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*Research article*

# Hilfer iterated-integro-differential equations and boundary conditions

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**Abstract:** In this research, a new class of fractional boundary value problems is introduced and studied, which combine Hilfer fractional derivatives with iterated Riemann-Liouville and Hadamard fractional integrals boundary conditions. Existence and uniqueness results are obtained by using standard tools from fixed point theory. The obtained results are well illustrated by numerical examples.

**Keywords:** Hilfer fractional derivative; Riemann-Liouville fractional integral; Hadamard fractional integral; boundary value problems; iterated fractional integrals boundary conditions
**Mathematics Subject Classification:** 34A08, 34B15

## 1. Introduction

Fractional calculus and fractional differential equations have been of great interest, because they describe many real world processes from applied sciences (biology, physics, chemistry, economics, ecology, control theory and so on) more accurately, as compared to classical order differential equations. For the basic theory on the topic see the monographs as [1–8], Many researchers have studied boundary value problems for fractional differential equations, see [9–18] and references cited therein. In the literature there are several kinds of fractional derivatives, such as Riemann-Liouville, Caputo, Erfely-Kober, Hadamard, Hilfer, Katugampola, to name a few. Hilfer fractional derivative [19] extends both Riemann-Liouville and Caputo fractional derivatives. Many applications of Hilfer fractional differential equations can be found in many fields of mathematics, physics, etc. (see [20–25]).

The study of boundary value problems for Hilfer-fractional differential equations of order in $(1, 2]$, and nonlocal boundary conditions were initiated in [26] by studying the boundary value problem of the form:

$$\begin{cases} {}^{H}D^{\alpha,\beta}u(z) = h(z, u(z)), \;\; z \in [c, d], \;\; 1 < \alpha \leq 2, \;\; 0 \leq \beta \leq 1, \\ u(c) = 0, \;\; u(d) = \sum_{i=1}^{m} \varepsilon_i I^{\phi_i} u(\xi_i), \;\; \phi_i > 0, \; \varepsilon_i \in \mathbb{R}, \; \xi_i \in [c, d], \end{cases} \tag{1.1}$$

where ${}^{H}D^{\alpha,\beta}$ is the Hilfer fractional derivative of order $\alpha$, and parameter $\beta$, $h : [c, d] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $m \in \mathbb{Z}^+$, $c \geq 0$, and $I^{\phi_i}$ is the Riemann-Liouville fractional integral of order $\phi_i$, $i = 1, 2, \ldots, m$. By using well known fixed point theorems, existence and uniqueness results were proved.

Recently, in [27], the authors initiated the study of boundary value problems containing sequential fractional derivatives of mixed Riemann-Liouville and Hadamard-Caputo type, subjected to iterated fractional integral boundary conditions of the form:

$$\begin{cases} {}^{RL}D^{p}\left({}^{HC}D^{q}x\right)(t) = f(t, x(t)), \quad t \in [0, T], \\ {}^{HC}D^{q}x(0) = 0, \\ x(T) = \lambda_1 \widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\beta_1,\alpha_1)}x(\xi_1) + \lambda_2 \widehat{R}^{(\delta_m,\gamma_m,\ldots,\delta_1,\gamma_1)}x(\xi_2), \end{cases} \tag{1.2}$$

where ${}^{RL}D^{p}$ and ${}^{HC}D^{q}$ are respectively the fractional derivatives of Riemann-Liouville and Hadamard-Caputo type of orders $p$ and $q$, $0 < p, q < 1$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $m, n \in \mathbb{Z}^+$, the given constants $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\widetilde{R}^{(\alpha_n,\ldots,\beta_1,\alpha_1)}x(t) = {}^{RL}I^{\alpha_n}\, {}^{H}I^{\beta_{n-1}}\, {}^{RL}I^{\alpha_{n-1}}\, {}^{H}I^{\beta_{n-2}} \ldots {}^{H}I^{\beta_2}\, {}^{RL}I^{\alpha_2}\, {}^{H}I^{\beta_1}\, {}^{RL}I^{\alpha_1} x(t)$$

and

$$\widehat{R}^{(\delta_m,\ldots,\delta_1,\gamma_1)}x(t) = {}^{H}I^{\delta_m}\, {}^{RL}I^{\gamma_m}\, {}^{H}I^{\delta_{m-1}}\, {}^{RL}I^{\gamma_{m-1}} \ldots {}^{H}I^{\delta_2}\, {}^{RL}I^{\gamma_2}\, {}^{H}I^{\delta_1}\, {}^{RL}I^{\gamma_1} x(t)$$

are the iterated fractional integrals, where $t = \xi_1$ and $t = \xi_2$, respectively, $\xi_1, \xi_2 \in (0, T)$, ${}^{RL}I^{\phi}$ and ${}^{H}I^{\psi}$ are the fractional integrals of Riemann-Liouville and Hadamard type of orders $\phi, \psi > 0$, respectively, $\phi \in \{\alpha_{(\cdot)}, \gamma_{(\cdot)}\}$, $\psi \in \{\beta_{(\cdot)}, \delta_{(\cdot)}\}$. Existence and uniqueness results are established by applying a variety of fixed point theorems.

Our goal in this paper, inspired by the above-mentioned papers, is to enrich the new research topic concerning boundary value problems for Hilfer fractional iterated-integro-differential equations, subjected to iterated boundary conditions. Thus, in this paper, we investigate the qualitative theory of existence to a nonlinear Hilfer iterated-integro-differential equation with iterated Riemann-Liouville and Hadamard fractional integrals of the form:

$$\begin{cases} ({}^{H}D^{\alpha,\beta}x)(t) + \lambda_1({}^{H}D^{\alpha-1,\beta}x)(t) = f(t, x(t), R^{(\delta_m,\cdots,\delta_1)}x(t)), \; t \in [0, T], \\ x(0) = 0, \qquad x(T) = \lambda_2 R^{(\mu_n,\cdots,\mu_1)}x(\xi), \qquad \xi \in (0, T), \end{cases} \tag{1.3}$$

where ${}^{H}D^{\alpha,\beta}$ is the fractional derivative of Hilfer of order $\alpha$, $1 < \alpha < 2$, $0 < \beta < 1$, $\gamma = \alpha + (2 - \alpha)\beta$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function, $m, n \in \mathbb{N}$ and

$$R^{(\phi_\rho,\ldots,\phi_1)}x(t) = \begin{cases} {}^{H}I^{\phi_\rho}\, I^{\phi_{\rho-1}}\, {}^{H}I^{\phi_{\rho-2}}\, I^{\phi_{\rho-3}} \ldots {}^{H}I^{\phi_4}\, I^{\phi_3}\, {}^{H}I^{\phi_2}\, I^{\phi_1} x(t), & \rho \text{ is even}, \\ I^{\phi_\rho}\, {}^{H}I^{\phi_{\rho-1}}\, I^{\phi_{\rho-2}}\, {}^{H}I^{\phi_{\rho-3}} \ldots {}^{H}I^{\phi_4}\, I^{\phi_3}\, {}^{H}I^{\phi_2}\, I^{\phi_1} x(t), & \rho \text{ is odd}, \end{cases}$$

is the iterated fractional integrals of mixed Riemann-Liouville and Hadamard type, $\phi \in \{\delta, \mu\}$, $\rho \in \{m, n\}$. $I^{\phi_{(\cdot)}}$, ${}^{H}I^{\phi_{(\cdot)}}$ are defined as fractional integrals of Riemann-Liouville and Hadamard type of order $\phi_{(\cdot)} > 0$, respectively.

