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Research article

Short time asymptotics for American maximum options with a dividend-paying asset

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Abstract: We investigate the asymptotic behaviors of American maximum options with dividend-paying assets near maturity. Using the exercise conditions of American options, we obtain the asymptotic forms of the two boundaries with respect to time-to-maturity. Furthermore, we derive the matched asymptotic expansion for the rescaled value function of American maximum option. The all results are provided with detailed computations and derivations. Numerical examples show that the asymptotic value function and exercise boundaries can provide an efficient alternative for the true ones, respectively.

Keywords: American option; minimum payoff; optimal exercise/free boundary; short time asymptotics
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1. Introduction

How to value options is a central problem of mathematical finance. For European call options, Black and Schole [1] derived a closed-form solution in their seminal paper. However, the formulas are not suitable for American options by analogy. Actually, the valuations of American options are the most difficult problems in the field of options' pricing. Using a simple arbitrage argument, Merton [2] recognized that the valuation of American options was a free boundary problem. The difficulty in pricing American option stems from ascertaining when to implement the early exercise right or deciding the optimal stopping time of the option, which embodies mathematically that the unknown exercise boundary involves in the solution of the free boundary problem. Although Roll [3], Geske [4], and Whaley [5] have derived the analytic solutions for American options on assets with discrete dividends, no analytic solution exists for American options if the underlying assets pay continuous dividends. The pricing of American options has been a challenging problem for decades.

To overcome the difficulty in pricing American options, many numerical methods and approximation methods have been developed. Binomial tree model, Monte Carlo simulation, finite difference method and other methods are commonly used numerical techniques. When the time is close to expiry, some of numerical method lose their accuracy and efficiency. To promote the theoretical analysis for American options, Kim [6], Jacka [7], and Carr et al. [8] divided American option value into the corresponding European option value and an early exercise right premium. The premium can be written as an integral involved the optimal exercise boundary. This makes it possible to find an approximation solution under proper conditions. The basic property of American option provides some matching conditions since the function of American options price is smooth at its exercise boundary. Goodman and Ostrov [9] and Chen and Chadam [10] found American option price must be independent of time-to-maturity whenever it is optimal to early exercise. Utilizing these matching conditions, we can get some integral equations. Various methods were adopted to solve the integral equations and then got the approximate analytic solutions. More information about the development of this subject can be found in a general review and summary given by Giovanni [11].

Much research has focused only on standard American options on single asset such as simple call or put options. However, options traded in modern financial markets are highly diversified, few studies have explored these complex American-style options. So there is an urgent need for pricing these options. Jiang [12] analyzed American options on the maximum (minimum) of two risk assets, he focused on the relationship between several factors and the option price as well as the monotonicity, convexity and limit behavior of the optimal exercise boundary. If we use the limits of the boundaries at the maturity to substitute them, it will produce unavoidable error of valuation. To the best of our knowledge, there are nearly no existing studies concerning the short time asymptotic behaviors of American options with multiple boundaries. Motivated by the work of Evans et al. [13], we extend the short time asymptotic studies from the case of single boundary to the case with double boundaries. The contribution of our paper is to study the short time asymptotic behaviors of double boundaries for American maximum options. For upper and lower boundaries of an American maximum option, we derive the behaviors of them and then derive the asymptotic expansion of the value function when the time approaches the maturity.

The American maximum option studied here holds the feature with the minimum payoff L and has the payoff function $\max(S, L)$, S is the underlying asset price satisfying geometric Brownian motion

$$\mathrm{d}S_t = S_t[(r - D)\mathrm{d}t + \sigma \mathrm{d}W_t], \tag{1.1}$$

where the constants r, D and σ are the riskless interest rate, dividend yield and volatility of the price, respectively. W_t is a standard Brown motion, which is defined in a complete probability space. According to Black-Scholes theory, American maximum option value $V(S, t)$ at time t and $S_t = S$ satisfies the following equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0, \ in\ 0 < t < T_F,\ S_a(t) < S < S_b(t), \tag{1.2}$$

where T_F is the time of expiry, $S_a(t)$ and $S_b(t)$ denote the lower and upper optimal boundaries, respectively. Different from standard American options studied by Kim [6] and Wilmott et al. [14], Eq (1.2) describes a pricing model for American maximum option with double exercise boundaries.

When the underlying asset price drops to the lower boundary $S_a(t)$ or rises to the upper boundary $S_b(t)$, the option will be exercised. By analogy, the smooth-fit conditions at two optimal exercise boundaries are given as

$$\begin{aligned} \frac{\partial V}{\partial S} &= 0,\ V = L,\ at\ S = S_a(t), \\ \frac{\partial V}{\partial S} &= 1,\ V = S,\ at\ S = S_b(t). \end{aligned} \tag{1.3}$$

At time T_F, the final value of V is the payoff,

$$V(S, T_F) = \max(S, L). \tag{1.4}$$

The limits of $S_a(t)$ and $S_b(t)$ as t tends to T_F from below can be obtained from Theorem 7.1 in Jiang [12] as

$$\begin{aligned} &S_a(t) \uparrow,\ S_a(T_F^-) = L, \\ &S_b(t) \downarrow,\ S_b(T_F^-) = L. \end{aligned} \tag{1.5}$$

To make the model and results more practical, we choose an analytical approximation method rather than a numerical method. By contrast, pricing options by approximation method is convenient and highly efficient. Evans et al. [13] proposed a wonderful way of approximating American put options, which showed great accuracy near expiry. This is also meaningful for market application. Referring to their studies, we expand their method to deal with the problem of double boundaries and obtain the following results for the optimal exercise boundaries for the American maximum option: as $T_F - t \ll 1$ or $t \to T_F$

$$S_a(t) = L - L\sigma\sqrt{(T_F - t)\ln\frac{\sigma^2}{8\pi(T_F - t)r^2}}, \tag{1.6}$$

$$S_b(t) = L + L\sigma\sqrt{(T_F - t)\ln\frac{\sigma^2}{8\pi(T_F - t)D^2}}. \tag{1.7}$$

We also derive the asymptotic expansion of the option value using matched asymptotic expansions method sketched by Bender and Orszag [15].

This paper is organized as follows. In Section 2, we use two different methods to get the representation of American maximum option value. In Section 3, we derive integral equations from a matching condition. In Section 4, we provide the solutions of the integral equations for the optimal boundaries. In Section 5, we use matched asymptotic expansions method to get the option value. In Section 6, we provide numerical examples to compare our results with exact result. Some concluding remarks are given in the end.

2. Representation of the maximum option value

In this section, we will give the representation of American maximum option value by two different methods.

