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Research article

Non-global nonlinear skew Lie triple derivations on factor von Neumann algebras

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Abstract: Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space H with dim $\mathcal{A} > 1$. We prove that if a map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies $\delta([[A, B]_*, C]_*) = [[\delta(A), B]_*, C]_* + [[A, \delta(B)]_*, C]_* + [[A, B]_*, \delta(C)]_*$ for any $A, B, C \in \mathcal{A}$ with $A^*B^*C = 0$, then δ is an additive $*$-derivation.

Keywords: non-global nonlinear skew Lie triple derivation; $*$-derivation; factor von Neumann algebra
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1. Introduction

Let $\mathcal{A}$ be an associative algebra. For $A, B \in \mathcal{A}$, denote by $[A, B] = AB - BA$ the Lie product of A and B. An additive (a linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is called a global Lie triple derivation if $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for all $A, B, C \in \mathcal{A}$. The study of global Lie triple derivations on various algebras has attracted several authors' attention, see for example [2, 11, 16, 17, 20]. Next, let $\delta : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity (linearity) assumption). δ is called a global nonlinear Lie triple derivation if δ satisfies $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for all $A, B, C \in \mathcal{A}$. Ji, Liu and Zhao [4] gave the concrete form of global nonlinear Lie triple derivations on triangular algebras. Chen and Xiao [3] investigated global nonlinear Lie triple derivations on parabolic subalgebras of finite-dimensional simple Lie algebras. Very recently, Zhao and Hao [21] paid attention to non-global nonlinear Lie triple derivations. Let $F : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a map and Q be a proper subset of $\mathcal{A}$. δ is called a non-global nonlinear Lie triple derivation if δ satisfies $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for any $A, B, C \in \mathcal{A}$ with $F(A, B, C) \in Q$. Let $\mathcal{M}$ be a finite von Neumann algebra with no central summands of type I_1. Zhao and Hao [21] proved that if $\delta : \mathcal{M} \to \mathcal{M}$ satisfies $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for any $A, B, C \in \mathcal{M}$ with $ABC = 0$, then $\delta = d + \tau$, where d is a derivation from $\mathcal{M}$ into itself and τ is a nonlinear map from $\mathcal{M}$ into its center such that

$\tau([[A,B],C]) = 0$ with $ABC = 0$.

Let $\mathcal{A}$ be an associative $*$-algebra. For $A, B \in \mathcal{A}$, denote by $[A, B]_* = AB - BA^*$ the skew Lie product of A and B. The skew Lie product arose in representability of quadratic functionals by sesquilinear functionals [12, 13]. In recent years, the study related to skew Lie product has attracted some authors' attention, see for example [1, 5–10, 14, 15, 18, 19, 22] and references therein. A map $\delta : \mathcal{A} \to \mathcal{A}$ (without the additivity (linearity) assumption) is called a global nonlinear skew Lie triple derivation if $\delta([[A, B]_*, C]_*) = [[\delta(A), B]_*, C]_* + [[A, \delta(B)]_*, C]_* + [[A, B]_*, \delta(C)]_*$ for all $A, B, C \in \mathcal{A}$. A map $\delta : \mathcal{A} \to \mathcal{A}$ is called an additive $*$-derivation if it is an additive derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$. Li, Zhao and Chen [5] proved that every global nonlinear skew Lie triple derivation on factor von Neumann algebras is an additive $*$-derivation. Taghavi, Nouri and Darvish [15] proved that every global nonlinear skew Lie triple derivation on prime $*$-algebras is additive. Similarly, let $F : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a map and Q be a proper subset of $\mathcal{A}$. If δ satisfies $\delta([[A, B]_*, C]_*) = [[\delta(A), B]_*, C]_* + [[A, \delta(B)]_*, C]_* + [[A, B]_*, \delta(C)]_*$ for any $A, B, C \in \mathcal{A}$ with $F(A, B, C) \in Q$, then δ is called a non-global nonlinear skew Lie triple derivation.

Motivated by the mentioned works, we will concentrate on characterizing a kind of non-global nonlinear skew Lie triple derivations δ on factor von Neumann algebras satisfying $\delta([[A, B]_*, C]_*) = [[\delta(A), B]_*, C]_* + [[A, \delta(B)]_*, C]_* + [[A, B]_*, \delta(C)]_*$ for any $A, B, C \in \mathcal{A}$ with $A^*B^*C = 0$.

As usual, $\mathbb{C}$ denotes the complex number field. Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. Let $\mathcal{A} \subseteq B(H)$ be a factor von Neumann algebra (i.e., the center of $\mathcal{A}$ is $\mathbb{C}I$, where I is the identity of $\mathcal{A}$). Recall that $\mathcal{A}$ is prime (i.e., for any $A, B \in \mathcal{A}$, $A\mathcal{A}B = \{0\}$ implies $A = 0$ or $B = 0$).

2. Main result

The main result is the following theorem.

Theorem 2.1. Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space H with $\dim\mathcal{A} > 1$. If a map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies

$$\delta([[A, B]_*, C]_*) = [[\delta(A), B]_*, C]_* + [[A, \delta(B)]_*, C]_* + [[A, B]_*, \delta(C)]_*$$

for any $A, B, C \in \mathcal{A}$ with $A^*B^*C = 0$, then δ is an additive $*$-derivation.

Let $P_1 \in \mathcal{A}$ be a nontrivial projection. Write $P_2 = I - P_1$, $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ $(i, j = 1, 2)$. Then $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. For any $A \in \mathcal{A}$, $A = A_{11} + A_{12} + A_{21} + A_{22}$, $A_{ij} \in \mathcal{A}_{ij}$ $(i, j = 1, 2)$.

Lemma 2.1. (a) $\delta(P_i)^* = \delta(P_i)$ $(i = 1, 2)$;

(b) $P_i\delta(P_i)P_j = -P_i\delta(P_j)P_j$ $(1 \le i \ne j \le 2)$.

Proof. (a) It is clear that $\delta(0) = 0$. For any $X_{21} \in \mathcal{A}_{21}$, it follows from $P_1^*P_1^*X_{21} = 0$ and $[[P_1, P_1]_*, X_{21}]_* = 0$ that

$$\begin{aligned} 0 =&\delta([[P_1, P_1]_*, X_{21}]_*)\\ =&[[\delta(P_1), P_1]_*, X_{21}]_* + [[P_1, \delta(P_1)]_*, X_{21}]_* + [[P_1, P_1]_*, \delta(X_{21})]_*\\ =&- P_1\delta(P_1)^*X_{21} - X_{21}\delta(P_1)^* + X_{21}\delta(P_1)P_1 + P_1\delta(P_1)X_{21} - X_{21}\delta(P_1)^*P_1 + X_{21}\delta(P_1)^*. \end{aligned} \tag{2.1}$$

Multiplying (2.1) by P_2 from the left and by P_1 from the right, we have $X_{21}(\delta(P_1)P_1 - \delta(P_1)^*P_1) = 0$. Then by the primeness of $\mathcal{A}$, we get

$$P_1\delta(P_1)^*P_1 = P_1\delta(P_1)P_1. \tag{2.2}$$

By $P_1^*P_2^*P_2 = 0$ and $[[P_1, P_2]_*, P_2]_* = 0$, we have

$$\begin{aligned} 0 &= \delta([[P_1, P_2]_*, P_2]_*) \\ &= [[\delta(P_1), P_2]_*, P_2]_* + [[P_1, \delta(P_2)]_*, P_2]_* + [[P_1, P_2]_*, \delta(P_2)]_* \\ &= \delta(P_1)P_2 - P_2\delta(P_1)^*P_2 - P_2\delta(P_1)^* + P_2\delta(P_1)P_2 + P_1\delta(P_2)P_2 - P_2\delta(P_2)^*P_1. \end{aligned} \tag{2.3}$$