Notice that: The iterated fractional integrals, $R^{(\phi_\rho,\dots,\phi_1)}(\cdot)$, can be reduced to a single Riemann-Liouville fractional integral when $\rho = 1$ as $R^{(\phi_1)}(\cdot) = I^{\phi_1}(\cdot)$ and to Hadamard integral integral when $\rho = 2$, $\phi_1 = 0$ as $R^{(\phi_2,0)}(\cdot) = {}^{H}I^{\phi_2}(\cdot)$.

In addition, we are now considering some special cases of the Hilfer iterated-integro-differential equation that appeared in the first equation of problem (1.3). If we put $\alpha = 2, \beta = 0$, $f(u, v, w) = u+v+w$ and the iterated integral term $R^{(\delta_m,\cdots,\delta_1)}x = R^{(\delta_1)}x$, then we have

$$x''(t) + \lambda_1 x'(t) = t + x(t) + \frac{1}{\Gamma(\delta_1)} \int_0^t (t-s)^{\delta_1 - 1} x(s) ds,$$

which is a well known integro-differential equation. If the iterated integral term presents $R^{(\delta_m,\cdots,\delta_1)}x = R^{(\delta_2,0)}x$, then we obtain a new integro-differential equation with Hadamard integral as

$$x''(t) + \lambda_1 x'(t) = t + x(t) + \frac{1}{\Gamma(\delta_2)} \int_0^t (\log_e t - \log_e s)^{\delta_2 - 1} x(s) \frac{ds}{s}.$$

By replacing the iterated integral term $R^{(\delta_m,\cdots,\delta_1)}x = R^{(\delta_2,\delta_1)}x$, we get

$$x''(t) + \lambda_1 x'(t) = t + x(t) + \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_0^t \int_0^s (\log_e t - \log_e s)^{\delta_2 - 1} (s-r)^{\delta_1 - 1} x(r) dr \frac{ds}{s},$$

which is an integro-differential equation with a mixed kernel of logarithm and power functions. Another one kernel can be interchanged by replacing $R^{(\delta_m,\cdots,\delta_1)}x = R^{(\delta_3,\delta_2,0)}x$ as

$$x''(t) + \lambda_1 x'(t) = t + x(t) + \frac{1}{\Gamma(\delta_2)\Gamma(\delta_3)} \int_0^t \int_0^s (t-s)^{\delta_3 - 1} (\log_e s - \log_e r)^{\delta_2 - 1} x(r) \frac{dr}{r} ds.$$

Both of the double integrals in the above two equations can not be reduced to a single integral because of differences of the kernels. These show some significance of studying the new iterated-integro-differential equation in (1.3), which is a new novel in literature.

By using standard tools from fixed point theory we establish existence and uniqueness results for the boundary value problem (1.3). More precisely, the existence is proved via Leray-Schauder nonlinear alternative, while the existence of a unique solution is established by using Banach's contraction mapping principle.

The remaining part of this manuscript is organized as follows: Section 2 contain some basic notations and definitions from fractional calculus. Section 3 presents the main results, while Section 4 contains illustrative examples. A brief conclusion closes the paper.

## 2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus in the sense of Riemann-Liouville, Hadamard and also Hilfer differential operators. Moreover, we present some lemmas that needed in main results later.

**Definition 2.1.** *[2] The Riemann-Liouville fractional integral of oreder $\alpha > 0$ to a function $f : [0,\infty) \to \mathbb{R}$ is defined by*

$$(I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \;\; 0 < t < \infty, \tag{2.1}$$

*where $\Gamma(\cdot)$ is the Euler Gamma function, provided that the right-hand side is point-wise defined on $(0,\infty)$.*

**Definition 2.2.** *[2] The Hadamard fractional integral of oreder $\alpha > 0$ is defined as*

$$({}^{H}I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \;\; 0 < t < \infty, \tag{2.2}$$

*where $\log(\cdot) = \log_e(\cdot)$, provided the integral exists.*

**Definition 2.3.** *[2] The Riemann-Liouville fractional derivative of order $\alpha$ for a function $f$ on $[0,\infty)$ is defined as follows:*

$$({}^{R}D^{\alpha} f)(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} f(s)ds, & 0 \le n-1 < \alpha < n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

*provided that the right-hand side is point-wise defined on $(0,\infty)$.*

**Definition 2.4.** *[2] The Caputo fractional derivative of order $\alpha$ of the nth-derivatives function $f$ is given on $[0,\infty)$ by*

$$({}^{C}D^{\alpha} f)(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-s)^{n-\alpha-1} \dfrac{d^n}{ds^n} f(s)ds, & 0 \le n-1 < \alpha < n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

*provided that the right-hand side is point-wise defined on $(0,\infty)$.*

**Definition 2.5.** *[19] The Hilfer fractional derivative of order $\alpha$ with parameter $\beta$ for a function $f$ on $[0,\infty)$ is defined by*

$$({}^{H}D^{\alpha,\beta} f)(t) = I^{\beta(n-\alpha)} \left(\frac{d}{dt}\right)^n I^{(1-\beta)(n-\alpha)} f(t), \tag{2.3}$$

*where $0 \le n-1 < \alpha < n$, and $0 \le \beta \le 1$ for $t > 0$.*

The Hilfer fractional derivative can be reduced to Riemann-Liouville and Caputo operators depending on the parameter $\beta$ $(0 \le \beta \le 1)$. If $\beta = 0$, (2.3) is reduced to the Riemann-Liouville fractional derivative as

$$\left({}^{H}D^{\alpha,0} f\right)(t) = ({}^{R}D^{\alpha} f)(t), \qquad t \in [0,\infty), \tag{2.4}$$

while if $\beta = 1$, (2.3) is reduced to the Caputo fractional derivative by

$$\left({}^{H}D^{\alpha,1} f\right)(t) = ({}^{C}D^{\alpha} f)(t), \qquad t \in [0,\infty). \tag{2.5}$$

**Lemma 2.1.** *[28] Let $f \in L^1(0,T)$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$, $j = 0, 1, \ldots, n-1$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $I^{(n-\gamma)} \in C^n([0,T], \mathbb{R})$. Then*

$$\left(I^{\alpha}\ {}^{H}D^{\alpha,\beta} f\right)(t) = f(t) - \sum_{j=0}^{n-1} \frac{t^{j-(n-\gamma)}}{\Gamma(j-(n-\gamma)+1)} \lim_{t \to 0^+} \frac{d^k}{dt^k} \left(I^{n-\gamma} f\right)(t). \tag{2.6}$$

