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Research article

Contractivity and expansivity of H-Toeplitz operators on the Bergman spaces

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Abstract: In this paper we consider the properties of H-Toeplitz operators B_φ on the Bergman space $L^2_a(\mathbb{D})$. We present some necessary and sufficient conditions for the contractive and expansive H-Toeplitz operators B_φ with various symbols φ.

Keywords: H-Toeplitz operators, contractive operators, expansive operators, Bergman space
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1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and dA the area measure on the complex plane $\mathbb{C}$. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f,\ g\rangle = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

The Bergman space $L^2_a(\mathbb{D})$ consists of all analytic functions on $\mathbb{D}$ and $L^\infty(\mathbb{D})$ is the space of the essentially bounded measurable function on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator M_φ on $L^2_a(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi \cdot f$ and the Toeplitz operator T_φ on $L^2_a(\mathbb{D})$ is defined by

$$T_\varphi(f) = P(\varphi \cdot f),$$

where P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto $L^2_a(\mathbb{D})$ and $f \in L^2_a(\mathbb{D})$. It is clear that those operators are bounded if $\varphi \in L^\infty(\mathbb{D})$.

The harmonic Bergman space $L^2_{harm}(\mathbb{D})$ denotes the space of all complex-valued harmonic functions in $L^2(\mathbb{D})$. The space $L^2_{harm}(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$ and it is a Hilbert space. Let P_{harm} be the orthogonal projection from the space $L^2(\mathbb{D})$ onto the space $L^2_{harm}(\mathbb{D})$.

Toeplitz operators on the Bergman space were studied by McDonald and Sundberg in [19]. Recently, lots of research about Toeplitz operators has been conducted in the Bergman space (see [2, 11]). In the Hardy space, the hyponormality of Toeplitz operators was studied in [7, 8, 12, 14, 20] ; refer to references therein for more details. Recently, many authors characterized the hyponormality of Toeplitz operators on the Bergman space and weighted Bergman space (see [7, 13, 15, 16, 18, 21]). In 2007, Arora and Paliwal [1] have introduced the notion of H-Toeplitz operators on the Hardy space. Recently, in [10], the authors studied H-Toeplitz operators on the Bergman space. The research of H-Toeplitz operators has arisen naturally in several fields of mathematics and in a variety problems. For example, an H-Toeplitz system comprises a matrix equation of the form $Tx = y$ where T is an n by n H-Toeplitz matrix with x, y in $\mathbb{C}^n$. The $n \times n$ H-Toeplitz matrix T has $2n - 1$ degrees of freedom rather than n^2. Thus for a large n, it is easier to solve the system of linear equations for an H-Toeplitz matrix(cf. [10]). In this paper we consider the algebraic properties of H-Toeplitz operators B_φ on the Bergman space $L^2_a(\mathbb{D})$. More concretely, we establish a tractable and explicit criterion for the contractivity and expansivity of H-Toeplitz operators. Several decades ago, many researchers began studying the contractive and expansive operators (see [3, 4, 5, 6]). In [5], the authors considered the invariant subspace problem for contractive operators. Recently, various results have been derived based on the papers (see [9, 17]).

The organization of this paper is as follows. In Section 2, we introduce the notion of H-Toeplitz operators on the Bergman space and provide various well-known properties of these operators. In Section 3, we focus on the contractive and expansive H-Toeplitz operators with analytic, coanalytic and harmonic symbols.

2. Preliminaries and auxiliary lemmas

Let s, t be nonnegative integers and P be the orthogonal projection from $L^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$. Then we have

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases}$$

The following lemmas will be used frequently in this paper.

Lemma 2.1. ([10]) *In the harmonic Bergman space $L^2_{harm}(\mathbb{D})$, for nonnegative integers s and t, the following:*

$$P_{harm}(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ \frac{t-s+1}{t+1} \overline{z}^{t-s} & \text{if } s < t. \end{cases}$$

Lemma 2.2. ([15]) *For $m \geq 0$, we have*

(i) $\|\overline{z}^m \sum_{n=0}^{\infty} c_i z^i\|^2 = \sum_{n=0}^{\infty} \frac{1}{i+m+1} |c_i|^2$,

(ii) $\|P(\overline{z}^m \sum_{n=0}^{\infty} c_i z^i)\|^2 = \sum_{i=m}^{\infty} \frac{i-m+1}{(i+1)^2} |c_i|^2$.