Multiplying (2.3) by P_2 from both sides, we see that

$$P_2\delta(P_1)^*P_2 = P_2\delta(P_1)P_2. \tag{2.4}$$

From $P_1^*P_1^*P_2 = 0$ and $[[P_1, P_1]_*, P_2]_* = 0$, we have

$$\begin{aligned} 0 &= \delta([[P_1, P_1]_*, P_2]_*) \\ &= [[\delta(P_1), P_1]_*, P_2]_* + [[P_1, \delta(P_1)]_*, P_2]_* + [[P_1, P_1]_*, \delta(P_2)]_* \\ &= -P_1\delta(P_1)^*P_2 + P_2\delta(P_1)P_1 + P_1\delta(P_1)P_2 - P_2\delta(P_1)^*P_1. \end{aligned} \tag{2.5}$$

Multiplying (2.5) by P_1 from the left and by P_2 from the right, then

$$P_1\delta(P_1)^*P_2 = P_1\delta(P_1)P_2. \tag{2.6}$$

Multiplying (2.5) by P_2 from the left and by P_1 from the right, then

$$P_2\delta(P_1)^*P_1 = P_2\delta(P_1)P_1. \tag{2.7}$$

It follows from (2.2), (2.4), (2.6) and (2.7) that $\delta(P_1)^* = \delta(P_1)$. Similarly, we can obtain that $\delta(P_2)^* = \delta(P_2)$.

(b) From $P_2^*P_1^*P_2 = 0$ and $[[P_2, P_1]_*, P_2]_* = 0$, we have

$$\begin{aligned} 0 &= \delta([[P_2, P_1]_*, P_2]_*) \\ &= [[\delta(P_2), P_1]_*, P_2]_* + [[P_2, \delta(P_1)]_*, P_2]_* + [[P_2, P_1]_*, \delta(P_2)]_* \\ &= -P_1\delta(P_2)^*P_2 + P_2\delta(P_2)P_1 + P_2\delta(P_1)P_2 - \delta(P_1)P_2 - P_2\delta(P_1)^*P_2 + P_2\delta(P_1)^*. \end{aligned} \tag{2.8}$$

Multiplying (2.8) by P_1 from the left and by P_2 from the right, we have $P_1\delta(P_1)P_2 = -P_1\delta(P_2)^*P_2$. Then $P_1\delta(P_1)P_2 = -P_1\delta(P_2)P_2$ by (a). Similarly, we can obtain that $P_2\delta(P_2)P_1 = -P_2\delta(P_1)P_1$.

Lemma 2.2. *For any $A_{ij} \in \mathcal{A}_{ij}$ $(1 \leq i \neq j \leq 2)$, we have*

$$P_j\delta(A_{ij})P_i = 0.$$

Proof. Let $A_{12} \in \mathcal{A}_{12}$. For any $X_{12} \in \mathcal{A}_{12}$, since $A_{12}^*X_{12}^*P_2 = 0$ and $[[A_{12}, X_{12}]_*, P_2]_* = 0$, we have

$$0 = \delta([[A_{12}, X_{12}]_*, P_2]_*)$$

$$
\begin{aligned}
=&[[\delta(A_{12}), X_{12}]_*, P_2]_* + [[A_{12}, \delta(X_{12})]_*, P_2]_* + [[A_{12}, X_{12}]_*, \delta(P_2)]_* \\
=&\delta(A_{12})X_{12} - X_{12}\delta(A_{12})^*P_2 - X_{12}^*\delta(A_{12})^* + P_2\delta(A_{12})X_{12}^* + A_{12}\delta(X_{12})P_2 \\
&- P_2\delta(X_{12})^*A_{12}^* - X_{12}A_{12}^*\delta(P_2) + \delta(P_2)A_{12}X_{12}^*.
\end{aligned} \tag{2.9}
$$

Multiplying (2.9) by P_2 from both sides, we have

$$0 = P_2\delta(A_{12})X_{12} - X_{12}^*\delta(A_{12})^*P_2. \tag{2.10}$$

Replacing X_{12} with iX_{12} in (2.10) yields that

$$0 = P_2\delta(A_{12})X_{12} + X_{12}^*\delta(A_{12})^*P_2. \tag{2.11}$$

Combining (2.10) and (2.11), we see that $P_2\delta(A_{12})X_{12} = 0$. Then $P_2\delta(A_{12})P_1 = 0$ by the primeness of $\mathcal{A}$. Similarly, we can obtain that $P_1\delta(A_{21})P_2 = 0$.

Lemma 2.3. *For any $A_{12} \in \mathcal{A}_{12}, B_{21} \in \mathcal{A}_{21}$, there exist $G_{A_{12},B_{21}} \in \mathcal{A}_{11}, K_{A_{12},B_{21}} \in \mathcal{A}_{22}$ such that*

$$\delta(A_{12} + B_{21}) = \delta(A_{12}) + \delta(B_{21}) + G_{A_{12},B_{21}} + K_{A_{12},B_{21}}.$$

Proof. Let $T = \delta(A_{12} + B_{21}) - \delta(A_{12}) - \delta(B_{21})$. From $P_2^*(A_{12} + B_{21})^*P_2 = P_2^*A_{12}^*P_2 = P_2^*B_{21}^*P_2 = 0$ and $[[P_2, B_{21}]_*, P_2]_* = 0$, we have

$$
\begin{aligned}
&[[\delta(P_2), A_{12} + B_{21}]_*, P_2]_* + [[P_2, \delta(A_{12} + B_{21})]_*, P_2]_* + [[P_2, A_{12} + B_{21}]_*, \delta(P_2)]_* \\
&= \delta([[P_2, A_{12} + B_{21}]_*, P_2]_*) \\
&= \delta([[P_2, A_{12}]_*, P_2]_*) + \delta([[P_2, B_{21}]_*, P_2]_*) \\
&= [[\delta(P_2), A_{12} + B_{21}]_*, P_2]_* + [[P_2, \delta(A_{12}) + \delta(B_{21})]_*, P_2]_* + [[P_2, A_{12} + B_{21}]_*, \delta(P_2)]_*,
\end{aligned}
$$

which implies

$$[[P_2, T]_*, P_2]_* = 0. \tag{2.12}$$

Multiplying (2.12) by P_1 from the left, we get $T_{12} = 0$. Similarly, $T_{21} = 0$. Let

$$G_{A_{12},B_{21}} = T_{11}, K_{A_{12},B_{21}} = T_{22}.$$

Then $G_{A_{12},B_{21}} \in \mathcal{A}_{11}, K_{A_{12},B_{21}} \in \mathcal{A}_{22}$, and so $\delta(A_{12} + B_{21}) = \delta(A_{12}) + \delta(B_{21}) + G_{A_{12},B_{21}} + K_{A_{12},B_{21}}$.

Lemma 2.4. (a) $P_j\delta(P_i)P_j = 0$ $(1 \le i \ne j \le 2)$;

(b) $P_i\delta(P_i)P_i = 0$ $(i = 1, 2)$.