To simplify the Eqs (1.2)–(1.5), we let

$$S = Le^x, \qquad \tau = \frac{\sigma^2}{2}(T_F - t), \tag{2.1}$$

$$V(S,t) = LM(x,\tau), \qquad S_a(t) = Le^{a(\tau)}, \qquad S_b(t) = Le^{b(\tau)}, \tag{2.2}$$

$$\rho = \frac{2r}{\sigma^2}, \qquad v = \frac{2D}{\sigma^2}, \qquad \lambda = \rho - v - 1. \tag{2.3}$$

Then (1.2)–(1.5) can be rewritten in the new form

$$\frac{\partial M}{\partial \tau} = \frac{\partial^2 M}{\partial x^2} + \lambda \frac{\partial M}{\partial x} - \rho M, \ in\ 0 < \tau < \frac{\sigma^2}{2} T_F, \ a(\tau) < x < b(\tau), \tag{2.4}$$

$$\begin{cases} \dfrac{\partial M}{\partial x} = 0, \ at\ x = a(\tau), \\ M = 1, \ at\ x \leq a(\tau), \end{cases} \tag{2.5}$$

$$\begin{cases} \dfrac{\partial M}{\partial x} = \mathrm{e}^x, \ at\ x = b(\tau), \\ M = \mathrm{e}^x, \ at\ x \geq b(\tau), \end{cases} \tag{2.5'}$$

$$M(x,0) = \max(1, \mathrm{e}^x), \tag{2.6}$$

$$\begin{aligned} a(\tau) \downarrow, \ a(0) = \ln \frac{S_a(T_F^-)}{L} = 0, \\ b(\tau) \uparrow, \ b(0) = \ln \frac{S_b(T_F^-)}{L} = 0. \end{aligned} \tag{2.7}$$

To solve partial differential Eqs (2.4), (2.5) and (2.5′), we use Green's theorem. By transforming (2.4) to a standard heat equation, we get Green's function $G(x,\tau)$ associated with Eq (2.4) is

$$G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \mathrm{e}^{-\rho\tau} \mathrm{e}^{-\frac{(x+\lambda\tau)^2}{4\tau}}, \ \tau > 0. \tag{2.8}$$

Then we define f and g by the following equalities

$$\begin{cases} M_\tau - M_{xx} - \lambda M_x + \rho M = f & \text{in } R \times (0, +\infty), \\ M = g & \text{on } R \times \{\tau = 0\}. \end{cases} \tag{2.9}$$

Noticing that $M(x,\tau) = 1$ for $x \leqslant a(\tau)$, $M(x,\tau) = \mathrm{e}^x$ for $x \geqslant b(\tau)$, and $M_\tau - M_{xx} - \lambda M_x + \rho M = 0$ for $a(\tau) < x < b(\tau)$, we get

$$\begin{cases} f(x,\tau) = \rho \cdot 1_{(x \leqslant a(\tau))} + v\mathrm{e}^x \cdot 1_{(x \geqslant b(\tau))}, \\ g(x) = \max\{1, \mathrm{e}^x\}, \end{cases} \tag{2.10}$$

where $1_{(\cdot)}$ is the characteristic function.

According to Green's theorem, in the domain bounded by two optimal exercise boundaries and the line $\tau = 0$, we can write

$$M(x,\tau) := I_1(x,\tau) + I_2(x,\tau). \tag{2.11}$$

with $I_1(x,\tau) = \int_{-\infty}^{+\infty} G(x-y,\tau) g(y) dy$ and $I_2(x,\tau) = \int_0^\tau \int_{-\infty}^{+\infty} G(x-y,\tau-u) f(y,u) dy du$. Further, we have that

$$\begin{aligned} I_1(x,\tau) =& e^{-\rho\tau} N\Big(-\frac{x+\lambda\tau}{\sqrt{2\tau}}\Big) + e^{x - v\tau} N\Big(\frac{x+(\lambda+2)\tau}{\sqrt{2\tau}}\Big), \\ I_2(x,\tau) =& \int_0^\tau \rho e^{-\rho(\tau-u)} N\Big(-\frac{x-a(u)+\lambda(\tau-u)}{\sqrt{2(\tau-u)}}\Big) du \\ &+ \int_0^\tau v e^{x - v(\tau-u)} N\Big(\frac{x-b(u)+(\lambda+2)(\tau-u)}{\sqrt{2(\tau-u)}}\Big) du, \end{aligned} \tag{2.12}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

In the following, we derive (2.12) by Itô calculus [16,17]. Back to $V(S,t)$, for the differential of the discounted value process, taking the integral from t to T_F and then compute the expectations of both sides, we have

$$\begin{aligned}\mathbb{E}\Big[\int_t^{T_F} d(\mathrm{e}^{-ru}V(S_u,u))\Big] &= \mathbb{E}[e^{-rT_F}V(S_{T_F},T_F) - e^{-rt}V(S_t,t)] \\ &= \mathbb{E}(\mathrm{e}^{-rT_F}\max\{S_{T_F},L\}) - e^{-rt}V(S_t,t).\end{aligned} \tag{2.13}$$

Using Itô calculus, we get

$$d(\mathrm{e}^{-rt}V(S_t,t)) = \mathrm{e}^{-rt}[(V_t + \frac{\sigma^2}{2}S_t^2V_{ss} + (r-D)SV_s - rV)dt + V_sS_t\sigma dW_t]. \tag{2.14}$$

where the simplified notations are defined by $V_s := \frac{\partial V}{\partial S}(S_t,t)$ and $V_{ss} := \frac{\partial^2 V}{\partial S^2}(S_t,t)$. Noticing that $V(S,t) = L$ for $S_t \leqslant S_a(t)$, $V(S,t) = S$ for $S \geqslant S_b(t)$, and $V_t + \frac{\sigma^2}{2}S^2V_{ss} + (r-D)SV_s - rV = 0$ for $S_a(t) < S < S_b(t)$, we get

$$\begin{aligned}\mathbb{E}\Big[\int_t^{T_F} d(e^{-ru}V(S_u,u))\Big] &= \mathbb{E}\Big[\int_t^{T_F} e^{-ru}[-rL\cdot 1_{(S_u\leqslant S_a(u))} - DS_u\cdot 1_{(S_u\geqslant S_b(u))}]du\Big] \\ &+ \mathbb{E}\Big[\int_t^{T_F} e^{-ru}\sigma S_u[V_S\cdot 1_{(S_a(u)<S_u<S_b(u))} + 1_{(S_u\geqslant S_b(u))}]dW_u\Big].\end{aligned} \tag{2.15}$$

The integral in the second term on the right side of (2.15) containing standard Brown motion is a martingale and its expectation equals to 0. The first term and the expectation on the right side of (2.13) are some integrals containing the transition probability density function

$$f(S_T|S_t) = \frac{1}{S_T\sigma\sqrt{2\pi(T-t)}}\mathrm{e}^{-\frac{[\ln\frac{S_T}{S_t} - (r-D-\frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)}}. \tag{2.16}$$

Then we get

$$\begin{aligned}V(S,t) =& S\,\mathrm{e}^{-D(T_F-t)}N(\frac{\ln\frac{S}{L} + (r-D+\frac{\sigma^2}{2})(T_F-t)}{\sigma\sqrt{T_F-t}}) \\ &+ Le^{-r(T_F-t)}N(\frac{\ln\frac{L}{S} - (r-D-\frac{\sigma^2}{2})(T_F-t)}{\sigma\sqrt{T_F-t}}) \\ &+ \int_0^{T_F-t} rLe^{-ru}N(\frac{\ln S_a(t+u) - \ln S - (r-D-\frac{\sigma^2}{2})u}{\sigma\sqrt{u}})du \\ &+ \int_0^{T_F-t} DS\,e^{-Du}N(\frac{\ln S - \ln S_b(t+u) + (r-D+\frac{\sigma^2}{2})u}{\sigma\sqrt{u}})du.\end{aligned} \tag{2.17}$$

Using variable substitution in (2.1)–(2.3), we know that (2.17) is the same as (2.12).