**Lemma 2.2.** *[2] Let $\alpha > 0$ and $m > 0$ be given constants. Then the following fomula*

$$ {}^{H}I^{\alpha} t^m = m^{-\alpha} t^m \tag{2.7}$$

*holds.*

**Lemma 2.3.** *[27] Let $m > -1$, $\mu_i > 0$, $i = 1, 2, \cdots, n$, be constants. Then the following equation holds:*

$$R^{(\mu_n,\ldots,\mu_1)} t^m = \Gamma(m+1) \frac{\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(m + \sum_{k=1}^{i} \mu_{2k-1}\right)^{-\mu_{2i}}}{\Gamma\left(m + 1 + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}\right)} t^{m + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}}. \tag{2.8}$$

*For $m = 0$ in Eq (2.8), we get*

$$R^{(\mu_n,\ldots,\mu_1)} 1 = \frac{\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{k=1}^{i} \mu_{2k-1}\right)^{-\mu_{2i}}}{\Gamma\left(1 + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}\right)} t^{\sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}}, \tag{2.9}$$

*where $\lceil n \rceil$, $\lfloor n \rfloor$ are the ceiling and floor functions of n, respectively.*

Now, we present some examples of our notations. The ceiling and floor functions of a number 2.4 are shown as $\lceil 2.4 \rceil = 3$ and $\lfloor 2.4 \rfloor = 2$, respectively. The odd and even iterations of Riemann-Liouville and Hadamard fractional integrals of functions $f(t) = t^3$ and $f(t) = 1$, respectively, can be seen as

$$\begin{aligned} R^{(\frac{4}{5},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{3}{4},\frac{1}{3},\frac{1}{2})} t^3 &= \frac{\Gamma(3+1)(3+\frac{1}{2})^{-\frac{1}{3}}(3+\frac{1}{2}+\frac{3}{4})^{-\frac{1}{4}}(3+\frac{1}{2}+\frac{3}{4}+\frac{1}{5})^{-\frac{1}{6}}}{\Gamma(3+1+\frac{1}{2}+\frac{3}{4}+\frac{1}{5}+\frac{4}{5})} \times t^{3+\frac{1}{2}+\frac{3}{4}+\frac{1}{5}+\frac{4}{5}} \\ &\approx 0.0116 \times t^{5.25} \end{aligned}$$

and

$$R^{(\frac{7}{4},\frac{1}{2},\frac{1}{3},\frac{3}{4})} 1 = \frac{(\frac{3}{4})^{-\frac{1}{3}}(\frac{3}{4}+\frac{1}{2})^{-\frac{7}{4}}}{\Gamma(1+\frac{3}{4}+\frac{1}{2})} \times t^{\frac{3}{4}+\frac{1}{2}} \approx 0.6574 \times t^{1.25}.$$

Let $f(t) = t$ be a given function. The iterations of Riemann-Liouville and Hadamard fractional integrals and its consecutive iterations can be considered as follows: $f(t) = (I^{\frac{1}{4}}\, I^{\frac{1}{3}}\, I^{\frac{1}{2}} s)(t)$ (RRR), $g(t) = ({}^{H}I^{\frac{1}{4}}\, {}^{H}I^{\frac{1}{3}}\, {}^{H}I^{\frac{1}{2}} s)(t)$ (HHH), $h(t) = (I^{\frac{1}{4}}\, {}^{H}I^{\frac{1}{3}}\, I^{\frac{1}{2}} s)(t)$ (RHR) and $q(t) = ({}^{H}I^{\frac{1}{4}}\, I^{\frac{1}{3}}\, {}^{H}I^{\frac{1}{2}} s)(t)$ (HRH). The graphs of functions $f(t)$, $g(t)$, $h(t)$ and $q(t)$ for $t \in [0,4]$ are shown in Figure 1.

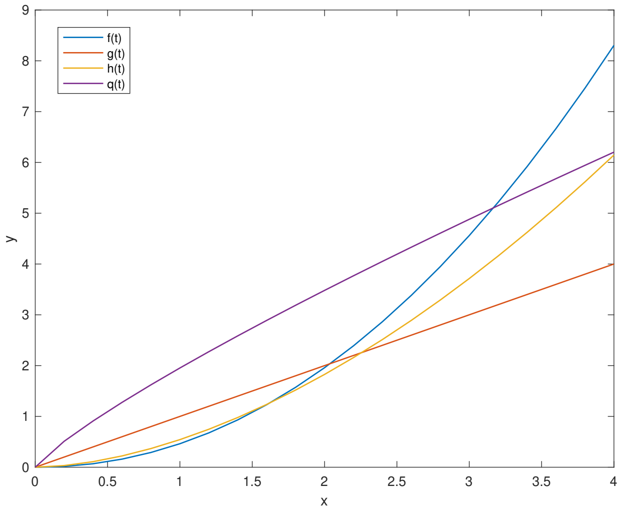


**Figure 1.** The graphs of functions $f(t)$, $g(t)$, $h(t)$ and $q(t)$.

The next lemma concerns a linear variant of the boundary value problem (1.3) and is useful to transform the boundary value problem (1.3) into an integral equation.

**Lemma 2.4.** *Let* $1 < \alpha < 2$, $0 < \beta < 1$, $\gamma = \alpha + (2-\alpha)\beta$, $\lambda_1, \lambda_2 \in \mathbb{R}$, *be constants and* $h : [0, T] \to \mathbb{R}$ *be a continuous function and*

$$\Omega := \frac{T^{\gamma-1}}{\Gamma(\gamma)} - \frac{\lambda_2}{\Gamma\left(\gamma + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}\right)} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\gamma - 1 + \sum_{k=1}^{i} \mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{\gamma-1+\sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}} \neq 0.$$

*Then the Hilfer iterated boundary value problem*

$$\begin{cases} ({}^H D^{\alpha,\beta} x)(t) + \lambda_1 ({}^H D^{\alpha-1,\beta} x)(t) = h(t), & t \in [0, T], \\ x(0) = 0, \quad x(T) = \lambda_2 R^{(\mu_n, \cdots, \mu_1)} x(\xi), & \xi \in (0, T), \end{cases} \tag{2.10}$$

*is equivalent the following integral equation:*

$$\begin{aligned} x(t) &= \frac{t^{\gamma-1}}{\Omega\Gamma(\gamma)} \Big[ -\lambda_1 \lambda_2 R^{(\mu_n, \cdots, \mu_1+1)} x(\xi) + \lambda_2 R^{(\mu_n, \cdots, \mu_1+\alpha)} h(\xi) \\ &\quad + \lambda_1 I^1 x(T) - I^\alpha h(T) \Big] - \lambda_1 I^1 x(t) + I^\alpha h(t). \end{aligned} \tag{2.11}$$