By using Lemmas 2.1 and 2.2, we have the following result.

Remark 2.3. *For $m \geq 0$, we have*

$$\|P_{harm}(\overline{z}^m \sum_{n=0}^{\infty} c_i z^i)\|^2 = \sum_{i=0}^{m-1} \frac{m-i+1}{(m+1)^2}|c_i|^2 + \sum_{i=m}^{\infty} \frac{i-m+1}{(i+1)^2}|c_i|^2.$$

In order to define the notion of an H-Toeplitz operator on $L_a^2(\mathbb{D})$, we first consider the operator $K : L_a^2(\mathbb{D}) \to L_{harm}^2(\mathbb{D})$ defined by

$$K(e_{2n}(z)) = e_n(z) = \sqrt{n+1}z^n \text{ and } K(e_{2n+1}(z)) = \overline{e_{n+1}(z)} = \sqrt{n+2}\overline{z}^{n+1}$$

for all $n \geq 0$ and $z \in \mathbb{D}$. It can be checked that the operator K is bounded linear on $L_a^2(\mathbb{D})$ with $\|K\| = 1$. Moreover, the adjoint K^* of the operator K is given by

$$K^*(e_n(z)) = e_{2n}(z) \text{ and } K^*(\overline{e_{n+1}(z)}) = e_{2n+1}(z)$$

for all $n \geq 0$. From the definition of K and K^*, we have that $KK^* = I_{L^2_{harm}(\mathbb{D})}$ and $K^*K = I_{L^2_a(\mathbb{D})}$.

Remark 2.4. *By the definitions of K and K^*, we can easily check that* $K(z^{2n}) = \frac{\sqrt{n+1}}{\sqrt{2n+1}}z^n$, $K(z^{2n+1}) = \frac{\sqrt{n+2}}{\sqrt{2n+2}}\overline{z}^{n+1}$, $K^*(z^n) = \frac{\sqrt{2n+1}}{\sqrt{n+1}}z^{2n}$ *and* $K^*(\overline{z}^n) = \frac{\sqrt{2n}}{\sqrt{n+1}}z^{2n-1}$.

Next, we define H-Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$ using the definition of the operator K.

Definition 2.5. ([10]) *For $\varphi \in L^\infty(\mathbb{D})$, the H-Toeplitz operator B_φ with the symbol φ is defined as the operator $B_\varphi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ such that $B_\varphi(f) = PM_\varphi K(f)$ for all $f \in L_a^2(\mathbb{D})$.*

The next proposition follows from the definition of the H-Toeplitz operators.

Proposition 2.6. ([10]) *For $\varphi, \psi \in L^\infty(\mathbb{D})$, the operator B_φ satisfies the following:*
(i) *B_φ is a bounded linear operator on $L_a^2(\mathbb{D})$ with $\|B_\varphi\| \leq \|\varphi\|_\infty$.*
(ii) *For any scalar α and β, $B_{\alpha\varphi+\beta\psi} = \alpha B_\varphi + \beta B_\psi$.*
(iii) *The adjoint of the H-Toeplitz operator B_φ is given by $B_\varphi^* = K^* P_{harm} M_{\overline{\varphi}}$.*

The following remark provides important information for adjoint operators. It shows the difference between adjoint Toeplitz operators and adjoint H-Toeplitz operators.

