}(\tau,\mathtt{x}(\tau))\mathrm{d}\tau.\end{aligned} \tag{6.2}$$

Detailed description of the problem context can be found in [2, 10, 34, 52]. The following theorem demonstrates that a solution exists to the nonlinear fractional differential Eq (6.1). Define integral operator $\mathfrak{J} : \mathcal{S} \to \mathcal{S}$ by

$$\begin{aligned}\mathfrak{J}\mathtt{u(s)} =& \frac{1}{\Gamma(\sigma)}\int_0^1 (1-\tau)^{\sigma-1}(1-s)\mathtt{f}(\tau,\mathtt{u}(\tau))\mathrm{d}\tau + \frac{1}{\Gamma(\sigma-1)}\int_0^1 (1-\tau)^{\sigma-2}(1-s)\mathtt{f}(\tau,\mathtt{u}(\tau))\mathrm{d}\tau \\ &+ \frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathtt{f}(\tau,\mathtt{u}(\tau))\mathrm{d}\tau.\end{aligned} \tag{6.3}$$

where $\mathcal{S}$ is an extended fuzzy b-metric space with extended fuzzy b-metric given by

$$\mathcal{M}(u,v,\ell) = \frac{\alpha\ell}{\alpha\ell + \beta\, sup_{s\in[0,1]} |u(s) - v(s)|},$$

$\forall\ \ell > 0$ and $u, v \in \mathcal{S}$. Here α and β are positive real numbers and continuous t-norm $*$ as the usual product.

Theorem 6.1. *Let $\{f_n\} \subset \mathcal{S}$, be defined by $f_{\mathfrak{n}+1} = \mathfrak{J}(f_{\mathfrak{n}})$, $\mathfrak{n} \in \mathbb{N} \cup \{0\}$, for $f \in \mathcal{S}$. Suppose there exists $\delta \in (0, \frac{1}{\lambda^2})$, with $\lambda = \limsup_{m,\mathfrak{n}\to\infty} \Omega(f_{\mathfrak{n}}, f_m)$, with $\Omega : \mathcal{S} \times \mathcal{S} \to [1,\infty)$ being a bounded function such that the following conditions are satisfied:*

- $|\mathtt{f}(\tau,\mathtt{u}(\tau)) - \mathtt{f}(\tau,\mathtt{v}(\tau))| \le |\mathtt{u}(\tau) - \mathtt{v}(\tau)|, \quad \forall u, v \in \mathcal{S}$
- $\sup_{\mathtt{t}\in[0,1]} \left\{\frac{1-s}{\Gamma(\sigma+1)} + \frac{1-s}{\Gamma(\sigma)} + \frac{s^{\sigma}}{\Gamma(\sigma+1)}\right\} \le \lambda < 1.$

Then the non-linear fractional differential Eq (6.1) has a unique solution

Proof.