Proof. (a) For any $X_{12} \in \mathcal{A}_{12}$, since $P_1^*X_{12}^*P_1 = 0$ and $[[P_1, X_{12}]_*, P_1]_* = 0$, we have

$$
\begin{aligned}
0 =&\delta([[P_1, X_{12}]_*, P_1]_*) \\
=&[[\delta(P_1), X_{12}]_*, P_1]_* + [[P_1, \delta(X_{12})]_*, P_1]_* + [[P_1, X_{12}]_*, \delta(P_1)]_* \\
=&- X_{12}\delta(P_1)P_1 + P_1\delta(P_1)X_{12}^* + P_1\delta(X_{12})P_1 - \delta(X_{12})P_1 - P_1\delta(X_{12})^*P_1 \\
&+ P_1\delta(X_{12})^* + X_{12}\delta(P_1) - \delta(P_1)X_{12}^*.
\end{aligned} \tag{2.13}
$$

Multiplying (2.13) by P_1 from the left and by P_2 from the right, we have

$$P_1\delta(X_{12})^*P_2 + X_{12}\delta(P_1)P_2 = 0.$$

It follows from Lemma 2.2 that $X_{12}\delta(P_1)P_2 = -(P_2\delta(X_{12})P_1)^* = 0$. Then $P_2\delta(P_1)P_2 = 0$. Similarly, $P_1\delta(P_2)P_1 = 0$.

(b) For any $X_{21} \in \mathcal{A}_{21}$, from $(iX_{21})^*P_1^*P_1 = 0$, $[[iX_{21}, P_1]_*, P_1]_* = iX_{21}^* + iX_{21}$, Lemma 2.1(a) and Lemma 2.3, there exist $G_{iX_{21}^*,iX_{21}} \in \mathcal{A}_{11}, K_{iX_{21}^*,iX_{21}} \in \mathcal{A}_{22}$ such that

$$\begin{aligned}
&\delta(iX_{21}^*) + \delta(iX_{21}) + G_{iX_{21}^*,iX_{21}} + K_{iX_{21}^*,iX_{21}}\\
=&\delta([[iX_{21}, P_1]_*, P_1]_*)\\
=&[[\delta(iX_{21}), P_1]_*, P_1]_* + [[iX_{21}, \delta(P_1)]_*, P_1]_* + [[iX_{21}, P_1]_*, \delta(P_1)]_*\\
=&\delta(iX_{21})P_1 - P_1\delta(iX_{21})^*P_1 - P_1\delta(iX_{21})^* + P_1\delta(iX_{21})P_1 + iX_{21}\delta(P_1)P_1 + iP_1\delta(P_1)X_{21}^*\\
&+ iX_{21}\delta(P_1) + iX_{21}^*\delta(P_1) + i\delta(P_1)X_{21}^* + i\delta(P_1)X_{21}.
\end{aligned} \tag{2.14}$$

Multiplying (2.14) by P_2 from the left and by P_1 from the right, we have

$$P_2\delta(iX_{21}^*)P_1 = 2iX_{21}\delta(P_1)P_1 + iP_2\delta(P_1)X_{21}. \tag{2.15}$$

By (2.15), Lemma 2.2 and the fact that $P_2\delta(P_1)P_2 = 0$, we obtain $X_{21}\delta(P_1)P_1 = 0$. Then $P_1\delta(P_1)P_1 = 0$. Similarly, $P_2\delta(P_2)P_2 = 0$.

Remark 2.1. *Let $S = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$. Then $S^* = -S$ by Lemma 2.1. We define a map $\Delta : \mathcal{A} \to \mathcal{A}$ by*

$$\Delta(X) = \delta(X) - [X, S]$$

for any $X \in \mathcal{A}$. It is easy to verify that Δ is a map satisfying

$$\Delta([[A, B]_*, C]_*) = [[\Delta(A), B]_*, C]_* + [[A, \Delta(B)]_*, C]_* + [[A, B]_*, \Delta(C)]_*$$

for any $A, B, C \in \mathcal{A}$ with $A^*B^*C = 0$. By Lemmas 2.1–2.4, it follows that

(a) $\Delta(P_i) = 0$ $(i = 1, 2)$;

(b) *For any $A_{ij} \in \mathcal{A}_{ij}$ $(1 \le i \ne j \le 2)$, we have $P_j\Delta(A_{ij})P_i = 0$;*

(c) *For any $A_{12} \in \mathcal{A}_{12}, B_{21} \in \mathcal{A}_{21}$, there exist $U_{A_{12},B_{21}} \in \mathcal{A}_{11}, V_{A_{12},B_{21}} \in \mathcal{A}_{22}$ such that*

$$\Delta(A_{12} + B_{21}) = \Delta(A_{12}) + \Delta(B_{21}) + U_{A_{12},B_{21}} + V_{A_{12},B_{21}}.$$

Lemma 2.5. $\Delta(\mathcal{A}_{ii}) \subseteq \mathcal{A}_{ii}$ $(i = 1, 2)$.

Proof. Let $A_{11} \in \mathcal{A}_{11}$. From $A_{11}^*P_2^*P_2 = 0$, $[[A_{11}, P_2]_*, P_2]_* = 0$ and $\Delta(P_2) = 0$, we have

$$\begin{aligned}
0 &= \Delta([[A_{11}, P_2]_*, P_2]_*)\\
&= [[\Delta(A_{11}), P_2]_*, P_2]_*\\
&= \Delta(A_{11})P_2 - P_2\Delta(A_{11})^*P_2 - P_2\Delta(A_{11})^* + P_2\Delta(A_{11})P_2.
\end{aligned} \tag{2.16}$$

Multiplying (2.16) by P_1 from the left, we get $P_1\Delta(A_{11})P_2 = 0$. Since $P_2^*A_{11}^*P_1 = 0$, $[[P_2, A_{11}]_*, P_1]_* = 0$ and $\Delta(P_1) = \Delta(P_2) = 0$, we have

$$0 = \Delta([[P_2, A_{11}]_*, P_1]_*) = [[P_2, \Delta(A_{11})]_*, P_1]_* = P_2\Delta(A_{11})P_1 - P_1\Delta(A_{11})^*P_2. \tag{2.17}$$

Multiplying (2.17) by P_2 from the left, we get $P_2\Delta(A_{11})P_1 = 0$. For any $X_{12} \in \mathcal{A}_{12}$, from $X_{12}^*A_{11}^*P_2 = 0$, $[[X_{12}, A_{11}]_*, P_2]_* = 0$ and $\Delta(P_2) = 0$, we have

$$\begin{aligned}0 =&\Delta([[X_{12}, A_{11}]_*, P_2]_*)\\ =&[[\Delta(X_{12}), A_{11}]_*, P_2]_* + [[X_{12}, \Delta(A_{11})]_*, P_2]_*\\ =&- A_{11}\Delta(X_{12})^*P_2 + P_2\Delta(X_{12})A_{11}^* + X_{12}\Delta(A_{11})P_2 - P_2\Delta(A_{11})^*X_{12}^*. \end{aligned} \tag{2.18}$$

Multiplying (2.18) by P_1 from the left, we get $-A_{11}\Delta(X_{12})^*P_2 + X_{12}\Delta(A_{11})P_2 = 0$. It follows from Remark 2.1(b) that $X_{12}\Delta(A_{11})P_2 = A_{11}(P_2\Delta(X_{12})P_1)^* = 0$. Then $P_2\Delta(A_{11})P_2 = 0$. Hence $\Delta(\mathcal{A}_{11}) \subseteq \mathcal{A}_{11}$. Similarly, $\Delta(\mathcal{A}_{22}) \subseteq \mathcal{A}_{22}$.