*Proof.* Operating Riemann-Liouville fractional integral of order $\alpha$ to both sides of Eq (2.10), we have

$$I^\alpha \left({}^H D^{\alpha,\beta} x\right)(t) + \lambda_1 I^\alpha \left({}^H D^{\alpha-1,\beta} x\right)(t) = I^\alpha h(t), \tag{2.12}$$

from which, by using Lemma 2.1, we get

$$x(t) - \frac{c_1}{\Gamma(\gamma)} t^{\gamma-1} - \frac{c_2}{\Gamma(\gamma-1)} t^{\gamma-2} + \lambda_1 I^1 x(t) = I^\alpha h(t),$$

where $c_1 = (I^{1-\gamma}x)(0)$ and $c_2 = (I^{2-\gamma}x)(0)$. From the first boundary condition $x(0) = 0$, we have $c_2 = 0$. Then we get

$$x(t) = \frac{c_1}{\Gamma(\gamma)} t^{\gamma-1} - \lambda_1 I^1 x(t) + I^\alpha h(t). \tag{2.13}$$

Next, by using Lemma 2.3, we obtain

$$\begin{aligned}
\lambda_2 R^{(\mu_n,\cdots,\mu_1)} x(\xi) =& \frac{c_1\lambda_2}{\Gamma(\gamma)} R^{(\mu_n,\cdots,\mu_1)} \xi^{\gamma-1} - \lambda_1\lambda_2 R^{(\mu_n,\cdots,\mu_1+1)} x(\xi) + \lambda_2 R^{(\mu_n,\cdots,\mu_1+\alpha)} h(\xi) \\
=& \frac{c_1\lambda_2}{\Gamma(\gamma)} \left( \Gamma(\gamma) \frac{\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left( \gamma - 1 + \sum_{k=1}^{i} \mu_{2k-1} \right)^{-\mu_{2i}}}{\Gamma\left( \gamma + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1} \right)} \times \xi^{\gamma-1+\sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}} \right) \\
& - \lambda_1\lambda_2 R^{(\mu_n,\cdots,\mu_1+1)} x(\xi) + \lambda_2 R^{(\mu_n,\cdots,\mu_1+\alpha)} h(\xi).
\end{aligned}$$

Applying the second boundary condition $x(T) = \lambda_2 R^{(\mu_n,\cdots,\mu_1)} x(\xi)$ with $x(T) = \frac{c_1}{\Gamma(\gamma)} T^{\gamma-1} - \lambda_1 I^1 x(T) + I^\alpha h(T)$, we have

$$\begin{aligned}
&\frac{c_1}{\Gamma(\gamma)} T^{\gamma-1} - c_1\lambda_2 \left( \frac{\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left( \gamma - 1 + \sum_{k=1}^{i} \mu_{2k-1} \right)^{-\mu_{2i}}}{\Gamma\left( \gamma + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1} \right)} \times \xi^{\gamma-1+\sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}} \right) \\
&= -\lambda_1\lambda_2 R^{(\mu_n,\cdots,\mu_1+1)} x(\xi) + \lambda_2 R^{(\mu_n,\cdots,\mu_1+\alpha)} h(\xi) + \lambda_1 I^1 x(T) - I^\alpha h(T),
\end{aligned}$$

form which we get

$$c_1 = \frac{1}{\Omega} \Big[ -\lambda_1\lambda_2 R^{(\mu_n,\cdots,\mu_1+1)} x(\xi) + \lambda_2 R^{(\mu_n,\cdots,\mu_1+\alpha)} h(\xi) + \lambda_1 I^1 x(T) - I^\alpha h(T) \Big].$$

Incerting the value of $c_1$ in (2.13), we obtain the solution (2.11). The converse of this lemma can be proved by direct computation. The proof is finished. □

## 3. Main results

Let $C = C([0,T], \mathbb{R})$ be a Banach space of all continuous function from $[0,T]$ to $\mathbb{R}$ endowed with the supremum norm defined as

$$\|x\| = \sup_{t\in[0,T]} |x(t)|.$$

In view of Lemma 2.4, we transform the problem (1.3) into a fixed point problem $x = Ax$, where $A : C \to C$ is defined by

$$
\begin{aligned}
(Ax)(t) \quad = \quad & \frac{t^{\gamma-1}}{\Omega\Gamma(\gamma)}\Big[-\lambda_1\lambda_2 R^{(\mu_n,\cdots,\mu_1+1)}x(\xi) + \lambda_2 R^{(\mu_n,\cdots,\mu_1+\alpha)} f(\xi, x(\xi), R^{(\delta_m,\cdots,\delta_1)}x(\xi)) \\
& +\lambda_1 I^1 x(T) - I^\alpha f(T, x(T), R^{(\delta_m,\cdots,\delta_1)}x(T))\Big] \\
& -\lambda_1 I^1 x(t) + I^\alpha f(t, x(t), R^{(\delta_m,\cdots,\delta_1)}x(t)).
\end{aligned} \tag{3.1}
$$

For computational convenience, we set

$$
\Lambda_0 \quad := \quad 1 + \frac{1}{\Lambda_3}\prod_{i=1}^{\lfloor \frac{m}{2} \rfloor}\left(\sum_{k=1}^{i}\delta_{2k-1}\right)^{-\delta_{2i}} \times T^{\sum_{k=1}^{\lceil \frac{m}{2} \rceil}\delta_{2k-1}}, \tag{3.2}
$$

$$
\Lambda_1 \quad := \quad \Gamma\left(\gamma + \sum_{k=1}^{\lceil \frac{n}{2} \rceil}\mu_{2k-1}\right), \qquad \Lambda_2 := \Gamma\left(1 + \alpha + \sum_{k=1}^{\lceil \frac{n}{2} \rceil}\mu_{2k-1}\right),
$$

$$
\Lambda_3 \quad := \quad \Gamma\left(1 + \sum_{k=1}^{\lceil \frac{m}{2} \rceil}\delta_{2k-1}\right), \qquad \Lambda_4 := \Gamma\left(2 + \sum_{k=1}^{\lceil \frac{n}{2} \rceil}\mu_{2k-1}\right),
$$

$$
\begin{aligned}
\Phi_1 \quad := \quad & \frac{|\lambda_2| T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_2}\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor}\left(\alpha + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{\alpha + \sum_{k=1}^{\lceil \frac{n}{2} \rceil}\mu_{2k-1}} \\
& + \frac{T^{\gamma+\alpha-1}}{|\Omega|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)},
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
\Phi_2 \quad := \quad & \frac{|\lambda_1||\lambda_2| T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_4}\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor}\left(1 + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{1 + \sum_{k=1}^{\lceil \frac{n}{2} \rceil}\mu_{2k-1}} \\
& + \frac{|\lambda_1| T^\gamma}{|\Omega|\Gamma(\gamma)} + \lambda_1 T.
\end{aligned} \tag{3.4}
$$

Now we present our main results for the boundary value problem (1.3). Our existence result is based on Leray-Schauder's nonlinear alternative [29].

