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Research article

Upper semicontinuous selections for fuzzy mappings in noncompact WPH-spaces with applications

Haishu Lu*, **Xiaoqiu Liu and Rong Li**

School of Business, Jiangsu University of Technology, Changzhou 213001, Jiangsu, China

* **Correspondence:** Email: luhaishu@126.com; Tel: +8651986953306; Fax: +8651986953300.

Abstract: In this paper, we introduce the concept of a WPH-space without linear structure and proceed to establish a new upper semicontinuous selection theorem for fuzzy mappings in the framework of noncompact WPH-spaces as well as a special form of this selection theorem in crisp settings. As applications, fuzzy collective coincidence point theorems, fuzzy collectively fixed point theorems, and existence theorems of equilibria for the generalized fuzzy games with three constraint set-valued mappings and generalized fuzzy qualitative games in WPH-spaces are obtained. As their special cases in crisp settings, we derive existence theorems of equilibria for generalized games and generalized qualitative games. Finally, we construct a multiobjective game model for water resource allocation and prove the existence of Pareto equilibria for this multiobjective game based on the existence theorem of equilibria for qualitative games.

Keywords: WPH-space; fuzzy collective coincidence point; generalized fuzzy game; generalized fuzzy qualitative game; equilibrium point; water resource allocation
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1. Introduction

Continuous selection theorems play an important role in various nonlinear problems arising in mathematics and applied science. In 1956, Michael [1] first established a well-known continuous selection theorem. In 1968, Browder [2] proved a continuous selection theorem in the setting of paracompact Hausdorff topological vector spaces and used this continuous selection theorem to obtain his famous fixed point theorem which is an essential tool in proving existence theorems of numerous nonlinear problems. Since then, Yannelis and Prabhakar [3], Ding et al. [4], Lee et al. [5], Wu and Shen [6], and Balaj and Lin [7] further studied the continuous selection theorems in Hausdorff topological vector spaces and gave applications to fixed point theorems, variational inequalities and various equilibrium problems. It is well known that the theory of fuzzy sets is widely used to deal with systems or phenomena which cannot be characterized precisely. Therefore, considering this fact, Kim et al. [8] and Kim and Lee [9] proved some continuous selection theorems for fuzzy mappings and gave their applications to fixed point theorems in Hausdorff topological vector spaces.

The linear and convex assumptions are important in the proof of the continuous selection theorems mentioned above, but at the same time these assumptions limit the application of these continuous selection theorems. Therefore, in order to further extend the application of continuous selection theorems, many authors made a lot of efforts to generalize continuous selection theorem to general spaces without any linear and convex structure. Horvath [10] generalized Michael's selection theorems for lower semicontinuous set-valued mappings from topological vector spaces to C-spaces. Ding and Park [11], Yu and Lin [12], Park [13], and Fakhar and Zafarani [14] studied continuous selection theorems in G-convex spaces and gave some applications to fixed point theorems and existence of equilibria. Ding [15] proved new continuous selection theorems in FC-spaces and applied these selection theorems to collectively fixed point theorems and coincidence theorems for two families of set-valued mappings defined on product FC-spaces. Recently, Khanh et al. [16] investigated continuous selection theory in GFC-spaces concluding FC-spaces in [15] as special cases and obtained a new continuous selection theorem with applications to fixed point theorems, section theorems, maximal elements, intersection theorems, and existence results of solutions of variational relations. Very recently, by using a continuous selection theorem which can be seen as a special case of Theorem 3.1 due to Khanh et al. [16], Khanh and Long [17] obtained a nonempty intersection theorem, an alternative theorem, and some general minimax inequalities in GFC-spaces.

However, taking into account of the above observations, it is known that almost all existing results provide sufficient conditions on the existence of selections for set-valued mappings in crisp settings, and they are also based on strong topological structures, in particular topological vector spaces or topological spaces with some abstract convex structure. Therefore, it is natural to ask whether it is possible to construct a more general space framework that can include the spaces mentioned above as special cases, and to use this space framework to study the existence of selections for fuzzy mappings and their related problems. From this viewpoint, we give the motivation of this paper, as well as the methodology and the conclusions as follows.

• **Motivations:** For any finite subset of a nonempty set, an upper semicontinuous set-valued mapping from a simplex to a topological space is constructed, on the basis of which we define the so-called WPH-space consisting of the topological space, nonempty set and set-valued mapping just mentioned. This space is more general and removes linear and convex structure, which unifies and covers many spaces in the existing literature. Therefore, it is necessary and meaningful to study the existence of upper semicontinuous selections for fuzzy mappings and related problems, such as the existence of fuzzy collective coincidence points, fuzzy collectively fixed points, and the existence of equilibria for generalized fuzzy games and generalized fuzzy qualitative games in noncompact WPH-spaces.

• **Methodology:** By using standard topological analysis, set-valued analysis, and the continuous partition of unity, we prove the existence of upper semicontinuous selections for fuzzy mappings in noncompact WPH-spaces when fuzzy mappings satisfy weaker conditions. The properties of WPH-spaces and the existence results of upper semicontinuous selections for fuzzy mappings are then used to provide full characterizations on the existence of fuzzy collective coincidence points, fuzzy collectively fixed points, and equilibria for generalized fuzzy games and generalized fuzzy qualitative games in the case where a family of fuzzy mappings, fuzzy constraint mappings, and fuzzy preference mappings satisfy weaker conditions.

- **Conclusions:** In the framework of WPH-spaces, we obtain a new existence result of upper semicontinuous selections for fuzzy mappings under the condition that fuzzy mappings satisfy weaker open cover property and convexity assumption, and as a corollary of this result, we also get an existence result of upper semicontinuous selections for set-valued mappings in crisp settings. Furthermore, based on the existence results of upper semicontinuous selections in fuzzy and crisp settings, we derive new existence results of fuzzy collective coincidence points, fuzzy collectively fixed points, and equilibria for generalized fuzzy games and generalized fuzzy qualitative games in noncompact WPH-spaces. The existence results mentioned above improve, extend and unify the main results in the existing literature.

The rest of this paper is organized as follows. Section 2 states notation and definitions. In Section 3, an upper semicontinuous selection theorem for fuzzy mappings is proved in WPH-spaces and as a direct consequence, an upper semicontinuous selection theorem for set-valued mappings is obtained. In Section 4, by using upper semicontinuous selection theorems for fuzzy mappings, we obtain fuzzy collective coincidence point theorems and fuzzy collectively fixed point theorems in noncompact WPH-spaces. In Section 5, by virtue of the upper semicontinuous selection theorem for set-valued mappings, we obtain existence theorems of equilibria for generalized fuzzy games and generalized fuzzy qualitative games, and as consequences, we also derive existence theorems of equilibria for generalized games, generalized qualitative games, and qualitative games in noncompact WPH-spaces. As applications, we use the existence result of equilibria for qualitative games to prove the existence of Pareto equilibria for a multiobjective water resource allocation game model. The concluding remarks highlight the main findings of the paper and future research trends.

2. Preliminaries

Let X be a nonempty set. We denote by 2^X the family of all subsets of X and by $\langle X \rangle$ the family of all nonempty finite subsets of X. For any $A \in \langle X \rangle$, we denote by $|A|$ the cardinality of A. Let Δ_n denote the standard n-dimensional simplex with vertices $\{e_0, e_1, \dots, e_n\}$, where e_i is the $(i+1)$th unit vector in $\mathbb{R}^{n+1}$. For every $J \subseteq \{0, 1, \dots, n\}$, let $\Delta_{|J|-1}$ denote the convex hull of $\{e_i : i \in J\}$.

A subset A of a topological space X is said to be compactly open if $A \bigcap C$ is open in C for every nonempty compact subset C of X. Note that there exists a nonempty subset A of a topological space X such that for each nonempty compact subset C of X, $A \bigcap C$ is open in C, but A is not open in X (see, for example, Kelley [18, page 240]). Therefore, the notion of a compactly open set generalizes the notion of a open set in general topological space. Moreover, let us define the compact closure and compact interior of A (see [19]) by $\mathrm{ccl}A = \bigcap\{D : A \subset D \text{ and } D \text{ is compactly closed in } X\}$ and $\mathrm{cint}A = \bigcup\{D : D \subset A \text{ and } D \text{ is compactly open in } X\}$, respectively. Obviously, $\mathrm{ccl}A$ (respectively, $\mathrm{cint}A$) is compactly closed (respectively, compactly open) in X and A is compactly closed (respectively, compactly open) if and only if $A = \mathrm{ccl}A$ (respectively, $A = \mathrm{cint}A$). If C is a nonempty compact subset of X with $A \bigcap C \neq \emptyset$, then we have $\mathrm{ccl}A \bigcap C = \mathrm{cl}_C(A \bigcap C)$ and $\mathrm{cint}A \bigcap C = \mathrm{int}_C(A \bigcap C)$.

Let X and Y be two nonempty sets. A function from X into $[0, 1]$ is called a fuzzy set on X. We denote by $\mathscr{F}(X)$ the family of all fuzzy sets on X. A mapping from Y into $\mathscr{F}(X)$ is called a fuzzy mapping. If $F : Y \to \mathscr{F}(X)$ is a fuzzy mapping, then for each $y \in Y$, $F(y)$ (denoted it by F_y in the sequel) is a fuzzy set in $\mathscr{F}(X)$ and the number $F_y(x)$ is called the degree of membership of point x in F_y. If $A \in \mathscr{F}(X)$ is a fuzzy set, then the set $(A)_\alpha = \{x \in X : A(x) > \alpha\}$, $\alpha \in [0, 1)$, is called strong α-cut, and $(A)_0$ is said to be the support of A. For more details on fuzzy sets, the reader is referred to [8, 9, 20–25] and the references therein.

Definition 2.1. Let X be a topological space and Z be a nonempty set. A triple $(X, Z; \sigma_N)$ is said to be a weakly pseudo H-space (for short, WPH-space) if for each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z\rangle$, there exists an upper semicontinuous set-valued mapping $\sigma_N : \Delta_n \to 2^X$ with nonempty values. A WPH-space $(X, Z; \sigma_N)$ is said to be a Hausdorff WPH-space if X is a Hausdorff topological space. In case $X = Z$, the triple $(X, Z; \sigma_N)$ can be written by $(X; \sigma_N)$.

Remark 2.2. It is worthwhile noting that in Definition 2.1, X and Z do not possess any linear and convex structure and $\sigma_N : \Delta_n \to 2^X$ is an upper semicontinuous mapping with nonempty values. Therefore, WPH-spaces include convex subsets of topological vector spaces, pseudo H-spaces introduced by Lai et al. [26], Lassonde's convex spaces in [27], H-spaces introduced by Horvath [28], G-convex spaces introduced by Park and Kim [29], L-spaces introduced by Ben-El-Mechaiekh et al. [30], G-H-spaces introduced by Verma [31], FC-spaces due to Ding [15], GFC-spaces due to Khanh et al. [16], and many other spaces (see, for example, [31–34] and the references therein) as special cases.

Definition 2.3. Let $(X, Z; \sigma_N)$ be a WPH-space, $A \subseteq Z$, and $B \subseteq X$. We say that B is WPH-convex relative to A if for each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z\rangle$ and each $\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subseteq A \bigcap \{z_0, z_1, \ldots, z_n\}$, we have $\sigma_N(\Delta_k) \subseteq B$, where Δ_k is the convex hull of $\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

Remark 2.4. (1) By Definition 2.3, we can see that if B is WPH-convex relative to A, then the triple $(B, A; \varphi_N)$ also forms a WPH-space. If A is nonempty and B is WPH-convex relative to A, then B is automatically nonempty. If $X = Z$ and $A = B$, then B is said to be a WPH-convex subspace of $(X; \sigma_N)$.

(2) Assume that $\{A_i\}$ be a family of subsets of Z and $\{B_i\}$ be a family of subsets of X with $\bigcap_{i\in I} B_i \neq \emptyset$, where I is an index set and B_i is WPH-convex relative to A_i for every $i \in I$. Then by Definition 2.3, we can show that $\bigcap_{i\in I} B_i$ is also WPH-convex relative to $\bigcap_{i\in I} A_i$. In fact, for each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z\rangle$ and each $\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subseteq (\bigcap_{i\in I} A_i) \bigcap \{z_0, z_1, \ldots, z_n\}$, we have $\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subseteq A_i \bigcap \{z_0, z_1, \ldots, z_n\}$ for every $i \in I$. Since B_i is WPH-convex relative to A_i for every $i \in I$, it follows that $\sigma_N(\Delta_k) \subseteq B_i$ and thus, $\sigma_N(\Delta_k) \subseteq \bigcap_{i\in I} B_i$.

(3) For any given subset B of X, let us define WPH-hull of B relative to A by

$$WPH(B, A) = \bigcap \{C \subseteq X : B \subseteq C,\ C \text{ is } WPH\text{-convex relative to } A\}.$$

It is easy to verify that $WPH(B, A)$ is WPH-convex relative to A.

Definition 2.5. Let $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ be a family of WPH-spaces, where Ω is a finite or infinite index set. Let $X = \prod_{i\in\Omega} X_i$ and Y be a topological space. The class $\widetilde{\mathscr{B}}(X, Y)$ of better admissible mappings is defined as follows: $T \in \widetilde{\mathscr{B}}(X, Y) \Leftrightarrow T : X \to 2^Y$ is a set-valued mapping with nonempty values and for each $i \in \Omega$, $N_i \in \langle Z_i\rangle$ with $|N_i| = n_i + 1$, each continuous mapping $\Psi : T(\prod_{i\in\Omega} \sigma_{N_i}(\Delta_{n_i})) \to C$, the composition mapping $\Psi \circ T|_{\prod_{i\in\Omega} \sigma_{N_i}(\Delta_{n_i})} \circ \Phi : C \to 2^C$ has a fixed point, where $C = \prod_{i\in\Omega} \Delta_{n_i}$, $\Phi(z) = \prod_{i\in\Omega} \sigma_{N_i}(\pi_i(z))$ for every $z \in C$ and π_i is the projection of C onto Δ_{n_i}.

Remark 2.6. Definition 2.5 improves and generalizes the corresponding definition due to Ding [35] in the following aspects: (a) From G-convex spaces to WPH-spaces; (b) The condition that T in the corresponding definition in Ding [35] is an upper semicontinuous set-valued mapping with compact values is removed. Even Ω is a singleton, the class $\widetilde{\mathscr{B}}(X, Y)$ unifies and extends many important classes of mappings, for example, the class $\mathscr{U}_C^K$ introduced by Park and Kim [29], the class $\mathscr{A}$ defined by Ben-Ei-Mechaiekh et al. [30], the class $\mathscr{B}$ due to Ding [34], and the class of mappings with KKM property in Chang and Yen [36].

Definition 2.7. Let $(X, Z; \varphi_N)$ be a WPH-space and Y be a nonempty set. Let $f : Y \times X \to \mathbb{R} \cup \{\pm\infty\}$ and $g : Y \times Z \to \mathbb{R} \cup \{\pm\infty\}$ be two functions. We say that f is WPH-g-quasiconcave on Y if for each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z\rangle$, each $\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subseteq \{z_0, z_1, \ldots, z_n\}$, and each $y \in Y$, we have $f(y, x) \geq \min_{0\leq j\leq k} g(y, z_{i_j})$ for every $x \in \sigma_N(\Delta_k)$. We say that f is WPH-g-quasiconvex on Y if $(-f)$ is WPH-$(-g)$-quasiconcave on Y. When $X = Z$ and $f = g$, f is said to be WPH-quasiconcave (respectively, WPH-quasiconvex) on Y.

Remark 2.8. Definition 2.7 generalizes Definition 4.1 of Zhang and Cheng [37] from FC-spaces to WPH-spaces. In addition, we can compare Definition 2.7 with Definition 3 of Kim [38] in the following aspects: (a) The condition that the mapping σ_N in Definition 2.7 is an upper semicontinuous set-valued mapping is weaker than the condition that the mapping ϕ_n in Definition 3 of Kim [38] is a single-valued continuous mapping; (b) There are two functions in Definition 2.7, but there is only one function in Definition 3 of Kim [38]; (c) In Definition 2.7, there are three nonempty sets and X does not need to be a subset of Y. In Definition 3 of Kim [38], there are two nonempty sets and one set is a subset of another one; (d) x and y in the left of the inequality in Definition 2.7 are independent of each other, but the corresponding parts in the left of the inequality in Definition 3 of Kim [38] are required to be same.

Definition 2.9. Let $(X, Z; \varphi_N)$ be a WPH-space and Y be a nonempty set. Let $F : Y \to \mathscr{F}(X)$ and $G : Y \to \mathscr{F}(Z)$ be two fuzzy mappings. We say that F is WPH-G-quasiconcave (respectively, WPH-G-quasiconvex) on Y if f is WPH-g-quasiconcave (respectively, WPH-g-quasiconvex) on Y, where the functions $f : Y \times X \to [0, 1]$ and $g : Y \times Z \to [0, 1]$ are defined by $f(y, x) = (F_y)(x)$ for every $(y, x) \in Y \times X$ and by $g(y, z) = (G_y)(z)$ for every $(y, z) \in Y \times Z$, respectively.

Definition 2.10. Let $\{(X_i, Z_i; \sigma_{N_i})|i \in \Omega\}$ be a family of WPH-spaces, where Ω is a finite or infinite index set. For each $i \in \Omega$ and each $\widetilde{N}_i \in \langle Z_i \times Z_i\rangle$, let $\pi^l(\widetilde{N}_i)$ (respectively, $\pi^r(\widetilde{N}_i)$) denote the projection of $\widetilde{N}_i$ onto the left (respectively, right) of $X_i \times X_i$. Let Y be a topological space. The class $\widetilde{\mathscr{DB}}(\Pi_{i\in\Omega}(X_i \times X_i), Y)$ of better admissible mappings is defined as follows: $T \in \widetilde{\mathscr{DB}}(\Pi_{i\in\Omega}(X_i \times X_i), Y) \Leftrightarrow T : \Pi_{i\in\Omega}(X_i \times X_i) \to 2^Y$ is a set-valued mapping with nonempty values and for each $i \in \Omega$, $\widetilde{N}_i \in \langle Z_i \times Z_i\rangle$ with $|\widetilde{N}_i| = n_i + 1$ and each continuous mapping $\Psi : T(\prod_{i\in\Omega}(\sigma_{\pi^l(\widetilde{N}_i)}(\Delta_{n_i}) \times \sigma_{\pi^r(\widetilde{N}_i)}(\Delta_{n_i}))) \to C$, the composition mapping $\Psi \circ T|_{\prod_{i\in\Omega}(\sigma_{\pi^l(\widetilde{N}_i)}(\Delta_{n_i}) \times \sigma_{\pi^r(\widetilde{N}_i)}(\Delta_{n_i}))} \circ \Phi : C \to 2^C$ has a fixed point, where $C = \prod_{i\in\Omega} \Delta_{n_i}$, $\Phi(z) = \prod_{i\in\Omega}(\sigma_{\pi^l(\widetilde{N}_i)}(\pi_i(z)) \times \sigma_{\pi^r(\widetilde{N}_i)}(\pi_i(z)))$ for every $z \in C$ and π_i is the projection of C onto Δ_{n_i}.

3. Upper semicontinuous selections for fuzzy mappings

The following upper semicontinuous selection theorem for fuzzy mappings is useful for proving our main results. The reader is referred to [39–42] for the concept of an upper semicontinuous set-valued mapping and its associated properties.