Lemma 2.6. $\Delta(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$ $(1 \le i \ne j \le 2)$.
Proof. Let $A_{12} \in \mathcal{A}_{12}$. Then $P_2\Delta(A_{12})P_1 = 0$ by Remark 2.1(b). For any $X_{12} \in \mathcal{A}_{12}$, from $X_{12}^*A_{12}^*P_1 = 0$ and $\Delta(P_1) = 0$, we have

$$\begin{aligned}\Delta(-A_{12}X_{12}^* + X_{12}A_{12}^*) =&\Delta([[X_{12}, A_{12}]_*, P_1]_*)\\ =&[[\Delta(X_{12}), A_{12}]_*, P_1]_* + [[X_{12}, \Delta(A_{12})]_*, P_1]_*\\ =&- A_{12}\Delta(X_{12})^*P_1 + P_1\Delta(X_{12})A_{12}^* + X_{12}\Delta(A_{12})P_1 - \Delta(A_{12})X_{12}^*\\ &- P_1\Delta(A_{12})^*X_{12}^* + X_{12}\Delta(A_{12})^*. \end{aligned} \tag{2.19}$$

Multiplying (2.19) by P_2 from the left and by P_1 from the right, then by Lemma 2.5, we get $P_2\Delta(A_{12})X_{12}^* = 0$. Hence $P_2\Delta(A_{12})P_2 = 0$. Since $A_{12}^*X_{12}^*P_2 = 0$, $[[A_{12}, X_{12}]_*, P_2]_* = 0$ and $\Delta(P_2) = 0$, we have

$$\begin{aligned}0 =&\Delta([[A_{12}, X_{12}]_*, P_2]_*)\\ =&[[\Delta(A_{12}), X_{12}]_*, P_2]_* + [[A_{12}, \Delta(X_{12})]_*, P_2]_*\\ =&\Delta(A_{12})X_{12} - X_{12}\Delta(A_{12})^*P_2 - X_{12}^*\Delta(A_{12})^* + P_2\Delta(A_{12})X_{12}^*\\ &+ A_{12}\Delta(X_{12})P_2 - P_2\Delta(X_{12})^*A_{12}^*. \end{aligned} \tag{2.20}$$

Multiplying (2.20) by P_1 from the left and by P_2 from the right, then by $P_2\Delta(A_{12})P_2 = P_2\Delta(X_{12})P_2 = 0$, we have $P_1\Delta(A_{12})X_{12} = 0$. It follows that $P_1\Delta(A_{12})P_1 = 0$. Therefore $\Delta(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. Similarly, $\Delta(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$.

Lemma 2.7. *For any $A_{ii} \in \mathcal{A}_{ii}, B_{ij} \in \mathcal{A}_{ij}, B_{ji} \in \mathcal{A}_{ji}$ $(1 \le i \ne j \le 2)$, we have*
(a) $\Delta(A_{ii} + B_{ij}) = \Delta(A_{ii}) + \Delta(B_{ij})$;
(b) $\Delta(A_{ii} + B_{ji}) = \Delta(A_{ii}) + \Delta(B_{ji})$.
Proof. (a) Let $T = \Delta(A_{ii} + B_{ij}) - \Delta(A_{ii}) - \Delta(B_{ij})$. Since $(iP_j)^*I^*(A_{ii} + B_{ij}) = (iP_j)^*I^*A_{ii} = (iP_j)^*I^*B_{ij} = 0$ and $[[iP_j, I]_*, A_{ii}]_* = 0$, we have

$$\begin{aligned}&[[\Delta(iP_j), I]_*, A_{ii} + B_{ij}]_* + [[iP_j, \Delta(I)]_*, A_{ii} + B_{ij}]_* + [[iP_j, I]_*, \Delta(A_{ii} + B_{ij})]_*\\ =&\Delta([[iP_j, I]_*, A_{ii} + B_{ij}]_*)\\ =&\Delta([[iP_j, I]_*, A_{ii}]_*) + \Delta([[iP_j, I]_*, B_{ij}]_*)\\ =&[[\Delta(iP_j), I]_*, A_{ii} + B_{ij}]_* + [[iP_j, \Delta(I)]_*, A_{ii} + B_{ij}]_* + [[iP_j, I]_*, \Delta(A_{ii}) + \Delta(B_{ij})]_*,\end{aligned}$$

which implies
$$[[iP_j, I]_*, T]_* = 0. \tag{2.21}$$
Multiplying (2.21) by P_i from the left, by P_i from the right, by P_j from both sides, respectively, we get $T_{ij} = T_{ji} = T_{jj} = 0$. Hence
$$\Delta(A_{ii} + B_{ij}) = \Delta(A_{ii}) + \Delta(B_{ij}) + T_{ii}. \tag{2.22}$$
For any $X_{ij} \in \mathcal{A}_{ij}$, from $(A_{ii} + B_{ij})^* X_{ij}^* P_j = A_{ii}^* X_{ij}^* P_j = B_{ij}^* X_{ij}^* P_j = 0$, $[[B_{ij}, X_{ij}]_*, P_j]_* = 0$ and (2.22), we have

$$\begin{aligned}
&[[\Delta(A_{ii}) + \Delta(B_{ij}) + T_{ii}, X_{ij}]_*, P_j]_* + [[A_{ii} + B_{ij}, \Delta(X_{ij})]_*, P_j]_* + [[A_{ii} + B_{ij}, X_{ij}]_*, \Delta(P_j)]_* \\
&=[[\Delta(A_{ii} + B_{ij}), X_{ij}]_*, P_j]_* + [[A_{ii} + B_{ij}, \Delta(X_{ij})]_*, P_j]_* + [[A_{ii} + B_{ij}, X_{ij}]_*, \Delta(P_j)]_* \\
&=\Delta([[A_{ii} + B_{ij}, X_{ij}]_*, P_j]_*) \\
&=\Delta([[A_{ii}, X_{ij}]_*, P_j]_*) + \Delta([[B_{ij}, X_{ij}]_*, P_j]_*) \\
&=[[\Delta(A_{ii}) + \Delta(B_{ij}), X_{ij}]_*, P_j]_* + [[A_{ii} + B_{ij}, \Delta(X_{ij})]_*, P_j]_* + [[A_{ii} + B_{ij}, X_{ij}]_*, \Delta(P_j)]_*.
\end{aligned}$$

This implies
$$[[T_{ii}, X_{ij}]_*, P_j]_* = 0. \tag{2.23}$$
Multiplying (2.23) by P_j from the right, we see that $T_{ii} X_{ij} = 0$. Hence $T_{ii} = 0$, and so we obtain (a).

Similarly, we can show that (b) holds.