3. Matching condition and the integral equation

In this section, we will derive integral equations from a matching condition. To find the matching conditions, for the variable $x = x(\tau)$, we take the τ derivative of M

$$\frac{\mathrm{d}M(x,\tau)}{\mathrm{d}\tau} = \frac{\partial M}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{\partial M}{\partial \tau}. \tag{3.1}$$

Substituting M_x in (2.5) and (2.5′) into (3.1), we obtain two equations on double boundaries

$$M_\tau(a(\tau),\tau) = 0,\ M_\tau(b(\tau),\tau) = 0. \tag{3.2}$$

Mathematically, we utilize the derivative formula in (3.1) to the compound functions $M(a(\tau),\tau)$ and $M(b(\tau),\tau)$ plus the boundaries (2.5) and (2.5′) and then obtain the equalities in (3.2). Besides, the equalities in (3.2) hold from the financial meaning that American maximum option has no time value at the exercise boundaries. By (2.11) and (3.2), we get the integral equations

$$\begin{aligned} &\frac{\partial I_1}{\partial \tau}[a(\tau),\tau] + \lim_{x\to a(\tau)}\frac{\partial I_2}{\partial \tau}[x,\tau] = 0,\\ &\frac{\partial I_1}{\partial \tau}[b(\tau),\tau] + \lim_{x\to b(\tau)}\frac{\partial I_2}{\partial \tau}[x,\tau] = 0. \end{aligned} \tag{3.3}$$

By Evans et al. [13], we can write I_1 and I_2 as follows

$$\begin{aligned} I_1(x,\tau) =& \mathrm{e}^{-\rho\tau}(1-\frac{1}{2}\mathrm{erfc}(-\frac{x+\lambda\tau}{2\sqrt{\tau}}))\\ &+\mathrm{e}^{x-\nu\tau}(1-\frac{1}{2}\mathrm{erfc}(\frac{x+(\lambda+2)\tau}{2\sqrt{\tau}})), \end{aligned} \tag{3.4}$$

$$\begin{aligned} I_2(x,\tau) =& \int_0^\tau \rho e^{-\rho(\tau-u)}(1-\frac{1}{2}\mathrm{erfc}(-\frac{x-a(u)+\lambda(\tau-u)}{2\sqrt{\tau-u}}))\mathrm{d}u\\ &+\int_0^\tau \nu e^{x-\nu(\tau-u)}(1-\frac{1}{2}\mathrm{erfc}(\frac{x-b(u)+(\lambda+2)(\tau-u)}{2\sqrt{\tau-u}}))\mathrm{d}u, \end{aligned} \tag{3.5}$$

where $\mathrm{erfc}(z)$ denotes the complementary error function defined as

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-t^2}\mathrm{d}t \sim \frac{e^{-z^2}}{\sqrt{\pi}z},\ \ as\ z\to+\infty. \tag{3.6}$$

The error function $\mathrm{erfc}(z)$ holds the following properties

$$\begin{aligned} &\mathrm{erfc}'(z) = -\frac{2}{\sqrt{\pi}}\mathrm{e}^{-z^2},\\ &\mathrm{erfc}(z)\to 0\ \ as\ z\to+\infty,\ \ \mathrm{erfc}(z)\to 2\ \ as\ z\to-\infty. \end{aligned} \tag{3.7}$$

To derive the short time asymptotic behaviors of the optimal exercise boundaries, we write $a(\tau)$ and $b(\tau)$ in terms of the following unknown functions $\alpha(\tau)(>0)$ and $\beta(\tau)(>0)$ as

$$\begin{aligned} a(\tau) &= a(0) - 2\sqrt{\tau}\alpha(\tau) = -2\sqrt{\tau}\alpha(\tau),\\ b(\tau) &= b(0) + 2\sqrt{\tau}\beta(\tau) = 2\sqrt{\tau}\beta(\tau). \end{aligned} \tag{3.8}$$

Actually, we can show that $\alpha(\tau) \to +\infty$ and $\beta(\tau) \to +\infty$ as $\tau \to 0$. Because of the equation $\frac{\partial M}{\partial x} = \frac{\partial I_1}{\partial x} + \frac{\partial I_2}{\partial x}$, and $\frac{\partial I_2}{\partial x}$ being an integral from 0 to τ, it holds that $\lim\limits_{\tau\to 0^+} \frac{\partial I_2}{\partial x} = 0$. The expression of $\frac{\partial I_1}{\partial x}$ is provided as follows:

$$\frac{\partial I_1}{\partial x}(a(\tau),\tau) = e^{-\rho\tau}\frac{1}{4\sqrt{\tau}}\text{erfc}'(-\frac{a(\tau)+\lambda\tau}{2\sqrt{\tau}})+e^{a(\tau)-v\tau}[(1-\frac{1}{2}\text{erfc}(\frac{a(\tau)+(\lambda+2)\tau}{2\sqrt{\tau}}))-\frac{1}{4\sqrt{\tau}}\text{erfc}'(\frac{a(\tau)+(\lambda+2)\tau}{2\sqrt{\tau}})].$$

Using $a(0) = 0$,$a(\tau) = -2\sqrt{\tau}\alpha(\tau)$ and $\text{erfc}'(z) = -\frac{2}{\sqrt{\pi}}e^{-z^2}$ and by (2.5), we have that $\lim\limits_{\tau\to 0^+} \frac{\partial I_1}{\partial x}(a(\tau),\tau) = \lim\limits_{\tau\to 0^+}(1 - \frac{1}{2}\text{erfc}(-\alpha(\tau)) = 0$, thus we have that $\lim\limits_{\tau\to 0^+} \text{erfc}(-\alpha(\tau)) = 2$, which implies $\lim\limits_{\tau\to 0^+} \alpha(\tau) = +\infty$ by the property of the error function. With the same discussions by (2.5′) for $b(\tau)$, we can prove that $\lim\limits_{\tau\to 0^+} \beta(\tau) = +\infty$.

The following asymptotic results will also confirm that both $\alpha(\tau)$ and $\beta(\tau)$ tend to $+\infty$ as $\tau \to 0$. For I_1 in (3.4), we take the partial derivative with respect to τ. On the lower boundary where $x = a(\tau)$, by the asymptotic form in (3.6) and the properties in (3.7), we have

$$\frac{\partial I_1}{\partial \tau}(a(\tau),\tau) \sim \frac{1}{2\sqrt{\pi\tau}}e^{-\frac{a^2(\tau)}{4\tau}} - \rho(1+\frac{\sqrt{\tau}}{\sqrt{\pi}a(\tau)}e^{-\frac{a^2(\tau)}{4\tau}}) + v\frac{\sqrt{\tau}}{\sqrt{\pi}a(\tau)}e^{-\frac{a^2(\tau)}{4\tau}}, \quad as\ \tau \to 0. \tag{3.9}$$

Keeping the leading order terms in (3.9), we obtain

$$\frac{\partial I_1}{\partial \tau}(a(\tau),\tau) \sim \frac{1}{2\sqrt{\pi\tau}}e^{-\frac{a^2(\tau)}{4\tau}} - \rho, \quad as\ \tau \to 0. \tag{3.10}$$

Letting $u = \tau z$, for ease of representations, we define two notations

$$\begin{aligned} B(x,z,\tau) &= \frac{a(u)-x}{2\sqrt{\tau-u}} = \frac{-x/(2\sqrt{\tau}) - \sqrt{z}\alpha(\tau z)}{\sqrt{1-z}}, \\ F(x,z,\tau) &= \frac{b(u)-x}{2\sqrt{\tau-u}} = \frac{-x/(2\sqrt{\tau}) + \sqrt{z}\beta(\tau z)}{\sqrt{1-z}}. \end{aligned} \tag{3.11}$$

The two arguments of erfc in (3.5) can be written as

$$\begin{aligned} -\frac{x-a(u)+\lambda(\tau-u)}{2\sqrt{\tau-u}} &= B(x,z,\tau) - \frac{\lambda}{2}\sqrt{\tau}\sqrt{1-z}, \\ \frac{x-b(u)+(\lambda+2)(\tau-u)}{2\sqrt{\tau-u}} &= \frac{\lambda+2}{2}\sqrt{\tau}\sqrt{1-z} - F(x,z,\tau). \end{aligned} \tag{3.12}$$