Theorem 3.1. *Let $(X, Z; \sigma_N)$ be a WPH-space, K be a nonempty compact subset of a Hausdorff topological space Y, and $\alpha : Y \to [0, 1)$ be a function. Let $H : Y \to \mathscr{F}(X)$ and $P : Y \to \mathscr{F}(Z)$ be two fuzzy mappings such that for each $y \in K$, $(H_y)_{\alpha(y)}$ is WPH-convex relative to $(P_y)_{\alpha(y)}$ and $K \subseteq \bigcup_{z\in Z} \text{cint}\{y \in Y : P_y(z) > \alpha(y)\}$. Then there exists an upper semicontinuous set-valued mapping $f : K \to 2^X$ such that $f = \sigma \circ \beta$ and $f(y) = \sigma(\beta(y)) \subseteq (H_y)_{\alpha(y)}$ for every $y \in K$, where $\sigma : \Delta_n \to 2^X$ is an upper semicontinuous set-valued mapping with nonempty values, $\beta : K \to \Delta_n$ is a continuous mapping, and n is a positive integer.*

Proof. Let us define a set-valued mapping $\Upsilon : K \to 2^Z$ by $\Upsilon(y) = (P_y)_{\alpha(y)}$ for every $y \in K$. Then we have $\Upsilon^{-1}(z) = \{y \in K : P_y(z) > \alpha(y)\}$ for every $z \in Z$. Since $K \subseteq \bigcup_{z\in Z} \text{cint}\{y \in Y : P_y(z) > \alpha(y)\}$, it follows that

$$\begin{aligned} K &= \bigcup_{z\in Z}\Big(K \bigcap \text{cint}\{y \in Y : P_y(z) > \alpha(y)\}\Big) \\ &= \bigcup_{z\in Z} \text{int}_K\{y \in K : P_y(z) > \alpha(y)\} \\ &= \bigcup_{z\in Z} \text{int}_K \Upsilon^{-1}(z). \end{aligned}$$

Since K is compact, there exists $\{\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_n\} \in \langle Z\rangle$ such that $K = \bigcup_{i=0}^n \text{int}_K \Upsilon^{-1}(\bar{z}_i)$. Therefore, there exists a continuous partition of unity $\{\beta_i\}_{i=0}^n$ subordinated to the open cover $\{\text{int}_K \Upsilon^{-1}(\bar{z}_i)\}_{i=0}^n$, that is, for each $i \in \{0, 1, \ldots, n\}$, $\beta_i : K \to [0, 1]$ is continuous such that $\sum_{i=0}^n \beta_i(y) = 1$ for every $y \in K$ and $\beta_i(y) > 0$ implies $y \in \text{int}_K \Upsilon^{-1}(\bar{z}_i)$. For each $y \in K$, let $J(y) := \{j \in \{0, 1, \ldots, n\} : \beta_j(y) > 0\}$. Then for each $j \in J(y)$, it follows that $y \in \text{int}_K \Upsilon^{-1}(\bar{z}_j)$ and so, $\bar{z}_j \in \Upsilon(y) = (P_y)_{\alpha(y)}$. Now, we define a single-valued continuous mapping $\beta : K \to \Delta_n$ by $\beta(y) = \sum_{i=0}^n \beta_i(y)e_i$ for every $y \in K$ and we have $\beta(y) = \sum_{j\in J(y)} \beta_j(y)e_j \in \Delta_{|J(y)|-1}$ for every $y \in K$. Since $(X, Z; \varphi_N)$ is a WPH-space, there exists an upper semicontinuous set-valued mapping $\sigma : \Delta_n \to 2^X$ associated with the finite set $\{\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_n\}$. Thus, the set-valued mapping $f : K \to 2^X$ defined by $f(y) = (\sigma \circ \beta)(y)$ for every $y \in K$, is upper semicontinuous on K. Since $(H_y)_{\alpha(y)}$ is WPH-convex relative to $(P_y)_{\alpha(y)}$ for every $y \in K$, it follows that $f(y) = (\sigma \circ \beta)(y) = \sigma(\beta(y)) \subseteq \sigma(\Delta_{|J(y)|-1}) \subseteq (H_y)_{\alpha(y)}$ for every $y \in K$. This completes the proof. □

Remark 3.2. Theorem 3.1 generalizes Theorem 1 of Kim et al. [8] in the following aspects: (a) From Hausdorff topological vector spaces to WPH-spaces without any linear and convex structure; (b) The compactness assumption on X of Theorem 1 of Kim et al. [8] is removed; (c) The continuous assumptions on α and the function $x \mapsto F_x(y)$ of Theorem 1 of Kim et al. [8] are dropped. Theorem 3.1 also extends and generalizes Theorem 2.2 of Tarafdar [42], Corollary 2.1 of Ding [15], and Lemma 4.2 of Khanh and Long [17] to WPH-spaces and fuzzy settings.

Example 3.3. Let $X = (0, 1)$ and $Y = (-2, 2)$ endowed with the Euclidean metric topology. Let $Z = \mathbb{R}$. Define two fuzzy mappings $H : Y \to \mathscr{F}(X)$ and $P : Y \to \mathscr{F}(Z)$ by

$$H_y(x) = \begin{cases} \frac{2}{3}, & \text{if } (y, x) \in (-2, -1) \times (0, 1) \text{ or } (y, x) \in (1, 2) \times (0, 1), \\ \frac{3}{4}, & \text{if } (y, x) \in [-1, 1] \times [\frac{1}{2}, 1), \\ 0, & \text{otherwise.} \end{cases}$$

$$P_y(z) = \begin{cases} \frac{4}{5}, & \text{if } (y,z) \in (-2,-1] \times (-\frac{1}{2}, 0] \text{ or } (y,z) \in [1,2) \times (-\frac{1}{2}, 0], \\ \frac{3}{5}, & \text{if } (y,z) \in (-1,-\frac{1}{2}) \times \{0\} \text{ or } (y,z) \in (-1,-\frac{1}{2}) \times (-2y-1, 1], \\ \frac{2}{3}, & \text{if } (y,z) \in [-\frac{1}{2}, \frac{1}{2}] \times (0,1], \\ \frac{7}{8}, & \text{if } (y,z) \in (\frac{1}{2}, 1) \times \{0\} \text{ or } (y,z) \in (\frac{1}{2}, 1) \times (2y-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha : Y \to [0,1)$ be defined by $\alpha(y) \equiv \frac{1}{2}$ for every $y \in Y$. Then we have the followings:

$$(H_y)_{\alpha(y)} = \begin{cases} (0,1), & \text{if } y \in (-2,-1) \text{ or } y \in (1,2), \\ [\frac{1}{2},1), & \text{if } y \in [-1,1]. \end{cases}$$

$$(P_y)_{\alpha(y)} = \begin{cases} (-\frac{1}{2}, 0], & \text{if } y \in (-2,-1] \text{ or } y \in [1,2), \\ \{0\} \bigcup (-2y-1, 1], & \text{if } y \in (-1, -\frac{1}{2}), \\ (0,1], & \text{if } y \in [-\frac{1}{2}, \frac{1}{2}], \\ \{0\} \bigcup (2y-1, 1], & \text{if } y \in (\frac{1}{2}, 1). \end{cases}$$

Now, we show that all the conditions of Theorem 3.1 are satisfied.

(1) For each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z \rangle$, define a continuous mapping $\rho_N : \Delta_n \to X$ as follows:

$$\rho_N(p) = \frac{1}{1 + \frac{\sum_{i=1}^n \lambda_i^2}{1 + \max_{0 \le j \le n}\{|z_j|\}}}, \quad \forall\ p = \sum_{i=0}^n \lambda_i e_i \in \Delta_n.$$

Define a set-valued mapping $L : X \to 2^X$ by $L(x) = [x, 1)$ for every $x \in X$, which is upper semicontinuous on X. Indeed, let $x \in X$ and for each open subset V of X with $L(x) = [x,1) \subseteq V$. Then there exists $\varepsilon > 0$ such that $x - \varepsilon > 0$ and $(x - \varepsilon, 1) \subseteq V$. Taking $U(x) = (x - \varepsilon, 1)$, we can see that $U(x)$ is an open neighborhood of x and $L(x') = [x', 1) \subseteq (x - \varepsilon, 1) \subseteq V$ for every $x' \in U(x)$. Thus, by the definition of an upper semicontinuous set-valued mapping (see Definition 7.1.1 of Klein and Thompson [39]), L is upper semicontinuous on X. Therefore, for each $N \in \langle Z \rangle$ with $|N| = n + 1$, the set-valued mapping $\sigma_N : \Delta_n \to 2^X$ defined by $\sigma_N(p) = L \circ \rho_N(p)$ for every $p \in \Delta_n$, is upper semicontinuous on Δ_n, which implies that $(X, Z; \sigma_N)$ forms a WPH-space. Then for each $y \in Y$, each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z \rangle$, and each $\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subseteq N \bigcap (P_y)_{\alpha(y)}$, it is easy to check that $\sigma_N(\Delta_k) \subseteq (H_y)_{\alpha(y)}$.

(2) Let $K = [-\frac{1}{2}, \frac{3}{2}]$ be a compact subset of Y. Since there exist $z_0 = 0 \in Z,\ z_1 = 1 \in Z$ such that $\mathrm{cint}\{y \in Y : P_y(0) > \alpha(y)\} = (-2, -\frac{1}{2}) \bigcup (\frac{1}{2}, 2)$ and $\mathrm{cint}\{y \in Y : P_y(1) > \alpha(y)\} = (-1, 1)$, it follows that

$$\begin{aligned} K &\subseteq \Big((-2, -\frac{1}{2}) \bigcup (\frac{1}{2}, 2)\Big) \bigcup (-1, 1) \\ &= \mathrm{cint}\{y \in Y : P_y(0) > \alpha(y)\} \bigcup \mathrm{cint}\{y \in Y : P_y(1) > \alpha(y)\} \\ &\subseteq \bigcup_{z \in Z} \mathrm{cint}\{y \in Y : P_y(z) > \alpha(y)\}. \end{aligned}$$

Hence, all the conditions of Theorem 3.1 are fulfilled. We can find a specific upper semicontinuous set-valued mapping $f : K \to 2^X$ such that $f = \sigma \circ \beta$ and $f(y) = \sigma(\beta(y)) \subseteq (H_y)_{\alpha(y)}$ for every $y \in K$, where $\sigma : \Delta_1 \to 2^X$ is an upper semicontinuous set-valued mapping with nonempty values and

$\beta : K \to \Delta_1$ is a continuous mapping. In fact, let us define a continuous mapping $\beta : K \to \Delta_1 \subseteq \mathbb{R}^2$ by $\beta(y) = (\frac{|y|}{6}, 1 - \frac{|y|}{6})$ for every $y \in K$ and an upper semicontinuous set-valued mapping $\sigma : \Delta_1 \to 2^X$ by

$$\sigma(p) = [\frac{2}{2 + \lambda_1^2 + \lambda_2^2}, 1), \ \ \forall \ p = (\lambda_1, \lambda_2) \in \Delta_1.$$

Then we can see that the upper semicontinuous set-valued mapping $f = \sigma \circ \beta$ defined on K satisfies the above requirement.

The following upper semicontinuous selection theorem for set-valued mappings is a direct consequence of Theorem 3.1 in crisp settings.

Theorem 3.4. *Let $(X, Z; \sigma_N)$ be a WPH-space and K be a nonempty compact subset of a Hausdorff topological space Y. Let $H : Y \to 2^X$ and $P : Y \to 2^Z$ be two set-valued mappings such that for each $y \in K$, $H(y)$ is WPH-convex relative to $P(y)$ and $K \subseteq \bigcup_{z \in Z} \mathrm{cint} P^{-1}(z)$. Then there exists an upper semicontinuous set-valued mapping $f : K \to 2^X$ such that $f = \sigma \circ \beta$ and $f(y) = \sigma(\beta(y)) \subseteq H(y)$ for every $y \in K$, where $\sigma : \Delta_n \to 2^X$ is an upper semicontinuous set-valued mapping with nonempty values, $\beta : K \to \Delta_n$ is a continuous mapping, and n is a positive integer.*

Proof. By using H and P, we can define two fuzzy mappings $\widetilde{H} : Y \to \mathscr{F}(X)$ and $\widetilde{P} : Y \to \mathscr{F}(Z)$ by $\widetilde{H}_y = \chi_{H(y)}$ and by $\widetilde{P}_y = \chi_{P(y)}$ for every $y \in Y$, respectively, where χ_E is the characteristic function of the subset $E \subseteq X$ or $E \subseteq Z$. Define $\alpha : Y \to [0, 1)$ by $\alpha(y) \equiv 0$ for every $y \in Y$. Then it is easy to see that $(\widetilde{H}_y)_{\alpha(y)} = H(y)$, $(\widetilde{P}_y)_{\alpha(y)} = P(y)$ for every $y \in Y$, and $P^{-1}(z) = \{y \in Y : \widetilde{P}_y(z) > \alpha(y)\}$. Thus, all the hypotheses of Theorem 3.1 are satisfied. Therefore, there exists an upper semicontinuous set-valued mapping $f : K \to 2^X$ such that $f = \sigma \circ \beta$ and $f(y) = \sigma(\beta(y)) \subseteq (\widetilde{H}_y)_{\alpha(y)} = H(y)$ for every $y \in K$, where $\sigma : \Delta_n \to 2^X$ is an upper semicontinuous set-valued mapping with nonempty values, $\beta : K \to \Delta_n$ is a continuous mapping, and n is a positive integer. This completes the proof. □

4. Fuzzy coincidence and fixed points

In this section, as applications of the upper semicontinuous selection theorems, we will establish fuzzy coincidence point theorems and fuzzy collectively fixed point theorems in noncompact *WPH*-spaces without any linear and convex structure.

Let Ω be a finite or infinite index set. For each $i \in \Omega$, let X_i be a nonempty set, $H_i : Y \to \mathscr{F}(X_i)$ be a fuzzy mapping, and $\alpha_i : Y \to [0, 1)$ be a function. Let $X = \prod_{i \in \Omega} X_i$ and $T : X \to 2^Y$ be a set-valued mapping, where Y is a topological space. We consider the fuzzy collective coincidence point problem (in short, FCCPP) as follows: finding $\widehat{x} \in X$ and $\widehat{y} \in Y$ such that $\widehat{y} \in T(\widehat{x})$ and $\widehat{x}_i \in (H_{i\widehat{y}})_{\alpha_i(\widehat{y})}$ for every $i \in \Omega$. Let $H_i : Y \to 2^{X_i}$ be a set-valued mapping for every $i \in \Omega$.

If $X = Y$ and T is the identity mapping on X, then the above problem deduces the fuzzy collectively fixed point problem (in short, FCFPP) as follows: Finding $\widehat{x} \in X$ such that $\widehat{x}_i \in (H_{i\widehat{x}})_{\alpha_i(\widehat{x})}$ for all $i \in \Omega$, which was studied by Kim and Lee [9] in locally convex Hausdorff topological vector spaces.

Theorem 4.1. *Let Ω be a finite or infinite index set and $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ be a family of WPH-spaces. Let $X = \prod_{i \in \Upsilon} X_i$ and $T \in \widetilde{\mathscr{B}}(X, Y)$, where Y is a Hausdorff topological space. For each $i \in \Omega$, let $\alpha_i : Y \to [0, 1)$ be a function and let $H_i : Y \to \mathscr{F}(X_i)$ and $P_i : Y \to \mathscr{F}(Z_i)$ be two fuzzy mappings such that the following conditions hold*:

(i) $T(X)$ *is a compact subset of* Y;

(ii) *For each* $y \in T(X)$, $(H_{i_y})_{\alpha_i(y)}$ *is WPH-convex relative to* $(P_{i_y})_{\alpha_i(y)}$;

(iii) $T(X) \subseteq \bigcup_{z_i \in Z_i} \operatorname{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$.

Then FCCPP has a solution.

Proof. By Theorem 3.1, for each $i \in \Omega$, there exists an upper semicontinuous set-valued mapping $f_i : T(X) \to 2^{X_i}$ such that $f_i = \sigma_i \circ \beta_i$ and $f_i(y) = \sigma_i(\beta_i(y)) \subseteq (H_{i_y})_{\alpha_i(y)}$ for every $y \in Y$, where $\sigma_i : \Delta_{n_i} \to 2^{X_i}$ is an upper semicontinuous set-valued mapping with nonempty values, $\beta_i : T(X) \to \Delta_{n_i}$ is a continuous mapping, and n_i is a positive integer. Let $C = \prod_{i\in\Omega} \Delta_{n_i}$ and $\pi_i : C \to \Delta_{n_i}$ be the projection of C onto Δ_{n_i}. Let us define a set-valued mapping $\Phi : C \to 2^X$ and a continuous mapping $\Psi : T(X) \to C$ by $\Phi(z) = \prod_{i\in\Omega} \sigma_i(\pi_i(z))$ for every $z \in C$ and $\Psi(y) = \prod_{i\in\Omega} \beta_i(y)$ for every $y \in T(X)$, respectively. We observe that $\sigma_i(\Delta_{n_i}) \subseteq X_i$ for every $i \in \Omega$. Hence, $\prod_{i\in\Omega} \sigma_i(\Delta_{n_i}) \subseteq \prod_{i\in\Omega} X_i = X$ and $T(\prod_{i\in\Omega} \sigma_i(\Delta_{n_i})) \subseteq T(X)$. Since $T \in \widetilde{\mathscr{B}}(X, Y)$, it follows that the composition $\Psi \circ T|_{\prod_{i\in\Omega} \sigma_i(\Delta_{n_i})} \circ \Phi : C \to 2^C$ has a fixed point, which implies that there exists $\widehat{z} \in C$ such that

$$\widehat{z} \in \Psi \circ T|_{\prod_{i\in\Omega} \sigma_i(\Delta_{n_i})} \circ \Phi(\widehat{z}).$$

Then there exists $\widehat{x} \in \Phi(\widehat{z})$ such that $\widehat{z} \in \Psi \circ T|_{\prod_{i\in\Omega} \sigma_i(\Delta_{n_i})}(\widehat{x})$. Let $\widehat{y} \in T(\widehat{x})$ such that

$$\widehat{x} \in \Phi(\widehat{z}) = \Phi \circ \Psi(\widehat{y}) = \Phi\Big(\prod_{i\in\Omega} \beta_i(\widehat{y})\Big) = \prod_{i\in\Omega} \sigma_i\Big(\pi_i\Big(\prod_{i\in\Omega} \beta_i(\widehat{y})\Big)\Big) = \prod_{i\in\Omega} (\sigma_i \circ \beta_i)(\widehat{y}).$$

It follows that $\widehat{x_i} \in (\sigma_i \circ \beta_i)(\widehat{y}) \subseteq (H_{i_{\widehat{y}}})_{\alpha_i(\widehat{y})}$ for every $i \in \Omega$. Hence, FCCPP has a solution. This completes the proof. □

Example 4.2. Let Ω be a singleton and $X = Y = (\frac{1}{2}, 1]$ with the Euclidean metric topology. Let $Z = (\frac{1}{2}, 1]$. Define two fuzzy mappings $H : Y \to \mathscr{F}(X)$ and $P : Y \to \mathscr{F}(Z)$ by

$$H_y(x) = (\frac{1}{3}y + \frac{1}{4})x, \;\; \forall\, x, y \in (\frac{1}{2}, 1],$$

$$P_y(z) = (\frac{2}{9}y + \frac{1}{6})z, \;\; \forall\, y, z \in (\frac{1}{2}, 1],$$

respectively. Let $\alpha : Y \to [0, 1)$ be defined by $\alpha(y) = \frac{1}{6}y + \frac{1}{8}$ for every $y \in (\frac{1}{2}, 1]$. Then we have $(H_y)_{\alpha(y)} = (\frac{1}{2}, 1]$ and $(P_y)_{\alpha(y)} = (\frac{3}{4}, 1]$ for every $y \in (\frac{1}{2}, 1]$. Define a set-valued mapping $T : X \to 2^Y$ by

$$T(x) = \begin{cases} [\frac{2}{3}, \frac{4}{5}], & \text{if } x \in [\frac{9}{10}, 1], \\ [\frac{11}{15}, \frac{4}{5}], & \text{if } x \in (\frac{1}{2}, \frac{9}{10}). \end{cases}$$

Now, we check all the conditions of Theorem 4.1 as follows:

(1) For each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z\rangle$, define a continuous function $\nu_N : \Delta_n \to X$ by $\nu_N(p) = \sum_{i=0}^n \lambda_i z_i$ for every $p = \sum_{i=0}^n \lambda_i e_i$. Furthermore, let $L : X \to 2^X$ be defined by

$$L(x) = \begin{cases} (\frac{1}{2}, \frac{3}{4}], & \text{if } x \in [\frac{3}{4}, 1], \\ [\frac{3}{5}, \frac{7}{10}], & \text{if } x \in (\frac{1}{2}, \frac{3}{4}). \end{cases}$$

We can prove that L is upper semicontinuous on X. Indeed, let $x \in X$ and for any open subset V of X with $L(x) \subseteq V$. Let $x = \frac{3}{4}$. Then we have $L(\frac{3}{4}) = (\frac{1}{2}, \frac{3}{4}] \subseteq V$ and thus, there exists $\varepsilon > 0$ such that

$\frac{3}{4}+\varepsilon<1$ and $(\frac{1}{2},\frac{3}{4}+\varepsilon)\subseteq V$. Let $U(\frac{3}{4})=(\frac{1}{2},\frac{3}{4}+\varepsilon)$ be an open neighborhood of $\frac{3}{4}$ in X. If $x^{'}\in(\frac{1}{2},\frac{3}{4})$, then we have $L(x^{'})=[\frac{3}{5},\frac{7}{10}]\subseteq(\frac{1}{2},\frac{3}{4}]\subseteq V$. If $x^{'}\in[\frac{3}{4},\frac{3}{4}+\varepsilon)$, then we have $L(x^{'})=(\frac{1}{2},\frac{3}{4}]\subseteq V$. Therefore, we have $L(x^{'})\subseteq V$ for every $x^{'}\in U(\frac{3}{4})$, which implies that L is upper semicontinuous at $\frac{3}{4}$. Now, let $x=1$. Then we have $L(1)=(\frac{1}{2},\frac{3}{4}]\subseteq V$. Taking $U(1)=(\frac{1}{2},1]$ as an open neighborhood of 1 in X, we know that $L(x^{'})=(\frac{1}{2},\frac{3}{4}]\subseteq V$ for each $x^{'}\in[\frac{3}{4},1]$ and $L(x^{'})=[\frac{3}{5},\frac{7}{10}]\subseteq(\frac{1}{2},\frac{3}{4}]\subseteq V$ for each $x^{'}\in(\frac{1}{2},\frac{3}{4})$. Therefore, we have $L(x^{'})\subseteq V$ for every $x^{'}\in U(1)$, which implies that L is upper semicontinuous at 1. By using the argument similar to that used above, we can prove that L is upper semicontinuous at every $x\in(\frac{3}{4},1)\bigcup(\frac{1}{2},\frac{3}{4})$. Thus, L is upper semicontinuous on X. For each $N=\{z_0,z_1\ldots,z_n\}\in\langle Z\rangle$, define $\sigma_N=L\circ\nu_N:\Delta_n\to 2^X$. Then σ_N is upper semicontinuous on Δ_n and thus, $(X;\sigma_N)$ forms a WPH-space. It is easy to see that $T(X)=[\frac{2}{3},\frac{4}{5}]$, which is a nonempty compact subset of Y. Therefore, condition (i) of Theorem 4.1 is fulfilled. For each $y\in T(X)=[\frac{2}{3},\frac{4}{5}]$, each $N=\{z_0,z_1\ldots,z_n\}\in\langle Z\rangle$, and each $\{z_{i_0},z_{i_1}\ldots,z_{i_k}\}\subseteq N\cap(P_y)_{\alpha(y)}$, we have $\sigma_N(\Delta_k)\subseteq(H_y)_{\alpha(y)}$, which implies that condition (ii) of Theorem 4.1 is satisfied.