Remark 2.7. *If f, g are in $L^\infty(\mathbb{D})$ then by the definition of Toeplitz operators T_f, we have that*

$$T_f^* = T_{\overline{f}} \quad \text{and} \quad T_{\overline{f}} T_g = T_{\overline{f}g} \text{ if } f \text{ or } g \text{ is analytic.}$$

But in the case of the H-Toeplitz operator,

$$B_z^*(az) = K^* P_{harm} M_{\overline{z}}(az) = K^* P_{harm}(a\overline{z}z) = K^*\left(\frac{a}{2}\right) = \frac{a}{2}$$

and

$$B_{\overline{z}}(az) = PM_{\overline{z}}K(az) = PM_{\overline{z}}a\overline{z} = P(a\overline{z}^2) = 0.$$

Therefore, $B_z^*(az) \neq B_{\overline{z}}(az)$. A straightforward calculation shows that $B_z B_z \neq B_{z^2}$ (cf. [10]).

3. Main results

A bounded linear operator T on a Hilbert space is said to be *expansive* if $T^*T \geq I$, *contractive* if $T^*T \leq I$, and *isometric* if $T^*T = I$.

For $k \in L_a^2(\mathbb{D})$, let $k(z) = k_e(z) + k_o(z)$, where

$$k_e(z) := \sum_{n=0}^{\infty} c_{2n} z^{2n} \quad \text{and} \quad k_o(z) := \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}.$$

3.1. H-Toeplitz operators with analytic symbols

In this subsection, we consider the properties of H-Toeplitz operators B_φ and B_φ^* with analytic symbols. First, we study the contractivity and expansivity of B_φ and B_φ^* with $\varphi = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Next, we extend the symbol φ of the form $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ with $a_i \in \mathbb{C}$.

Theorem 3.1. *Let $\varphi(z) = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then B_φ is contractive if and only if $|a| \leq 1$.*

Proof. For any $k \in L_a^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi k(z) &= PM_\varphi K(k(z)) = PM_\varphi K(k_e(z) + k_o(z)) \\ &= PM_\varphi \left[\sum_{n=0}^{\infty} \left(\frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^n + \frac{\sqrt{n+2}}{\sqrt{2n+2}} c_{2n+1} \overline{z}^{n+1} \right) \right] \\ &= a \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^{n+N} + P\left(az^N \sum_{n=0}^{\infty} \frac{\sqrt{n+2}}{\sqrt{2n+2}} c_{2n+1} \overline{z}^{n+1} \right) \\ &= a \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^{n+N} + a \sum_{n=0}^{N-1} \frac{\sqrt{n+2}}{\sqrt{2n+2}} \cdot \frac{N-n}{N+1} c_{2n+1} z^{N-n-1}, \end{aligned}$$

and we have that

$$\|B_\varphi k(z)\|^2 = |a|^2 \left(\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(n+N+1)} |c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n+2)(N-n)}{2(n+1)(N+1)^2} |c_{2n+1}|^2 \right).$$

According to the definition for the contractivity of B_φ, the inequality $B_\varphi^* B_\varphi \leq I$ is equivalent to $\|B_\varphi k(z)\|^2 \leq \|k(z)\|^2$ for any $k \in L_a^2(\mathbb{D})$. Thus, B_φ on $L_a^2(\mathbb{D})$ is contractive if and only if

$$\begin{aligned} &|a|^2 \left(\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(n+N+1)} |c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n+2)(N-n)}{2(n+1)(N+1)^2} |c_{2n+1}|^2 \right) \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j+1} |c_j|^2. \end{aligned} \tag{3.1}$$

There are two cases to consider. If $c_\ell \neq 0$ for ℓ is even and $c_\ell = 0$ for ℓ is odd, from (3.1), we have