Taking the derivative of I_2 about τ and keeping the leading order terms, we get

$$\begin{aligned} \frac{\partial I_2}{\partial \tau} \sim& \rho e^{-\rho\tau}\int_0^1 e^{\rho\tau z}(1-\frac{1}{2}\text{erfc}(B-\frac{\lambda}{2}\sqrt{\tau}\sqrt{1-z}))dz \\ &+ \tau\rho e^{-\rho\tau}\int_0^1 e^{\rho\tau z}\frac{1}{\sqrt{\pi}}e^{-(B-\frac{\lambda}{2}\sqrt{\tau}\sqrt{1-z})^2}(B_\tau - \frac{\lambda}{2}\frac{1}{2\sqrt{\tau}}\sqrt{1-z})dz \\ &+ ve^{x-v\tau}\int_0^1 e^{v\tau z}(1-\frac{1}{2}\text{erfc}(\frac{\lambda+2}{2}\sqrt{\tau}\sqrt{1-z}-F))dz \\ &+ \tau ve^{x-v\tau}\int_0^1 e^{v\tau z}\frac{1}{\sqrt{\pi}}e^{-(\frac{\lambda+2}{2}\sqrt{\tau}\sqrt{1-z}-F)^2}(\frac{\lambda+2}{2}\frac{1}{2\sqrt{\tau}}\sqrt{1-z}-F_\tau)dz. \end{aligned} \tag{3.13}$$

Letting $\tau \to 0$ and $x \to a(\tau)$, expanding the exponential functions and erfc to order $\sqrt{\tau}$, we obtain

$$\begin{aligned}\lim_{x\to a(\tau)}\frac{\partial I_2}{\partial\tau}(x,\tau) \sim \rho + \lim_{x\to a(\tau)}\Big[&\frac{\nu e^x}{2}(\int_0^1 \mathrm{erfc}F\mathrm{d}z - \tau\frac{2}{\sqrt{\pi}}\int_0^1 F_\tau e^{-F^2}\mathrm{d}z)\\ &-\frac{\rho}{2}(\int_0^1 \mathrm{erfc}B\mathrm{d}z - \tau\frac{2}{\sqrt{\pi}}\int_0^1 B_\tau e^{-B^2}\mathrm{d}z)\\ &+\nu e^x(\lambda+2)(\frac{3\sqrt{\tau}}{4\sqrt{\pi}}\int_0^1 \sqrt{1-z}e^{-F^2}\mathrm{d}z - \frac{\tau^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1 \sqrt{1-z}FF_\tau e^{-F^2}\mathrm{d}z)\\ &-\rho\lambda(\frac{3\sqrt{\tau}}{4\sqrt{\pi}}\int_0^1 \sqrt{1-z}e^{-B^2}\mathrm{d}z - \frac{\tau^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1 \sqrt{1-z}BB_\tau e^{-B^2}\mathrm{d}z)\\ &+\tau\nu e^x(\frac{\lambda+2}{2})^2\frac{1}{\sqrt{\pi}}\int_0^1(1-z)Fe^{-F^2}\mathrm{d}z - \tau\rho(\frac{\lambda}{2})^2\frac{1}{\sqrt{\pi}}\int_0^1(1-z)Be^{-B^2}\mathrm{d}z\Big].\end{aligned} \tag{3.14}$$

At the upper boundary $x = b(\tau)$, as $\tau \to 0$, we have

$$\frac{\partial I_1}{\partial\tau}(b(\tau),\tau) \sim \frac{1}{2\sqrt{\pi\tau}}e^{-\frac{b^2(\tau)}{4\tau}} - \nu, \tag{3.15}$$

$$\begin{aligned}\lim_{x\to b(\tau)}\frac{\partial I_2}{\partial\tau}(x,\tau) \sim \nu + \lim_{x\to b(\tau)}\Big[&\frac{\nu e^x}{2}(-\int_0^1 \mathrm{erfc}(-F)\mathrm{d}z - \tau\frac{2}{\sqrt{\pi}}\int_0^1 F_\tau e^{-F^2}\mathrm{d}z)\\ &-\frac{\rho}{2}(-\int_0^1 \mathrm{erfc}(-B)\mathrm{d}z - \tau\frac{2}{\sqrt{\pi}}\int_0^1 B_\tau e^{-B^2}\mathrm{d}z)\\ &+\nu e^x(\lambda+2)(\frac{3\sqrt{\tau}}{4\sqrt{\pi}}\int_0^1 \sqrt{1-z}e^{-F^2}\mathrm{d}z - \frac{\tau^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1 \sqrt{1-z}FF_\tau e^{-F^2}\mathrm{d}z)\\ &-\rho\lambda(\frac{3\sqrt{\tau}}{4\sqrt{\pi}}\int_0^1 \sqrt{1-z}e^{-B^2}\mathrm{d}z - \frac{\tau^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1 \sqrt{1-z}BB_\tau e^{-B^2}\mathrm{d}z)\\ &+\tau\nu e^x(\frac{\lambda+2}{2})^2\frac{1}{\pi}\int_0^1(1-z)Fe^{-F^2}\mathrm{d}z - \tau\rho(\frac{\lambda}{2})^2\frac{1}{\pi}\int_0^1(1-z)Be^{-B^2}\mathrm{d}z\Big].\end{aligned} \tag{3.16}$$

4. Solution of the integral equations

In this section, we will provide the solutions of the integral equations for the optimal boundaries.

We shall deal with the lower boundary at first. As $\tau \to 0$, in the following, we will prove the ten terms containing integrals on the right side of (3.14) tend to 0 except one term tends to $-\frac{\rho}{2}\times 2$.

The results (A.6) and (A.7) in Appendix A in Evans et al. [13] show that the third integral tends to 0, the fourth integral multiplied by $-2\tau/\sqrt{\pi}$ tends to 2, and the seventh integral tends to $O(\tau)$ when $\tau \to 0$. Applying the formula (A.3) to the tenth integral, as $\tau \to 0$, we get

$$\int_0^1 (1-z)Be^{-B^2}\mathrm{d}z \sim (1-z_0)B(z_0)\frac{\sqrt{\pi}}{|B_z(z_0)|}. \tag{4.1}$$

Then the tenth integral tends to 0 for $B(z_0) = 0$.

Comparing the seventh integral and the eighth integral, we find

$$\frac{d}{d\tau}(\int_0^1 \sqrt{1-z}\mathrm{e}^{-B^2}\mathrm{d}z) = -2\int_0^1 \sqrt{1-z}BB_\tau\mathrm{e}^{-B^2}\mathrm{d}z. \tag{4.2}$$

So the eighth integral is of order $O(1)$.

As $x \to a(\tau)$, $F \to \frac{\alpha(\tau)+\sqrt{z}\beta(\tau z)}{\sqrt{1-z}}$. As $\tau \to 0$, $F \to +\infty$. So the first, fifth and ninth integral tends to 0.

To evaluate the second integral in (3.14), we let

$$\tau\int_0^1 F_\tau\mathrm{e}^{-F^2}\mathrm{d}z = \int_0^1 (-\frac{\alpha(\tau)/2 - z^{\frac{3}{2}}\tau\beta'(\tau z)}{\sqrt{1-z}})\mathrm{e}^{-(\frac{\alpha(\tau)+\sqrt{z}\beta(\tau z)}{\sqrt{1-z}})^2}\mathrm{d}z. \tag{4.3}$$

Notice that $\alpha(\tau)$ and $-\beta'(\tau z)$ both tend to $+\infty$ as $\tau \to 0$. If $\alpha(\tau)$ has higher order than $-\beta'(\tau z)$, we have

$$(-\frac{1}{2\sqrt{1-z}}\alpha(\tau))\mathrm{e}^{-(\frac{\alpha(\tau)+\sqrt{z}\beta(\tau z)}{\sqrt{1-z}})^2} \to 0, \ as\ \tau \to 0. \tag{4.4}$$

If $-\beta'(\tau z)$ has higher order than $\alpha(\tau)$, by the monotonicity of upper boundary, we can produce $(\beta(\tau z)\sqrt{\tau z})' > 0$ and, for $\forall z \in (0, 1)$,

$$-z^{\frac{3}{2}}\tau\beta'(\tau z) < \frac{1}{2}\sqrt{z}\beta(\tau z). \tag{4.5}$$

Hence, for $\forall z \in (0, 1)$,

$$[-\frac{1}{\sqrt{1-z}}(-z^{\frac{3}{2}}\tau\beta'(\tau z))]\mathrm{e}^{-(\frac{\alpha(\tau)+\sqrt{z}\beta(\tau z)}{\sqrt{1-z}})^2} \to 0, \ as\ \tau \to 0. \tag{4.6}$$

So in both cases the second integral multiplied by τ tends to 0.