(2) Since α and the function $y\mapsto(P_y)(z)$ for each $z\in Z$ are continuous on Y, it follows that the set $\{y\in Y:P_y(z)>\alpha(y)\}$ is an open subset of Y for every $z\in Z$. Therefore, for each $z\in Z$, we have $\{y\in Y:P_y(z)>\alpha(y)\}=\text{cint}\{y\in Y:P_y(z)>\alpha(y)\}$. For each $y\in Y$, there exists $\{1\}\subseteq Z$ such that $1\in(P_y)_{\alpha(y)}=(\frac{3}{4},1]$ and hence, we get $T(X)=[\frac{2}{3},\frac{4}{5}]\subseteq Y=\text{cint}\{y\in Y:P_y(1)>\alpha(y)\}$, which implies that condition (iii) of Theorem 4.1 is satisfied.

(3) We show that $T\in\widetilde{\mathscr{B}}(X,Y)$. We can prove that T is upper semicontinuous on X. Indeed, let $x\in X$ and for any open subset V of Y with $T(x)\subseteq V$. Let $x=\frac{9}{10}$. Then we have $T(\frac{9}{10})=[\frac{2}{3},\frac{4}{5}]\subseteq V$. Let $U(\frac{9}{10})=(\frac{4}{5},1)$ be an open neighborhood of $\frac{9}{10}$ in X. If $x^{'}\in(\frac{4}{5},\frac{9}{10})$, then we have $T(x^{'})=[\frac{11}{15},\frac{4}{5}]\subseteq[\frac{2}{3},\frac{4}{5}]\subseteq V$. If $x^{'}\in[\frac{9}{10},1)$, then we have $T(x^{'})=[\frac{2}{3},\frac{4}{5}]\subseteq V$. Therefore, we have $T(x^{'})\subseteq V$ for every $x^{'}\in U(\frac{9}{10})$. Now, let $x=1$. Then we have $T(1)=[\frac{2}{3},\frac{4}{5}]\subseteq V$. Let $U(1)=(\frac{1}{2},1]$ be an open neighborhood of 1 in X and thus, we can see that if $x^{'}\in[\frac{9}{10},1]$, then we have $T(x^{'})=[\frac{2}{3},\frac{4}{5}]\subseteq V$ and if $x^{'}\in(\frac{1}{2},\frac{9}{10})$, then we have $T(x^{'})=[\frac{11}{15},\frac{4}{5}]\subseteq[\frac{2}{3},\frac{4}{5}]\subseteq V$. Therefore, we have $T(x^{'})\subseteq V$ for every $x^{'}\in U(1)$. By using the argument similar to that used above, we can prove that T is upper semicontinuous at every $x\in(\frac{9}{10},1)\bigcup(\frac{1}{2},\frac{9}{10})$. Thus, T is upper semicontinuous on X. For each $N=\{z_0,z_1,\ldots,z_n\}\in\langle D\rangle$ and each $p\in\Delta_n$, we can see that $T(\sigma_N)(p)=[\frac{11}{15},\frac{4}{5}]$. In fact, for each $p\in\Delta_n$ and each $N=\{z_0,z_1\ldots,z_n\}\in\langle Z\rangle$, by the definition of the function ν, we have $\frac{1}{2}<\nu_N(p)=\sum_{i=0}^{n}\lambda_i d_i\leq 1$. If $\frac{3}{4}\leq\nu_N(p)\leq\frac{1}{2}$, then $\sigma_N(p)=L\circ\nu_N(p)=(\frac{1}{2},\frac{3}{4}]\subseteq(\frac{1}{2},\frac{9}{10})$. If $\frac{1}{2}<\nu_N(p)<\frac{3}{4}$, then $\sigma_N(p)=L\circ\nu_N(p)=[\frac{3}{5},\frac{7}{10}]\subseteq(\frac{1}{2},\frac{9}{10})$. By the definition of T, we have $T(\sigma_N)(p)=T(\sigma_N(\Delta_n))=[\frac{11}{15},\frac{4}{5}]$. Therefore, the composition $T|_{\sigma_N}\circ\sigma_N:\Delta_n\to 2^{T(\sigma_N(\Delta_n))}$ is an upper semicontinuous set-valued mapping with nonempty compact convex values. By Lemma 1 in Yuan [40], for each continuous function $\psi:T(\sigma_N(\Delta_n))\to\Delta_n$, the composition $\psi\circ T|_{\sigma_N}\circ\sigma_N:\Delta_n\to 2^{\Delta_n}$ has a fixed point, which implies that $T\in\widetilde{\mathscr{B}}(X,Y)$. Let $x^*=\frac{9}{10}$ and $y^*=\frac{7}{10}$. Then we can see that $y^*\in T(x^*)$ and $x^*\in(H_{y^*})_{\alpha(y^*)}$.

Corollary 4.3. *Let Ω be a finite or infinite index set and $\{(X_i,Z_i;\sigma_{N_i})|i\in\Omega\}$ be a family of WPH-spaces. Let $X=\prod_{i\in\Upsilon}X_i$ and $T\in\widetilde{\mathscr{B}}(X,Y)$, where T is upper semicontinuous set-values mapping with compact values and Y is a Hausdorff topological space. For each $i\in\Omega$, let $\alpha_i:Y\to[0,1)$ be a function and let $H_i:Y\to\mathscr{F}(X_i)$ and $P_i:Y\to\mathscr{F}(Z_i)$ be two fuzzy mappings such that the following conditions hold*:

(i) *For each $y\in T(X)$, $(H_{i_y})_{\alpha_i(y)}$ is WPH-convex relative to $(P_{i_y})_{\alpha_i(y)}$;*

(ii) $T(X)\subseteq\bigcup_{z_i\in Z_i}\text{cint}\{y\in Y:P_{i_y}(z_i)>\alpha_i(y)\}$.

Then FCCPP has a solution.

Proof. It suffices to prove that $T(X)$ is a compact subset of Y. Indeed, by Proposition 3.1.11 of Aubin and Ekeland [43], $T(X)$ is compact subset of Y. Therefore, all the requirements of Theorem 4.1 are satisfied. So, by Theorem 4.1, FCCPP has a solution. This completes the proof. □

Theorem 4.4. *Let Y be a Hausdorff topological space and $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ be a family of WPH-spaces, where Ω is a finite or infinite index set. Let $X = \prod_{i\in\Omega} X_i$ and $T \in \widetilde{\mathscr{B}}(X, Y)$. For each $i \in \Omega$, let $\alpha_i : Y \to [0, 1)$ be a function and let $H_i : Y \to \mathscr{F}(X_i)$ and $P_i : Y \to \mathscr{F}(Z_i)$ be two fuzzy mappings such that the following conditions hold:*

(i) *For each $y \in T(X)$, $(H_{i_y})_{\alpha_i(y)}$ is WPH-convex relative to $(P_{i_y})_{\alpha_i(y)}$;*

(ii) *There exists a nonempty subset Z_i^0 of Z_i such that $\bigcup_{z_i\in Z_i} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$ contains $\bigcap_{z_i\in Z_i^0}[\mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c$ which is a nonempty compact subset of Y or an empty set, and for each $Q_i \in \langle Z_i\rangle$, there exists a nonempty subset L_{Q_i} of X_i, which is WPH-convex relative to some $Z_i^{'} \subseteq Z_i$ such that $Z_i^0 \bigcup Q_i \subseteq Z_i^{'}$ and $T(L_Q)$ is compact, where $L_Q = \prod_{i\in\Omega} L_{Q_i}$.*

Then FCCPP has a solution.

Proof. For each $i \in \Omega$, set $K_i := \bigcap_{z_i\in Z_i^0}[\mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c$. Then it follows from (ii) that $K_i \subseteq \bigcup_{z_i\in Z_i} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$. If K_i is nonempty compact for every $i \in \Omega$, then there exists $Q_i \in \langle Z_i\rangle$ such that

$$K_i \subseteq \bigcup_{z_i\in Q_i} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}. \tag{4.1}$$

Thus, by (4.1), we have

$$\begin{aligned} Y &= (Y \setminus K_i)\bigcup K_i \\ &= \Big(\bigcup_{z_i\in Z_i^0} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}\Big)\bigcup\Big(\bigcup_{z_i\in Q_i} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}\Big) \\ &= \bigcup_{z_i\in Z_i^0\cup Q_i} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}. \end{aligned}$$

If $K_i = \emptyset$, then for every $Q_i \in \langle Z_i\rangle$, we have

$$Y = \bigcup_{z_i\in Z_i^0} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\} = \bigcup_{z_i\in Z_i^0\cup Q_i} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}. \tag{4.2}$$

Therefore, (4.2) holds for each $i \in \Omega$ and the finite set Q_i in (4.1). By (ii) again, for each $i \in \Omega$ and the finite set Q_i in (4.1), there exists a nonempty subset $L_{Q_i} \subseteq X_i$ which is WPH-convex relative to some $Z_i^{'} \subseteq Z_i$ such that $Z_i^0 \bigcup Q_i \subseteq Z_i^{'}$. Hence, by (4.2), we have

$$T(L_Q) \subseteq Y = \bigcup_{z_i\in Z_i^{'}} \mathrm{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}. \tag{4.3}$$

As remarked before, we can see that $\{(L_{Q_i}, Z_i^{'}; \sigma_{N_i}) | i \in \Omega\}$ is a family of WPH-spaces. For each $i \in \Omega$, define two fuzzy mappings $H_i' : Y \to \mathscr{F}(L_{Q_i})$ and $P_i' : Y \to \mathscr{F}(Z_i')$ by

$$H_{i_y}' = H_{i_y}|_{L_{Q_i}} \text{ and } P_{i_y}' = P_{i_y}|_{Z_i'}, \ \ \forall\, y \in Y.$$

Since $T \in \widetilde{\mathscr{B}}(X, Y)$, it follows that $T|_{L_Q} \in \widetilde{\mathscr{B}}(L_Q, Y)$. By (4.3), we have

$$T(L_Q) \subseteq \bigcup_{z_i \in Z_i'} \text{cint}\{y \in Y : P'_{i_y}(z_i) > \alpha_i(y)\}, \ \ \forall\, i \in \Omega. \tag{4.4}$$

Then by (4.4) and the assumption that $T(L_Q)$ is compact, we know that (i) and (iii) of Theorem 4.1 are fulfilled, respectively. We now prove that (ii) of Theorem 4.1 for H_i' and P_i' holds. In fact, by (i), for each $y \in T(L_Q)$, each $i \in \Omega$, each $N_i = \{z_{i_0}, z_{i_1}, \ldots, z_{i_{n_i}}\} \in \langle Z_i' \rangle$, and each $\{z_{i_{j_0}}, z_{i_{j_1}}, \ldots, z_{i_{j_{k_i}}}\} \subseteq N_i \bigcap (P'_{i_y})_{\alpha_i(y)} = N_i \bigcap \{z_i \in Z_i' : P'_{i_y}(z_i) > \alpha_i(y)\} = N_i \bigcap (P_{i_y})_{\alpha_i(y)} \bigcap Z_i'$, we have

$$\sigma_{N_i}(\Delta_{k_i}) \subseteq (H_{i_y})_{\alpha_i(y)} \bigcap L_{Q_i} = (H'_{i_y})_{\alpha_i(y)},$$

which implies that (ii) of Theorem 4.1 is satisfied. Therefore, it follows from Theorem 4.1 that FCCPP for $T|_{L_Q}$ and $\{H_i'\}_{i\in\Omega}$ has a solution, which is also a solution to FCCPP for T and $\{H_i\}_{i\in\Omega}$. This completes the proof. □

Remark 4.5. (1) Theorem 3.3 of Lu and Hu [44] and Theorem 4.4 cannot be deduced from each other for the following reasons: (a) Although the FWC-spaces involved in Theorem 3.3 of Lu and Hu [44] are more general than the WPH-spaces involved in Theorem 4.4, (ii) of Theorem 3.3 due to Lu and Hu [44] is not required in Theorem 4.4; (b) The condition that $\bigcap_{z_i \in Z_i^0}[\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c$ is a nonempty compact subset of Y or an empty set in Theorem 4.4 is weaker than the condition that $\bigcap_{d_i \in D_i^0}[\text{int}\{y \in Y : G_{i_y}(d_i) > \alpha_i(y)\}]^c$ is a nonempty compact subset of $T(X)$ or an empty set in Theorem 3.3 of Lu and Hu [44]; (c) It is obvious that in Theorem 3.3 due to Lu and Hu [44], (ii) and the condition that $\bigcap_{d_i \in D_i^0}[\text{int}\{y \in Y : G_{i_y}(d_i) > \alpha_i(y)\}]^c$ is a nonempty compact subset of $T(X)$ or an empty set implies that the condition that $\bigcap_{z_i \in Z_i^0}[\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c$ is a nonempty compact subset of Y or an empty set and $\bigcap_{z_i \in Z_i^0}[\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c \subseteq \bigcup_{z_i \in Z_i} \text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$ holds in Theorem 4.4.

(2) Theorem 4.4 generalizes Theorem 3.2 of Khanh et al. [45] in the following aspects: (a) From WPH-spaces to GFC-spaces; (b) From FCCPP to CFP; (c) (ii′) Of Theorem 3.2 of Khanh et al. [45] is dropped. Therefore, by Remark 3.2 of Khanh et al. [45], Theorem 3.4 extends and generalizes Theorem 3.4 of Ding [15], Theorem 3 of Park [13], and Theorem 1 of Ansari and Yao [46].

Example 4.6. Let Ω be a singleton, $X = Y = (0, 1]$ with the Euclidean metric topology, and let $Z = [\frac{1}{3}, 1]$. Define two fuzzy mappings $H : Y \to \mathscr{F}(X)$ and $P : Y \to \mathscr{F}(Z)$ by

$$H_y(x) = \begin{cases} (\frac{y}{2} + \frac{1}{8})x, & \text{if } y \in (0, \frac{1}{2}],\ x \in (0, 1], \\ (\frac{1}{3}y^2 + \frac{1}{8})x, & \text{if } y \in (\frac{1}{2}, 1],\ x \in (0, 1], \end{cases}$$

$$P_y(z) = \begin{cases} (\frac{1}{4}y + \frac{1}{16})z, & \text{if } y \in (0, \frac{1}{2}],\ z \in [\frac{1}{3}, 1], \\ (\frac{1}{6}y^2 + \frac{7}{48})z, & \text{if } y \in (\frac{1}{2}, 1],\ z \in [\frac{1}{3}, 1]. \end{cases}$$

The function $\alpha : Y \to [0, 1)$ is defined by

$$\alpha(y) = \begin{cases} \frac{1}{9}y + \frac{1}{36}, & \text{if } y \in (0, \frac{1}{2}], \\ \frac{2}{27}y^2 + \frac{7}{108}, & \text{if } y \in (\frac{1}{2}, 1]. \end{cases}$$

Then we have

$$(H_y)_{\alpha(y)} = \begin{cases} (\frac{2}{9}, 1] & \text{if } y \in (0, \frac{1}{2}], \\ (\frac{16y^2+14}{72y^2+27}, 1] & \text{if } y \in (\frac{1}{2}, 1], \end{cases}$$

and $(P_y)_{\alpha(y)} = (\frac{4}{9}, 1]$ for each $y \in Y$. Define a set-valued mapping $T : X \to 2^Y$ by

$$T(x) = \begin{cases} [\frac{7}{10}, \frac{4}{5}], & \text{if } x = 1, \\ (\frac{3}{4}, \frac{4}{5}], & \text{if } x \in (0, 1). \end{cases}$$

Now, we show that all the conditions of Theorem 4.4 are fulfilled.

(1) For each each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z \rangle$, let us define a set-valued mapping $\sigma_N : \Delta_n \to 2^X$ by $\sigma_N(p) = [\sum_{j=0}^{n} \lambda_j z_j, 1]$ for every $p = \sum_{j=0}^{n} \lambda_j e_j \in \Delta_n$. Then by using the same method as in Example 3.2, we can verify that $(X, Z; \sigma_N)$ forms a WPH-space. For each $y \in T(X) = [\frac{7}{10}, \frac{4}{5}] \subseteq (\frac{1}{2}, 1]$, each $N = \{z_0, z_1, \ldots, z_n\} \in \langle Z \rangle$, each $\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subseteq N \bigcap (P_y)_{\alpha(y)}$, and each $p = \sum_{j=0}^{k} \lambda_{i_j} e_{i_j} \in \Delta_k$, we have

$$\frac{4}{9} < \sum_{j=0}^{k} \lambda_{i_j} z_{i_j} \leq 1.$$

Therefore, combining the fact that $\frac{4}{9} > \frac{16y^2+14}{72y^2+27}$ if $y > \frac{1}{2}$, we get the following:

$$\sigma_N(p) = [\sum_{j=0}^{k} \lambda_{i_j} z_{i_j}, 1] \subseteq (\frac{4}{9}, 1] \subseteq (\frac{16y^2 + 14}{72y^2 + 27}, 1] = (H_y)_{\alpha(y)},$$

which implies $\sigma_N(\Delta_k) \subseteq (H_y)_{\alpha(y)}$. Therefore, (i) of Theorem 4.4 is satisfied. Since α is upper semicontinuous on Y and for each $z \in Z$, the function $y \mapsto P_y(z)$ is lower semicontinuous on Y, it follows that the set $\{y \in Y : P_y(z) > \alpha(y)\}$ is open in Y for every $z \in Z$ and thus, we have $\{y \in Y : P_y(z) > \alpha(y)\} = \text{cint}\{y \in Y : P_y(z) > \alpha(y)\}$.

(2) Let $Z^0 = \{1\} \subseteq Z$. Since $\bigcup_{z \in Z^0} \text{cint}\{y \in Y : P_y(z) > \alpha(y)\} = Y$, $\bigcap_{z \in Z^0} [\text{cint}\{P_y(z) > \alpha(y)\}]^c$ is empty. For each $Q = \{m_0, m_1, \ldots, m_n\} \in \langle Z \rangle$, let $\overline{m} = \min_{0 \leq j \leq n} \{m_j\}$, $L_Q = \text{co}(\{1\} \bigcup Q)$, and $Z' = L_Q$. Then we have $L_Q = [\overline{m}, 1]$ and $Z^0 \bigcup Q = \{1\} \bigcup Q \subseteq Z'$. Now, we verify that L_Q is WPH-convex relative to Z'. In fact, for each $N' = \{z'_0, z'_1, \ldots, z'_n\} \in \langle Z \rangle$ and each $\{z'_{i_0}, z'_{i_1} \ldots, z'_{i_k}\} \subseteq N' \bigcap Z'$, we have $\overline{m} \leq \sum_{j=0}^{k} \lambda_{i_j} z'_{i_j} \leq 1$ for every $p = \sum_{j=0}^{k} \lambda_{i_j} e_{i_j} \in \Delta_k$. Hence, $\sigma_{N'}(p) = [\sum_{j=0}^{k} \lambda_{i_j} z'_{i_j}, 1] \subseteq [\overline{m}, 1] = L_Q$, which implies that L_Q is is WPH-convex relative to Z'. By the definition of T, we have $T(L_Q) = [\frac{7}{10}, \frac{4}{5}]$, which is compact.

(3) For each $N \in \langle Z \rangle$ with $|N| = n + 1$, by the definitions of T and σ_N, we can see that $T(z) = T(\sigma_N(\Delta_n)) = [\frac{7}{10}, \frac{4}{5}]$ for every $z \in \Delta_n$. Therefore, the composition $T|_{\sigma_N} \circ \sigma_N : \Delta_n \to 2^{T(\sigma_N(\Delta_n))}$ is an upper semicontinuous set-valued mapping with nonempty compact convex values. By Lemma 1 in Yuan [40], for each single-valued continuous function $\psi : T(\sigma_N(\Delta_n)) \to \Delta_n$, the composition $\psi \circ T|_{\sigma_N} \circ \sigma_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point, which implies that $T \in \widetilde{\mathscr{B}}(X, Y)$. In summary, all the hypotheses of Theorem 4.4 are fulfilled and hence, the FCCPP in Example 4.6 has a solution. Let $x^* = \frac{3}{5}$ and $y^* = \frac{4}{5}$. Then we can check that $y^* \in T(x^*)$ and $x^* \in (H_{y^*})_{\alpha(y^*)}$.

Corollary 4.7. *Let Y be a Hausdorff topological space and $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ be a family of WPH-spaces, where Ω is a finite or infinite index set. Let $X = \prod_{i \in \Omega} X_i$ and $T \in \widetilde{\mathscr{B}}(X, Y)$, where T is upper* *semicontinuous set-values mapping with compact values. For each $i \in \Omega$, let $G_i : Z_i \to 2^{X_i}$ be a set-valued mapping, $\alpha_i : Y \to [0,1)$ be a function, $H_i : Y \to \mathscr{F}(X_i)$ and $P_i : Y \to \mathscr{F}(Z_i)$ be two fuzzy mappings such that the following conditions hold*:

(i) *For each $y \in T(X)$, $WPH(G_i((P_{i_y})_{\alpha_i(y)}), (P_{i_y})_{\alpha_i(y)}) \subseteq (H_{i_y})_{\alpha_i(y)}$;*

(ii) *There exists a nonempty subset Z_i^0 of Z_i such that $\bigcup_{z_i \in Z_i} \text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$ contains $\bigcap_{z_i \in Z_i^0} [\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c$ which is a nonempty compact subset of Y or an empty set, and for each $Q_i \in \langle Z_i \rangle$, there exists a nonempty compact subset L_{Q_i} of X_i, which is WPH-convex relative to some $Z_i' \subseteq Z_i$ such that $Z_i^0 \bigcup Q_i \subseteq Z_i'$.*

Then FCCPP has a solution.