$$
\begin{aligned}
\left|\mathfrak{J}\mathrm{u}(\mathrm{s})-\mathfrak{J}\mathrm{v}(\mathrm{s})\right| =& \Bigg|\frac{1-s}{\Gamma(\sigma)}\int_0^1(1-\tau)^{\sigma-1}\left[\mathrm{f}(\tau,\mathrm{u}(\tau))-\mathrm{f}(\tau,\mathrm{v}(\tau))\right]\mathrm{d}\tau\\
&+\frac{1-s}{\Gamma(\sigma-1)}\int_0^1(1-\tau)^{\sigma-2}\left[\mathrm{f}(\tau,\mathrm{u}(\tau))-\mathrm{f}(\tau,\mathrm{v}(\tau))\right]\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\left[\mathrm{f}(\tau,\mathrm{u}(\tau))-\mathrm{f}(\tau,\mathrm{v}(\tau))\right]\mathrm{d}\tau\Bigg|\\
\leq& \frac{1-s}{\Gamma(\sigma)}\int_0^1(1-\tau)^{\sigma-1}\left|\mathrm{f}(\tau,\mathrm{u}(\tau))-\mathrm{f}(\tau,\mathrm{v}(\tau))\right|\mathrm{d}\tau\\
&+\frac{1-s}{\Gamma(\sigma-1)}\int_0^1(1-\tau)^{\sigma-2}\left|\mathrm{f}(\tau,\mathrm{u}(\tau))-\mathrm{f}(\tau,\mathrm{v}(\tau))\right|\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\left|\mathrm{f}(\tau,\mathrm{u}(\tau))-\mathrm{f}(\tau,\mathrm{v}(\tau))\right|\mathrm{d}\tau\\
\leq& \frac{1-s}{\Gamma(\sigma)}\int_0^1(1-\tau)^{\sigma-1}\left|\mathrm{u}(\tau)-\mathrm{v}(\tau)\right|\mathrm{d}\tau\\
&+\frac{1-s}{\Gamma(\sigma-1)}\int_0^1(1-\tau)^{\sigma-2}\left|\mathrm{u}(\tau)-\mathrm{v}(\tau)\right|\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\left|\mathrm{u}(\tau)-\mathrm{v}(\tau)\right|\mathrm{d}\tau\\
\leq& \sup_{\mathrm{s}\in[0,1]}\left|\mathrm{u}(s)-\mathrm{v}(s)\right|\Bigg(\frac{1-s}{\Gamma(\sigma)}\int_0^1(1-\tau)^{\sigma-1}\mathrm{d}\tau+\frac{1-s}{\Gamma(\sigma-1)}\int_0^1(1-\tau)^{\sigma-2}\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\mathrm{d}\tau\Bigg)\\
=& \sup_{\mathrm{s}\in[0,1]}\left|\mathrm{u}(s)-\mathrm{v}(s)\right|\left(\frac{1-s}{\Gamma(\sigma+1)}+\frac{1-s}{\Gamma(\sigma)}+\frac{s^\sigma}{\Gamma(\sigma+1)}\right)\\
=& \delta\sup_{\mathrm{s}\in[0,1]}\left|\mathrm{u}(s)-\mathrm{v}(s)\right|,\ \text{where } \delta=\frac{1-s}{\Gamma(\sigma+1)}+\frac{1-s}{\Gamma(\sigma)}+\frac{s^\sigma}{\Gamma(\sigma+1)}.
\end{aligned}
$$

From the above inequality, it turns out that

$$
\begin{aligned}
&\sup_{\mathrm{s}\in[0,1]}\left|\mathfrak{J}\mathrm{u}(\mathrm{s})-\mathfrak{J}\mathrm{v}(\mathrm{s})\right|\leq\delta\sup_{\mathrm{s}\in[0,1]}\left|\mathrm{u}(s)-\mathrm{v}(s)\right|\\
\Rightarrow\ &\alpha s+\frac{\beta}{\delta}\sup_{\mathrm{s}\in[0,1]}\left|\mathfrak{J}\mathrm{u}(\mathrm{s})-\mathfrak{J}\mathrm{v}(\mathrm{s})\right|\leq\alpha s+\beta\sup_{\mathrm{s}\in[0,1]}\left|\mathrm{u}(s)-\mathrm{v}(s)\right|\\
\Rightarrow\ &\frac{\alpha(\delta s)}{\alpha(\delta s)+\beta\sup_{\mathrm{s}\in[0,1]}\left|\mathfrak{J}\mathrm{u}(\mathrm{s})-\mathfrak{J}\mathrm{v}(\mathrm{s})\right|}\geq\frac{\alpha s}{\alpha s+\beta\sup_{\mathrm{s}\in[0,1]}\left|\mathrm{u}(s)-\mathrm{v}(s)\right|}\\
\Rightarrow\ &\mathcal{M}(\mathfrak{J}\mathrm{u},\mathfrak{J}\mathrm{v},\delta s)\geq\mathcal{M}(\mathrm{u},\mathrm{v},s),
\end{aligned}
$$

Thus by Theorem 3.1, the operator $\mathfrak{J}$ has a fixed point in $\mathcal{S}$, consequently the non-linear fractional differential Eq (6.1) has a unique solution in $\mathcal{S}$. □

Remark 6.1. *Using similar arguments as above, with little modification, Theorems 3.2–3.4 can also be applied to the solution of the non-linear fractional differential Eq (6.1).*

7. Conclusions

In this work we established an important lemma for showing a sequence to be Caushy in $\mathcal{EF}_b\mathcal{MS}$. Utilizing this lemma we have established some fixed point results in the context of $\mathcal{EF}_b\mathcal{MS}$. As application, we apply the established theory for the existence of solution to a type of integral equations and a nonlinear fractional differential equation. Our results generalize some well-known fixed point results in the literature. Our established results may lead to further research and investigation.

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Conflict of interest

The authors declare that they have no competing interest regarding this manuscript

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