**Theorem 3.1.** *Let* $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that* $\Phi_2 < 1$ *where* $\Phi_2$ *is defined by* (3.4). *In addition, we suppose that:*

($H_1$) *There exist a function* $p \in C([0, T], \mathbb{R}^+)$ *and a continuous nondecreasing function* $\psi : [0, \infty) \to [0, \infty)$ *which is subhomogeneous (that is,* $\psi(\mu x) \le \mu\psi(x)$*, for all* $\mu \ge 1$ *and* $x \in C$*), such that*

$$
|f(t, u, v)| \le p(t)\psi(|u| + |v|) \quad \text{for each} \quad (t, u, v) \in [0, T] \times \mathbb{R}^2;
$$

($H_2$) *There exists a constant* $M > 0$*, such that*

$$
\frac{(1 - \Phi_2)M}{\|p\|\Lambda_0\psi(M)\Phi_1} > 1, \tag{3.5}
$$

*where* $\Lambda_0$ *and* $\Phi_1$ *are defined in* (3.2) *and* (3.3) *respectively.*

*Then there exists at least one solution of the boundary value problem* (1.3) *on* $[0, T]$.

*Proof.* Leray-Schauder's nonlinear alternative will be used to prove that the operator $A$, defined by (3.1), has a fixed point. Firstly, we shall show that $A$ is continuous. Let $\{x_q\}$ be a sequence such that $x_q \to x$ as $q \to \infty$ in $C$. Then, for each $t \in [0,T]$,

$$
\begin{aligned}
&|(Ax_q)(t) - (Ax)(t)| \\
\leq\ & \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_1||\lambda_2|R^{(\mu_n,\dots,\mu_1+1)}|x_q - x|(\xi) \\
&+ |\lambda_2|R^{(\mu_n,\dots,\mu_1+\alpha)}\left|f(\cdot,x_q,R^{(\delta_m,\cdots,\delta_1)}x_q)(\xi) - f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x)(\xi)\right| \\
&+ |\lambda_1|I^1|x_q - x|(T) + I^{\alpha}|f(\cdot,x_q,R^{(\delta_m,\cdots,\delta_1)}x_q) \\
&- f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x)|(T)\Big] + |\lambda_1|I^1|x_q - x|(t) \\
&+ I^{\alpha}|f(\cdot,x_q,R^{(\delta_m,\cdots,\delta_1)}x_q) - f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x)|(t) \\
\leq\ & \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_2|R^{(\mu_n,\dots,\mu_1+\alpha)}\left|f(\cdot,x_q,R^{(\delta_m,\cdots,\delta_1)}x_q) - f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x)\right|(\xi) \\
&+ I^{\alpha}|f(\cdot,x_q,R^{(\delta_m,\cdots,\delta_1)}x_q) - f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x)|(T)\Big] \\
&+ I^{\alpha}|f(\cdot,x_q,R^{(\delta_m,\cdots,\delta_1)}x_q) - f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x)|(T) \\
&+ \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_1||\lambda_2|R^{(\mu_n,\dots,\mu_1+1)}|x_q - x|(\xi) + |\lambda_1|I^1|x_q - x|(T)\Big] \\
&+ |\lambda_1|I^1|x_q - x|(T) \\
\leq\ & \left|f(\cdot,x_q,R^{(\delta_m,\cdots,\delta_1)}x_q) - f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x)\right|\Bigg(\frac{T^{\gamma+\alpha-1}}{|\Omega|\Gamma(\gamma)\Gamma(\alpha+1)} \\
&+ \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_2}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\alpha + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{\alpha+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}}\Bigg) \\
&+ \|x_q - x\|\left[\frac{|\lambda_1|T^{\gamma}}{|\Omega|\Gamma(\gamma)} + \lambda_1 T + \frac{|\lambda_1||\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_4}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(1 + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{1+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}}\right].
\end{aligned}
$$

Since $f$ is a continuous function, it implies that

$$
\left\|Ax_q - Ax\right\| \to 0, \qquad q \to \infty. \tag{3.6}
$$

Next, we show that $A$ maps bounded sets into bounded set in $C$. For any $r > 0$, let $B_r := \{x \in C : \|x\| \leq r\}$. For convenience, putting $\tilde{f}_x(t) := f(t, x(t), R^{(\delta_m,\cdots,\delta_1)}x(t))$ and using $(H_1)$, we have that

$$
\begin{aligned}
|\tilde{f}_x(t)| &\leq \|p\|\left[\psi\left(|x(t)| + |R^{(\delta_m,\cdots,\delta_1)}x(t)|\right)\right] \\
&\leq \|p\|\left[\psi\left(\|x\| + \|x\|R^{(\delta_m,\cdots,\delta_1)}(1)\right)\right] \\
&= \|p\|\left[\psi(\|x\|\Lambda_0)\right] \\
&\leq \|p\|\Lambda_0\psi(\|x\|),
\end{aligned}
$$

where $\Lambda_0$ is defined by (3.2).

Therefore, we obtain

$$
\begin{aligned}
|(Ax)(t)| &\leq \frac{t^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_1||\lambda_2|R^{(\mu_n,\cdots,\mu_1+1)}|x|(\xi) + |\lambda_2|R^{(\mu_n,\cdots,\mu_1+\alpha)}|\tilde{f}_x|(\xi) + |\lambda_1|I^1|x|(T) \\
&\quad + I^\alpha|\tilde{f}_x|(T)\Big] + |\lambda_1|I^1|x|(t) + I^\alpha|\tilde{f}_x|(t) \\
&\leq \|p\|\Lambda_0\psi(\|x\|)\bigg(\frac{|\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\left(R^{(\mu_n,\cdots,\mu_1+\alpha)}1\right)(\xi) + \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}(I^\alpha 1)(T) + (I^\alpha 1)(T)\bigg) \\
&\quad + \|x\|\bigg(\frac{|\lambda_1||\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}(R^{(\mu_n,\cdots,\mu_1+1)}1)(\xi) + \frac{|\lambda_1|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}I^1(1)(T) + |\lambda_1|I^1(1)(T)\bigg) \\
&\leq \|p\|\Lambda_0\psi(\|x\|)\Bigg[\frac{|\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_2}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\alpha + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{\alpha+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}} \\
&\quad + \frac{T^{\gamma+\alpha-1}}{|\Omega|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}\Bigg] \\
&\quad + \|x\|\Bigg[\frac{|\lambda_1|T^\gamma}{|\Omega|\Gamma(\gamma)} + \lambda_1 T + \frac{|\lambda_1||\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_4}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(1 + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{1+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}}\Bigg] \\
&= \|p\|\Lambda_0\psi(\|x\|)\Phi_1 + \|x\|\Phi_2,
\end{aligned}
$$

and consequently

$$\|Ax\| \leq \|p\|\Lambda_0\psi(r)\Phi_1 + r\Phi_2,$$

which means that the set $(AB_r)$ is uniformly bounded.