To evaluate the sixth integral multiplied by $\tau^{\frac{3}{2}}$, we write

$$\tau^{\frac{3}{2}}\int_0^1 \sqrt{1-z}FF_\tau\mathrm{e}^{-F^2}\mathrm{d}z = \int_0^1 (\sqrt{1-z}\sqrt{\tau}F)\tau F_\tau\mathrm{e}^{-F^2}\mathrm{d}z. \tag{4.7}$$

Following (4.3), (4.6) and the fact

$$\sqrt{\tau}F = \frac{\sqrt{\tau}\alpha(\tau) + \sqrt{\tau z}\beta(\tau z)}{\sqrt{1-z}} \to 0, \ as\ \tau \to 0. \tag{4.8}$$

we have that the sixth integral multiplied by $\tau^{\frac{3}{2}}$ tends to 0.

Through the above discussions, we obtain the results that all unknown terms on the right side of (3.14) tend to 0 except the fourth term tends to $-\frac{\rho}{2} \times 2$.

Substituting (3.10) and (3.14) into (3.3), the integral equation yields to leading order

$$\frac{1}{2\sqrt{\pi\tau}}\mathrm{e}^{-\alpha^2(\tau)} \sim \rho. \tag{4.9}$$

From (4.9), we have

$$\alpha^2(\tau) \sim \ln\frac{1}{2\rho\sqrt{\pi\tau}}. \tag{4.10}$$

For the upper boundary, the discussion on (3.16) is similar as (3.14). The second term tends to $\frac{\nu}{2} \times (-2)$ and other terms tend to 0, as $\tau \to 0$. Substituting (3.15) and (3.16) into (3.3), we obtain the integral equation and its solution

$$\frac{1}{2\sqrt{\pi\tau}} \mathrm{e}^{-\beta^2(\tau)} \sim \nu. \tag{4.11}$$

From (4.11), we have

$$\beta^2(\tau) \sim \ln \frac{1}{2\nu\sqrt{\pi\tau}}. \tag{4.12}$$

When $T_F - t \ll 1$ or $\tau \to 0$, the two boundaries $S_a(t)$ and $S_b(t)$ in (2.2) have the following asymptotic forms,

$$S_a(t) = L\mathrm{e}^{a(\tau)} \sim L + L \cdot a(\tau) \sim L - 2L\tau^{1/2}[\ln 1/2\rho\sqrt{\pi\tau}]^{1/2}. \tag{4.13}$$

$$S_b(t) = L\mathrm{e}^{b(\tau)} \sim L + L \cdot b(\tau) \sim L + 2L\tau^{1/2}[\ln 1/2\nu\sqrt{\pi\tau}]^{1/2}. \tag{4.14}$$

5. Matched asymptotic expansions for the maximum option

To derive the asymptotic expansion expression of the option value, we analyze the small time behavior of system (2.4)–(2.7). When T and X are bounded variables, let θ be an artificial small parameter and

$$\tau = \theta T, \ x = \theta^{1/2} X, \tag{5.1}$$

then (2.4) and (2.6) become

$$\frac{\partial M}{\partial T} = \frac{\partial^2 M}{\partial X^2} + \theta^{1/2}\lambda \frac{\partial M}{\partial X} - \theta\rho M, \tag{5.2}$$

$$M(x, 0) = \max(1, \mathrm{e}^{\theta^{1/2} X}). \tag{5.3}$$

Then we can get the expansion of M as

$$M(x, \tau) = 1 + \theta^{1/2} M_0(X, T) + \theta M_1(X, T) + \theta^{3/2} M_2(X, T) + O(\theta^2). \tag{5.4}$$

Matching the terms with the same order of θ on two sides of (5.2) and (5.3), we get three partial differential equations for M_0, M_1 and M_2 as follows.

Problem 1. M_0 satisfies a PDE problem with initial conditions

$$\frac{\partial M_0}{\partial T} = \frac{\partial^2 M_0}{\partial X^2}, \ \ in \ -\infty < X < +\infty, \ T > 0,$$

$$M_0(X, 0) = \max(X, 0); \ \text{ as } X \to -\infty, \ M_0 \to 0; \ \text{as } X \to +\infty, \ M_0 \sim X.$$

Using Green's formula, it admits the following representation

$$M_0(X, T) = \int_0^{+\infty} \frac{1}{\sqrt{4\pi T}} \mathrm{e}^{-\frac{(X-\xi)^2}{4T}} \xi \mathrm{d}\xi. \tag{5.5}$$

By direct computations, we can obtain its expression as

$$M_0(X, T) = \frac{\sqrt{T}}{\sqrt{\pi}} e^{-(\frac{X}{2\sqrt{T}})^2} + \frac{X}{2} \mathrm{erfc}(-\frac{X}{2\sqrt{T}}). \tag{5.6}$$

Setting $\zeta = \frac{X}{2\sqrt{T}}$ and

$$h_0(\zeta) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-\zeta^2} + \zeta\mathrm{erfc}(-\zeta), \tag{5.7}$$

we get $M_0(X,T) = \sqrt{T}h_0(\zeta)$.

Problem 2. M_1 satisfies a PDE problem with initial conditions

$$\frac{\partial M_1}{\partial T} = \frac{\partial^2 M_1}{\partial X^2} + \lambda\frac{\partial M_0}{\partial X} - \rho, \ \ in \ -\infty < X < +\infty, \ T > 0,$$

$$M_1(X,0) = \begin{cases} \frac{1}{2}X^2, & X \geqslant 0, \\ 0, & X < 0; \end{cases} \ \text{as } X \to -\infty, \ \frac{\partial M_1}{\partial X} \to 0; \ \text{as } X \to +\infty, \ M_1 \sim \frac{X^2}{2}.$$

Since $\frac{\partial M_0}{\partial X} = \frac{1}{2}\mathrm{erfc}(-\frac{X}{2\sqrt{T}})$, by Green's formula, $M_1(X,T)$ has the following solution

$$\begin{aligned} M_1(X,T) = &\int_0^{+\infty} \frac{1}{\sqrt{4\pi T}}\mathrm{e}^{-\frac{(X-\xi)^2}{4T}}\frac{\xi^2}{2}d\xi \\ &+ \int_0^T\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(T-s)}}\mathrm{e}^{-\frac{(X-\xi)^2}{4(T-s)}}\Big(\frac{\lambda}{2}\mathrm{erfc}(-\frac{\xi}{2\sqrt{s}}) - \rho\Big)d\xi ds. \end{aligned} \tag{5.8}$$