Lemma 2.8. *For any $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ $(1 \le i \ne j \le 2)$, we have*
$$\Delta(A_{ij} + B_{ij}) = \Delta(A_{ij}) + \Delta(B_{ij}).$$

Proof. For any $A_{12}, B_{12} \in \mathcal{A}_{12}$, it follows that
$$[[P_1 + A_{12}, P_2 + B_{12}]_*, P_2]_* = A_{12} + B_{12} - A_{12}^* - B_{12}^*. \tag{2.24}$$
Then by (2.24) and Remark 2.1(c), there exist $U_{A_{12}+B_{12},-A_{12}^*-B_{12}^*} \in \mathcal{A}_{11}$, $V_{A_{12}+B_{12},-A_{12}^*-B_{12}^*} \in \mathcal{A}_{22}$ such that
$$\begin{aligned}
\Delta([[P_1 + A_{12}, P_2 + B_{12}]_*, P_2]_*) =&\Delta(A_{12} + B_{12}) + \Delta(-A_{12}^* - B_{12}^*) + U_{A_{12}+B_{12},-A_{12}^*-B_{12}^*} \\
&+ V_{A_{12}+B_{12},-A_{12}^*-B_{12}^*}.
\end{aligned} \tag{2.25}$$
From $(P_1 + A_{12})^*(P_2 + B_{12})^* P_2 = 0$, $\Delta(P_1) = \Delta(P_2) = 0$, (2.25), Lemmas 2.6 and 2.7, we have
$$\begin{aligned}
&\Delta(A_{12} + B_{12}) + \Delta(-A_{12}^* - B_{12}^*) + U_{A_{12}+B_{12},-A_{12}^*-B_{12}^*} + V_{A_{12}+B_{12},-A_{12}^*-B_{12}^*} \\
&=\Delta([[P_1 + A_{12}, P_2 + B_{12}]_*, P_2]_*) \\
&=[[\Delta(A_{12}), P_2 + B_{12}]_*, P_2]_* + [[P_1 + A_{12}, \Delta(B_{12})]_*, P_2]_* \\
&=\Delta(A_{12}) + \Delta(B_{12}) - \Delta(A_{12})^* - \Delta(B_{12})^*.
\end{aligned} \tag{2.26}$$

Multiplying (2.26) by P_1 from the left and by P_2 from the right, then by Lemma 2.6 and the fact that $U_{A_{12}+B_{12},-A_{12}^*-B_{12}^*} \in \mathcal{A}_{11}, V_{A_{12}+B_{12},-A_{12}^*-B_{12}^*} \in \mathcal{A}_{22}$, we see that $\Delta(A_{12} + B_{12}) = \Delta(A_{12}) + \Delta(B_{12})$. Similarly, we can show that $\Delta(A_{21} + B_{21}) = \Delta(A_{21}) + \Delta(B_{21})$.

Lemma 2.9. *For any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ $(i = 1, 2)$, we have*

$$\Delta(A_{ii} + B_{ii}) = \Delta(A_{ii}) + \Delta(B_{ii}).$$

Proof. For any $A_{11}, B_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}$, from $A_{11}^* B_{12}^* P_2 = 0$, $\Delta(P_2) = 0$, $[[A_{11}, B_{12}]_*, P_2]_* = A_{11}B_{12} - B_{12}^* A_{11}^*$, Lemmas 2.5, 2.6 and 2.8, we have

$$\begin{aligned}\Delta(A_{11}B_{12}) + \Delta(-B_{12}^* A_{11}^*) =& \Delta([[A_{11}, B_{12}]_*, P_2]_*)\\ =& [[\Delta(A_{11}), B_{12}]_*, P_2]_* + [[A_{11}, \Delta(B_{12})]_*, P_2]_*\\ =& \Delta(A_{11})B_{12} + A_{11}\Delta(B_{12}) - B_{12}^*\Delta(A_{11})^* - \Delta(B_{12})^* A_{11}^*. \end{aligned} \tag{2.27}$$

Multiplying (2.27) by P_1 from the left and by P_2 from the right, we have

$$\Delta(A_{11}B_{12}) = \Delta(A_{11})B_{12} + A_{11}\Delta(B_{12}). \tag{2.28}$$

Similarly, we can show that

$$\Delta(A_{22}B_{21}) = \Delta(A_{22})B_{21} + A_{22}\Delta(B_{21}). \tag{2.29}$$

For any $X_{12} \in \mathcal{A}_{12}$, it follows from Lemma 2.8 and (2.28) that

$$\begin{aligned}\Delta(A_{11} + B_{11})X_{12} + (A_{11} + B_{11})\Delta(X_{12}) &= \Delta((A_{11} + B_{11})X_{12})\\ &= \Delta(A_{11}X_{12}) + \Delta(B_{11}X_{12})\\ &= \Delta(A_{11})X_{12} + A_{11}\Delta(X_{12}) + \Delta(B_{11})X_{12} + B_{11}\Delta(X_{12}).\end{aligned}$$

It follows that $(\Delta(A_{11} + B_{11}) - \Delta(A_{11}) - \Delta(B_{11}))X_{12} = 0$. Then $\Delta(A_{11} + B_{11}) = \Delta(A_{11}) + \Delta(B_{11})$. Similarly, we can show that $\Delta(A_{22} + B_{22}) = \Delta(A_{22}) + \Delta(B_{22})$.

Lemma 2.10. *For any $A_{12} \in \mathcal{A}_{12}, B_{21} \in \mathcal{A}_{21}$, we have*

$$\Delta(A_{12} + B_{21}) = \Delta(A_{12}) + \Delta(B_{21}).$$

Proof. For any $X_{12} \in \mathcal{A}_{12}$, by $X_{12}^*(A_{12} + B_{21})^* P_1 = X_{12}^* A_{12}^* P_1 = X_{12}^* B_{21}^* P_1 = 0$,

$$[[X_{12}, A_{12} + B_{21}]_*, P_1]_* = [[X_{12}, A_{12}]_*, P_1]_* + [[X_{12}, B_{21}]_*, P_1]_* \in \mathcal{A}_{11},$$

Remark 2.1(c) and Lemma 2.9, there exist $U_{A_{12},B_{21}} \in \mathcal{A}_{11}, V_{A_{12},B_{21}} \in \mathcal{A}_{22}$ such that

$$\begin{aligned}&[[\Delta(X_{12}), A_{12} + B_{21}]_*, P_1]_* + [[X_{12}, \Delta(A_{12}) + \Delta(B_{21}) + U_{A_{12},B_{21}} + V_{A_{12},B_{21}}]_*, P_1]_*\\ &\quad + [[X_{12}, A_{12} + B_{21}]_*, \Delta(P_1)]_*\\ =&\Delta([[X_{12}, A_{12} + B_{21}]_*, P_1]_*)\\ =&\Delta([[X_{12}, A_{12}]_*, P_1]_*) + \Delta([[X_{12}, B_{21}]_*, P_1]_*)\\ =&[[\Delta(X_{12}), A_{12} + B_{21}]_*, P_1]_* + [[X_{12}, \Delta(A_{12}) + \Delta(B_{21})]_*, P_1]_* + [[X_{12}, A_{12} + B_{21}]_*, \Delta(P_1)]_*.\end{aligned}$$

Then

$$0 = [[X_{12}, U_{A_{12},B_{21}} + V_{A_{12},B_{21}}]_*, P_1]_* = -V_{A_{12},B_{21}}X_{12}^* + X_{12}V_{A_{12},B_{21}}^*. \tag{2.30}$$

Multiplying (2.30) by P_1 from the right, we get $V_{A_{12},B_{21}}X_{12}^* = 0$. Hence $V_{A_{12},B_{21}} = 0$. Then by Remark 2.1(c), we get

$$\Delta(A_{12} + B_{21}) = \Delta(A_{12}) + \Delta(B_{21}) + U_{A_{12},B_{21}}. \tag{2.31}$$