Proof. By the definition of a WPH-hull, we can see that $WPH(G_i((P_{i_y})_{\alpha_i(y)}), (P_{i_y})_{\alpha_i(y)})$ is WPH-convex relative to $(P_{i_y})_{\alpha_i(y)}$ for every $i \in \Omega$ and every $y \in T(X)$. Then it follows from (i) that $(H_{i_y})_{\alpha_i(y)}$ is WPH-convex relative to $(P_{i_y})_{\alpha_i(y)}$ for every $i \in \Omega$ and every $y \in T(X)$. By (ii), for each $i \in \Omega$ and each $Q_i \in \langle Z_i \rangle$, there exists a nonempty compact subset L_{Q_i} of X_i. By this fact, we can see that the set $L_Q = \prod_{i \in \Omega} L_{Q_i}$ is compact. Since T is upper semicontinuous set-values mapping with compact values, it follows from Proposition 3.1.11 of Aubin and Ekeland [43] that $T(L_Q)$ is compact subset of Y. Thus, all the conditions of Theorem 4.4 are fulfilled. Therefore, by Theorem 4.4, the conclusion of Corollary 4.7 holds. The proof is complete. □

Remark 4.8. (ii) of Corollary 4.7 can be replaced by the following conditions:

(ii)′ There exist a compact subset K of Y and a nonempty subset Z_i^0 of Z_i such that $Y \setminus K \subseteq \bigcup_{z_i \in Z_i^0} \text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$, and for each $Q_i \in \langle Z_i \rangle$, there exists a nonempty compact subset L_{Q_i} of X_i, which is WPH-convex relative to some $Z_i' \subseteq Z_i$ such that $Z_i^0 \bigcup Q_i \subseteq Z_i'$;

(ii)′′ For each compact subset K of Y, $K \subseteq \bigcup_{z_i \in Z_i} \text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$.

In fact, by (ii)′, we get

$$\bigcap_{z_i \in Z_i^0} [\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c = Y \setminus \bigcup_{z_i \in Z_i^0} \text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\} \subseteq K.$$

If $\bigcap_{z_i \in Z_i^0} [\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c \neq \emptyset$, then it is compact by the fact that it is a compactly closed subset of the compact set K. Therefore, it follows from (ii)′′ that $\bigcap_{z_i \in Z_i^0} [\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c \subseteq \bigcup_{z_i \in Z_i} \text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}$, which implies that condition (ii) of Corollary 4.7 holds. If $\bigcap_{z_i \in Z_i^0} [\text{cint}\{y \in Y : P_{i_y}(z_i) > \alpha_i(y)\}]^c = \emptyset$, then (ii) of Corollary 4.7 holds automatically.

Theorem 4.9. *Let Y be a Hausdorff topological space and $\{X_i | i \in \Omega\}$ be a family of topological vector spaces, where Ω is a finite or infinite index set. Let $X = \prod_{i \in \Omega} X_i$ and $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping with nonempty compact values. For each $i \in \Omega$, let $\alpha_i : Y \to [0,1)$ be a function and let $H_i : Y \to \mathscr{F}(X_i)$ and $P_i : Y \to \mathscr{F}(X_i)$ be two fuzzy mappings such that the following conditions are fulfilled*:

(i) *For each $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$ and for each single-valued continuous mapping $\Psi : T(\prod_{i \in \Omega} \tau_{N_i}(\Delta_{n_i})) \to C$, the composition mapping $\Psi \circ T_{\prod_{i \in \Omega} \tau_{N_i}(\Delta_{n_i})} \circ \prod_{i \in \Omega} \tau_{N_i} : C \to 2^C$ has a fixed point, where $C = \prod_{i \in \Omega} \Delta_{n_i}$, $\tau_{N_i} : \Delta_{n_i} \to X_i$ is a linear function such that $\tau_{N_i}(e_{i_j}) = x_{i_j}$ for every $j \in \{0, 1, \ldots, n_i\}$, $\prod_{i \in \Omega} \tau_{N_i} : C \to X$ is defined by $\prod_{i \in \Omega} \tau_{N_i}(z) = \prod_{i \in \Omega} \tau_{N_i}(\pi_i(z))$ for every $z \in C$, and π_i is the projection of C onto Δ_{n_i}.;*

(ii) *For each* $y \in T(X)$, $\mathrm{co}((P_{i_y})_{\alpha_i(y)}) \subseteq (H_{i_y})_{\alpha_i(y)}$;

(iii) *There exists a nonempty subset* X_i^0 *which is contained in a compact convex subset* $\widetilde{X}_i$ *of* X_i *such that* $\bigcap_{x_i \in X_i^0}[\mathrm{cint}\{y \in Y : P_{i_y}(x_i) > \alpha_i(y)\}]^c \subseteq \bigcup_{x_i \in X_i} \mathrm{cint}\{y \in Y : P_{i_y}(x_i) > \alpha_i(y)\}$, *where* $\bigcap_{x_i \in X_i^0}[\mathrm{cint}\{y \in Y : P_{i_y}(x_i) > \alpha_i(y)\}]^c$ *is a nonempty compact subset of* Y *or an empty set.*

Then FCCPP has a solution.

Proof. For each $i \in \Omega$ and each $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$, we define a continuous mapping $\sigma_{N_i} : \Delta_{n_i} \to \mathrm{co}(N_i) \subseteq X_i$ as follows:

$$\sigma_{N_i}(\sum_{j=0}^{n_i} t_{i_j} e_{i_j}) = \sum_{j=0}^{n_i} t_{i_j} x_{i_j}, \ \ \forall (t_{i_0}, t_{i_1}, \ldots, t_{i_{n_i}}) = \sum_{j=0}^{n_i} t_{i_j} e_{i_j} \in \Delta_{n_i}. \tag{4.5}$$

Then $(X_i; \sigma_{N_i})$ forms a WPH-space. We claim that $\sigma_{N_i} = \tau_{N_i}$ for every $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$ and every $i \in \Omega$. Indeed, let $(t_{i_0}, t_{i_1}, \ldots, t_{i_{n_i}}) = \sum_{j=0}^{n_i} t_{i_j} e_{i_j} \in \Delta_{n_i}$ be any given. By (i) and (4.5), we have

$$\tau_{N_i}(\sum_{j=0}^{n_i} t_{i_j} e_{i_j}) = \sum_{j=0}^{n_i} t_{i_j} \tau_{N_i}(e_{i_j}) = \sum_{j=0}^{n_i} t_{i_j} x_{i_j} = \sigma_{N_i}(\sum_{j=0}^{n_i} t_{i_j} e_{i_j}).$$

This implies that $\sigma_{N_i} = \tau_{N_i}$ for every $i \in \Omega$. By (4.5) again, we can see that $\sigma_{N_i}(\Delta_{n_i}) = \tau_{N_i}(\Delta_{n_i}) = \mathrm{co}(N_i)$ for every $i \in \Omega$ and every $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$. Define $\Phi : C \to X$ by $\Phi(z) = \prod_{i \in \Omega} \sigma_{N_i}(\pi_i(z))$ for every $z \in C$. Then by (i), for each $i \in \Omega$, each $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$, and each single-valued continuous mapping $\Psi : T(\prod_{i \in \Omega} \sigma_{N_i}(\Delta_{n_i})) \to C$, the composition $\Psi \circ T|_{\prod_{i \in \Omega} \sigma_{N_i}(\Delta_{n_i})} \circ \Phi : C \to 2^C$ has a fixed point, which implies that $T \in \widetilde{\mathscr{B}}(X, Y)$. By (ii) and (4.5), for each $y \in T(X)$, each $i \in \Omega$, each $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$, and each $\{x_{i_{j_0}}, x_{i_{j_1}}, \ldots, x_{i_{j_{k_i}}}\} \subseteq N_i \bigcap (P_{i_y})_{\alpha_i(y)}$, we have

$$\begin{aligned} \sigma_{N_i}(\Delta_{k_i}) &= \mathrm{co}(\{x_{i_{j_0}}, x_{i_{j_1}}, \ldots, x_{i_{j_{k_i}}}\}) \\ &\subseteq \mathrm{co}((P_{i_y})_{\alpha_i(y)}) \\ &\subseteq (H_{i_y})_{\alpha_i(y)}. \end{aligned}$$

This shows that $(H_{i_y})_{\alpha_i(y)}$ is WPH-convex relative to $(P_{i_y})_{\alpha_i(y)}$ for every $i \in \Omega$ and every $y \in T(X)$. Hence, (i) of Theorem 4.4 is satisfied. Now, we prove that (ii) of Theorem 4.4 is fulfilled. Indeed, for each $i \in \Omega$ and each $Q_i \in \langle X_i \rangle$, let $X_i^0 = Z_i^0$ and $L_{Q_i} = Z_i^{'} = \mathrm{co}(\widetilde{X}_i \bigcup Q_i)$. Since each $\widetilde{X}_i$ is compact convex and $X_i^0 \subseteq \widetilde{X}_i$, it follows that each L_{Q_i} is compact convex subset of X_i containing $Z_i^0 \bigcup Q_i$ and hence, L_{Q_i} is a WPH-subspace of $(X_i; \sigma_{N_i})$ such that $Z_i^0 \bigcup Q_i \subseteq L_{Q_i} = Z_i^{'}$. Furthermore, let $L_Q = \prod_{i \in \Omega} L_{Q_i}$. Then L_Q is a compact subset of X. Since T is an upper semicontinuous set-valued mapping with nonempty compact values, it follows from Proposition 3.1.11 of Aubin and Ekeland [43] that $T(L_Q)$ is compact subset of Y. Therefore, all the requirements of Theorem 4.4 are satisfied. Hence, by Theorem 4.4, FCCPP has a solution. This completes the proof. □

Remark 4.10. (ii) of Theorem 4.9 can be replaced by one of the following conditions:

(ii)′ For each $y \in T(X)$ and each $N_i \in \langle (P_{i_y})_{\alpha_i(y)} \rangle$, $\mathrm{co}(N_i) \subseteq (H_{i_y})_{\alpha_i(y)}$;

(ii)′′ For each $y \in T(X)$, $(P_{i_y})_{\alpha_i(y)} \subseteq (H_{i_y})_{\alpha_i(y)}$ and the set $(H_{i_y})_{\alpha_i(y)}$ is convex.

Theorem 4.11. *Let* Y *be a Hausdorff topological space and* $\{X_i | i \in \Omega\}$ *be a family of topological vector spaces, where* Ω *is a finite or infinite index set. Let* $X = \prod_{i \in \Omega} X_i$ *and* $T : X \to 2^Y$ *be an upper* *semicontinuous set-valued mapping with nonempty compact values. For each $i \in \Omega$, let $\alpha_i : Y \to [0,1)$ be a function and let $H_i : Y \to \mathscr{F}(X_i)$ and $P_i : Y \to \mathscr{F}(X_i)$ be two fuzzy mappings such that the following conditions hold*:

(i) *For each $N_i \in \langle X_i \rangle$ with $|N_i| = n_i + 1$ and for each single-valued continuous mapping $\Psi : T(\prod_{i\in\Omega} \mathrm{co}(N_i)) \to \prod_{i\in\Omega} \Delta_{n_i}$, the composition mapping $\Psi \circ T|_{\prod_{i\in\Omega} \mathrm{co}(N_i)}$ has convex values*;

(ii) *For each $y \in T(X)$, $\mathrm{co}((P_{i_y})_{\alpha_i(y)}) \subseteq (H_{i_y})_{\alpha_i(y)}$*;

(iii) *There exists a nonempty subset X_i^0 which is contained in a compact convex subset $\widetilde{X}_i$ of X_i such that $\bigcap_{x_i\in X_i^0}[\mathrm{cint}\{y \in Y : P_{i_y}(x_i) > \alpha_i(y)\}]^c \subseteq \bigcup_{x_i\in X_i} \mathrm{cint}\{y \in Y : P_{i_y}(x_i) > \alpha_i(y)\}$, where $\bigcap_{x_i\in X_i^0}[\mathrm{cint}\{y \in Y : P_{i_y}(x_i) > \alpha_i(y)\}]^c$ is a nonempty compact subset of Y or an empty set.*

Then FCCPP has a solution.

Proof. For each $i \in \Omega$ and each $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$, we define a continuous mapping $\sigma_{N_i} : \Delta_{n_i} \to \mathrm{co}(N_i) \subseteq X_i$ by

$$\sigma_{N_i}(\sum_{j=0}^{n_i} t_{i_j} e_{i_j}) = \sum_{j=0}^{n_i} t_{i_j} x_{i_j}, \ \ \forall (t_{i_0}, t_{i_1}, \ldots, t_{i_{n_i}}) = \sum_{j=0}^{n_i} t_{i_j} e_{i_j} \in \Delta_{n_i}. \tag{4.6}$$

Therefore, $(X_i; \sigma_{N_i})$ forms a WPH-space. By (4.6), we have $\sigma_{N_i}(\Delta_{n_i}) = \mathrm{co}(N_i)$ for each $i \in \Omega$ and each $N_i = \{x_{i_0}, x_{i_1}, \ldots, x_{i_{n_i}}\} \in \langle X_i \rangle$. Thus, we get

$$T\Big(\prod_{i\in\Omega} \sigma_{N_i}(\Delta_{n_i})\Big) = T\Big(\prod_{i\in\Omega} \mathrm{co}(N_i)\Big).$$

Now, we prove that $T \in \widetilde{B}(X,Y)$. Indeed, for each $i \in \Omega$, each $N_i \in \langle X_i \rangle$ with $|N_i| = n_i + 1$ and for each single-valued continuous mapping $\Psi : T(\prod_{i\in\Omega} \sigma_{N_i}(\Delta_{n_i})) \to \prod_{i\in\Omega} \Delta_{n_i}$, by the assumption of T, we know that the composition $\Psi \circ T|_{\prod_{i\in\Upsilon} \sigma_{N_i}(\Delta_{n_i})} \circ \Phi : C \to 2^C$ is an upper semicontinuous set-valued mapping with nonempty compact convex values, where $C = \prod_{i\in\Omega} \Delta_{n_i}$, $\Phi(z) = \prod_{i\in\Omega} \sigma_{N_i}(\pi_i(z))$ for every $z \in C$ and π_i is the projection of C onto Δ_{n_i}. For each $i \in \Omega$, let E_i be the linear hull of the set $\{e_{i_j} : j = 0, 1, \ldots, n_i\}$. Then E_i is a locally convex Hausdorff topological vector space as it is finite dimensional and Δ_{n_i} is compact convex subset of E_i. Let $E = \prod_{i\in\Omega} E_i$. Then E is also a Hausdorff locally convex topological vector space and $C = \prod_{i\in\Omega} \Delta_{n_i}$ is a compact convex subset of E. So, by Fan-Glicksberg fixed point theorem (see Theorem 1 in Fan [41]), there exists $\widehat{z} \in C$ such that $\widehat{z} \in \Psi \circ T|_{\prod_{i\in\Omega} \sigma_{N_i}(\Delta_{n_i})} \circ \Phi(\widehat{z})$, which implies that $T \in \widetilde{\mathscr{B}}(X,Y)$. The rest of proof is the same as the proof of Theorem 4.9. Therefore, by Theorem 4.4, FCCPP has a solution. This completes the proof. □

Taking $X = Y$ and $T(x) = \{x\}$ for every $x \in X$, we can derive the following existence theorem of solutions to FCFPP from Theorem 4.4.

Theorem 4.12. *Let $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ be a family of WPH-spaces such that $X =: \prod_{i\in\Omega} X_i$ is a Hausdorff topological space, where Ω is a finite or infinite index set. Let I_X be the identity mapping on X and $I_X \in \widetilde{\mathscr{B}}(X,X)$. For each $i \in \Omega$, let $\alpha_i : X \to [0,1)$ be a function and let $H_i : X \to \mathscr{F}(X_i)$ and $P_i : X \to \mathscr{F}(Z_i)$ be two fuzzy mappings such that the following conditions hold*:

(i) *For each $x \in X$, $(H_{i_x})_{\alpha_i(x)}$ is WPH-convex relative to $(P_{i_x})_{\alpha_i(x)}$*;

(ii) *There exists a nonempty subset Z_i^0 of Z_i such that $\bigcap_{z_i\in Z_i^0}[\mathrm{cint}\{x \in X : P_{i_x}(z_i) > \alpha_i(x)\}]^c \subseteq \bigcup_{z_i\in Z_i} \mathrm{cint}\{x \in X : P_{i_x}(z_i) > \alpha_i(x)\}$, where $\bigcap_{z_i\in Z_i^0}[\mathrm{cint}\{x \in X : P_{i_x}(z_i) > \alpha_i(x)\}]^c$ is a nonempty compact* *subset of X or an empty set, and for each* $Q_i \in \langle Z_i \rangle$*, there exists a nonempty subset* L_{Q_i} *of* X_i*, which is* WPH*-convex relative to some* $Z_i' \subseteq Z_i$ *such that* $Z_i^0 \bigcup Q_i \subseteq Z_i'$ *and* L_Q *is compact, where* $L_Q = \prod_{i\in\Omega} L_{Q_i}$.

Then FCFPP has a solution.

Remark 4.13. By Lemma 2.1 of Al-Homidan and Ansari [33], one can see that Theorem 4.12 generalizes Theorem 3.3 of Al-Homidan and Ansari [33] from topological semilattice spaces to WPH-spaces and from set-valued mappings to fuzzy mappings. So, combining Remark 3.4 of Al-Homidan and Ansari [33], it is known that Theorem 4.12 extends Theorem 3.2 of Lan and Webb [47], Theorem 3.1 of Lin et al. [48], and Theorem 2.1 of Singh et al. [49] to WPH-spaces and fuzzy mappings.

If $X = Y$ and $T(x) = \{x\}$ for every $x \in X$, then we have the following theorem as a common consequence of Theorems 4.9 and 4.11.

Theorem 4.14. *Let* $\{X_i | i \in \Omega\}$ *be a family of Hausdorff topological vector spaces, where* Ω *is a finite or infinite index set. Let* $X = \prod_{i\in\Omega} X_i$*. For each* $i \in \Omega$*, let* $\alpha_i : X \to [0, 1)$ *be a function and let* $H_i : X \to \mathscr{F}(X_i)$ *and* $P_i : X \to \mathscr{F}(X_i)$ *be two fuzzy mappings such that the following conditions hold*:

(i) *For each* $x \in X$, $\mathrm{co}((P_{i_x})_{\alpha_i(x)}) \subseteq (H_{i_x})_{\alpha_i(x)}$;

(ii) *There exists a nonempty subset* X_i^0 *which is contained in a compact convex subset* $\widetilde{X}_i$ *of* X_i *such that* $\bigcap_{y_i\in X_i^0}[\mathrm{cint}\{x \in X : P_{i_x}(y_i) > \alpha_i(x)\}]^c \subseteq \bigcup_{y_i\in X_i} \mathrm{cint}\{x \in X : P_{i_x}(y_i) > \alpha_i(x)\}$*, where* $\bigcap_{y_i\in X_i^0}[\mathrm{cint}\{x \in X : P_{i_x}(y_i) > \alpha_i(x)\}]^c$ *is a nonempty compact subset of X or an empty set.*

Then FCFPP has a solution.

In Corollary 4.3, if $X = Y$ and $T(x) = \{x\}$ for every $x \in X$, then Corollary 4.3 reduces to the following corollary.

Corollary 4.15. *Let* $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ *be a family of WPH-spaces such that* $X = \prod_{i\in\Omega} X_i$ *is a compact Hausdorff topological space, where* Ω *is a finite or infinite index set. Let* $I_X \in \widetilde{\mathscr{B}}(X, X)$ *be the identity mapping on X. For each* $i \in \Omega$*, let* $\alpha_i : X \to [0, 1)$ *be a function and let* $H_i : X \to \mathscr{F}(X_i)$ *and* $P_i : X \to \mathscr{F}(Z_i)$ *be two fuzzy mappings such that the following conditions hold*:

(i) *For each* $x \in X$, $(H_{i_x})_{\alpha_i(x)}$ *is WPH-convex relative to* $(P_{i_x})_{\alpha_i(x)}$;

(ii) $X = \bigcup_{z_i\in Z_i} \mathrm{cint}\{x \in X : P_{i_x}(z_i) > \alpha_i(x)\}$.

Then FCFPP has a solution.

By Corollary 4.15, we have the following existence theorem of solutions to FCFPP.

Theorem 4.16. *Let* $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ *be a family of WPH-spaces such that* $X = \prod_{i\in\Omega} X_i$ *is a compact Hausdorff topological space, where* Ω *is a finite or infinite index set. Let* $I_X \in \widetilde{\mathscr{B}}(X, X)$ *be the identity mapping on X. For each* $i \in \Omega$*, let* $\alpha_i : X \to [0, 1)$ *be a function and let* $H_i : X \to \mathscr{F}(X_i)$ *and* $P_i : X \to \mathscr{F}(Z_i)$ *be a fuzzy mapping such that the following conditions hold*:

(i) H_i *is WPH-*P_i*-quasiconcave on X*;

(ii) $X = \bigcup_{z_i\in Z_i} \mathrm{cint}\{x \in X : P_{i_x}(z_i) > \alpha_i(x)\}$.