The first integral in (5.8) can be computed as $\frac{X^2}{4}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) + \frac{T}{\sqrt{\pi}}\frac{X}{2\sqrt{T}}e^{-\frac{X^2}{4T}} + \frac{T}{2}\mathrm{erfc}(-\frac{X}{2\sqrt{T}})$. Let $y = \frac{X-\xi}{2\sqrt{T-s}}$, we can compute the second integral in (5.8) as

$$\begin{aligned} &\int_0^T\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(T-s)}}\mathrm{e}^{-\frac{(X-\xi)^2}{4(T-s)}}\Big(\frac{\lambda}{2}\mathrm{erfc}(-\frac{\xi}{2\sqrt{s}}) - \rho\Big)d\xi ds \\ = &\int_0^T\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}}\mathrm{e}^{-y^2}\Big(\frac{\lambda}{2}\mathrm{erfc}(\sqrt{\frac{T-s}{s}}y - \frac{X}{2\sqrt{s}}) - \rho\Big)dyds. \end{aligned} \tag{5.9}$$

Setting parameters $a = \sqrt{\frac{T-s}{s}}$ and $b = -\frac{X}{2\sqrt{s}}$, the inner integral in (5.9) containing the error function becomes

$$f(a,b) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}}\mathrm{e}^{-y^2}\mathrm{erfc}(ay+b)\mathrm{d}y. \tag{5.10}$$

Taking the derivative of f with respect to b, we get

$$\begin{aligned} \frac{\partial f}{\partial b} &= -\int_{-\infty}^{+\infty} \frac{2}{\pi}e^{-y^2}e^{-(ay+b)^2}dy \\ &= -\frac{2}{\pi}e^{-\frac{b^2}{1+a^2}}\int_{-\infty}^{+\infty} e^{-(\sqrt{1+a^2}y + \frac{ab}{\sqrt{1+a^2}})^2}dy \\ &= -\frac{2}{\sqrt{\pi}}e^{-\frac{b^2}{1+a^2}}\frac{1}{\sqrt{1+a^2}} \\ &= \frac{\partial\mathrm{erfc}(\frac{b}{\sqrt{1+a^2}})}{\partial b}, \end{aligned} \tag{5.11}$$

where the third equality comes from the fact that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. Thus, we deduce that

$$f(a,b) = \text{erfc}(\frac{b}{\sqrt{1+a^2}}) + g(a). \tag{5.12}$$

Since

$$\begin{cases} g'(a) = f_a(a,0) = -\int_{-\infty}^{+\infty} \frac{2}{\pi} y\text{e}^{-(1+a^2)y^2}\text{d}y = 0, \\ g(0) = f(0,0) - \text{erfc}(0) = 0. \end{cases} \tag{5.13}$$

So $g(a) = 0$ and

$$f(a,b) = \text{erfc}(\frac{b}{\sqrt{1+a^2}}). \tag{5.14}$$

With the formula (5.14), we can compute the integral containing error function in (5.9) as

$$\begin{aligned} &\frac{\lambda}{2}\int_0^T \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}}\text{e}^{-y^2}\text{erfc}(-\sqrt{\frac{T-s}{s}}y - \frac{X}{2\sqrt{s}})dyds \\ =& \frac{\lambda}{2}\int_0^T \text{erfc}(-\frac{\frac{X}{2\sqrt{s}}}{\sqrt{1+\frac{T-s}{s}}})ds = \frac{\lambda}{2}T\text{erfc}(-\frac{X}{2\sqrt{T}}). \end{aligned}$$

Thus, we have the expression of M_1 in (5.8) as

$$M_1(X,T) = \frac{X^2}{4}\text{erfc}(-\frac{X}{2\sqrt{T}}) + \frac{T}{\sqrt{\pi}}\frac{X}{2\sqrt{T}}e^{-\frac{X^2}{4T}} + \frac{T}{2}\text{erfc}(-\frac{X}{2\sqrt{T}}) + \frac{\lambda}{2}T\text{erfc}(-\frac{X}{2\sqrt{T}}) - \rho T.$$

Set $\zeta = \frac{X}{2\sqrt{T}}$, we get $M_1(X,T) = Th_1(\zeta)$ with

$$h_1(\zeta) = \zeta^2\text{erfc}(-\zeta) + \frac{1}{\sqrt{\pi}}\zeta\text{e}^{-\zeta^2} + \frac{\lambda+1}{2}\text{erfc}(-\zeta) - \rho \tag{5.15}$$

and h_1 satisfies

$$\begin{cases} h_1'' + 2\zeta h_1' - 4h_1 = -2\lambda h_0' + 4\rho, \ \text{in} \ -\infty < \zeta < \infty; \\ \text{as } \zeta \to -\infty, \ h_1 \to -\rho, \ h_1' \to 0; \quad \text{as } \zeta \to +\infty, \ h_1 \sim 2\zeta^2 - v. \end{cases}$$

Problem 3. M_2 satisfies a PDE problem with initial conditions

$$\frac{\partial M_2}{\partial T} = \frac{\partial^2 M_2}{\partial X^2} + \lambda\frac{\partial M_1}{\partial X} - \rho M_0, \ \ in \ -\infty < X < +\infty, \ T > 0,$$

$$M_2(X,0) = \max(\frac{1}{6}X^3, 0); \ \text{ as } X \to -\infty, \ \frac{\partial M_2}{\partial X} \to 0; \ \text{as } X \to +\infty, \ M_2 \sim \frac{X^3}{6} - vXT.$$

From $\lambda\frac{\partial M_1}{\partial X} - \rho M_0 = \sqrt{\frac{T}{\pi}}(\frac{\lambda(\lambda+2)}{2} - \rho)\text{e}^{-\frac{X^2}{4T}} - (v+1)\frac{X}{2}\text{erfc}(-\frac{X}{2\sqrt{T}})$, using Green's formula, we have

$$\begin{aligned} M_2(X,T) =& \int_0^{+\infty} \frac{1}{\sqrt{4\pi T}}\text{e}^{-\frac{(X-\xi)^2}{4T}}\frac{\xi^3}{6}\text{d}\xi \\ &+ \int_0^T \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(T-s)}}\text{e}^{-\frac{(X-\xi)^2}{4(T-s)}}\Big[\sqrt{\frac{s}{\pi}}(\frac{\lambda(\lambda+2)}{2} - \rho)\text{e}^{-\frac{\xi^2}{4s}} - (v+1)\frac{\xi}{2}\text{erfc}(-\frac{\xi}{2\sqrt{s}})\Big]d\xi ds. \end{aligned} \tag{5.16}$$

For the first integral in (5.16), direct computations imply

$$
\begin{aligned}
&\int_0^{+\infty} \frac{1}{\sqrt{4\pi T}} \mathrm{e}^{-\frac{(X-\xi)^2}{4T}} \frac{\xi^3}{6} d\xi \\
&= \frac{X^3}{12}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) + \frac{XT}{2}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) + \frac{2}{3}\frac{T\sqrt{T}}{\sqrt{\pi}}(\frac{X}{2\sqrt{T}})^2 e^{-(\frac{X}{2\sqrt{T}})^2} + \frac{2}{3}\frac{T\sqrt{T}}{\sqrt{\pi}} e^{-(\frac{X}{2\sqrt{T}})^2}.
\end{aligned}
$$