$$|a|^2 \frac{n+1}{(2n+1)(n+N+1)} |c_{2n}|^2 \leq \frac{1}{2n+1} |c_{2n}|^2$$

or equivalently,
$$|a|^2 \le \frac{n+N+1}{n+1}$$
for any nonnegative integer n. Since $\frac{n+N+1}{n+1}$ is decreasing for n, we have
$$|a|^2 \le \min_{n\ge 0} \frac{n+N+1}{n+1} = \lim_{n\to\infty} \frac{n+N+1}{n+1} = 1. \tag{3.2}$$
If $c_\ell \neq 0$ for ℓ is odd, and $c_\ell = 0$ for ℓ is even, from (3.1), we have
$$|a|^2 \frac{(n+2)(N-n)}{2(n+1)(N+1)^2}|c_{2n+1}|^2 \le \frac{1}{2(n+1)}|c_{2n+1}|^2$$
or equivalently,
$$|a|^2 \le \frac{(N+1)^2}{(n+2)(N-n)}$$
for any $0 \le n \le N-1$. Put $f(n) = \frac{(N+1)^2}{(n+2)(N-n)}$, then f is increasing for $\frac{N-2}{2} \le n \le N-1$, and decreasing for $0 \le n < \frac{N-2}{2}$. Moreover, if N is even, then B_φ is contractive if and only if
$$|a|^2 \le \min_{0\le n\le N-1} \frac{(N+1)^2}{(n+2)(N-n)} = f\left(\frac{N-2}{2}\right) = \frac{4(N+1)^2}{(N+2)^2}. \tag{3.3}$$
If N is odd, then B_φ is contractive if and only if
$$|a|^2 \le \min_{0\le n\le N-1} \frac{(N+1)^2}{(n+2)(N-n)} = f\left(\frac{N-1}{2}\right) = f\left(\frac{N-3}{2}\right) = \frac{4(N+1)}{N+3}. \tag{3.4}$$
Since $\frac{4(N+1)^2}{(N+2)^2} \ge 1$ and $\frac{4(N+1)}{N+3} \ge 1$ for any $N \in \mathbb{N}$, from (3.2)–(3.4), B_φ is contractive if and only if $|a| \le 1$. This completes the proof. □

Corollary 3.2. *Let $\varphi(z) = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then B_φ is neither expansive nor isometric.*

Proof. From the proof of Theorem 3.1, B_φ is expansive if and only if
$$|a|^2 \left(\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(n+N+1)}|c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n+2)(N-n)}{2(n+1)(N+1)^2}|c_{2n+1}|^2 \right) \ge \sum_{j=0}^{\infty} \frac{1}{j+1}|c_j|^2. \tag{3.5}$$
Set $c_{2N+1} \neq 0$ and $c_i = 0$ for $i \neq 2N+1$. Then from (3.5), $\frac{1}{2(N+1)} \le 0$; it is a contradiction. □

In the next result, we have the sufficient condition for the contractivity and expansivity of the H-Toeplitz operators B_φ with symbols $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ where $a_i \in \mathbb{C}$ on $L_a^2(\mathbb{D})$.

Theorem 3.3. *Let $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ and $a_i \in \mathbb{C}$.*
(i) *If B_φ is contractive then*
$$\sum_{i=0}^{\infty} \frac{1}{s+i+1}|a_i|^2 \le \frac{1}{s+1} \quad and \quad \sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2}|a_i|^2 \le \frac{1}{s+2}$$

for any nonnegative integer s.
(ii) *If B_φ is expansive then*

$$\sum_{i=0}^{\infty}\frac{1}{s+i+1}|a_i|^2 \geq \frac{1}{s+1} \quad \text{and} \quad \sum_{i=s+1}^{\infty}\frac{i-s}{(i+1)^2}|a_i|^2 \geq \frac{1}{s+2} \tag{3.6}$$

for any nonnegative integer s.

Proof. For any $k \in L_a^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi k(z) &= PM_\varphi K(k(z)) = PM_\varphi K(k_e(z)+k_o(z)) \\ &= PM_\varphi\left[\sum_{n=0}^{\infty}\left(\frac{\sqrt{n+1}}{\sqrt{2n+1}}c_{2n}z^n + \frac{\sqrt{n+2}}{\sqrt{2n+2}}c_{2n+1}\overline{z}^{n+1}\right)\right] \\ &= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sqrt{n+1}}{\sqrt{2n+1}}a_i c_{2n}z^{n+i} + \sum_{i=1}^{\infty}\sum_{n=0}^{i-1}\frac{\sqrt{n+2}}{\sqrt{2n+2}}\cdot\frac{i-n}{i+1}a_i c_{2n+1}z^{i-n-1} \end{aligned} \tag{3.7}$$