Then FCFPP has a solution. Furthermore, if, for each $i \in \Omega$*, the function* $(x, y_i) \mapsto H_{i_x}(y_i)$ *is upper* *semicontinuous on $X \times X_i$ and for each $i \in \Omega$ and each fixed $y_i \in X_i$, the function $x \mapsto H_{i_x}(y_i)$ is lower semicontinuous on X, then there exists $\widehat{x} \in X$ such that, for each $i \in \Omega$, $H_{i_{\widehat{x}}}(\widehat{x_i}) = \max_{y_i \in X_i} H_{i_{\widehat{x}}}(y_i)$.*

Proof. In order to apply Corollary 4.15, we only show that $(H_{i_x})_{\alpha_i(x)}$ is WPH-convex relative to $(P_{i_x})_{\alpha_i(x)}$ for every $i \in \Omega$ and every $x \in X$. We proceed by contradiction. Suppose that there exist $i \in \Omega$, $x \in X$, $N_i = \{z_{i_0}, z_{i_1}, \ldots, z_{i_{n_i}}\} \in \langle Z_i \rangle$ and $\{z_{i_{j_0}}, z_{i_{j_1}}, \ldots, z_{i_{j_{k_i}}}\} \subseteq N_i \bigcap (P_{i_x})_{\alpha_i(x)}$ such that $\sigma_{N_i}(\Delta_{k_i}) \nsubseteq (H_{i_x})_{\alpha_i(x)}$. Then there exists $q_i \in \sigma_{N_i}(\Delta_{k_i})$ such that $y_i \notin (H_{i_x})_{\alpha_i(x)}$; i.e., $H_{i_x}(y_i) \leq \alpha_i(x)$. Since $\{z_{i_{j_0}}, z_{i_{j_1}}, \ldots, z_{i_{j_{k_i}}}\} \subseteq N_i \bigcap (P_{i_x})_{\alpha_i(x)}$, it follows that $P_{i_x}(z_{i_{j_l}}) > \alpha_i(x)$ for every $l \in \{0, 1, \ldots, k_i\}$. By (i), we obtain the following contradiction

$$\alpha_i(x) \geq H_{i_x}(y_i) \geq \min_{0 \leq l \leq k_i} P_{i_x}(z_{i_{j_l}}) > \alpha_i(x),$$

which implies that $(H_{i_x})_{\alpha_i(x)}$ is WPH-convex relative to $(P_{i_x})_{\alpha_i(x)}$ for every $i \in \Omega$ and every $x \in X$. Therefore, it follows from Corollary 4.15 that the first part of the conclusion of Theorem 4.16 holds.

Now, we prove that the second part of the conclusion of Theorem 4.16 holds. For each $i \in \Omega$, let us define a function $\rho_i : X \to [0, 1]$ by $\rho_i(x) = \max_{y_i \in X_i} H_{i_x}(y_i)$ for every $x \in X$, which is well defined since for each $i \in \Omega$ and each $x \in X$, $H_{i_x}(y_i)$ is upper semicontinuous function on the compact set X_i. By Theorem 1 of Aubin [50, p. 67] and the fact that the function $(x, y_i) \mapsto H_{i_x}(y_i)$ is upper semicontinuous on $X \times X_i$ for every $i \in \Omega$, we know that ρ_i is upper semicontinuous on X. For each $i \in \Omega$ and each $n \in \{1, 2, \ldots, \}$, we define a set-valued mapping $W_{(i,n)} : X \to 2^{X_i}$ as follows:

$$W_{(i,n)}(x) = \{y_i \in X_i : H_{i_x}(y_i) > \rho_i(x) - \frac{1}{n}\}, \quad \forall\, x \in X.$$

Then $W_{(i,n)}(x)$ is nonempty for every $i \in \Omega$, every $n \in \{1, 2, \ldots, \}$ and every $x \in X$. Therefore, we have

$$X = \bigcup_{y_i \in X_i} W^{-1}_{(i,n)}(y_i). \tag{4.7}$$

Applying (i) and the argument similar to that in the proof of the first part, we can prove that $W_{(i,n)}(x)$ is a WPH-convex subspace of $(X_i; \sigma_{N_i})$ for every $i \in \Omega$, every $n \in \{1, 2, \ldots, \}$ and every $x \in X$. For each $i \in \Omega$, each $n \in \{1, 2, \ldots, \}$ and each $y_i \in X_i$, we have

$$W^{-1}_{(i,n)}(y_i) = \{x \in X : H_{i_x}(y_i) > \rho_i(x) - \frac{1}{n}\}.$$

Since for each $i \in \Omega$ and each fixed $y_i \in X_i$, the function $x \mapsto H_{i_x}(y_i)$ is lower semicontinuous on X and ρ_i is upper semicontinuous on X, it follows that $W^{-1}_{(i,n)}(y_i)$ is open in X. By (4.7) and the compactness of X, there exists $\{y^0_{(i,n)}, y^1_{(i,n)}, \ldots, y^{m(i,n)}_{(i,n)}\} \in \langle X_i \rangle$ such that

$$X = \bigcup_{j=0}^{m(i,n)} W^{-1}_{(i,n)}(y^j_{(i,n)}),$$

where $m(i, n)$ is a positive integer. Let $\{\eta_j\}_{j=0}^{m(i,n)}$ be the continuous partition of unity subordinated to the open cover $\{W^{-1}_{(i,n)}(y^j_{(i,n)}) | j = 0, 1, \ldots, m(i, n)\}$; i.e.,

$$\begin{cases} \eta_j : X \to [0, 1] \text{ is continuous for every } j \in \{0, 1, \ldots, m(i, n)\}; \\ \eta_j(x) > 0 \Rightarrow x \in W^{-1}_{(i,n)}(y^j_{(i,n)}); \\ \sum_{j=0}^{m(i,n)} \eta_j(x) = 1 \text{ for every } x \in X. \end{cases}$$

Subsequently, we define a continuous mapping $\eta_{(i,n)} : X \to \Delta_{m(i,n)}$ by $\eta_{(i,n)}(x) = \sum_{j=0}^{m(i,n)} \eta_j(x)e_j$ for every $x \in X$. Then we have $\eta_{(i,n)}(x) = \sum_{j\in J(x)} \eta_j(x)e_j \in \Delta_{|J(x)|-1}$ for every $x \in X$, where $J(x) := \{j \in \{0, 1, \ldots, m(i, n)\} : \eta_j(x) > 0\}$. For the finite set $\{y^0_{(i,n)}, y^1_{(i,n)}, \ldots, y^{m(i,n)}_{(i,n)}\}$, by the definition of a WPH-space, there exists an upper semicontinuous set-valued mapping $\sigma_{(i,n)} : \Delta_{m(i,n)} \to 2^{X_i}$. Consider an upper semicontinuous set-valued mapping $f_{(i,n)} : X \to 2^{X_i}$ defined by $f_{(i,n)}(x) = \sigma_{(i,n)}(\eta_{(i,n)}(x))$ for every $x \in X$. By the fact that $W_{(i,n)}(x)$ is a WPH-convex subspace of $(X_i; \sigma_{N_i})$, we can easily verify that $f_{(i,n)}(x) = \sigma_{(i,n)}(\eta_{(i,n)}(x)) \subseteq W_{(i,n)}(x)$ for every $i \in \Omega$, every $n \in \{1, 2, \ldots, \}$, and every $x \in X$.

For each $n \in \{1, 2, \ldots, \}$, a set-valued mapping $\Phi_n : C(n) \to 2^X$ and a continuous mapping $\Psi_n : X \to C(n)$ are defined by $\Phi_n(z) = \prod_{i\in\Omega} \sigma_{(i,n)}(\pi_i(z))$ for every $z \in C(n)$ and by $\Psi_n(x) = \prod_{i\in\Omega} \eta_{(i,n)}(x)$ for every $x \in X$, respectively, where $C(n) = \prod_{i\in\Omega} \Delta_{m_{(i,n)}}$ and $\pi_i : C(n) \to \Delta_{m_{(i,n)}}$ is the projection of $C(n)$ onto $\Delta_{m_{(i,n)}}$. Since $I_X \in \widetilde{\mathcal{B}}(X, X)$, it follows that, for each $n \in \{1, 2, \ldots, \}$, the composition $\Psi_n \circ \Phi_n : C(n) \to 2^{C(n)}$ has a fixed point, which implies that there exists $\widehat{z_n} \in C(n)$ such that $\widehat{z_n} \in \Psi_n \circ \Phi_n(\widehat{z_n})$. Then there exists $\widehat{x_n} \in \Phi_n(\widehat{z_n})$ such that $\widehat{z_n} = \Psi_n(\widehat{x_n})$ for every $n \in \{1, 2, \ldots, \}$ and thus, for each $n \in \{1, 2, \ldots, \}$, we have

$$\begin{aligned}
\widehat{x_n} &= (\widehat{x}_{(i,n)})_{i\in\Omega} \in \Phi_n(\widehat{z_n}) = \Phi_n \circ \Psi_n(\widehat{x_n}) = \Phi_n\Big(\prod_{i\in\Omega} \eta_{(i,n)}(\widehat{x_n})\Big) \\
&= \prod_{i\in\Omega} \sigma_{(i,n)}\Big(\pi_i\Big(\prod_{i\in\Omega} \eta_{(i,n)}(\widehat{x_n})\Big)\Big) \\
&= \prod_{i\in\Omega} (\sigma_{(i,n)} \circ \eta_{(i,n)})(\widehat{x_n}).
\end{aligned}$$

It follows that $\widehat{x}_{(i,n)} \in (\sigma_{(i,n)} \circ \eta_{(i,n)})(\widehat{x_n}) \subseteq W_{(i,n)}(\widehat{x_n})$ for every $i \in \Omega$ and every $n \in \{1, 2, \ldots, \}$. By the definition of $W_{(i,n)}$, we get $H_{i_{\widehat{x}_n}}(\widehat{x}_{(i,n)}) > \rho_i(\widehat{x_n}) - \frac{1}{n}$. By the compactness of X, without loss of generality, we may assume that $\widehat{x_n} \to \widehat{x}$, that is, $\widehat{x}_{(i,n)} \to \widehat{x}_i$ for every $i \in \Omega$. Since the function $x \mapsto H_{i_x}(y_i)$ is lower semicontinuous on X for every $i \in \Omega$ and every fixed $y_i \in X_i$, it follows that ρ_i is also lower semicontinuous on X. Therefore, by this fact and the condition that the function $(x, y_i) \mapsto H_{i_x}(y_i)$ is upper semicontinuous on $X \times X_i$, we have, for each $i \in \Omega$,

$$\begin{aligned}
H_{i_{\widehat{x}}}(\widehat{x}_i) &\geq \overline{\lim_{n\to\infty}}\, H_{i_{\widehat{x}_n}}(\widehat{x}_{(i,n)}) \geq \overline{\lim_{n\to\infty}}(\rho_i(\widehat{x_n}) - \frac{1}{n}) \\
&\geq \underline{\lim_{n\to\infty}}(\rho_i(\widehat{x_n}) - \frac{1}{n}) \\
&\geq \rho_i(\widehat{x}) = \max_{y_i\in X_i} H_{i_{\widehat{x}}}(y_i).
\end{aligned}$$

Hence, $H_{i_{\widehat{x}}}(\widehat{x}_i) = \max_{y_i\in X_i} H_{i_{\widehat{x}}}(y_i)$ for every $i \in \Omega$. The proof is complete. □

5. Generalized fuzzy games

In this section, if not specified, for convenience we may assume from now on that every upper semicontinuous set-valued mapping σ_{N_i} from a standard simplex to a topological space involved in WPH-spaces has nonempty compact values. By Theorem 3.4, we establish new existence theorems of equilibria for the generalized fuzzy games and generalized fuzzy qualitative games in the framework of noncompact WPH-spaces. Before doing so, we need to explain the relevant issues as follows.

Let Ω be a finite or an infinite set of agents. For each $i \in \Omega$, let X_i and Z_i be the sets of actions available to the agent i. Let $X = \prod_{i\in\Omega} X_i$. A generalized fuzzy game is a family of ordered quintuple $\Gamma = (X_i, Z_i, A_i, B_i, F_i, P_i, a_i, b_i, c_i, p_i)_{i\in\Omega}$, where $A_i : X \to \mathscr{F}(Z_i)$, $B_i, F_i : X \to \mathscr{F}(X_i)$ are fuzzy constraint mappings and $P_i : X \times X \to \mathscr{F}(Z_i)$ is a fuzzy preference mapping. An equilibrium of Γ is a point $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $i \in \Omega$, $\widehat{x}_i \in (B_{i_{\widehat{x}}})_{a_i(\widehat{x})}$, $\widehat{y}_i \in (F_{i_{\widehat{x}}})_{b_i(\widehat{x})}$ and $(A_{i_{\widehat{x}}})_{c_i(\widehat{x})} \bigcap (P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})} = \emptyset$, where $a_i, b_i, c_i : X \to [0, 1)$ and $p_i : X \times X \to [0, 1)$ are fuzzy constraint functions. For each $i \in \Omega$, let $A^{-1}_{i_{c_i}} : Z_i \to 2^X$ be defined by $A^{-1}_{i_{c_i}}(z_i) = \{x \in X : A_{i_x}(z_i) > c_i(x)\}$ for every $z_i \in Z_i$. Using the same approach, we can define $F^{-1}_{i_{b_i}} : X_i \to 2^X$ and $P^{-1}_{i_{p_i}} : Z_i \to 2^{X\times X}$. We call $\Gamma = (X_i, Z_i, F_i, P_i, b_i, p_i)_{i\in\Omega}$ a generalized fuzzy qualitative game if for each $i \in \Omega$, X_i and Z_i are the sets of actions available to the agent i, $F_i : X \to \mathscr{F}(X_i)$ is a fuzzy constraint mapping, and $P_i : X \times X \to \mathscr{F}(Z_i)$ is a fuzzy preference mapping. An equilibrium of $\Gamma = (X_i, Z_i, F_i, b_i, P_i)_{i\in\Omega}$ is a point $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $i \in \Omega$, $\widehat{y}_i \in (F_{i_{\widehat{x}}})_{b_i(\widehat{x})}$ and $(P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})} = \emptyset$. If the constraint set-valued mappings A_i, B_i, F_i and the preference mapping P_i are defined by using the ith characteristic functions,and $a_i \equiv b_i \equiv c_i \equiv p_i \equiv 0$, then the equilibrium for generalized fuzzy games (respectively, generalized fuzzy qualitative games) implies the equilibrium for generalized games (respectively, generalized qualitative games).

Remark 5.1. Our definitions of generalized fuzzy games extend the corresponding definitions of the games in [16, 33, 42, 48–53] to fuzzy settings, and also generalize the definition of the generalized fuzzy games due to Kim and Lee [9].

Remark 5.2. As remarked in [9], the fuzzy constraint and preference mappings are very useful in the process of analysing real economic models. In fact, the constraint and preference sets of each player have fuzzy behavioral characteristics in the generalized games in strategic markets. In order to obtain the possible small cores, Aubin [50] introduced the concepts of fuzzy market cooperative games and fuzzy coalitions by embedding set of coalitions identified with $\{0, 1\}^n$ into the subset $[0, 1]^n$ of fuzzy coalitions. Therefore, as the same way, the concepts of generalized fuzzy game and generalized fuzzy qualitative game can provide useful frameworks for analysing fuzzy behaviors of the market economies.

As an application of Theorem 3.4, we are ready to prove the following existence theorem of equilibria for generalized fuzzy games in noncompact WPH-spaces.

Theorem 5.3. *Let Ω be a set of agents (finite or infinite) and $\{(X_i, Z_i; \sigma_{N_i}) | i \in \Omega\}$ be a family of Hausdorff WPH-spaces. Let $\Gamma = (X_i, Z_i, A_i, B_i, F_i, P_i, a_i, b_i, c_i, p_i)_{i\in\Omega}$ be a generalized fuzzy game and $I_{\Pi_{i\in\Omega}(X_i\times X_i)} \in \widetilde{\mathfrak{D}\mathfrak{B}}(\Pi_{i\in\Omega}(X_i \times X_i), \Pi_{i\in\Omega}(X_i \times X_i))$, where $I_{\Pi_{i\in\Omega}(X_i\times X_i)}$ is the identity mapping on $\Pi_{i\in\Omega}(X_i \times X_i)$. For each $i \in \Omega$, let $\overline{P}_i : X \times X \to \mathscr{F}(X_i)$, $\overline{F}_i : X \to \mathscr{F}(Z_i)$ be two fuzzy mappings and let $l_i : X \to [0, 1)$, $m_i : X \times X \to [0, 1)$ be two fuzzy sets. For each $i \in \Omega$, assume that the following conditions hold*:

(i) *For each $x \in X$, $(B_{i_x})_{a_i(x)}$ is WPH-convex relative to $(A_{i_x})_{c_i(x)}$*;

(ii) *For each $x \in X$, $(F_{i_x})_{b_i(x)}$ is WPH-convex relative to $(\overline{F}_{i_x})_{l_i(x)}$*;

(iii) *For each $(x, y) \in W_i$, $x_i \notin WPH((\overline{P}_{i_{(x,y)}})_{m_i(x,y)}, (P_{i_{(x,y)}})_{p_i(x,y)})$, where $W_i = \{(x, y) \in X \times X : (A_{i_x})_{c_i(x)} \bigcap (P_{i_{(x,y)}})_{p_i(x,y)} \neq \emptyset\}$*;

(iv) *There exist two nonempty subsets Z_i^0 and Z_i^1 of Z_i such that*

$$\bigcup_{(u_i,v_i)\in Z_i\times Z_i} \text{cint}\{((A^{-1}_{i_{c_i}}(u_i) \bigcap \overline{F}^{-1}_{i_{l_i}}(v_i)) \times X) \bigcap (P^{-1}_{i_{p_i}}(u_i) \bigcup W_i^c)\}$$

contains $\bigcap_{(u_i,v_i)\in Z_i^0\times Z_i^1} [\text{cint}\{((A^{-1}_{i_{c_i}}(u_i) \cap \overline{F}^{-1}_{i_{l_i}}(v_i)) \times X) \bigcap (P^{-1}_{i_{p_i}}(u_i) \bigcup W_i^c)\}]^c$ *which is a nonempty compact* *subset of $X \times X$ or an empty set, and for each $Q_i, M_i \in \langle Z_i \rangle$, there exist nonempty compact subsets L_{Q_i}, L_{M_i} of X_i such that $L_{Q_i} \times L_{M_i}$ is WPH-convex relative to some $Z_i' \subseteq Z_i \times Z_i$ and $(Z_i^0 \times Z_i^1) \bigcup (Q_i \times M_i) \subseteq Z_i'$.*

Then there exists a point $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $i \in \Omega$, $\widehat{x}_i \in (B_{i_{\widehat{x}}})_{a_i(\widehat{x})}$, $\widehat{y}_i \in (F_{i_{\widehat{x}}})_{b_i(\widehat{x})}$ and $(A_{i_{\widehat{x}}})_{c_i(\widehat{x})} \bigcap (P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})} = \emptyset$.

Proof. For each $i \in \Omega$ and each $\widetilde{N}_i = \{(z_0^i, w_0^i), (z_1^i, w_1^i), \ldots, (z_{n_i}^i, w_{n_i}^i)\} \in \langle Z_i \times Z_i \rangle$, it follows from the definition of a WPH-space that there exist two upper semicontinuous set-valued mappings $\sigma_{\pi^l(\widetilde{N}_i)} : \Delta_{n_i} \to 2^{X_i}$ and $\sigma_{\pi^r(\widetilde{N}_i)} : \Delta_{n_i} \to 2^{X_i}$, which have nonempty compact values. For each $i \in \Omega$, we define a set-valued mapping $\sigma_{\pi^l(\widetilde{N}_i)} \times \sigma_{\pi^r(\widetilde{N}_i)} : \Delta_{n_i} \to 2^{X_i \times X_i}$ by

$$(\sigma_{\pi^l(\widetilde{N}_i)} \times \sigma_{\pi^r(\widetilde{N}_i)})(p) = \sigma_{\pi^l(\widetilde{N}_i)}(p) \times \sigma_{\pi^r(\widetilde{N}_i)}(p), \quad \forall\, p \in \Delta_{n_i},$$

where $\pi^l(\widetilde{N}_i)$ (respectively, $\pi^r(\widetilde{N}_i)$) denotes the projection of $\widetilde{N}_i$ onto the left (respectively, right) of $Z_i \times Z_i$. Then it follows from Lemma 3 due to Fan [41] that the set-valued mapping $\sigma_{\pi^l(\widetilde{N}_i)} \times \sigma_{\pi^r(\widetilde{N}_i)}$ is upper semicontinuous and has nonempty compact values. Thus, one can see that $\{(X_i \times X_i, Z_i \times Z_i; \sigma_{\pi^l(\widetilde{N}_i)} \times \sigma_{\pi^r(\widetilde{N}_i)}) | i \in \Omega\}$ forms a family of WPH-spaces.