Let $y = \frac{X-\xi}{2\sqrt{T-s}}$, we compute the second integral of (5.16) as

$$
\begin{aligned}
&\int_0^T \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(T-s)}} e^{-\frac{(X-\xi)^2}{4(T-s)}} \sqrt{\frac{s}{\pi}}\Big(\frac{\lambda(\lambda+2)}{2} - \rho\Big) e^{-\frac{\xi^2}{4s}} d\xi ds \\
&= \int_0^T \Big(\frac{\lambda(\lambda+2)}{2} - \rho\Big)\sqrt{\frac{s}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} e^{-\frac{(X-2\sqrt{T-s}y)^2}{4s}} dyds \\
&= \int_0^T \Big(\frac{\lambda(\lambda+2)}{2} - \rho\Big)\sqrt{\frac{s}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} e^{-(\frac{X}{2\sqrt{s}} - \sqrt{\frac{T-s}{s}}y)^2} dyds \\
&= \int_0^T \Big(\frac{\lambda(\lambda+2)}{2} - \rho\Big)\sqrt{\frac{s}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-\frac{X^2}{4T}} e^{-(\sqrt{\frac{T}{s}}y - \frac{X}{2\sqrt{s}}\sqrt{\frac{T-s}{T}})^2} dyds \\
&= \int_0^T (\frac{\lambda(\lambda+2)}{2} - \rho)\sqrt{\frac{s}{\pi}}\sqrt{\frac{s}{T}} e^{-\frac{X^2}{4T}} ds \\
&= \Big(\frac{\lambda(\lambda+2)}{2} - \rho\Big)\frac{T^{\frac{3}{2}}}{2\sqrt{\pi}} e^{-\frac{X^2}{4T}}
\end{aligned}
$$

and

$$
\begin{aligned}
&-(v+1)\int_0^T \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(T-s)}} e^{-\frac{(X-\xi)^2}{4(T-s)}} \frac{\xi}{2}\mathrm{erfc}(-\frac{\xi}{2\sqrt{s}}) d\xi ds \\
&= -(v+1)\int_0^T \int_{-\infty}^{+\infty} \frac{\sqrt{s}}{\sqrt{\pi}} e^{-y^2}\Big(\frac{X}{2\sqrt{s}} - \sqrt{\frac{T-s}{s}}y\Big)\mathrm{erfc}(-\frac{X}{2\sqrt{s}} + \sqrt{\frac{T-s}{s}}y) dyds \\
&= -(v+1)\int_0^T \sqrt{s}\Big[\frac{X}{2\sqrt{s}}\mathrm{erfc}(-\frac{\frac{X}{2\sqrt{s}}}{\sqrt{1+\frac{T-s}{s}}}) - \frac{T-s}{s\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2} e^{-(-\frac{X}{2\sqrt{s}} + \sqrt{\frac{T-s}{s}}y)^2} dy\Big] ds \\
&= -(v+1)\int_0^T \sqrt{s}\Big[\frac{X}{2\sqrt{s}}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) - \frac{T-s}{s\sqrt{\pi}}\frac{\sqrt{s}}{\sqrt{T}} e^{-\frac{X^2}{4T}}\Big] ds \\
&= -T^{\frac{3}{2}}(v+1)\Big(\frac{X}{2\sqrt{T}}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) - \frac{1}{2}\frac{1}{\sqrt{\pi}} e^{-\frac{X^2}{4T}}\Big),
\end{aligned}
$$

where the first equality comes from variable change, the second equality comes from (5.14) and integration by parts, the third equality comes from

$$
\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}(b+ay)\mathrm{erfc}(b+ay)dy = b\mathrm{erfc}(\frac{b}{\sqrt{a^2+1}}) - \frac{a^2}{\sqrt{\pi}}\frac{e^{-\frac{b^2}{a^2+1}}}{\sqrt{a^2+1}}.
$$

Thus, we obtain the expression of M_2 as

$$\begin{aligned} M_2(X,T) = &\frac{X^3}{12}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) + \frac{XT}{2}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) + \frac{2}{3}\frac{T^{\frac{3}{2}}}{\sqrt{\pi}}(\frac{X}{2\sqrt{T}})^2 e^{-(\frac{X}{2\sqrt{T}})^2} + \frac{2}{3}\frac{T^{\frac{3}{2}}}{\sqrt{\pi}}e^{-(\frac{X}{2\sqrt{T}})^2} \\ &+\Big(\frac{\lambda(\lambda+2)}{2} - \rho\Big)\frac{T^{\frac{3}{2}}}{2\sqrt{\pi}}e^{-\frac{X^2}{4T}} - T^{\frac{3}{2}}(\nu+1)\Big(\frac{X}{2\sqrt{T}}\mathrm{erfc}(-\frac{X}{2\sqrt{T}}) - \frac{1}{2}\frac{1}{\sqrt{\pi}}e^{-\frac{X^2}{4T}}\Big). \end{aligned} \tag{5.17}$$

Using (5.14) and (5.17), we can write $M_2(X,T) = T^{\frac{3}{2}}h_2(\zeta)$ with

$$h_2(\zeta) = \frac{2}{3}\zeta^3\mathrm{erfc}(-\zeta) + \frac{2}{3}\frac{1}{\sqrt{\pi}}\zeta^2\mathrm{e}^{-\zeta^2} - \nu\zeta\mathrm{erfc}(-\zeta) + (\frac{2}{3} + \frac{\lambda^2}{4})\frac{1}{\sqrt{\pi}}e^{-\zeta^2} \tag{5.18}$$

and h_2 statisfies

$$\begin{cases} h_2'' + 2\zeta h_2' - 6h_2 = -2\lambda h_1' + 4\rho h_0, \ \text{in } -\infty < \zeta < \infty; \\ \text{as } \zeta \to -\infty, \ h_2 \to 0, \ h_2' \to 0; \quad \text{as } \zeta \to +\infty, \ h_2 \sim \frac{4}{3}\zeta^3 - 2\nu\zeta. \end{cases}$$

Thus we obtain the asymptotic expansion of the option value as

$$M(x,\tau) \sim \tau^{1/2}h_0(\frac{x}{2\sqrt{\tau}}) + \tau h_1(\frac{x}{2\sqrt{\tau}}) + \tau^{3/2}h_2(\frac{x}{2\sqrt{\tau}}) + O(\tau^2), \quad x = O(\sqrt{\tau}). \tag{5.19}$$

In addition, the following asymptotic forms of h_0, h_1 and h_2 is needed for matching h_τ near the boundaries,

$$\begin{aligned} &h_0(\zeta) \sim \frac{1}{2\sqrt{\pi}}\frac{1}{\zeta^2}e^{-\zeta^2} + O(\zeta^{-4}e^{-\zeta^2}), \quad \text{as } \zeta \to -\infty; \ h_0(\zeta) \sim 2\zeta + O(\zeta^{-2}e^{-\zeta^2}), \ \text{as } \zeta \to +\infty; \\ &h_1(\zeta) \sim -\rho + O(\zeta^{-1}e^{-\zeta^2}), \quad \text{as } \zeta \to -\infty; \ h_1(\zeta) \sim 2\zeta^2 - \nu + O(\zeta^{-1}e^{-\zeta^2}) \ \text{as } \zeta \to +\infty; \\ &h_2(\zeta) \sim -\frac{2}{3\sqrt{\pi}}\zeta^2 e^{-\zeta^2} + \frac{1}{\sqrt{\pi}}(\frac{\lambda^2}{4} - \frac{1}{3})e^{-\zeta^2} + O(\zeta^{-2}e^{-\zeta^2}), \quad \text{as } \zeta \to -\infty; \\ &h_2(\zeta) \sim \frac{4}{3}\zeta^3 - 2\nu\zeta + O(\zeta^{-1}e^{-\zeta^2}), \ \text{as } \zeta \to +\infty. \end{aligned}$$

When approaching the boundaries $x = a(\tau) + O(\tau)$ and $x = b(\tau) + O(\tau)$ respectively, and leaving the higher order, we can use the above asymptotic forms of h_0, h_1, h_2 to obtain the equations

$$\frac{1}{2\sqrt{\pi}\sqrt{\tau}}e^{-\alpha^2(\tau)} - \rho \sim 0 \ \text{ and } \ \frac{1}{2\sqrt{\pi}\sqrt{\tau}}e^{-\beta^2(\tau)} - \nu \sim 0,$$

which correspond to the Eqs (4.9) and (4.11).