For any $X_{21} \in \mathcal{A}_{21}$, from $X_{21}^*(A_{12} + B_{21})^*P_2 = X_{21}^*A_{12}^*P_2 = X_{21}^*B_{21}^*P_2 = 0$,

$$[[X_{21}, A_{12} + B_{21}]_*, P_2]_* = [[X_{21}, A_{12}]_*, P_2]_* + [[X_{21}, B_{21}]_*, P_2]_* \in \mathcal{A}_{22},$$

Lemma 2.9 and (2.31), we have

$$\begin{aligned}
&[[\Delta(X_{21}), A_{12} + B_{21}]_*, P_2]_* + [[X_{21}, \Delta(A_{12}) + \Delta(B_{21}) + U_{A_{12},B_{21}}]_*, P_2]_* \\
&\quad + [[X_{21}, A_{12} + B_{21}]_*, \Delta(P_2)]_* \\
=&\Delta([[X_{21}, A_{12} + B_{21}]_*, P_2]_*) \\
=&\Delta([[X_{21}, A_{12}]_*, P_2]_*) + \Delta([[X_{12}, B_{21}]_*, P_2]_*) \\
=&[[\Delta(X_{21}), A_{12} + B_{21}]_*, P_2]_* + [[X_{21}, \Delta(A_{12}) + \Delta(B_{21})]_*, P_2]_* + [[X_{21}, A_{12} + B_{21}]_*, \Delta(P_2)]_*,
\end{aligned}$$

which implies

$$0 = [[X_{21}, U_{A_{12},B_{21}}]_*, P_2]_* = -U_{A_{12},B_{21}}X_{21}^* + X_{21}U_{A_{12},B_{21}}^*. \tag{2.32}$$

Multiplying (2.32) by P_2 from the right, we obtain $U_{A_{12},B_{21}}X_{21}^* = 0$. Then $U_{A_{12},B_{21}} = 0$. Hence we obtain the desired result.

Lemma 2.11. *For any* $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, *we have*

(a) $\Delta(A_{11} + B_{12} + C_{21}) = \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21})$;

(b) $\Delta(B_{12} + C_{21} + D_{22}) = \Delta(B_{12}) + \Delta(C_{21}) + \Delta(D_{22})$.

Proof. (a) Let $T = \Delta(A_{11} + B_{12} + C_{21}) - \Delta(A_{11}) - \Delta(B_{12}) - \Delta(C_{21})$. From $P_2^*(A_{11} + B_{12} + C_{21})^*P_2 = P_2^*A_{11}^*P_2 = P_2^*B_{12}^*P_2 = P_2^*C_{21}^*P_2 = 0$ and $[[P_2, A_{11}]_*, P_2]_* = [[P_2, C_{21}]_*, P_2]_* = 0$, we have

$$\begin{aligned}
&[[\Delta(P_2), A_{11} + B_{12} + C_{21}]_*, P_2]_* + [[P_2, \Delta(A_{11} + B_{12} + C_{21})]_*, P_2]_* + [[P_2, A_{11} + B_{12} + C_{21}]_*, \Delta(P_2)]_* \\
=&\Delta([[P_2, A_{11} + B_{12} + C_{21}]_*, P_2]_*) \\
=&\Delta([[P_2, A_{11}]_*, P_2]_*) + \Delta([[P_2, B_{12}]_*, P_2]_*) + \Delta([[P_2, C_{21}]_*, P_2]_*) \\
=&[[\Delta(P_2), A_{11} + B_{12} + C_{21}]_*, P_2]_* + [[P_2, \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21})]_*, P_2]_* \\
&\quad + [[P_2, A_{11} + B_{12} + C_{21}]_*, \Delta(P_2)]_*.
\end{aligned}$$

This implies

$$[[P_2, T]_*, P_2]_* = 0. \tag{2.33}$$

Multiplying (2.33) by P_1 from the left, we have $T_{12} = 0$. For any $X_{12} \in \mathcal{A}_{12}$, from $P_1^*X_{12}^*(A_{11} + B_{12} + C_{21}) = P_1^*X_{12}^*A_{11} = P_1^*X_{12}^*B_{12} = P_1^*X_{12}^*C_{21} = 0$, $[[P_1, X_{12}]_*, A_{11} + B_{12} + C_{21}]_* = X_{12}C_{21} - B_{12}X_{12}^*$ and Lemma 2.9, we have

$$\begin{aligned}
&[[\Delta(P_1), X_{12}]_*, A_{11} + B_{12} + C_{21}]_* + [[P_1, \Delta(X_{12})]_*, A_{11} + B_{12} + C_{21}]_* \\
&\quad + [[P_1, X_{12}]_*, \Delta(A_{11} + B_{12} + C_{21})]_* \\
=&\Delta([[P_1, X_{12}]_*, A_{11} + B_{12} + C_{21}]_*) \\
=&\Delta(X_{12}C_{21}) + \Delta(-B_{12}X_{12}^*)
\end{aligned}$$

$$
\begin{aligned}
&=\Delta([[P_1, X_{12}]_*, A_{11}]_*) + \Delta([[P_1, X_{12}]_*, B_{12}]_*) + \Delta([[P_1, X_{12}]_*, C_{21}]_*) \\
&=[[\Delta(P_1), X_{12}]_*, A_{11} + B_{12} + C_{21}]_* + [[P_1, \Delta(X_{12})]_*, A_{11} + B_{12} + C_{21}]_* \\
&\quad + [[P_1, X_{12}]_*, \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21})]_*,
\end{aligned}
$$

which implies

$$[[P_1, X_{12}]_*, T]_* = 0. \tag{2.34}$$

Multiplying (2.34) by P_1 from both sides, we obtain $X_{12}TP_1 - P_1TX_{12}^* = 0$. Then $X_{12}TP_1 = 0$ by $T_{12} = 0$. Hence $T_{21} = 0$. Multiplying (2.34) by P_2 from the right, we have $X_{12}TP_2 = 0$ and so $T_{22} = 0$. Let $S_{A_{11},B_{12},C_{21}} = T_{11}$. Then $S_{A_{11},B_{12},C_{21}} \in \mathcal{A}_{11}$ and

$$\Delta(A_{11} + B_{12} + C_{21}) = \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21}) + S_{A_{11},B_{12},C_{21}}.$$

Similarly, there exists a $R_{B_{12},C_{21},D_{22}} \in \mathcal{A}_{22}$ such that

$$\Delta(B_{12} + C_{21} + D_{22}) = \Delta(B_{12}) + \Delta(C_{21}) + \Delta(D_{22}) + R_{B_{12},C_{21},D_{22}}. \tag{2.35}$$

For any $X_{21} \in \mathcal{A}_{21}$, by $[[P_2, X_{21}]_*, A_{11} + B_{12} + C_{21}]_* = -A_{11}X_{21}^* + X_{21}A_{11} + X_{21}B_{12} - C_{21}X_{21}^*$ and (2.35), there exist a $R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*} \in \mathcal{A}_{22}$ such that

$$
\begin{aligned}
\Delta([[P_2, X_{21}]_*, A_{11} + B_{12} + C_{21}]_*) =&\Delta(-A_{11}X_{21}^*) + \Delta(X_{21}A_{11}) + \Delta(X_{21}B_{12} - C_{21}X_{21}^*) \\
&+ R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*}.
\end{aligned} \tag{2.36}
$$