for any $c_j \in \mathbb{C}$ $(j = 0, 1, 2, \cdots)$. Then on comparing the coefficient of z^m, by the equation (3.7) we have that

$$a_m c_0 + \frac{\sqrt{2}}{\sqrt{3}}a_{m-1}c_2 + \cdots + \frac{\sqrt{m+1}}{\sqrt{2m+1}}a_0 c_{2m} + \sum_{n=0}^{\infty}\frac{\sqrt{n+2}}{\sqrt{2n+2}}\cdot\frac{m+1}{n+m+2}a_{n+m+1}c_{2n+1}.$$

Set $c_\ell \neq 0$ for some ℓ and $c_j = 0$ for any $j \neq \ell$. Then we consider that the following two cases arise:
Case 1: If $\ell = 2s$ for any nonnegative integer s, then

$$B_\varphi k(z) = \sum_{i=0}^{\infty}\frac{\sqrt{s+1}}{\sqrt{2s+1}}a_i c_{2s}z^{s+i}.$$

If B_φ on $L_a^2(\mathbb{D})$ is contractive then

$$\sum_{i=0}^{\infty}\frac{s+1}{(2s+1)(s+i+1)}|a_i|^2|c_{2s}|^2 \leq \frac{1}{2s+1}|c_{2s}|^2.$$

Thus, $\sum_{i=0}^{\infty}\frac{1}{s+i+1}|a_i|^2 \leq \frac{1}{s+1}$ for any nonnegative integer s. Similarly, if B_φ on $L_a^2(\mathbb{D})$ is expansive then $\sum_{i=0}^{\infty}\frac{1}{s+i+1}|a_i|^2 \geq \frac{1}{s+1}$ for any nonnegative integer s.
Case 2: If $\ell = 2s+1$ for any nonnegative integer s, then

$$B_\varphi k(z) = \sum_{i=s+1}^{\infty}\frac{\sqrt{s+2}}{\sqrt{2s+2}}\cdot\frac{i-s}{i+1}a_i c_{2s+1}z^{i-s-1}.$$

If B_φ on $L_a^2(\mathbb{D})$ is contractive then

$$\sum_{i=s+1}^{\infty}\frac{(s+2)(i-s)}{2(s+1)(i+1)^2}|a_i|^2|c_{2s+1}|^2 \leq \frac{1}{2(s+1)}|c_{2s+1}|^2.$$

Thus, $\sum_{i=s+1}^{\infty}\frac{i-s}{(i+1)^2}|a_i|^2 \leq \frac{1}{s+2}$ for any nonnegative integer s. Similarly, if B_φ on $L_a^2(\mathbb{D})$ is expansive then $\sum_{i=s+1}^{\infty}\frac{i-s}{(i+1)^2}|a_i|^2 \geq \frac{1}{s+2}$ for any nonnegative integer s. This completes the proof. □

Example 3.4. *Let* $\varphi(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{i+1}} z^i$. *Then*

$$\sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = \frac{\pi^2}{6} > 1,$$

and so, B_φ *is not contractive.*

The following example shows that the converse of Theorem 3.3 (ii) is not true.

Example 3.5. *Consider the polynomial* $\varphi(z) = z + \sqrt{3}z^2$. *Then the conditions in* (3.6) *hold. Put* $k(z) = -\frac{\sqrt{2}}{3} + \frac{1}{\sqrt{6}}z + z^2$. *A straightforward calculation shows that* $B_\varphi k(z) = \frac{1}{2\sqrt{6}} + \sqrt{2}z^3$. *Thus* $\|B_\varphi k(z)\|^2 = \frac{13}{24}$ *and* $\|k(z)\|^2 = \frac{23}{36}$. *Therefore,* B_φ *is not expansive.*

We obtained the contractivity and expansivity of the adjoint H-Toeplitz operators B_φ^* on $L_a^2(\mathbb{D})$.