For each $i \in \Omega$, defne two set-valued mappings $S_i : X \times X \to 2^{Z_i \times Z_i}$ and $T_i : X \times X \to 2^{X_i \times X_i}$ by setting, for every $(x, y) \in X \times X$,

$$S_i(x,y) = \begin{cases} [(P_{i_{(x,y)}})_{p_i(x,y)} \bigcap (A_{i_x})_{c_i(x)}] \times (\overline{F}_{i_x})_{l_i(x)}, & \text{if } (x,y) \in W_i, \\ (A_{i_x})_{c_i(x)} \times (\overline{F}_{i_x})_{l_i(x)}, & \text{if } (x,y) \in W_i^c, \end{cases}$$

and

$$T_i(x,y) = \begin{cases} [WPH((\overline{P}_{i_{(x,y)}})_{m_i(x,y)}, (P_{i_{(x,y)}})_{p_i(x,y)}) \bigcap (B_{i_x})_{a_i(x)}] \times (F_{i_x})_{b_i(x)}, & \text{if } (x,y) \in W_i, \\ (B_{i_x})_{a_i(x)} \times (F_{i_x})_{b_i(x)}, & \text{if } (x,y) \in W_i^c, \end{cases}$$

respectively. Now, we show that $T_i(x, y)$ is WPH-convex relative to $S_i(x, y)$ for every $i \in \Omega$ and every $(x, y) \in W_i$. In fact, for each $i \in \Omega$, each $(x, y) \in W_i$, each $\widetilde{N}_i = \{(z_0^i, w_0^i), (z_1^i, w_1^i), \ldots, (z_{n_i}^i, w_{n_i}^i)\} \in \langle Z_i \times Z_i \rangle$, and each $\{(z_{j_0}^i, w_{j_0}^i), (z_{j_1}^i, w_{j_1}^i), \ldots, (z_{j_{k_i}}^i, w_{j_{k_i}}^i)\} \subseteq S_i(x, y) \bigcap \{(z_0^i, w_0^i), (z_1^i, w_1^i), \ldots, (z_{n_i}^i, w_{n_i}^i)\}$, we get

$$\{z_{j_0}^i, z_{j_1}^i, \ldots, z_{j_{k_i}}^i\} \subseteq (P_{i_{(x,y)}})_{p_i(x,y)} \bigcap \{z_0^i, z_1^i, \ldots, z_{n_i}^i\} = (P_{i_{(x,y)}})_{p_i(x,y)} \bigcap \pi^l(\widetilde{N}_i),$$

$$\{z_{j_0}^i, z_{j_1}^i, \ldots, z_{j_{k_i}}^i\} \subseteq (A_{i_x})_{c_i(x)} \bigcap \{z_0^i, z_1^i, \ldots, z_{n_i}^i\} = (A_{i_x})_{c_i(x)} \bigcap \pi^l(\widetilde{N}_i), \text{ and}$$

$$\{w_{j_0}^i, w_{j_1}^i, \ldots, w_{j_{k_i}}^i\} \subseteq (\overline{F}_{i_x})_{l_i(x)} \bigcap \{w_0^i, w_1^i, \ldots, w_{n_i}^i\} = (\overline{F}_{i_x})_{l_i(x)} \bigcap \pi^r(\widetilde{N}_i).$$

Then by (i), (ii), and the definition of a WPH-hull, we have

$$\sigma_{\pi^l(\widetilde{N}_i)}(\Delta_{k_i}) \subseteq WPH((\overline{P}_{i_{(x,y)}})_{m_i(x,y)}, (P_{i_{(x,y)}})_{p_i(x,y)}) \bigcap (B_{i_x})_{a_i(x)} \text{ and } \sigma_{\pi^r(\widetilde{N}_i)}(\Delta_{k_i}) \subseteq (F_{i_x})_{b_i(x)}.$$

It follows that $\sigma_{\widetilde{N}_i}(\Delta_{k_i}) = \sigma_{\pi^l(\widetilde{N}_i)}(\Delta_{k_i}) \times \sigma_{\pi^r(\widetilde{N}_i)}(\Delta_{k_i})$. Thus, we have

$$\sigma_{\widetilde{N}_i}(\Delta_{k_i}) \subseteq (WPH((\overline{P}_{i_{(x,y)}})_{m_i(x,y)}, (P_{i_{(x,y)}})_{p_i(x,y)}) \bigcap (B_{i_x})_{a_i(x)}) \times (F_{i_x})_{b_i(x)}$$

which implies that $T_i(x,y)$ is WPH-convex relative to $S_i(x,y)$ for every $i \in \Omega$ and every $(x,y) \in W_i$. The same method can be used to prove that $T_i(x,y)$ is WPH-convex relative to $S_i(x,y)$ for every $i \in \Omega$ and every $(x,y) \in W_i^c$. Therefore, these two scenarios lead to the conclusion that $T_i(x,y)$ is WPH-convex relative to $S_i(x,y)$ for every $i \in \Omega$ and every $(x,y) \in X \times X$.

For each $i \in \Omega$ and each $(u_i, v_i) \in Z_i \times Z_i$, we have

$$\begin{aligned} S_i^{-1}(u_i, v_i) &= (P_{i_{p_i}}^{-1}(u_i) \bigcap (A_{i_{c_i}}^{-1}(u_i) \times X) \bigcap (\overline{F}_{i_{b_i}}^{-1}(v_i) \times X)) \\ &\quad \bigcup (W_i^c \bigcap (A_{i_{c_i}}^{-1}(u_i) \times X) \bigcap (\overline{F}_{i_{b_i}}^{-1}(v_i) \times X)) \\ &= ((A_{i_{c_i}}^{-1}(u_i) \bigcap \overline{F}_{i_{b_i}}^{-1}(v_i)) \times X) \bigcap (P_{i_{p_i}}^{-1}(u_i) \bigcup W_i^c). \end{aligned}$$

According to (iv), for each $i \in \Omega$, there exists a nonempty subset $Z_i^0 \times Z_i^1$ of $Z_i \times Z_i$ such that

$$\bigcap_{(u_i,v_i)\in Z_i^0\times Z_i^1} (\mathrm{cint} S_i^{-1}(u_i, v_i))^c \subseteq \bigcup_{(u_i,v_i)\in Z_i\times Z_i} \mathrm{cint} S_i^{-1}(u_i, v_i).$$

For each $i \in \Omega$, setting $\widehat{K_i} := \bigcap_{(u_i,v_i)\in Z_i^0\times Z_i^1} (\mathrm{cint} S_i^{-1}(u_i, v_i))^c$, we have

$$\widehat{K_i} \subseteq \bigcup_{(u_i,v_i)\in Z_i\times Z_i} \mathrm{cint} S_i^{-1}(u_i, v_i),$$

where $\widehat{K_i}$ is a nonempty compact subset of $X \times X$ or an empty set. If $\widehat{K_i}$ is nonempty compact for every $i \in \Omega$, then there exist $Q_i \in \langle Z_i \rangle$ and $M_i \in \langle Z_i \rangle$ such that

$$\widehat{K_i} \subseteq \bigcup_{(u_i,v_i)\in Q_i\times M_i} \mathrm{cint} S_i^{-1}(u_i, v_i). \tag{5.1}$$

Thus, by (5.1), we have

$$\begin{aligned} X \times X &= (X \times X \setminus \widehat{K_i}) \bigcup \widehat{K_i} \\ &= \Big(\bigcup_{(u_i,v_i)\in Z_i^0\times Z_i^1} \mathrm{cint} S_i^{-1}(u_i, v_i) \Big) \bigcup \Big(\bigcup_{(u_i,v_i)\in Q_i\times M_i} \mathrm{cint} S_i^{-1}(u_i, v_i) \Big) \\ &= \bigcup_{(u_i,v_i)\in (Z_i^0\times Z_i^1)\cup(Q_i\times M_i)} \mathrm{cint} S_i^{-1}(u_i, v_i). \end{aligned}$$

If $\widehat{K_i} = \emptyset$, then for every $Q_i \in \langle Z_i \rangle$ and every $M_i \in \langle Z_i \rangle$, we have

$$X \times X = \bigcup_{(u_i,v_i)\in Z_i^0\times Z_i^1} \mathrm{cint} S_i^{-1}(u_i, v_i) = \bigcup_{(u_i,v_i)\in (Z_i^0\times Z_i^1)\cup(Q_i\times M_i)} \mathrm{cint} S_i^{-1}(u_i, v_i). \tag{5.2}$$

Therefore, (5.2) holds for each $i \in \Omega$ and the finite sets Q_i and M_i in (5.1). By (iv) again, for each $i \in \Omega$ and the finite sets Q_i and M_i, there exist nonempty compact subsets L_{Q_i}, L_{M_i} of $X_i \times X_i$ such that $L_{Q_i} \times L_{M_i}$ is WPH-convex relative to some $Z_i' \subseteq Z_i \times Z_i$ and $(Z_i^0 \times Z_i^1) \bigcup (Q_i \times M_i) \subseteq Z_i'$. Let $L_Q = \prod_{i\in I} L_{Q_i}$ and $L_M = \prod_{i\in I} L_{M_i}$. Hence, by (5.2), we have

$$L_Q \times L_M \subseteq X \times X = \bigcup_{(u_i,v_i)\in Z_i'} \mathrm{cint} S_i^{-1}(u_i, v_i). \tag{5.3}$$

Observe that $\{(L_{Q_i} \times L_{M_i}, Z_i'; \sigma_{\pi^l(\widetilde{N}_i)} \times \sigma_{\pi^r(\widetilde{N}_i)})|i \in \Omega\}$ is a family of WPH-spaces. For each $i \in \Omega$, define two set-valued mappings $S_i' : L_Q \times L_M \to 2^{Z_i'}$ and $T_i' : L_Q \times L_M \to 2^{L_{Q_i} \times L_{M_i}}$ by

$$S_i'(x,y) = S_i(x,y) \bigcap Z_i' \text{ and } T_i'(x,y) = T_i(x,y) \bigcap (L_{Q_i} \times L_{M_i}), \quad \forall\, (x,y) \in L_Q \times L_M.$$

Now, we check all the conditions of Theorem 3.4 for S_i' and T_i'. It is obvious that $L_Q \times L_M$ is a nonempty compact subset of $X \times X$. By (5.3), we have

$$\begin{aligned} L_Q \times L_M &= \Big(\bigcup_{(u_i,v_i)\in Z_i'} \operatorname{cint} S_i^{-1}(u_i,v_i)\Big) \bigcap \Big(L_Q \times L_M\Big) \\ &= \bigcup_{(u_i,v_i)\in Z_i'} \operatorname{int}_{L_Q\times L_M}(S_i^{-1}(u_i,v_i) \bigcap (L_Q \times L_M)) \\ &= \bigcup_{(u_i,v_i)\in Z_i'} \operatorname{int}_{L_Q\times L_M} S_i'^{-1}(u_i,v_i). \end{aligned}$$

We show that $T_i'(x,y)$ is WPH-convex relative to $S_i'(x,y)$ for every $i \in \Omega$ and every $(x,y) \in L_Q \times L_M$. In fact, since $(L_{Q_i} \times L_{M_i}, Z_i'; \sigma_{\pi^l(\widetilde{N}_i)} \times \sigma_{\pi^r(\widetilde{N}_i)})$ is a WPH-space and $T_i(x,y)$ is WPH-convex relative to $S_i(x,y)$ for every $i \in \Omega$ and every $(x,y) \in X \times X$, it follows that for each $(x,y) \in L_Q \times L_M$, each $\widetilde{N}_i = \{(z_0^i, w_0^i), (z_1^i, w_1^i), \ldots, (z_{n_i}^i, w_{n_i}^i)\} \in \langle Z_i' \rangle$, and each $\{(z_{j_0}^i, w_{j_0}^i), (z_{j_1}^i, w_{j_1}^i), \ldots, (z_{j_{k_i}}^i, w_{j_{k_i}}^i)\} \subseteq S_i'(x,y) \bigcap \widetilde{N}_i = S_i(x,y) \bigcap Z_i' \bigcap \widetilde{N}_i$, we have $\sigma_{\widetilde{N}_i}(\Delta_{k_i}) = \sigma_{\pi^l(\widetilde{N}_i)}(\Delta_{k_i}) \times \sigma_{\pi^r(\widetilde{N}_i)}(\Delta_{k_i}) \subseteq T_i(x,y) \bigcap (L_{Q_i} \times L_{M_i}) = T_i'(x,y)$. So, by Theorem 3.4, for each $i \in \Omega$, there exist $\widetilde{N}_i^* = \{(z_0^i, w_0^i), (z_1^i, w_1^i), \ldots, (z_{n_i}^i, w_{n_i}^i)\} \in \langle Z_i' \rangle$ and an upper semicontinuous set-valued mapping $f_i : L_Q \times L_M \to 2^{L_{Q_i} \times L_{M_i}}$ such that $f_i = \sigma_{\widetilde{N}_i^*} \circ \beta_i$ and $f_i(x,y) = \sigma_{\widetilde{N}_i^*}(\beta_i(x,y)) \subseteq T_i'(x,y)$ for every $(x,y) \in L_Q \times L_M$, where $\sigma_{\widetilde{N}_i^*} : \Delta_{n_i} \to 2^{L_{Q_i} \times L_{M_i}}$ is an upper semicontinuous set-valued mapping with nonempty compact values, $\beta_i : L_Q \times L_M \to \Delta_{n_i}$ is a continuous single-valued mapping, and n_i is a positive integer. Further, we define a set-valued mapping $\Phi : C \to 2^{\prod_{i\in\Omega}(L_{Q_i} \times L_{M_i})}$ by

$$\Phi(z) = \prod_{i\in\Omega} \Big(\sigma_{\pi^l(\widetilde{N}_i^*)} \pi_i(z) \times \sigma_{\pi^r(\widetilde{N}_i^*)} \pi_i(z)\Big) \tag{5.4}$$

for every $z \in C$ and a continuous mapping $\Psi : \prod_{i\in\Omega}(L_{Q_i} \times L_{M_i}) \to C$ by

$$\Psi((x_i,y_i)_{i\in\Omega}) = (\beta_i((x_i)_{i\in\Omega}, (y_i)_{i\in\Omega}))_{i\in\Omega}, \quad \forall\, (x,y) \in \prod_{i\in\Omega}(L_{Q_i} \times L_{M_i}), \tag{5.5}$$

where $C = \prod_{i\in\Omega} \Delta_{n_i}$ and $\pi_i : C \to \Delta_{n_i}$ is the projection of C onto Δ_{n_i}. Since $I_{\Pi_{i\in\Omega}(X_i \times X_i)} \in \widetilde{\mathcal{DB}}(\Pi_{i\in\Omega}(X_i \times X_i), \Pi_{i\in\Omega}(X_i \times X_i))$, it follows that $I|_{\prod_{i\in\Omega}(L_{Q_i} \times L_{M_i})} \in \widetilde{\mathscr{DB}}(\prod_{i\in\Omega}(L_{Q_i} \times L_{M_i}), \prod_{i\in\Omega}(L_{Q_i} \times L_{M_i}))$ and thus, the composition $\Psi \circ \Phi : C \to 2^C$ has a fixed point, which implies that there exists $\widehat{z} \in C$ such that $\widehat{z} \in \Psi \circ \Phi(\widehat{z})$. Hence, there exists $(\widehat{x_i}, \widehat{y_i})_{i\in\Omega} \in \Phi(\widehat{z})$ such that $\widehat{z} = \Psi((\widehat{x_i}, \widehat{y_i})_{i\in\Omega})$ and thus, by (5.4) and (5.5), we have

$$\begin{aligned} (\widehat{x_i}, \widehat{y_i})_{i\in\Omega} &\in \Phi(\widehat{z}) = \Phi(\Psi((\widehat{x_i}, \widehat{y_i})_{i\in\Omega})) = \Phi((\beta_i((\widehat{x_i})_{i\in\Omega}, (\widehat{y_i})_{i\in\Omega}))_{i\in\Omega}) \\ &= \prod_{i\in\Omega} \Big(\sigma_{\pi^l(\widetilde{N}_i^*)}(\beta_i((\widehat{x_i})_{i\in\Omega}, (\widehat{y_i})_{i\in\Omega})) \times \sigma_{\pi^r(\widetilde{N}_i^*)}(\beta_i((\widehat{x_i})_{i\in\Omega}, (\widehat{y_i})_{i\in\Omega}))\Big) \\ &= \prod_{i\in\Omega} (\sigma_{\pi^l(\widetilde{N}_i^*)} \times \sigma_{\pi^r(\widetilde{N}_i^*)})(\beta_i((\widehat{x_i})_{i\in\Omega}, (\widehat{y_i})_{i\in\Omega})) \end{aligned}$$

$$= \prod_{i\in\Omega}\sigma_{\widetilde{N_i^*}}(\beta_i(\widehat{x},\widehat{y}))\subseteq\prod_{i\in\Omega}T_i'(\widehat{x},\widehat{y}).$$

This, together with the definition of T_i', implies that $(\widehat{x_i},\widehat{y_i})\in T_i(\widehat{x},\widehat{y})$ for every $i\in\Omega$. If $(\widehat{x},\widehat{y})\in W_i$ for some $i\in\Omega$, then we have

$$(\widehat{x_i},\widehat{y_i})\in(WPH((\overline{P}_{i_{(\widehat{x},\widehat{y})}})_{m_i(\widehat{x},\widehat{y})},(P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})})\bigcap(B_{i_{\widehat{x}}})_{a_i(\widehat{x})})\times(F_{i_{\widehat{x}}})_{b_i(\widehat{x})}. \tag{5.6}$$

By (5.6), $\widehat{x_i}\in WPH((\overline{P}_{i_{(\widehat{x},\widehat{y})}})_{m_i(\widehat{x},\widehat{y})},(P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})})\bigcap(B_{i_{\widehat{x}}})_{a_i(\widehat{x})}$ and thus, $\widehat{x_i}\in WPH((\overline{P}_{i_{(\widehat{x},\widehat{y})}})_{m_i(\widehat{x},\widehat{y})},(P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})})$, which contradicts (iii). Therefore, we must have that $(\widehat{x},\widehat{y})\in W_i^c$ for every $i\in\Omega$. By the definitions of W_i^c and $T_i(x,y)$, we know that for each $i\in\Omega$, $\widehat{x_i}\in(B_{i_{\widehat{x}}})_{a_i(\widehat{x})}$, $\widehat{y_i}\in(F_{i_{\widehat{x}}})_{b_i(\widehat{x})}$ and $(A_{i_{\widehat{x}}})_{c_i(\widehat{x})}\bigcap(P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})}=\emptyset$. This completes the proof. □

Corollary 5.4. *Let Ω be a set of agents (finite or infinite) and $\{(X_i,Z_i;\sigma_{N_i})|i\in\Omega\}$ be a family of Hausdorff WPH-spaces. Let $\Gamma=(X_i,Z_i,A_i,B_i,F_i,P_i,a_i,b_i,c_i,p_i)_{i\in\Omega}$ be a generalized fuzzy game and $I_{\Pi_{i\in\Omega}(X_i\times X_i)}\in\widetilde{\mathfrak{DB}}(\Pi_{i\in\Omega}(X_i\times X_i),\Pi_{i\in\Omega}(X_i\times X_i))$, where $I_{\Pi_{i\in\Omega}(X_i\times X_i)}$ is the identity mapping on $\Pi_{i\in\Omega}(X_i\times X_i)$. For each $i\in\Omega$, let $\overline{P}_i:X\times X\to\mathscr{F}(X_i)$, $\overline{F}_i:X\to\mathscr{F}(Z_i)$ be two fuzzy mappings and let $l_i:X\to[0,1)$, $m_i:X\times X\to[0,1)$ be two fuzzy sets. For each $i\in\Omega$, assume that the following conditions hold*:

(i) *For each $x\in X$, $(B_{i_x})_{a_i(x)}$ is WPH-convex relative to $(A_{i_x})_{c_i(x)}$;*

(ii) *For each $x\in X$, $(F_{i_x})_{b_i(x)}$ is WPH-convex relative to $(\overline{F}_{i_x})_{l_i(x)}$;*

(iii) *For each $(x,y)\in W_i$, $x_i\notin WPH((\overline{P}_{i_{(x,y)}})_{m_i(x,y)},(P_{i_{(x,y)}})_{p_i(x,y)})$, where $W_i=\{(x,y)\in X\times X:(A_{i_x})_{c_i(x)}\bigcap(P_{i_{(x,y)}})_{p_i(x,y)}\neq\emptyset\}$;*

(iv) *There exist two nonempty subsets Z_i^0 and Z_i^1 of Z_i such that*

$$\bigcup_{(u_i,v_i)\in Z_i\times Z_i}\text{cint}\{((A_{i_{c_i}}^{-1}(u_i)\bigcap\overline{F}_{i_{l_i}}^{-1}(v_i))\times X)\bigcap(P_{i_{p_i}}^{-1}(u_i)\bigcup W_i^c)\}$$

contains $\bigcap_{(u_i,v_i)\in Z_i^0\times Z_i^1}[\text{cint}\{((A_{i_{c_i}}^{-1}(u_i)\cap\overline{F}_{i_{l_i}}^{-1}(v_i))\times X)\bigcap(P_{i_{p_i}}^{-1}(u_i)\bigcup W_i^c)\}]^c$ *which is a nonempty compact subset of $X\times X$ or an empty set, and for each $Q_i,M_i\in\langle Z_i\rangle$, there exist nonempty compact subsets L_{Q_i},L_{M_i} of X_i such that $L_{Q_i}\times L_{M_i}$ is WPH-convex relative to some $Z_i'\subseteq Z_i\times Z_i$ and $(Z_i^0\times Z_i^1)\bigcup(Q_i\times M_i)\subseteq Z_i'$.*

Then there exists a point $(\widehat{x},\widehat{y})\in X\times X$ such that for each $i\in\Omega$, $\widehat{x_i}\in(B_{i_{\widehat{x}}})_{a_i(\widehat{x})}$, $\widehat{y_i}\in(F_{i_{\widehat{x}}})_{b_i(\widehat{x})}$ and $(A_{i_{\widehat{x}}})_{c_i(\widehat{x})}\bigcap(P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})}=\emptyset$.