From $P_2^*X_{21}^*(A_{11} + B_{12} + C_{21}) = P_2^*X_{21}^*A_{11} = P_2^*X_{21}^*B_{12} = P_2^*X_{21}^*C_{21} = 0$, (2.36), Lemmas 2.9 and 2.10, we have

$$
\begin{aligned}
&[[\Delta(P_2), X_{21}]_*, A_{11} + B_{12} + C_{21}]_* + [[P_2, \Delta(X_{21})]_*, A_{11} + B_{12} + C_{21}]_* \\
&\quad + [[P_2, X_{21}]_*, \Delta(A_{11} + B_{12} + C_{21})]_* \\
&=\Delta([[P_2, X_{21}]_*, A_{11} + B_{12} + C_{21}]_*) \\
&=\Delta(-A_{11}X_{21}^*) + \Delta(X_{21}A_{11}) + \Delta(X_{21}B_{12} - C_{21}X_{21}^*) + R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*} \\
&=\Delta(-A_{11}X_{21}^* + X_{21}A_{11}) + \Delta(X_{21}B_{12}) + \Delta(-C_{21}X_{21}^*) + R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*} \\
&=\Delta([[P_2, X_{21}]_*, A_{11}]_*) + \Delta([[P_2, X_{21}]_*, B_{12}]_*) + \Delta([[P_2, X_{21}]_*, C_{21}]_*) \\
&\quad + R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*} \\
&=[[\Delta(P_2), X_{21}]_*, A_{11} + B_{12} + C_{21}]_* + [[P_2, \Delta(X_{21})]_*, A_{11} + B_{12} + C_{21}]_* \\
&\quad + [[P_2, X_{21}]_*, \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21})]_* + R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*}.
\end{aligned}
$$

It follows that

$$[[P_2, X_{21}]_*, T]_* = R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*}. \tag{2.37}$$

Multiplying (2.37) by P_1 from the right, then by $R_{-A_{11}X_{21}^*,X_{21}A_{11},X_{21}B_{12}-C_{21}X_{21}^*} \in \mathcal{A}_{22}$, we obtain $X_{21}TP_1 = 0$. Hence $S_{A_{11},B_{12},C_{21}} = T_{11} = 0$, and so $\Delta(A_{11} + B_{12} + C_{21}) = \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21})$.

Similarly, we can show that (b) holds.

Lemma 2.12. *For any $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have*

$$\Delta(A_{11} + B_{12} + C_{21} + D_{22}) = \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21}) + \Delta(D_{22}).$$

Proof. Let $T = \Delta(A_{11}+B_{12}+C_{21}+D_{22}) - \Delta(A_{11}) - \Delta(B_{12}) - \Delta(C_{21}) - \Delta(D_{22})$. From $(A_{11}+B_{12}+C_{21}+D_{22})^*P_1^*P_2 = A_{11}^*P_1^*P_2 = B_{12}^*P_1^*P_2 = C_{21}^*P_1^*P_2 = D_{22}^*P_1^*P_2 = 0$ and $[[A_{11}+B_{12}+C_{21}+D_{22}, P_1]_*, P_2]_* = -C_{21}^* + C_{21}$, we have

$$\begin{aligned}
&[[\Delta(A_{11}+B_{12}+C_{21}+D_{22}), P_1]_*, P_2]_* + [[A_{11}+B_{12}+C_{21}+D_{22}, \Delta(P_1)]_*, P_2]_* \\
&\quad + [[A_{11}+B_{12}+C_{21}+D_{22}, P_1]_*, \Delta(P_2)]_* \\
&=\Delta([[A_{11}+B_{12}+C_{21}+D_{22}, P_1]_*, P_2]_*) \\
&=\Delta([[A_{11}, P_1]_*, P_2]_*) + \Delta([[B_{12}, P_1]_*, P_2]_*) + \Delta([[C_{21}, P_1]_*, P_2]_*) + \Delta([[D_{22}, P_1]_*, P_2]_*) \\
&=[[\Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21}) + \Delta(D_{22}), P_1]_*, P_2]_* + [[A_{11}+B_{12}+C_{21}+D_{22}, \Delta(P_1)]_*, P_2]_* \\
&\quad + [[A_{11}+B_{12}+C_{21}+D_{22}, P_1]_*, \Delta(P_2)]_*.
\end{aligned}$$

This implies

$$[[T, P_1]_*, P_2]_* = 0. \tag{2.38}$$

Multiplying (2.38) by P_2 from the left, we have $T_{21} = 0$. Similarly, $T_{12} = 0$. For any $X_{12} \in \mathcal{A}_{12}$, from $P_1^*X_{12}^*(A_{11}+B_{12}+C_{21}+D_{22}) = P_1^*X_{12}^*A_{11} = P_1^*X_{12}^*B_{12} = P_1^*X_{12}^*C_{21} = P_1^*X_{12}^*D_{22} = 0$, $[[P_1, X_{12}]_*, A_{11}+B_{12}+C_{21}+D_{22}]_* = X_{12}C_{21} - B_{12}X_{12}^* + X_{12}D_{22} - D_{22}X_{12}^*$, Lemmas 2.9 and 2.12, we have

$$\begin{aligned}
&[[\Delta(P_1), X_{12}]_*, A_{11}+B_{12}+C_{21}+D_{22}]_* + [[P_1, \Delta(X_{12})]_*, A_{11}+B_{12}+C_{21}+D_{22}]_* \\
&\quad + [[P_1, X_{12}]_*, \Delta(A_{11}+B_{12}+C_{21}+D_{22})]_* \\
&=\Delta([[P_1, X_{12}]_*, A_{11}+B_{12}+C_{21}+D_{22}]_*) \\
&=\Delta(X_{12}C_{21}) + \Delta(-B_{12}X_{12}^*) + \Delta(X_{12}D_{22} - D_{22}X_{12}^*) \\
&=\Delta([[P_1, X_{12}]_*, A_{11}]_*) + \Delta([[P_1, X_{12}]_*, B_{12}]_*) + \Delta([[P_1, X_{12}]_*, C_{21}]_*) + \Delta([[P_1, X_{12}]_*, D_{22}]_*) \\
&=[[\Delta(P_1), X_{12}]_*, A_{11}+B_{12}+C_{21}+D_{22}]_* + [[P_1, \Delta(X_{12})]_*, A_{11}+B_{12}+C_{21}+D_{22}]_* \\
&\quad + [[P_1, X_{12}]_*, \Delta(A_{11}) + \Delta(B_{12}) + \Delta(C_{21}) + \Delta(D_{22})]_*.
\end{aligned}$$

This implies

$$[[P_1, X_{12}]_*, T]_* = 0. \tag{2.39}$$

Multiplying (2.39) by P_2 from the right, we obtain $X_{12}TP_2 = 0$. Then $T_{22} = 0$. Similarly, $T_{11} = 0$. Hence we obtain the desired result.