Next, we show that the operator $A$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $x \in B_r$. Then we get

$$
\begin{aligned}
&|(Ax)(t_2) - (Ax)(t_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]\tilde{f}_x(s)ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\tilde{f}_x(s)ds\right| \\
&+ |\lambda_1||I^1x(t_2) - I^1x(t_1)| + \frac{(t_2^{\gamma-1} - t_1^{\gamma-1})|}{|\Omega|\Gamma(\gamma)}\Big(|\lambda_1||\lambda_2|R^{(\mu_n,\ldots,\mu_1+1)}|x|(\xi) + I^\alpha|\tilde{f}_x|(T) + |\lambda_2|R^{(\mu_n,\ldots,\mu_1+\alpha)}|\tilde{f}_x|(\xi)\Big) \\
&+ \frac{|\lambda_1|}{|\Omega|\Gamma(\gamma)}|t_2^{\gamma-1}I^1x(t_2) - t_1^{\gamma-1}I^1x(t_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha+1)}\|p\|\Lambda_0\psi(r)\Big[2(t_2-t_1)^\alpha + |t_2^\alpha - t_1^\alpha|\Big] + |\lambda_1|r(t_2-t_1) \\
&+ \frac{r}{|\Omega|\Gamma(\gamma)}\Bigg(|\lambda_1|(t_2^\gamma - t_1^\gamma) + \frac{(t_2^{\gamma-1} - t_1^{\gamma-1})}{\Lambda_2}|\lambda_1||\lambda_2|\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(1 + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{1+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}}\Bigg) \\
&+ \frac{(t_2^{\gamma-1} - t_1^{\gamma-1})}{|\Omega|\Gamma(\gamma)}\|p\|\Lambda_0\psi(r)\Bigg(\frac{|\lambda_2|}{\Lambda_2}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\alpha + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{\alpha+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}} + \frac{T^\alpha}{\Gamma(\alpha+1)}\Bigg).
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right hand side of the above inequality tends to zero, independently of $x$, which implies that the set $(AB_r)$ is an equicontinuous set. Hence, we can conclude that $(AB_r)$ is relatively compact. By application of Arzelá-Ascoli theorem, the operator $A$ is completely continuous.

The result will follow from the Leray-Schauder's nonlinear alternative once we have proved the boundedness of the set of all solutions to the equations $x = \theta Ax$ for $\theta \in (0, 1)$.

Let $x$ be a solution of (1.3). Then, for $t \in [0, T]$, and following calculations similar to the second step above, we obtain

$$\begin{aligned} |x(t)| &= |\theta Ax(t)| \leq |Ax(t)| \\ &\leq \|p\|\Lambda_0\psi(\|x\|)\Phi_1 + \|x\|\Phi_2, \end{aligned}$$

which lends to

$$\|x\| \leq \|p\|\Lambda_0\psi(\|x\|)\Phi_1 + \|x\|\Phi_2,$$

or

$$\frac{(1-\Phi_2)\|x\|}{\|p\|\Lambda_0\psi(\|x\|)\Phi_1} \leq 1.$$

In view of $(H_2)$, there exists a constant $M > 0$ such that $\|x\| \neq M$. Let us set

$$K := \{x \in C : \|x\| < M\}.$$

We see that the operator $A : \overline{K} \to C$ is continuous and completely continuous. From the choice of $\overline{K}$, there is no $x \in \partial K$ such that $x = \theta Ax$ for some $\theta \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that the operator $A$ has a fixed point $x \in \overline{K}$, which is a solution of the problem (1.3). The proof is completed. □

**Corollary 3.1.** *If the function $\psi$ in $(H_1)$ is replaced by the following three special cases, then we obtain some interesting results.*

(*i*) *If $\psi(x) = Q$, where $Q > 0$, then the problem* (1.3) *has at least one solution with*

$$M > \frac{\|p\|\Lambda_0 Q\Phi_1}{1-\Phi_2}. \tag{3.7}$$

(*ii*) *If $\psi(x) = Gx+Q$, where $G, Q$ are positive constants and if $\Phi_2+\|p\|\Lambda_0\Phi_1 G < 1$, then the boundary value problem* (1.3) *has at least one solution provided that*

$$M > \frac{\|p\|\Lambda_0 Q\Phi_1}{(1-\Phi_2-\|p\|\Lambda_0\Phi_1 G)}. \tag{3.8}$$

(*iii*) *If $\psi(x) = Gx^2 + Q$, where $G, Q > 0$ and if $\frac{4GQ\|p\|^2\Lambda_0^2\Phi_1^2}{(1-\Phi_2)^2} < 1$, then the problem* (1.3) *has at least one solution with*

$$M \in \left( \frac{1-\sqrt{1-\frac{4GQ\|p\|^2\Lambda_0^2\Phi_1^2}{(1-\Phi_2)^2}}}{2\left(\frac{\|p\|\Lambda_0 G\Phi_1}{1-\Phi_2}\right)}, \frac{1+\sqrt{1-\frac{4GQ\|p\|^2\Lambda_0^2\Phi_1^2}{(1-\Phi_2)^2}}}{2\left(\frac{\|p\|\Lambda_0 G\Phi_1}{1-\Phi_2}\right)} \right). \tag{3.9}$$

Now we prove an existence and uniqueness result via Banach's contraction mapping principle.

**Theorem 3.2.** *Assume that the nonlinear function $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following condition:*

$(H_3)$ *There exists a constant* $L > 0$ *such that*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq L(|x_1 - y_1| + |x_2 - y_2|),$$

*for each* $t \in [0, T]$ *and* $x_i, y_i \in \mathbb{R}$, $i = 1, 2$.

*Then the boundary value problem* (1.3) *has a unique solution on* $[0, T]$, *provided that*

$$L\Lambda_0\Phi_1 + \Phi_2 < 1, \tag{3.10}$$

*where* $\Lambda_0$, $\Phi_1$ *and* $\Phi_2$ *are defined by* (3.2), (3.3) *and* (3.4), *respectively.*

*Proof.* By using the benefit of Banach contraction mapping principle, we will show that the operator $A$, defined by (3.1), has a unique fixed point, which is the unique solution of the problem (1.3).

Let $N$ be a constant as $N = \sup_{t\in[0,T]} |f(t, 0, 0)|$. Next, we give $B_r := \{x \in C : \|x\| \leq r\}$ with $r$ satisfies

$$r \geq \frac{N\Phi_1}{1 - (L\Lambda_0\Phi_1 + \Phi_2)}. \tag{3.11}$$

Observe that $B_r$ is a bounded, closed, and convex subset of $C$. The proof is divided into two steps:

**Step I**. We show that $(AB_r) \subset B_r$.

From $|\tilde{f}_x(t)| := |f(t, x(t), R^{(\delta_m,\cdots,\delta_1)}x(t))|$ and $(H_3)$, we obtain

$$\begin{aligned} |\tilde{f}_x(t)| &\leq |\tilde{f}_x(t) - \tilde{f}_0(t)| + |\tilde{f}_0(t)| \\ &\leq L\Big(\|x\|\Big(1 + R^{(\delta_m,\cdots,\delta_1)}(1)\Big)\Big) + N \\ &= L\Lambda_0\|x\| + N. \end{aligned}$$