Proof. By (iii), we have $x_i\notin WPH((\overline{P}_{i_{(x,y)}})_{m_i(x,y)},(P_{i_{(x,y)}})_{p_i(x,y)})$ for every $(x,y)\in W_i$, where $W_i=\{(x,y)\in X\times X:(A_{i_x})_{c_i(x)}\bigcap(P_{i_{(x,y)}})_{p_i(x,y)}\neq\emptyset\}$. Therefore, by Theorem 5.3, the conclusion of Corollary 5.4 holds. This completes the proof. □

As a direct consequence of Theorem 5.3, we have the following existence theorem of equilibria for generalized fuzzy qualitative games in noncompact WPH-spaces.

Theorem 5.5. *Let Ω be a set of agents (finite or infinite) and $\{(X_i,Z_i;\sigma_{N_i})|i\in\Omega\}$ be a family of Hausdorff WPH-spaces. Let $\Gamma=(X_i,Z_i,F_i,P_i,b_i,p_i)_{i\in\Omega}$ be a generalized fuzzy game and $I_{\Pi_{i\in\Omega}(X_i\times X_i)}\in\widetilde{\mathfrak{DB}}(\Pi_{i\in\Omega}(X_i\times X_i),\Pi_{i\in\Omega}(X_i\times X_i))$, where $I_{\Pi_{i\in\Omega}(X_i\times X_i)}$ is the identity mapping on $\Pi_{i\in\Omega}(X_i\times X_i)$. For each $i\in\Omega$, let $\overline{P}_i:X\times X\to\mathscr{F}(X_i)$, $\overline{F}_i:X\to\mathscr{F}(Z_i)$ be two fuzzy mappings and let $l_i:X\to[0,1)$, $m_i:X\times X\to[0,1)$ be two fuzzy sets. For each $i\in\Omega$, assume that the following conditions hold*:

(i) *For each* $x \in X$, $(F_{i_x})_{b_i(x)}$ *is WPH-convex relative to* $(\overline{F}_{i_x})_{l_i(x)}$;
(ii) *For every* $(x, y) \in W_i = \{(x, y) \in X \times X : (P_{i_{(x,y)}})_{p_i(x,y)} \neq \emptyset\}$, $x_i \notin WPH((\overline{P}_{i_{(x,y)}})_{m_i(x,y)}, (P_{i_{(x,y)}})_{p_i(x,y)})$;
(iii) *There exist two nonempty subsets* Z_i^0 *and* Z_i^1 *of* Z_i *such that*

$$\bigcup_{(u_i,v_i)\in Z_i\times Z_i} \mathrm{cint}\{(\overline{F}_{i_{b_i}}^{-1}(v_i) \times X) \bigcap (P_{i_{p_i}}^{-1}(u_i) \bigcup W_i^c)\}$$

contains $\bigcap_{(u_i,v_i)\in Z_i^0\times Z_i^1} [\mathrm{cint}\{(\overline{F}_{i_{b_i}}^{-1}(v_i) \times X) \bigcap (P_{i_{p_i}}^{-1}(u_i) \bigcup W_i^c)\}]^c$ *which is a nonempty compact subset of* $X \times X$ *or an empty set, and for each* $Q_i, M_i \in \langle Z_i\rangle$, *there exist nonempty compact subsets* L_{Q_i}, L_{M_i} *of* X_i *such that* $L_{Q_i} \times L_{M_i}$ *is WPH-convex relative to some* $Z_i' \subseteq Z_i \times Z_i$ *and* $(Z_i^0 \times Z_i^1) \bigcup (Q_i \times M_i) \subseteq Z_i'$.

Then there exists a point $(\widehat{x}, \widehat{y}) \in X \times X$ *such that for each* $i \in \Omega$, $\widehat{y}_i \in (F_{i_{\widehat{x}}})_{b_i(\widehat{x})}$ *and* $(P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})} = \emptyset$.

Proof. For each $i \in I$, let $(A_{i_x})_{c_i(x)} \equiv Z_i$ and $(B_{i_x})_{a_i(x)} \equiv X_i$ for every $x \in X$. Then we have $A_{i_{c_i}}^{-1}(z_i) = B_{i_{a_i}}^{-1}(x_i) = X$ for every $i \in I$, every $z_i \in Z_i$, and every $x_i \in X_i$. Therefore, by Theorem 5.3, the conclusion of Theorem 5.5 holds. This completes the proof. □

As a special case of Theorem 5.3, we can obtain the following existence theorem of equilibria for generalized fuzzy games in noncompact WPH-spaces.

Theorem 5.6. *Let* Ω *be a set of agents (finite or infinite) and* $\{(X_i; \sigma_{N_i}) | i \in \Omega\}$ *be a family of Hausdorff WPH-spaces. Let* $\Gamma = (X_i, A_i, B_i, F_i, P_i, a_i, b_i, c_i, p_i)_{i\in\Omega}$ *be a generalized fuzzy game and* $I_{\Pi_{i\in\Omega}(X_i\times X_i)} \in \widetilde{\mathfrak{DB}}(\Pi_{i\in\Omega}(X_i \times X_i), \Pi_{i\in\Omega}(X_i \times X_i))$, *where* $I_{\Pi_{i\in\Omega}(X_i\times X_i)}$ *is the identity mapping on* $\Pi_{i\in\Omega}(X_i \times X_i)$. *For each* $i \in \Omega$, *assume that the following conditions hold*:

(i) *For each* $x \in X$, $(B_{i_x})_{a_i(x)}$ *is WPH-convex relative to* $(A_{i_x})_{c_i(x)}$;
(ii) *For each* $x \in X$, $(F_{i_x})_{b_i(x)}$ *is a WPH-convex subspace of* $(X_i; \sigma_{N_i})$;
(iii) *For each* $(x, y) \in W_i$, $x_i \notin WPH((P_{i_{(x,y)}})_{p_i(x,y)}, (P_{i_{(x,y)}})_{p_i(x,y)})$, *where* $W_i = \{(x, y) \in X \times X : (A_{i_x})_{c_i(x)} \bigcap (P_{i_{(x,y)}})_{p_i(x,y)} \neq \emptyset\}$;
(iv) *There exist two nonempty subsets* X_i^0 *and* X_i^1 *of* X_i *such that*

$$\bigcup_{(u_i,v_i)\in X_i\times X_i} \mathrm{cint}\{((A_{i_{c_i}}^{-1}(u_i) \bigcap F_{i_{b_i}}^{-1}(v_i)) \times X) \bigcap (P_{i_{p_i}}^{-1}(u_i) \bigcup W_i^c)\}$$

contains $\bigcap_{(u_i,v_i)\in X_i^0\times X_i^1} [\mathrm{cint}\{((A_{i_{c_i}}^{-1}(u_i) \bigcap F_{i_{b_i}}^{-1}(v_i)) \times X) \bigcap (P_{i_{p_i}}^{-1}(u_i) \bigcup W_i^c)\}]^c$ *which is a nonempty compact subset of* $X \times X$ *or an empty set, and for each* $Q_i, M_i \in \langle X_i\rangle$, *there exist nonempty compact subsets* L_{Q_i}, L_{M_i} *of* X_i *such that* $L_{Q_i} \times L_{M_i}$ *is WPH-convex relative to some* $X_i' \subseteq X_i \times X_i$ *and* $(X_i^0 \times X_i^1) \bigcup (Q_i \times M_i) \subseteq X_i'$.

Then there exists a point $(\widehat{x}, \widehat{y}) \in X \times X$ *such that for each* $i \in \Omega$, $\widehat{x}_i \in (B_{i_{\widehat{x}}})_{a_i(\widehat{x})}$, $\widehat{y}_i \in (F_{i_{\widehat{x}}})_{b_i(\widehat{x})}$ *and* $(A_{i_{\widehat{x}}})_{c_i(\widehat{x})} \bigcap (P_{i_{(\widehat{x},\widehat{y})}})_{p_i(\widehat{x},\widehat{y})} = \emptyset$.

Proof. For each $i \in I$, let $X_i = Z_i$, $\overline{P}_i = P_i$, $\overline{F}_i = F_i$, $p_i = m_i$, and $b_i = l_i$. Then the conclusion of Theorem 5.6 holds from Theorem 5.3. This completes the proof. □

Remark 5.7. (1) If X_i is a compact Hausdorff WPH-space for every $i \in \Omega$, then we can see that (iv) of Theorem 5.6 is trivially fulfilled by letting $X_i^0 = X_i^1 = L_{Q_i} = L_{M_i} = X_i$ and $X_i' = X_i \times X_i$.

(2) If $(P_{i_{(x,y)}})_{p_i(x,y)}$ is a WPH-convex subspace of $(X_i; \sigma_{N_i})$ for every $i \in \Omega$ and every $(x, y) \in X \times X$,

then based on the fact that $WPH((P_{i_{(x,y)}})_{p_i(x,y)}, (P_{i_{(x,y)}})_{p_i(x,y)}) = (P_{i_{(x,y)}})_{p_i(x,y)}$, (iii) of Theorem 5.6 can be replaced by the following condition:

(iii)′ For each $(x, y) \in W_i = \{(x, y) \in X \times X : (A_{i_x})_{c_i(x)} \bigcap (P_{i_{(x,y)}})_{p_i(x,y)} \neq \emptyset\}$, $x_i \notin (P_{i_{(x,y)}})_{p_i(x,y)}$.

(3) Theorem 5.6 generalizes Theorem 4.2 of Al-Homidan and Ansari [33] in the following aspects: (a) From crisp settings to fuzzy settings; (b) From topological semilattice spaces to WPH-spaces without any linear and convex structure. In fact, by Lemma 2.1 of Al-Homidan and Ansari [33], we can see that every topological semilattice space is a WPH-space; (c) (ii) of Theorem 4.2 due to Al-Homidan and Ansari [33] is dropped; (d) (iii) of Theorem 5.6 is weaker than (iii) of Theorem 4.2 of Al-Homidan and Ansari [33].

(4) Theorem 5.6 improves and extends Theorem 9 of Khanh and Quan [51] in the following aspects: (a) From crisp settings to fuzzy settings; (b) From KKM-structure with compact Hausdorff topological spaces to noncompact Hausdorff WPH-spaces. In fact, from Definition 1 of Khanh and Quan [51], it is easy to see that KKM-structures introduced by Khanh and Quan [51] are special cases of WPH-spaces; (c) From the games with one constraint set-valued mapping to the generalized fuzzy games with three constraint set-valued mappings; (d) (iii) of Theorem 5.6 is weaker than (iii) of Theorem 9 of Khanh and Quan [51]. In addition, it should be pointed out that the proof of Theorem 5.6 is based on the upper semicontinuous selection and collectively fixed point methods, while the proof of Theorem 9 due to Khanh and Quan [51] is based on systems of variational relation method.

In Theorems 5.3, 5.5 and 5.6, if the fuzzy constrain set-valued mappings are defined by the characteristic functions and the fuzzy constraint functions are defined by the functions whose values are constant zero, then we have the following existence theorems of equilibria for the generalized games and generalized qualitative games in crisp settings. We omit their proofs.

Theorem 5.8. *Let Ω be a set of agents (finite or infinite), $\{(X_i, Z_i; \sigma_{N_i})|i \in \Omega\}$ be a family of Hausdorff WPH-spaces, and let $\Gamma = (X_i, Z_i, A_i, B_i, F_i, P_i)_{i\in\Omega}$ be a generalized game. Let $I_{\Pi_{i\in\Omega}(X_i\times X_i)} \in \widetilde{\mathcal{DB}}(\Pi_{i\in\Omega}(X_i\times X_i), \Pi_{i\in\Omega}(X_i \times X_i))$, where $I_{\Pi_{i\in\Omega}(X_i\times X_i)}$ is the identity mapping on $\Pi_{i\in\Omega}(X_i \times X_i)$. For each $i \in \Omega$, let $\overline{P}_i : X \times X \to 2^{X_i}$ and $\overline{F}_i : X \to 2^{Z_i}$ be two set-valued mappings. For each $i \in \Omega$, assume that the following conditions are fulfilled*:

(i) *For each $x \in X$, $B_i(x)$ is WPH-convex relative to $A_i(x)$*;

(ii) *For each $x \in X$, $F_i(x)$ is WPH-convex relative to $\overline{F}_i(x)$*;

(iii) *For each $(x, y) \in W_i$, $x_i \notin WPH(\overline{P}_i(x, y), P_i(x, y))$, where $W_i = \{(x, y) \in X \times X : A_i(x) \bigcap P_i(x, y) \neq \emptyset\}$*;

(iv) *There exist two nonempty subsets Z_i^0 and Z_i^1 of Z_i such that*

$$\bigcup_{(u_i,v_i)\in Z_i\times Z_i} \mathrm{cint}\{((A_i^{-1}(u_i) \bigcap \overline{F}_i^{-1}(v_i)) \times X) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}$$

contains $\bigcap_{(u_i,v_i)\in Z_i^0\times Z_i^1}[\mathrm{cint}\{((A_i^{-1}(u_i) \cap \overline{F}_i^{-1}(v_i)) \times X) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}]^c$ *which is a nonempty compact subset of $X \times X$ or an empty set, and for each $Q_i, M_i \in \langle Z_i\rangle$, there exist nonempty compact subsets L_{Q_i}, L_{M_i} of X_i such that $L_{Q_i}\times L_{M_i}$ is WPH-convex relative to some $Z_i' \subseteq Z_i\times Z_i$ and $(Z_i^0\times Z_i^1) \bigcup (Q_i\times M_i) \subseteq Z_i'$.*

Then there exists a point $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $i \in \Omega$, $\widehat{x}_i \in B_i(\widehat{x})$, $\widehat{y}_i \in F_i(\widehat{x})$ and $A_i(\widehat{x}) \bigcap P_i(\widehat{x}, \widehat{y}) = \emptyset$.

Theorem 5.9. *Let Ω be a set of agents (finite or infinite) and $\{(X_i, Z_i; \sigma_{N_i})|i \in \Omega\}$ be a family of Hausdorff WPH-spaces. Let $\Gamma = (X_i, Z_i, F_i, P_i)_{i\in\Omega}$ be a generalized qualitative game and $I_{\Pi_{i\in\Omega}(X_i\times X_i)} \in \widetilde{\mathfrak{DB}}(\Pi_{i\in\Omega}(X_i \times X_i), \Pi_{i\in\Omega}(X_i \times X_i))$, where $I_{\Pi_{i\in\Omega}(X_i\times X_i)}$ is the identity mapping on $\Pi_{i\in\Omega}(X_i \times X_i)$. For each $i \in \Omega$, let $\overline{P}_i : X \times X \to 2^{X_i}$ and $\overline{F}_i : X \to 2^{Z_i}$ be two set-valued mappings. For each $i \in \Omega$, assume that the following conditions hold*:

(i) *For each $x \in X$, $F_i(x)$ is WPH-convex relative to $\overline{F}_i(x)$*;
(ii) *For each $(x, y) \in W_i$, $x_i \notin WPH(\overline{P}_i(x, y), P_i(x, y))$, where $W_i = \{(x, y) \in X \times X : P_i(x, y) \neq \emptyset\}$*;
(iii) *There exist two nonempty subsets Z_i^0 and Z_i^1 of Z_i such that*

$$\bigcup_{(u_i,v_i)\in Z_i\times Z_i} \mathrm{cint}\{(\overline{F}_i^{-1}(v_i) \times X) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}$$

contains $\bigcap_{(u_i,v_i)\in Z_i^0\times Z_i^1}[\mathrm{cint}\{(\overline{F}_i^{-1}(v_i) \times X) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}]^c$ which is a nonempty compact subset of $X \times X$ or an empty set, and for each $Q_i, M_i \in \langle Z_i\rangle$, there exist nonempty compact subsets L_{Q_i}, L_{M_i} of X_i such that $L_{Q_i} \times L_{M_i}$ is WPH-convex relative to some $Z_i' \subseteq Z_i \times Z_i$ and $(Z_i^0 \times Z_i^1) \bigcup (Q_i \times M_i) \subseteq Z_i'$.

Then there exists a point $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $i \in \Omega$, $\widehat{y}_i \in F_i(\widehat{x})$ and $P_i(\widehat{x}, \widehat{y}) = \emptyset$.

Theorem 5.10. *Let Ω be a set of agents (finite or infinite) and $\{(X_i; \sigma_{N_i})|i \in \Omega\}$ be a family of Hausdorff WPH-spaces. Let $\Gamma = (X_i, A_i, B_i, F_i, P_i)_{i\in\Omega}$ be a generalized game and $I_{\Pi_{i\in\Omega}(X_i\times X_i)} \in \widetilde{\mathfrak{DB}}(\Pi_{i\in\Omega}(X_i \times X_i), \Pi_{i\in\Omega}(X_i \times X_i))$, where $I_{\Pi_{i\in\Omega}(X_i\times X_i)}$ is the identity mapping on $\Pi_{i\in\Omega}(X_i \times X_i)$. For each $i \in \Omega$, assume that the following conditions hold*:

(i) *For each $x \in X$, $B_i(x)$ is WPH-convex relative to $A_i(x)$*;
(ii) *For each $x \in X$, $F_i(x)$ is a nonempty WPH-convex subspace of $(X_i; \sigma_{N_i})$*;
(iii) *For each $(x, y) \in W_i$, $x_i \notin WPH(P_i(x, y), P_i(x, y))$, where $W_i = \{(x, y) \in X \times X : A_i(x) \bigcap P_i(x, y) \neq \emptyset\}$*;
(iv) *There exist two nonempty subsets X_i^0 and X_i^1 of X_i such that*

$$\bigcup_{(u_i,v_i)\in X_i\times X_i} \mathrm{cint}\{((A_i^{-1}(u_i) \bigcap F_i^{-1}(v_i)) \times X) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}$$

contains $\bigcap_{(u_i,v_i)\in X_i^0\times X_i^1}[\mathrm{cint}\{((A_i^{-1}(u_i) \bigcap F_i^{-1}(v_i)) \times X) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}]^c$ which is a nonempty compact subset of $X \times X$ or an empty set, and for each $Q_i, M_i \in \langle X_i\rangle$, there exist nonempty compact subsets L_{Q_i}, L_{M_i} of X_i such that $L_{Q_i}\times L_{M_i}$ is WPH-convex relative to some $X_i' \subseteq X_i\times X_i$ and $(X_i^0\times X_i^1) \bigcup (Q_i\times M_i) \subseteq X_i'$.

Then there exists a point $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $i \in \Omega$, $\widehat{x}_i \in B_i(\widehat{x})$, $\widehat{y}_i \in F_i(\widehat{x})$ and $A_i(\widehat{x}) \bigcap P_i(\widehat{x}, \widehat{y}) = \emptyset$.

In Theorem 5.10, when $P_i(x, y) = P_i(x)$ and $F_i(x) \equiv X_i$ for every $i \in \Omega$ and every $(x, y) \in X \times X$, we can easily obtain the following existence theorem of equilibria for generalized games in the framework of noncompact WPH-spaces.

Theorem 5.11. *Let Ω be a set of agents (finite or infinite) and $\{(X_i; \sigma_{N_i})|i \in \Omega\}$ be a family of Hausdorff WPH-spaces. Let $X = \prod_{i\in\Omega} X_i$, $\Gamma = (X_i, A_i, B_i, P_i)_{i\in\Omega}$ be a generalized game, and $I_X \in \widetilde{\mathfrak{B}}(X, X)$, where I_X is the identity mapping on X. For each $i \in \Omega$, assume that the following conditions hold*:

(i) *For each* $x \in X$, $B_i(x)$ *is* WPH*-convex relative to* $A_i(x)$;

(ii) *For each* $x \in W_i$, $x_i \notin WPH(P_i(x), P_i(x))$, *where* $W_i = \{x \in X : A_i(x) \bigcap P_i(x) \neq \emptyset\}$;

(iii) *There exists a nonempty subset* X_i^0 *of* X_i *such that*

$$\bigcup_{u_i \in X_i} \mathrm{cint}\{A_i^{-1}(u_i) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}$$

contains $\bigcap_{u_i \in X_i^0}[\mathrm{cint}\{A_i^{-1}(u_i) \bigcap (P_i^{-1}(u_i) \bigcup W_i^c)\}]^c$ *which is a nonempty compact subset of* X *or an empty set, and for each* $Q_i \in \langle X_i \rangle$, *there exists a nonempty compact subset* L_{Q_i} *of* X_i, *which is* WPH*-convex relative to some* $X_i' \subseteq X_i$ *such that* $X_i^0 \bigcup Q_i \subseteq X_i'$.

Then there exists a point $\widehat{x} \in X$ *such that for each* $i \in \Omega$, $\widehat{x_i} \in B_i(\widehat{x})$ *and* $A_i(\widehat{x}) \bigcap P_i(\widehat{x}) = \emptyset$.

Remark 5.12. (1) The proof of Theorem 5.11 is similar to that of Theorem 5.3 in crisp settings. Since hypothesis that $(X_i \times X_i; \sigma_{\pi^l(\widetilde{N}_i)} \times \sigma_{\pi^r(\widetilde{N}_i)})$ is a WPH-space for every $i \in \Omega$ is not needed in the proof process of theorem 5.11, it follows that this hypothesis, together with the hypothesis that σ_{N_i} has nonempty compact values for every $i \in \Omega$, does not appear in Theorem 5.11.