Theorem 3.6. *Let* $\varphi(z) = az^N$ *for* $N \in \mathbb{N}$ *and* $a \in \mathbb{C}$. *Then* B_φ^* *is contractive if and only if* $|a| \leq 1$.

Proof. For any $k \in L_a^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi^* k(z) &= K^* P_{harm} M_{\overline{\varphi}} k(z) \\ &= K^* P_{harm} \left(\overline{az}^N \sum_{n=0}^{\infty} c_n z^n \right) \\ &= \overline{a} K^* \left(\sum_{n=0}^{N-1} \frac{N-n+1}{N+1} c_n \overline{z}^{N-n} + \sum_{n=N}^{\infty} \frac{n-N+1}{n+1} c_n z^{n-N} \right) \\ &= \overline{a} \sum_{n=0}^{N-1} \frac{\sqrt{N-n+1}\,\sqrt{2N-2n}}{N+1} c_n z^{2N-2n-1} + \overline{a} \sum_{n=N}^{\infty} \frac{\sqrt{n-N+1}\,\sqrt{2n-2N+1}}{n+1} c_n z^{2n-2N}. \end{aligned}$$

Thus

$$\|B_\varphi^* k(z)\|^2 = |a|^2 \left(\sum_{n=0}^{N-1} \frac{N-n+1}{(N+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right).$$

Thus, B_φ^* on $L_a^2(\mathbb{D})$ is contractive if and only if

$$|a|^2 \left(\sum_{n=0}^{N-1} \frac{N-n+1}{(N+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right) \leq \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2. \tag{3.8}$$

If $0 \leq n \leq N-1$, then $|a|^2 \leq \frac{(N+1)^2}{(n+1)(N-n+1)}$; so,

$$|a|^2 \leq \min_{0 \leq n \leq N-1} \frac{(N+1)^2}{(n+1)(N-n+1)} = \frac{(N+1)^2}{2N},$$

since $\frac{(N+1)^2}{(n+1)(N-n+1)}$ is decreasing. If $n \geq N$, then $|a|^2 \leq \frac{n+1}{n-N+1}$; so,

$$|a|^2 \leq \min_{n \geq N} \frac{n+1}{n-N+1} = 1,$$

since $\frac{n+1}{n-N+1}$ is decreasing. Hence, for any arbitrary c_i $(i = 0, 1, 2, \cdots)$, the inequality given by (3.8) holds if and only if $|a|^2 \leq \min\left\{\frac{(N+1)^2}{2N}, 1\right\} = 1$. This completes the proof. □

From Theorem 3.6, we get the following corollaries and example.

Corollary 3.7. *Let $\varphi(z) = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then B_φ^* is expansive if and only if $|a|^2 \geq N + 1$.*

Proof. From the proof of Theorem 3.6, B_φ^* is expansive if and only if

$$|a|^2 \left(\sum_{n=0}^{N-1} \frac{N-n+1}{(N+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right) \geq \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2. \tag{3.9}$$

If $0 \leq n \leq N-1$, then $|a|^2 \geq \frac{(N+1)^2}{(n+1)(N-n+1)}$; thus $|a|^2 \geq N+1$ since $\frac{(N+1)^2}{(n+1)(N-n+1)}$ is decreasing. If $n \geq N$, then $|a|^2 \geq \frac{n+1}{n-N+1}$; thus $|a|^2 \geq N+1$ since $\frac{n+1}{n-N+1}$ is decreasing. Hence, the inequality given by (3.9) holds for any arbitrary c_i $(i = 0, 1, 2, \cdots)$ if and only if $|a|^2 \geq N + 1$. □