For $x \in B_r$, we have

$$\begin{aligned} &|(Ax)(t)| \\ \leq\; & \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_1||\lambda_2|R^{(\mu_n,\cdots,\mu_1+1)}|x|(\xi) + |\lambda_2|R^{(\mu_n,\cdots,\mu_1+\alpha)}|\tilde{f}_x|(\xi) + |\lambda_1|I^1|x|(T) \\ & + I^\alpha|\tilde{f}_x|(T)\Big] + |\lambda_1|I^1|x|(t) + I^\alpha|\tilde{f}_x|(t) \\ \leq\; & (L\Lambda_0\|x\| + N)\left(\frac{|\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\left(R^{(\mu_n,\cdots,\mu_1+\alpha)}1\right)(\xi) + \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}(I^\alpha 1)(T) + (I^\alpha 1)(T)\right) \\ & + \|x\|\left(\frac{|\lambda_1||\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}(R^{(\mu_n,\cdots,\mu_1+1)}1)(\xi) + \frac{|\lambda_1|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}I^1(1)(T) + |\lambda_1|I^1(1)(T)\right) \\ \leq\; & (L\Lambda_0\|x\| + N)\left[\frac{|\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_2}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\alpha + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{\alpha+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}} + \frac{T^{\gamma+\alpha-1}}{|\Omega|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}\right] \\ & + \|x\|\left[\frac{|\lambda_1|T^\gamma}{|\Omega|\Gamma(\gamma)} + \lambda_1 T + \frac{|\lambda_1||\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_4}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(1 + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{1+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}}\right] \\ \leq\; & (L\Lambda_0 r + N)\Phi_1 + r\Phi_2 \leq r, \end{aligned}$$

which implies that $\|Ax\| \leq r$. This confirms $(AB_r) \subset B_r$.

**Step II**. We show that $A : C \to C$ is a contraction operator.

For any $x, y \in C$ and for each $t \in [0, T]$, we have

$$
\begin{aligned}
& |(Ax)(t) - (Ay)(t)| \\
\leq\ & \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_1||\lambda_2|R^{(\mu_n,\ldots,\mu_1+1)}|x-y|(\xi) \\
& + |\lambda_2|R^{(\mu_n,\ldots,\mu_1+\alpha)}\left|f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x) - f(\cdot,y,R^{(\delta_m,\cdots,\delta_1)}y)\right|(\xi) \\
& + |\lambda_1|I^1|x-y|(T) + I^\alpha|f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x) \\
& - f(\cdot,y,R^{(\delta_m,\cdots,\delta_1)}y)|(T)\Big] + |\lambda_1|I^1|x-y|(t) \\
& + I^\alpha|f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x) - f(\cdot,y,R^{(\delta_m,\cdots,\delta_1)}y)|(t) \\
\leq\ & \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_2|R^{(\mu_n,\ldots,\mu_1+\alpha)}\left|f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x) - f(\cdot,y,R^{(\delta_m,\cdots,\delta_1)}y)\right|(\xi) \\
& + I^\alpha|f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x) - f(\cdot,y,R^{(\delta_m,\cdots,\delta_1)}y)|(T)\Big] \\
& + I^\alpha|f(\cdot,x,R^{(\delta_m,\cdots,\delta_1)}x) - f(\cdot,y,R^{(\delta_m,\cdots,\delta_1)}y)|(T) \\
& + \frac{T^{\gamma-1}}{|\Omega|\Gamma(\gamma)}\Big[|\lambda_1||\lambda_2|R^{(\mu_n,\ldots,\mu_1+1)}|x-y|(\xi) + |\lambda_1|I^1|x-y|(T)\Big] \\
& + |\lambda_1|I^1|x-y|(T) \\
\leq\ & L\Lambda_0\|x-y\|\Bigg[\frac{T^{\gamma+\alpha-1}}{|\Omega|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \\
& + \frac{|\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_2}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\alpha + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{\alpha+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}}\Bigg] \\
& + \|x-y\|\Bigg[\frac{|\lambda_1|T^\gamma}{|\Omega|\Gamma(\gamma)} + \lambda_1 T + \frac{|\lambda_1||\lambda_2|T^{\gamma-1}}{|\Omega|\Gamma(\gamma)\Lambda_4}\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(1 + \sum_{k=1}^{i}\mu_{2k-1}\right)^{-\mu_{2i}} \times \xi^{1+\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\mu_{2k-1}}\Bigg] \\
=\ & (L\Lambda_0\Phi_1 + \Phi_2)\|x-y\|.
\end{aligned}
$$

Thus, we obtain the relation that

$$\|Ax - Ay\| \leq (L\Lambda_0\Phi_1 + \Phi_2)\|x-y\|.$$

Since $L\Lambda_0\Phi_1 + \Phi_2 < 1$, the operator $A$ is a contraction. Therefore, by applying Banach contraction mapping principle, the operator $A$ has a fixed point, which implies that the boundary problem (1.3) has a unique solution on $[0, T]$. The proof is completed. □

## 4. Examples

**Example 4.1.** Consider the following boundary value problem containing Hilfer fractional derivative and iterated integrals as

$$
\begin{cases}
({}^H D^{\frac{3}{2},\frac{1}{2}}x)(t) + \dfrac{1}{9}({}^H D^{\frac{1}{2},\frac{1}{2}}x)(t) = f(t, x(t), (R^{(\frac{9}{8},\frac{2}{7},\frac{3}{5},\frac{7}{3})}x)(t)),\ t \in \left[0, \dfrac{13}{4}\right], \\
x(0) = 0, \qquad x\left(\dfrac{13}{4}\right) = \dfrac{1}{15}R^{(\frac{7}{4},\frac{5}{4},\frac{1}{6},\frac{1}{2},\frac{1}{3})}x\left(\dfrac{6}{5}\right).
\end{cases}
\tag{4.1}
$$

Here, $\alpha = 3/2$, $T = 13/4$, $\beta = 1/2$, $\lambda_1 = 1/9$, $m = 4$, $\delta_1 = 7/3$, $\delta_2 = 3/5$, $\delta_3 = 2/7$, $\delta_4 = 9/8$, $\lambda_2 = 1/15$, $\xi = 6/5$, $n = 5$, $\mu_1 = 1/3$, $\mu_2 = 1/2$, $\mu_3 = 1/6$, $\mu_4 = 5/4$, $\mu_5 = 7/4$, and

$$\begin{aligned} R^{\left(\frac{9}{8},\frac{2}{7},\frac{3}{5},\frac{7}{3}\right)}x(t) &= {}^{H}I^{\frac{9}{8}}I^{\frac{2}{7}}\,{}^{H}I^{\frac{3}{5}}I^{\frac{7}{3}}x(t), \\ R^{\left(\frac{7}{4},\frac{5}{4},\frac{1}{6},\frac{1}{2},\frac{1}{3}\right)}x\left(\frac{6}{5}\right) &= I^{\frac{7}{4}}\,{}^{H}I^{\frac{5}{4}}\,I^{\frac{1}{6}}\,{}^{H}I^{\frac{1}{2}}\,I^{\frac{1}{3}}x\left(\frac{6}{5}\right). \end{aligned}$$

From the given data, we find $\Omega \approx 2.6192$, $\Lambda_0 \approx 1.4925$, $\Phi_1 \approx 0.0034$ and $\Phi_2 \approx 0.0021$.