(2) Theorem 5.11 extends and generalizes Corollary 4.1 of Al-Homidan and Ansari [33] to WPH-spaces. Theorem 5.11 is also an improved variant of Corollary 5.1 of Lin et al. [48], Theorem 2 of Ansari and Yao [46], and Theorem 4.1 of Tarafdar [42] in noncompact WPH-spaces.

If $A_i(x) = B_i(x) = X_i$ for every $i \in \Omega$ and every $x \in X$, then by Theorem 5.11, we have the following corollary.

Corollary 5.13. *Let* Ω *be a set of agents (finite or infinite) and* $\{(X_i; \sigma_{N_i}) | i \in \Omega\}$ *be a family of Hausdorff* WPH*-spaces. Let* $X = \prod_{i \in \Omega} X_i$, $\Gamma = (X_i, P_i)_{i \in \Omega}$ *be a qualitative game, and* $I_X \in \widetilde{\mathcal{B}}(X, X)$, *where* I_X *is the identity mapping on* X. *For each* $i \in \Omega$, *assume that the following conditions hold*:

(i) *For each* $x \in W_i$, $x_i \notin WPH(P_i(x), P_i(x))$, *where* $W_i = \{x \in X : P_i(x) \neq \emptyset\}$;

(ii) *There exists a nonempty subset* X_i^0 *of* X_i *such that*

$$\bigcup_{u_i \in X_i} \mathrm{cint}(P_i^{-1}(u_i) \bigcup W_i^c)$$

contains $\bigcap_{u_i \in X_i^0}[\mathrm{cint}(P_i^{-1}(u_i) \bigcup W_i^c)]^c$ *which is a nonempty compact subset of* X *or an empty set, and for each* $Q_i \in \langle X_i \rangle$, *there exists a nonempty compact subset* L_{Q_i} *of* X_i, *which is* WPH*-convex relative to some* $X_i' \subseteq X_i$ *such that* $X_i^0 \bigcup Q_i \subseteq X_i'$.

Then there exists a point $\widehat{x} \in X$ *such that for each* $i \in \Omega$, $P_i(\widehat{x}) = \emptyset$.

Example 5.14. We are in position to apply Corollary 5.13 to the equilibrium analysis of water allocation problems characterized by multiobjective non-cooperative games. Suppose that there are n water users ($n \geq 2$) that can draw water from a common water body for domestic and agricultural needs. For each $i \in \{1, 2, \ldots, n\}$, let us denote (q_i, r_i) the combination of quantities collected by the ith water user, which corresponds to two different uses, where $(q_i, r_i) \in [0, k_i] \times [0, l_i] = S_i \subseteq \mathbb{R}^2$ and $k_i, l_i > 0$. In fact, from here we can consider S_i to be the strategy space of the ith water user, and (q_i, r_i) can be seen as a strategy in that strategy space.

The payoff function for each water user is a two-dimensional vector-valued function, where the first component corresponds to the amount of water for domestic needs and represents improvements in health and agricultural production. We may assume that the first part of the vector-valued payoff function for every water user i is expressed as $b_iq_i - c_i(\sum_{i=1}^n q_i + \sum_{i=1}^n r_i)$, where b_iq_i represents the benefit of water user i, which is proportional to b_i with $b_i > 0$; $c_i(\sum_{i=1}^n q_i + \sum_{i=1}^n r_i)$ is the cost function of water user i. The second part of the vector-valued payoff function for every water user i is the market profit, which is proportional to the total water quantity $q_i + r_i$ and to the inverse demand function $D(\sum_{i=1}^n q_i + \sum_{i=1}^n r_i)$. Considering a marginal cost function that is the same for every water user i and is denoted by m, the vector-valued payoff function for every water user i is:

$$f_i((q_1, r_1), (q_2, r_2), \ldots, (q_i, r_i), \ldots, (q_n, r_n)) = \begin{pmatrix} b_iq_i - c_i(\sum_{i=1}^n q_i + \sum_{i=1}^n r_i) \\ (q_i + r_i)(D(\sum_{i=1}^n q_i + \sum_{i=1}^n r_i) - m) \end{pmatrix}.$$

Before introducing the equilibrium concepts of the multiobjective water resource allocation game model as above, we need several notation for later use. For each $i \in \{1, 2, \ldots, n\}$, we denote $S_{\hat{i}} := \prod_{j\neq i} S_j$. If $(q, r) = ((q_1, r_1), (q_2, r_2), \ldots, (q_i, r_i), \ldots, (q_n, r_n)) \in S$, then we write

$$(q_{\hat{i}}, r_{\hat{i}}) := ((q_1, r_1), \ldots, (q_{i-1}, r_{i-1}), (q_{i+1}, r_{i+1}), \ldots, (q_n, r_n)) \in S_{\hat{i}}$$

for every $i \in \{1, 2, \ldots, n\}$. If $(q_i, r_i) \in S_i$, $(z_i, p_i) \in S_i$ and $(q_{\hat{i}}, r_{\hat{i}}) \in S_{\hat{i}}$, then we use the following notation:

$$((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) := ((q_1, r_1), (q_2, r_2), \ldots, (q_i, r_i), \ldots, (q_n, r_n)) = (q, r) \in S$$

and

$$((q_{\hat{i}}, r_{\hat{i}}), (z_i, p_i)) := ((q_1, r_1), \ldots, (q_{i-1}, r_{i-1}), (z_i, p_i), (q_{i+1}, r_{i+1}), \ldots, (q_n, r_n)) \in S.$$

We denote by $\mathbb{R}^2_+ := \{x := (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. For each $u, v \in \mathbb{R}^2$, $u \cdot v$ denotes the standard Euclidian inner product.

Now, we introduce the concepts of equilibrium for the above multi-objective game model. A strategy n-tuple $(q^*, r^*) = ((q_1^*, r_1^*), (q_2^*, r_2^*), \ldots, (q_n^*, r_n^*)) \in S$ is said to be a Pareto equilibrium of this game model if for every water user $i \in \{1, 2, \ldots, n\}$, there is no strategy $(q_i, r_i) \in S_i$ such that

$$\begin{pmatrix} b_i(q_i - q_i^*) + c_i(\sum_{i=1}^n q_i^* + \sum_{i=1}^n r_i^*) - c_i(\sum_{j\neq i} q_j^* + \sum_{j\neq i} r_j^* + q_i + r_i) \\ m(q_i^* - q_i + r_i^* - r_i) + (q_i + r_i)D(\sum_{j\neq i} q_j^* + \sum_{j\neq i} r_j^* + q_i + r_i) - (q_i^* + r_i^*)D(\sum_{i=1}^n q_i^* + \sum_{i=1}^n r_i^*) \end{pmatrix}$$

$$= f_i((q_{\hat{i}}^*, r_{\hat{i}}^*), (q_i, r_i)) - f_i(q^*, r^*) \in \mathbb{R}^2_+ \setminus \{0\}.$$

For each $i \in \{1, 2, \ldots, n\}$, let $\omega_i = \begin{pmatrix}\alpha_i \\ \beta_i\end{pmatrix}$ be a weight vector with $\alpha_i + \beta_i = 1$ and $\alpha_i, \beta_i > 0$, the components of which correspond to the components of the i-th water user's vector-valued payoff function f_i, respectively. A strategy n-tuple $(q^*, r^*) = ((q_1^*, r_1^*), (q_2^*, r_2^*), \ldots, (q_n^*, r_n^*)) \in S$ is called a normal weight Nash equilibrium with respect to weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ if for each water user i and each $(q_i, r_i) \in S_i$, we have the following:

$$\begin{aligned} \omega_i \cdot f_i(q^*, r^*) &= \alpha_i[b_iq_i^* - c_i(\sum_{i=1}^n q_i^* + \sum_{i=1}^n r_i^*)] + \beta_i(q_i^* + r_i^*)(D(\sum_{i=1}^n q_i^* + \sum_{i=1}^n r_i^*) - m) \\ &\geq \alpha_i[b_iq_i - c_i(\sum_{j\neq i} q_j^* + \sum_{j\neq i} r_j^* + q_i + r_i)] + \beta_i(q_i + r_i)(D(\sum_{j\neq i} q_j^* + \sum_{j\neq i} r_j^* + q_i + r_i) - m) \end{aligned}$$

$$= \omega_i \cdot f_i((q^*_{\hat{i}}, r^*_{\hat{i}}), (q_i, r_i)),$$

where $\cdot$ denotes the inner product in $\mathbb{R}^2$.

We give the following assumptions:

Assumption 1: For each $i \in \{1, 2, \ldots, n\}$ and each $(q, r) = ((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) \in S$, the set of points $(z_i, p_i) \in S_i$ satisfying the following condition is a convex subset of S_i:

$$\begin{aligned}
\omega_i \cdot f_i(q, r) &= \alpha_i[b_i q_i - c_i(\sum_{i=1}^{n} q_i + \sum_{i=1}^{n} r_i)] + \beta_i(q_i + r_i)(D(\sum_{i=1}^{n} q_i + \sum_{i=1}^{n} r_i) - m) \\
&< \alpha_i[b_i z_i - c_i(\sum_{j \neq i} q_j + \sum_{j \neq i} r_j + z_i + p_i)] + \beta_i(z_i + p_i)(D(\sum_{j \neq i} q_j + \sum_{j \neq i} r_j + z_i + p_i) - m) \\
&= \omega_i \cdot f_i((q_{\hat{i}}, r_{\hat{i}}), (z_i, p_i)).
\end{aligned}$$

Assumption 2: For each $i \in \{1, 2, \ldots, n\}$ and each $(q, r) = ((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) \in S$, there exist a point $(z_i, p_i) \in S_i$ and an open neighborhood $U((q, r)) \subseteq \mathbb{R}^2$ such that for every $(g, h) = ((g_{\hat{i}}, h_{\hat{i}}), (g_i, h_i)) \in U((q, r)) \bigcap S$, there is no $(z_i, p_i) \in S_i$ such that

$$\begin{aligned}
\omega_i \cdot f_i(g, h) &= \alpha_i[b_i g_i - c_i(\sum_{i=1}^{n} g_i + \sum_{i=1}^{n} h_i)] + \beta_i(g_i + h_i)(D(\sum_{i=1}^{n} g_i + \sum_{i=1}^{n} h_i) - m) \\
&< \alpha_i[b_i z_i - c_i(\sum_{j \neq i} g_j + \sum_{j \neq i} h_j + z_i + p_i)] + \beta_i(z_i + p_i)(D(\sum_{j \neq i} g_j + \sum_{j \neq i} h_j + z_i + p_i) - m) \\
&= \omega_i \cdot f_i((g_{\hat{i}}, h_{\hat{i}}), (z_i, p_i))
\end{aligned}$$

or there exists $(z_i, p_i) \in S_i$ such that the above inequality holds.

Assumption 1 states that for any water user $i \in \{1, 2, \ldots, n\}$ and any strategy combination (q, r), judging from the weighted combination of water user i's vector-valued payoff function, the set of his/her strategy that strictly deviates from the strategy combination $((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i))$ is a convex subset of S_i. Assumption 2 means that for each water user $i \in \{1, 2, \ldots, n\}$ and each strategy combination $((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i))$, there exists a relatively open neighborhood of $((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i))$ such that for every point in this neighborhood, according to the judgment standard after the weighted combination of water user i's vector-valued payoff function, either there is no deviation strategy, or there is at least one deviation strategy.

We are ready to verify the conclusion that the multiobjective water resource allocation game model mentioned above has a Pareto equilibrium if Assumptions 1 and 2 are fulfilled. For each $i \in \{1, 2, \ldots, n\}$ and each $N_i = \{(q_{i_0}, r_{i_0}), (q_{i_1}, r_{i_1}), \ldots, (q_{i_{n_i}}, r_{i_{n_i}})\} \in \langle S_i \rangle$, we define a continuous mapping

$$\sigma_{N_i} : \Delta_{n_i} \to \mathrm{co}(N_i) \subseteq S_i$$

by $\sigma_{N_i}(\sum_{j=0}^{n_i} t_{i_j} e_{i_j}) = \sum_{j=0}^{n_i} t_{i_j}(q_{i_j}, r_{i_j})$ for every $(t_{i_0}, t_{i_1}, \ldots, t_{i_{n_i}}) = \sum_{j=0}^{n_i} t_{i_j} e_{i_j} \in \Delta_{n_i}$. Clearly, one can see that $(S_i; \sigma_{N_i})$ forms a special case of WPH-space for every $i \in \{1, 2, \ldots, n\}$. For each $i \in \{1, 2, \ldots, n\}$ and each $N_i = \{(q_{i_0}, r_{i_0}), (q_{i_1}, r_{i_1}), \ldots, (q_{i_{n_i}}, r_{i_{n_i}})\} \in \langle S_i \rangle$, there is a continuous mapping $\sigma_{N_i} : \Delta_{n_i} \to \mathrm{co}(N_i) \subseteq S_i$ as defined above. Let $C = \prod_{i=1}^{n} \Delta_{n_i}$. For each $i \in \{1, 2, \ldots, n\}$, let V_i be be the linear hull of the set $\{e_{i_0}, e_{i_1}, \ldots, e_{i_{n_i}}\}$. Then for every $i \in \{1, 2, \ldots, n\}$, by the finite dimensionality of V_i, it is a Hausdorff locally convex topological vector space. Thus, $V = \prod_{i=1}^{n} V_i$ is also a Hausdorff locally convex topological vector space and C is a compact convex subset of V. Let I_S be the identity mapping on S. For each $i \in \{1, 2, \ldots, n\}$, each $N_i = \{(q_{i_0}, r_{i_0}), (q_{i_1}, r_{i_1}), \ldots, (q_{i_{n_i}}, r_{i_{n_i}})\} \in \langle S_i \rangle$, and each continuous mapping $\Psi : I_S(\prod_{i\in\Omega} \sigma_{N_i}(\Delta_{n_i})) \to C$. Define a continuous mapping $\Phi : C \to S$ by $\Phi(\xi) = \prod_{i=1}^{n} \sigma_{N_i}(\pi_i(\xi))$ for every $\xi \in C$, where π_i is the projection of C onto Δ_{n_i}. Then it follows from the Tychonof fixed point theorem that the continuous mapping $\Psi \circ I_S|_{\prod_{i\in\Omega} \sigma_{N_i}(\Delta_{n_i})} \circ \Phi : C \to C$ has a fixed point, which implies that $I_S \in \widetilde{\mathscr{B}}(S, S)$. For each $i \in \{1, 2, \ldots, n\}$, we define the i-th water user's deviation set-valued mapping $\Theta_i : S \to 2^{S_i}$ by

$$\Theta_i(q, r) = \{(z_i, p_i) \in S_i : \omega_i \cdot f_i((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) < \omega_i \cdot f_i((q_{\hat{i}}, r_{\hat{i}}), (z_i, p_i))\}$$

for every $(q, r) = ((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) \in S$. Then by Assumption 1, we can see that the set $\Theta_i(q, r)$ is a convex subset of S_i for every $i \in \{1, 2, \ldots, n\}$ and every $(q, r) = ((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) \in S$. For each $(q, r) = ((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) \in S$, each $i \in \{1, 2, \ldots, n\}$, each $N_i = \{(q_{i_0}, r_{i_0}), (q_{i_1}, r_{i_1}), \ldots, (q_{i_{n_i}}, r_{i_{n_i}})\} \in \langle S_i \rangle$, and each $((q_{j_0}, r_{j_0}), (q_{j_1}, r_{j_1}), \ldots, (q_{j_{k_i}}, r_{j_{k_i}})) \subseteq N_i \bigcap \Theta_i(q, r)$, we have

$$\begin{aligned}
\sigma_{N_i}(\Delta_{k_i}) &= \operatorname{co}(\{(q_{j_0}, r_{j_0}), (q_{j_1}, r_{j_1}), \ldots, (q_{j_{k_i}}, r_{j_{k_i}})\}) \\
&\subseteq \operatorname{co}(\Theta_i(q, r)) \\
&\subseteq \Theta_i(q, r).
\end{aligned}$$

This shows that $\Theta_i(q, r)$ is a WPH-convex subspace of $(S_i; \sigma_{N_i})$. Therefore, we have$(q_i, r_i) \notin \Theta_i(q, r) = WPH(\Theta_i(q, r), \Theta_i(q, r))$ for every $(q, r) = ((q_{\hat{i}}, r_{\hat{i}}), (q_i, r_i)) \in S$ and thus, (i) of Corollary 5.13 holds. Finally, we check that (ii) of Corollary 5.13 is satisfied. In fact, it follows from Assumption 2 that $S = \bigcup_{(z_i,p_i)\in S_i} \operatorname{int}(\Theta_i^{-1}((z_i, p_i)) \bigcup W_i^c) = \bigcup_{(z_i,p_i)\in S_i} \operatorname{cint}(\Theta_i^{-1}((z_i, p_i)) \bigcup W_i^c)$ for every $i \in \{1, 2, \ldots, n\}$, where $W_i = \{(q, r) \in S : \Theta_i(q, r) \neq \emptyset\}$. For each $i \in \{1, 2, \ldots, n\}$ and each $Q_i \in \langle S_i \rangle$, let $L_{Q_i} = S_i = S_i' = S_i^0$ and then, we can see that $S_i^0 \bigcup Q_i \subseteq S_i'$ and L_{Q_i} is WPH-convex relative to some $S_i' \subseteq S_i$. It is obvious that $\bigcap_{(z_i,p_i)\in S_i^0} [\operatorname{cint}(\Theta_i^{-1}((z_i, p_i)) \bigcup W_i^c)]^c \subseteq S = \bigcup_{(z_i,p_i)\in S_i} \operatorname{cint}(\Theta_i^{-1}((z_i, p_i)) \bigcup W_i^c)$. If $\bigcap_{(z_i,p_i)\in S_i^0} [\operatorname{cint}(\Theta_i^{-1}((z_i, p_i)) \bigcup W_i^c)]^c \neq \emptyset$, then it is a nonempty compact subset of S by the fact that it is a compactly closed subset of the compact set S. If $\bigcap_{(z_i,p_i)\in S_i^0} [\operatorname{cint}(\Theta_i^{-1}((z_i, p_i)) \bigcup W_i^c)]^c = \emptyset$, then (ii) of Corollary 5.13 holds automatically. So far, we have verified that all the hypotheses of Corollary 5.13 are fulfilled. Thus, it follows from Corollary 5.13 that there exists a strategy n-tuple $(q^*, r^*) = ((q_1^*, r_1^*), (q_2^*, r_2^*), \ldots, (q_n^*, r_n^*)) \in S$ such that for each $i \in \{1, 2, \ldots, n\}$, $\Theta_i(q^*, r^*) = \emptyset$, which implies that for each water user i and each $(q_i, r_i) \in S_i$,

$$\begin{aligned}
\omega_i \cdot f_i(q^*, r^*) &= \alpha_i[b_i q_i^* - c_i(\sum_{i=1}^{n} q_i^* + \sum_{i=1}^{n} r_i^*)] + \beta_i(q_i^* + r_i^*)(D(\sum_{i=1}^{n} q_i^* + \sum_{i=1}^{n} r_i^*) - m) \\
&\geq \alpha_i[b_i q_i - c_i(\sum_{j\neq i} q_j^* + \sum_{j\neq i} r_j^* + q_i + r_i)] + \beta_i(q_i + r_i)(D(\sum_{j\neq i} q_j^* + \sum_{j\neq i} r_j^* + q_i + r_i) - m) \\
&= \omega_i \cdot f_i((q_{\hat{i}}^*, r_{\hat{i}}^*), (q_i, r_i)),
\end{aligned}$$

that is, $(q^*, r^*) = ((q_1^*, r_1^*), (q_2^*, r_2^*), \ldots, (q_n^*, r_n^*)) \in S$ is a normal weight Nash equilibrium with respect to weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Further, by using a method similar to that used to prove Lemma 5.1 due to Lu et al. [54], we can show that $(q^*, r^*) = ((q_1^*, r_1^*), (q_2^*, r_2^*), \ldots, (q_n^*, r_n^*)) \in S$ is a Pareto equilibrium of the multiobjective water resource allocation game model. We omit the proof.

6. Conclusions

A WPH-space is characterized by the existence of an upper semicontinuous set-valued mapping from a simplex to a topological space, which is established through the medium of finite subsets of a nonempty set. Since the applications of selection theorems in crisp settings and in the framework of Hausdorff topological vector spaces are narrow, in order to eliminate the limitations and extend the applications of selection theorems, we prove an upper semicontinuous selection theorem for fuzzy mappings in WPH-spaces without linear structure. The validity of the upper semicontinuous selection theorem for fuzzy mappings is illustrated by an example. Furthermore, by using our upper semicontinuous selection results, we establish fuzzy collective coincidence point theorems, fuzzy collectively fixed point theorems, and existence theorems of equilibria for the generalized fuzzy games with three constraint set-valued mappings and generalized fuzzy qualitative games in noncompact WPH-spaces. In our opinions, future theoretical research should be focused on applying the upper semicontinuous selection theorems for fuzzy mappings to deal with the existence of solutions to systems of generalized fuzzy vector quasi-equilibrium problems, while in terms of practical applied research, the concepts of generalized fuzzy games and generalized fuzzy qualitative games can be applied to the model of socio-economic systems with coordination and hence, our existence results of equilibria for generalized fuzzy games and generalized fuzzy qualitative games can be useful tools for analyzing general equilibrium models of economic institutions, which contain most known types of institutions, for example, trade market, bilateral exchanges, supply chains, etc., as special cases.

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Conflict of interest

The authors declare that they have no competing interests.

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