Example 3.8. *Let $\varphi(z) = 2z^4$. By a direct calculation,*

$$\|B_\varphi^* k(z)\|^2 = 4 \left(\sum_{n=0}^{3} \frac{5-n}{25} |c_n|^2 + \sum_{n=4}^{\infty} \frac{n-3}{(n+1)^2} |c_n|^2 \right)$$

and

$$\|k(z)\|^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2.$$

Since c_i's are arbitrary, set $c_0 \neq 0$ and $c_i = 0$ for $i > 0$; then $\|B_\varphi^* k(z)\|^2 = \frac{4}{5}|c_0|^2$ and $\|k(z)\|^2 = |c_0|^2$. Thus $\|B_\varphi^* k(z)\|^2 < \|k(z)\|^2$. Set $c_5 \neq 0$ and $c_i = 0$ for $i \neq 5$; then $\|B_\varphi^* k(z)\|^2 = \frac{2}{9}|c_5|^2$ and $\|k(z)\|^2 = \frac{1}{6}|c_5|^2$. Thus, $\|B_\varphi^* k(z)\|^2 > \|k(z)\|^2$. Hence $B_{2z^4}^*$ is neither contractive nor expansive.

Corollary 3.9. *Let $\varphi(z) = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then B_φ is not self-adjoint.*

Proof. In the proof of Theorems 3.1 and 3.6,

$$B_\varphi k(z) = a \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^{n+N} + a \sum_{n=0}^{N-1} \frac{\sqrt{n+2}}{\sqrt{2n+2}} \cdot \frac{N-n}{N+1} c_{2n+1} z^{N-n-1}$$

and

$$B_\varphi^* k(z) = \overline{a} \sum_{n=0}^{N-1} \frac{\sqrt{N-n+1}\,\sqrt{2N-2n}}{N+1} c_n z^{2N-2n-1} + \overline{a} \sum_{n=N}^{\infty} \frac{\sqrt{n-N+1}\,\sqrt{2n-2N+1}}{n+1} c_n z^{2n-2N}.$$

Then, on comparing the coefficient of z^0, we get

$$\frac{a}{\sqrt{2N(N+1)}} c_{2N-1} \quad \text{and} \quad \frac{\overline{a}}{N+1} c_N.$$

Since c_{2N-1} and c_N are arbitrary, B_φ is not self-adjoint. □

Corollary 3.10. *Let $\varphi(z) = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then B_φ is not normal.*

Proof. For any $k \in L_a^2(\mathbb{D})$ such that $k(z) = \sum_{n=0}^{\infty} c_n z^n$, B_φ is normal if and only if $B_\varphi^* B_\varphi k(z) = B_\varphi B_\varphi^* k(z)$ or equivalently, $\|B_\varphi k(z)\|^2 = \|B_\varphi^* k(z)\|^2$. As in the proof of Theorems 3.1 and 3.6, we have

$$\|B_\varphi k(z)\|^2 = |a|^2 \left(\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(n+N+1)} |c_{2n}|^2 + \sum_{n=0}^{N-1} \frac{(n+2)(N-n)}{2(n+1)(N+1)^2} |c_{2n+1}|^2 \right)$$

and

$$\|B_\varphi^* k(z)\|^2 = |a|^2 \left(\sum_{n=0}^{N-1} \frac{N-n+1}{(N+1)^2} |c_n|^2 + \sum_{n=N}^{\infty} \frac{n-N+1}{(n+1)^2} |c_n|^2 \right).$$

Since c_i's are arbitrary, set $c_{2N+1} \neq 0$ and $c_i = 0$ for $i \neq 2N+1$. Then $\|B_\varphi k(z)\|^2 = 0$ and $\|B_\varphi^* k(z)\|^2 = \frac{|a|^2 (N+2)}{4(N+1)^2} |c_{2N+1}|^2$; thus, $\|B_\varphi k(z)\|^2 \neq \|B_\varphi^* k(z)\|^2$. □

In the next result, we investigated a sufficient condition for the contractivity and expansivity of the adjoint H-Toeplitz operators B_φ^* with symbols $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ where $a_i \in \mathbb{C}$ on $L_a^2(\mathbb{D})$.

Theorem 3.11. *Let $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ and $a_i \in \mathbb{C}$.*
(i) *If B_φ^* is contractive then*

$$\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 \leq \frac{1}{s+1}$$

for any nonnegative integer s.
(ii) *If B_φ^* is expansive then*

$$\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 \geq \frac{1}{s+1} \tag{3.10}$$

for any nonnegative integer s.