(i) Let the function $f(t,\cdot,\cdot)$ in (4.1) be given by

$$f(t,x,R^{(\delta_m,\cdots,\delta_1)}x) = \frac{1}{(t+\sqrt{20})^2}\left[\frac{e^{-t}}{2}\left(\frac{x^{30}}{|x|^{29}+2}\right) + \frac{e^{-t}}{2}\left(\frac{\left|R^{\left(\frac{9}{8},\frac{2}{7},\frac{3}{5},\frac{7}{3}\right)}x\right|^{27}}{1+\left(R^{\left(\frac{9}{8},\frac{2}{7},\frac{3}{5},\frac{7}{3}\right)}x\right)^{26}}\right) + 1\right]. \tag{4.2}$$

By direct computation, we have

$$\begin{aligned} |f(t,u,v)| &\le \frac{e^{-t}}{(t+\sqrt{20})^2}\left[\frac{1}{2}\left(|u|+|v|\right)+1\right] \\ &:= p(t)\psi\left(|u|+|v|\right), \end{aligned}$$

which yields that $(H_1)$ is satisfied when $p(t) = \dfrac{e^{-t}}{(t+\sqrt{20})^2}$ and $\psi(y) = \dfrac{y}{2}+1$. Therefore, there exists a constant $M > 0.9$ that satisfies $(H_2)$. By Theorem 3.1, the boundary value problem (4.1), with $f$ defined by (4.2), has at least one solution on $\left[0, \frac{13}{4}\right]$.

(ii) If $f : [0, 13/4] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t,x,R^{(\delta_m,\cdots,\delta_1)}x) = \frac{\cos^2 \pi t}{2(t+70)}\left[2\left(x + R^{\left(\frac{9}{8},\frac{2}{7},\frac{3}{5},\frac{7}{3}\right)}x\right)^2 + 1\right], \tag{4.3}$$

then we have

$$|f(t,u,v)| \le \frac{1}{2(t+70)}\left[2(u+v)^2+1\right].$$

We choose $p(t) = \dfrac{1}{2(t+70)}$ and $\psi(y) = 2y^2+1$. Then $\|p\| = 1/140$, $G = 2$ and $Q = 1$. Hence, the inequality $\frac{4GQ\|p\|^2\Lambda_0^2\Phi_1^2}{(1-\Phi_2)^2} \approx 0.9494 < 1$ holds. Therefore, by Corollary 3.1 (*iii*), the boundary value problem (4.1), with $f$ defined by (4.3), has at least one solution on $\left[0, \frac{13}{4}\right]$.

(iii) Assume the function $f : [0, 13/4] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t,x,R^{(\delta_m,\cdots,\delta_1)}x) = \frac{e^{-t^2}}{(t+25)^2}\left[\frac{2\,x^2+|x|}{1+2|x|} + R^{\left(\frac{9}{8},\frac{2}{7},\frac{3}{5},\frac{7}{3}\right)}x\right] + \frac{1}{4}.$$

It is obvious that

$$\begin{aligned} |f(t,x_1,x_2) - f(t,y_1,y_2)| &\le \frac{e^{-t^2}}{(t+25)^2}\left(|x_1-y_1|+|x_2-y_2|\right) \qquad (4.4) \\ &\le \frac{1}{625}\left(|x_1-y_1|+|x_2-y_2|\right). \qquad (4.5) \end{aligned}$$

Thus, $(H_3)$ is satisfied with $L = \frac{1}{625}$. By the given data we can compute that $L\Lambda_0\Phi_1 + \Phi_2 \approx 0.7475 < 1$. Therefore, by Theorem 3.2, the boundary value problem (4.1), with $f$ defined by (4.4), has a unique solution on $\left[0, \frac{13}{4}\right]$.

**Example 4.2.** The functions $y(t) = \left(\frac{t}{6}\right)^4$, $t \in [0,4]$ and $x(t) = t^3$, $t \in [0,3]$, are analytic solutions of the following boundary value problems with iterated fractional integral (RHR) of the form:

$$\begin{cases} ({}^H D^{\frac{3}{2},\frac{1}{2}} y)(t) = \dfrac{1}{36} R^{(\frac{1}{4},\frac{1}{2},\frac{5}{4})} y^{\frac{1}{4}}(t), \qquad t \in [0,4], \\ y(0) = 0, \quad y(4) = 1050 I^{\frac{3}{4}} {}^H I^{\frac{1}{2}} I^{\frac{9}{4}} y(2), \end{cases} \tag{4.6}$$

and (HRH):

$$\begin{cases} ({}^H D^{\frac{3}{2},\frac{1}{2}} x)(t) = \dfrac{18}{\sqrt[3]{20}} R^{(\frac{3}{20},\frac{1}{3},\frac{7}{20})} x^{\frac{1}{3}}(t), \qquad t \in [0,3], \\ x(0) = 0, \quad x(3) = \dfrac{4}{\sqrt{3}} \Gamma\left(\dfrac{35}{8}\right) {}^H I^{\frac{1}{8}} I^{\frac{3}{8}} {}^H I^{\frac{1}{2}} x\left(\dfrac{3}{2}\right), \end{cases} \tag{4.7}$$

respectively.

Indeed, by direct computations, we have

$$({}^H D^{\frac{3}{2},\frac{1}{2}} y)(t) = \frac{\Gamma(5)}{\Gamma(7/2)6^4} t^{\frac{5}{2}} = \frac{1}{36} R^{(\frac{1}{4},\frac{1}{2},\frac{5}{4})} y^{\frac{1}{4}}(t)$$

and

$$({}^H D^{\frac{3}{2},\frac{1}{2}} x)(t) = \frac{\Gamma(4)}{\Gamma(5/2)} t^{\frac{3}{2}} = \frac{18}{\sqrt[3]{20}} R^{(\frac{3}{20},\frac{1}{3},\frac{7}{20})} x^{\frac{1}{3}}(t),$$

which satisfy the first equation of (4.6) and (4.7), respectively. Clearly, the conditions $y(0) = 0$ and $x(0) = 0$ are satisfied. Finally,

$$1050 I^{\frac{3}{4}} {}^H I^{\frac{1}{2}} I^{\frac{9}{4}} y(2) = \frac{\Gamma(5)}{\Gamma(8)} 1050 (25/4)^{-1/2} 2^7 = 4^4 = y(4),$$

and

$$\frac{4}{\sqrt{3}} \Gamma\left(\frac{35}{8}\right) {}^H I^{\frac{1}{8}} I^{\frac{3}{8}} {}^H I^{\frac{1}{2}} x\left(\frac{3}{2}\right) = 27 = x(3).$$

Therefore, the analytic solutions of problems (4.6) and (4.7) are claimed.

## 5. Conclusions

In this paper we studied a fractional boundary value problem, in which a differential equation with Hilfer fractional derivative is combined with iterated fractional integral boundary conditions of

Riemann-Liouville and Hadamard type. As far as we know, this combination appears in the literature for the first time. Firstly, we transformed the given nonlinear fractional boundary value problem into a fixed point problem. Then, by applying Banach's contraction mapping principle and Leray-Schauder nonlinear alternative, we established our main existence and uniqueness results. Furthermore, some numerical examples are illustrated to support the theoretical results. Our results are new in the given configuration and enrich the literature on the new topic of boundary value problems for fractional differential equations of Hilfer type with iterated boundary conditions of Riemann-Liouville and Hadamard type.

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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