Lemma 2.13. *For any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}, A_{ij}, B_{ij} \in \mathcal{A}_{ij}, B_{ji} \in \mathcal{A}_{ji}, B_{jj} \in \mathcal{A}_{jj}$ $(1 \le i \ne j \le 2)$, we have*

(a) $\Delta(A_{ii}B_{ij}) = \Delta(A_{ii})B_{ij} + A_{ii}\Delta(B_{ij})$;
(b) $\Delta(A_{ij}B_{jj}) = \Delta(A_{ij})B_{jj} + A_{ij}\Delta(B_{jj})$;
(c) $\Delta(A_{ii}B_{ii}) = \Delta(A_{ii})B_{ii} + A_{ii}\Delta(B_{ii})$;
(d) $\Delta(A_{ij}B_{ji}) = \Delta(A_{ij})B_{ji} + A_{ij}\Delta(B_{ji})$.

Proof. (a) It follows from (2.28) and (2.29) that (a) holds.

(b) Let $A_{12} \in \mathcal{A}_{12}, B_{22} \in \mathcal{A}_{22}$. From $A_{12}^*B_{22}^*P_2 = 0$, $\Delta(P_2) = 0$, $[[A_{12}, B_{22}]_*, P_2]_* = A_{12}B_{22} - B_{22}^*A_{12}^*$, Lemmas 2.5, 2.6 and 2.12, we have

$$\Delta(A_{12}B_{22}) + \Delta(-B_{22}^*A_{12}^*) =\Delta([[A_{12}, B_{22}]_*, P_2]_*)$$

$$=[[\Delta(A_{12}), B_{22}]_*, P_2]_*) + [[A_{12}, \Delta(B_{22})]_*, P_2]_*$$
$$=\Delta(A_{12})B_{22} + A_{12}\Delta(B_{22}) - B_{22}^*\Delta(A_{12})^* - \Delta(B_{22})^*A_{12}^*. \quad (2.40)$$

Multiplying (2.40) by P_1 from the left and by P_2 from the right, we have $\Delta(A_{12}B_{22}) = \Delta(A_{12})B_{22} + A_{12}\Delta(B_{22})$. Similarly, $\Delta(A_{21}B_{11}) = \Delta(A_{21})B_{11} + A_{21}\Delta(B_{11})$.

(c) Let $A_{11}, B_{11} \in \mathcal{A}_{11}, X_{12} \in \mathcal{A}_{12}$. It follows from (a) that

$$\begin{aligned}\Delta(A_{11}B_{11})X_{12} + A_{11}B_{11}\Delta(X_{12}) &= \Delta(A_{11}B_{11}X_{12})\\ &= \Delta(A_{11})B_{11}X_{12} + A_{11}\Delta(B_{11}X_{12})\\ &= \Delta(A_{11})B_{11}X_{12} + A_{11}\Delta(B_{11})X_{12} + A_{11}B_{11}\Delta(X_{12}).\end{aligned}$$

It follows that $(\Delta(A_{11}B_{11}) - \Delta(A_{11})B_{11} - A_{11}\Delta(B_{11}))X_{12} = 0$. Hence $\Delta(A_{11}B_{11}) = \Delta(A_{11})B_{11} + A_{11}\Delta(B_{11})$. Similarly, $\Delta(A_{22}B_{22}) = \Delta(A_{22})B_{22} + A_{22}\Delta(B_{22})$.

(d) Let $A_{12} \in \mathcal{A}_{12}, B_{21} \in \mathcal{A}_{21}$. From $B_{21}^*P_1^*A_{12} = 0$, $\Delta(P_1) = 0$, $[[B_{21}, P_1]_*, A_{12}]_* = B_{21}A_{12} + A_{12}B_{21}$, Lemmas 2.6 and 2.12, we have

$$\begin{aligned}\Delta(B_{21}A_{12}) + \Delta(A_{12}B_{21}) =&\Delta([[B_{21}, P_1]_*, A_{12}]_*)\\ =&[[\Delta(B_{21}), P_1]_*, A_{12}]_* + [[B_{21}, P_1]_*, \Delta(A_{12})]_*\\ =&\Delta(B_{21})A_{12} + A_{12}\Delta(B_{21}) + B_{21}\Delta(A_{12}) + \Delta(A_{12})B_{21}. \quad (2.41)\end{aligned}$$

Multiplying (2.41) by P_1 from both sides, we have $\Delta(A_{12}B_{21}) = \Delta(A_{12})B_{21} + A_{12}\Delta(B_{21})$. Similarly, $\Delta(A_{21}B_{12}) = \Delta(A_{21})B_{12} + A_{21}\Delta(B_{12})$.

Now, we give the proof of Theorem 2.1 in the following.

Proof of Theorem 2.1. By Lemmas 2.5, 2.6, 2.8, 2.9, 2.12 and 2.13, it is easy to verify that Δ is an additive derivation on $\mathcal{A}$. Let $A_{ij} \in \mathcal{A}_{ij}$ $(1 \le i \ne j \le 2)$. By $A_{ij}^*P_j^*P_j = 0$, $\Delta(P_j) = 0$ and Lemma 2.6, we have

$$\Delta(A_{ij}) - \Delta(A_{ij}^*) = \Delta([[A_{ij}, P_j]_*, P_j]_*) = [[\Delta(A_{ij}), P_j]_*, P_j]_* = \Delta(A_{ij}) - \Delta(A_{ij})^*.$$

It follows that

$$\Delta(A_{ij}^*) = \Delta(A_{ij})^*. \quad (2.42)$$

Let $A_{ii} \in \mathcal{A}_{ii}, X_{ji} \in \mathcal{A}_{ji}$ $(1 \le i \ne j \le 2)$. Since $A_{ii}^*P_i^*X_{ji} = 0$, $\Delta(P_i) = 0$, $[[A_{ii}, P_i]_*, X_{ji}]_* = X_{ji}A_{ii} - X_{ji}A_{ii}^*$, Lemmas 2.5, 2.6 and 2.13(b), we have

$$\begin{aligned}&\Delta(X_{ji})A_{ii} + X_{ji}\Delta(A_{ii}) - \Delta(X_{ji})A_{ii}^* - X_{ji}\Delta(A_{ii}^*)\\ &= \Delta(X_{ji}A_{ii}) - \Delta(X_{ji}A_{ii}^*)\\ &= \Delta([[A_{ii}, P_i]_*, X_{ji}]_*)\\ &= [[\Delta(A_{ii}), P_i]_*, X_{ji}] + [[A_{ii}, P_i]_*, \Delta(X_{ji})]\\ &= \Delta(X_{ji})A_{ii} + X_{ji}\Delta(A_{ii}) - \Delta(X_{ji})A_{ii}^* - X_{ji}\Delta(A_{ii})^*.\end{aligned}$$

It follows that $X_{ji}(\Delta(A_{ii}^*) - \Delta(A_{ii})^*) = 0$. Then

$$\Delta(A_{ii}^*) = \Delta(A_{ii})^*. \quad (2.43)$$

For any $A \in \mathcal{A}$, we have $A = \sum_{i,j=1}^{2} A_{ij}$. By (2.42), (2.43) and the additivity of Δ on $\mathcal{A}$, it follows that

$$\Delta(A^*) = \sum_{i,j=1}^{2} \Delta(A_{ij}^*) = \sum_{i,j=1}^{2} \Delta(A_{ij})^* = \Delta(A)^*.$$

Hence Δ is an additive $*$-derivation. Therefore, δ is an additive $*$-derivation on $\mathcal{A}$ by Remark 2.1.

3. Conclusions

In this paper, we gave the characterization of a kind of non-global nonlinear skew Lie triple derivations on factor von Neumann algebras.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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