6. Numerical examples

In this section, we study some numerical examples with the parameters $S_0 = 50$, $L = 49$, $r = 0.08$, $q = 0.02$ and $\sigma = 0.2$ We use difference method and asymptotic expansion approach to compute the prices of American maximum options. For comparison, we consider the options with two different maturities $\tau = 0.05$ and $\tau = 0.15$, respectively. For $\tau = 0.15$, the exact upper and lower boundaries are computed as 56.31 and 44.28, the asymptotic upper and lower boundaries are computed as 55.87 and 46.30. For $\tau = 0.05$, the exact upper and lower boundaries are computed as 53.88 and 45.71, the asymptotic upper and lower boundaries are computed as 53.58 and 46.22. The following Figure 1 shows the results for two methods.

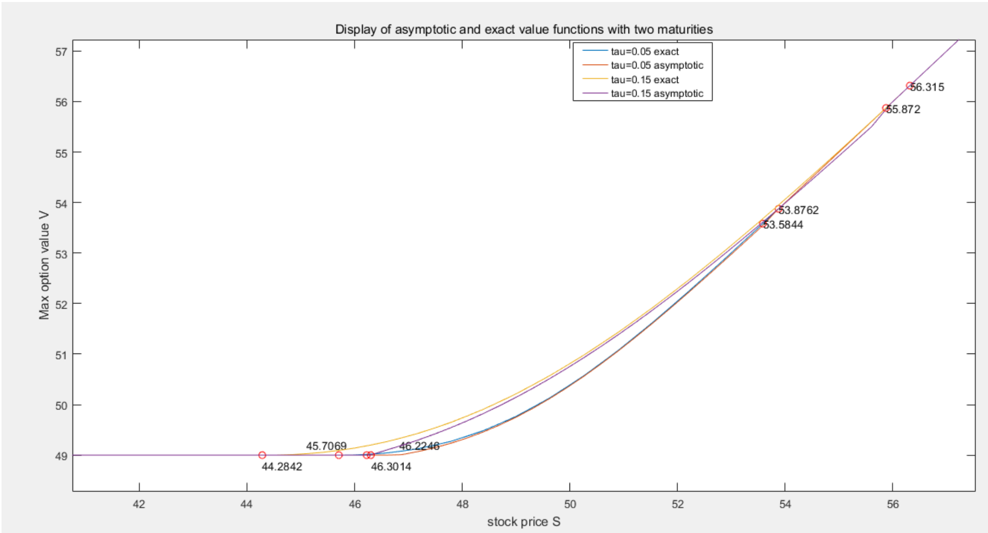

Figure 1. Maximum options' exact and asymptotic values with two maturities.

From the above figure, it is clear that our results based on the derived asymptotic formula have obtained obvious effect.

7. Conclusions

In this paper, we study American maximum options with dividend near expiry. Different from single boundary considered in vanilla American put and call options, we consider option with two boundaries. Firstly, we give the asymptotic expressions of the boundaries, and find that the lower boundary is related to the interest rate and the upper boundary is related to the dividend yield. Then, by asymptotic expansions, we give asymptotic formula for the value of American maximum option with short maturity. The analytic asymptotic formulas provide more efficient and more accurate features at the time near expiry. Furthermore, the formulas obtained in our paper can be extended to the option with multiple boundaries, so it is expected that our methods have feasibility and practicality in dealing to the option with multiple boundaries in real financial market.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.*, **81** (1973), 637–654. http://doi.org/10.1086/260062
2. R. C. Merton, Theory of rational option pricing, *Bell J. Econ. Manag. Sci.*, **4** (1973), 141–183. http://doi.org/10.2307/3003143
3. R. Roll, An analytic valuation formula for unprotected American call options on stocks with known dividends, *J. Financ. Econ.*, **5** (1977), 251–258. http://doi.org/10.1016/0304-405X(77)90021-6
4. R. Geske, A note on an analytical valuation formula for unprotected American options on stocks with known dividends, *J. Financ. Econ.*, **7** (1979), 375–380. http://doi.org/10.1016/0304-405X(79)90004-7
5. R. E. Whaley, On the valuation of American call options on stocks with known dividends, *J. Financ. Econ.*, **9** (1981), 207–211. http://doi.org/10.1016/0304-405X(81)90013-1
6. I. J. Kim, The analytical valuation of American options, *Rev. Financ. Stud.*, **3** (1990), 547–572. http://doi.org/10.1093/rfs/3.4.547
7. S. D. Jacka, Optimal stopping and the Ameican put, *Math. Financ.*, **1** (1991), 1–14. http://doi.org/10.1111/j.1467-9965.1991.tb00007.x
8. P. Carr, R. Jarrow, R. Myneni, Alternative characterization of American puts, *Math. Financ.*, **2** (1992), 87–106. http://doi.org/10.1111/j.1467-9965.1992.tb00040.x
9. J. Goodman, D. N. Ostrov, On the early exercise boundary of the American put Option, *SIAM J. Appl. Math.*, **62** (2002), 1823–1835. https://doi.org/10.1137/S0036139900378293
10. X. Chen, J. Chadam, Analytical and numerical approximations for the early exercise boundary for American put option, *Dynam. Cont. Dis. Ser. A*, **10** (2003), 649–660.
11. G. Barone-Adesi, The saga of the American put, *J. Bank. Financ.*, **29** (2005), 2909–2918. http://doi.org/10.1016/j.jbankfin.2005.02.001
12. L. S. Jiang, Analysis of pricing American options on the maximum(minimum) of two risk assets, *Interface. Free Bound.*, **4** (2002), 27–46. http://doi.org/10.4171/IFB/51
13. J. D. Evans, R. Kuske, J. B. Keller, American options on assets with dividends near expiry, *Math. Financ.*, **12** (2002), 219–237. http://doi.org/10.1111/1467-9965.02008
14. P. Wilmott, J. Dewynne, S. Howison, *Option pricing: Mathematical models and computation*, Oxford: Oxford Financial Press, 1993.
15. C. M. Bender, S. A. Orszag, *Advanced mathematical methods for scientists and engineers*, New York: McGrawHill, 1978.
16. K. Itô, Stochastic integral, *Proc. Imp. Acad.*, **20** (1944), 519–524. http://doi.org/10.3792/pia/1195572786
17. K. Itô, On stochastic differential equations, *Mem. Am. Math. Soc.*, **4** (1951), 1–51. http://doi.org/10.1090/memo/0004
18. R. Kuske, J. B. Keller, Optimal exercise boundary for an American put option, *Appl. Math. Financ.*, **5** (1998), 107–116. http://doi.org/10.1080/135048698334673

19. S. D. Jacka, A. Mijatovi, On the policy improvement algorithm in continuous time, *Stochastics*, **89** (2016), 348–359. http://doi.org/10.1080/17442508.2016.1187609

20. E. Magirou, P. Vassalos, N. Barakitis, A policy iteration algorithm for the American put option and free boundary control problems, *J. Comput. Appl. Math.*, **373** (2020), 1–14. http://doi.org/10.1016/j.cam.2019.112544

21. T. Zaevski, A new approach for pricing discounted American options, *Commun. Nonlinear Sci.*, **97** (2021), 1–19. http://doi.org/10.1016/j.cnsns.2021.105752

22. X. He, W. Chen, Pricing foreign exchange options under a hybrid Heston-Cox-Ingersoll-Ross model with regime switching, *IMA J. Manag. Math.*, **33** (2022), 255–272. http://doi.org/10.1093/imaman/dpab013

23. X. He, S. Lin, An analytical approximation formula for barrier option prices under the Heston model, *Comput. Econ.*, 2021. http://doi.org/10.1007/s10614-021-10186-7

24. X. He, W. Chen, A closed-form pricing formula for European options under a new stochastic volatility model with a stochastic long-term mean, *Math. Financ. Econ.*, **15** (2021), 381–396. http://doi.org/10.1007/s11579-020-00281-y