Proof. For any $k \in L_a^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi^* k(z) &= K^* P_{harm} M_{\overline{\varphi}} k(z) \\ &= K^* P_{harm} \left(\sum_{n=0}^{i-1} \sum_{i=1}^{\infty} \overline{a_i} c_n z^n \overline{z}^i + \sum_{n=i}^{\infty} \sum_{i=0}^{\infty} \overline{a_i} c_n z^n \overline{z}^i \right) \\ &= K^* \left(\sum_{n=0}^{i-1} \sum_{i=1}^{\infty} \frac{i-n+1}{i+1} \overline{a_i} c_n \overline{z}^{i-n} + \sum_{n=i}^{\infty} \sum_{i=0}^{\infty} \frac{n-i+1}{n+1} \overline{a_i} c_n z^{n-i} \right) \\ &= \sum_{n=0}^{i-1} \sum_{i=1}^{\infty} \frac{i-n+1}{i+1} \cdot \frac{\sqrt{2i-2n}}{\sqrt{i-n+1}} \overline{a_i} c_n z^{2i-2n-1} + \sum_{n=i}^{\infty} \sum_{i=0}^{\infty} \frac{n-i+1}{n+1} \cdot \frac{\sqrt{2n-2i+1}}{\sqrt{n-i+1}} \overline{a_i} c_n z^{2n-2i}. \end{aligned}$$

Set $c_s \neq 0$ for some s and $c_j = 0$ for any $j \neq s$. Then

$$B_\varphi^* k(z) = \sum_{i=s+1}^{\infty} \frac{i-s+1}{i+1} \cdot \frac{\sqrt{2(i-s)}}{\sqrt{i-s+1}} \overline{a_i} c_s z^{2i-2s-1} + \sum_{i=0}^{s} \frac{s-i+1}{s+1} \cdot \frac{\sqrt{2s-2i+1}}{\sqrt{s-i+1}} \overline{a_i} c_s z^{2s-2i}.$$

If B_φ^* on $L_a^2(\mathbb{D})$ is contractive then

$$\|B_\varphi^* k(z)\|^2 = \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2}|a_i|^2|c_s|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2}|a_i|^2|c_s|^2 \le \frac{1}{s+1}|c_s|^2.$$

Thus,

$$\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2}|a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2}|a_i|^2 \le \frac{1}{s+1}.$$

Similarly, if B_φ^* on $L_a^2(\mathbb{D})$ is expansive then

$$\sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2}|a_i|^2 + \sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2}|a_i|^2 \ge \frac{1}{s+1}.$$

This completes the proof. □

The following example shows that the converse of Theorem 3.11 (ii) is not true.

Example 3.12. *Consider the polynomial $\varphi(z) = \sqrt{2}z + \sqrt{2}z^2$. Then the condition given by* (3.10) *holds. Put $k(z) = \frac{4}{9} - \frac{2}{3}z + z^2$. A straightforward calculation shows that $B_\varphi^* k(z) = \frac{2\sqrt{3}}{3}z^2 + \frac{8\sqrt{6}}{27}z^3$. Then $\|B_\varphi^* k(z)\|^2 = \frac{140}{243}$ and $\|k(z)\|^2 = \frac{183}{243}$. Therefore, B_φ^* is not expansive.*

Corollary 3.13. *Let $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i$ and $a_i \in \mathbb{C}$. If B_φ^* is contractive then $\sum_{i=0}^{\infty} \frac{1}{i+1}|a_i|^2 \le 1$ and if B_φ^* is expansive then $\sum_{i=0}^{\infty} \frac{1}{i+1}|a_i|^2 \ge 1$.*

Proof. We have the result by putting $s = 0$ in Theorem 3.11. □

Example 3.14. *Let $\varphi(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{i+1}} z^i$. Then*

$$\sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = \frac{\pi^2}{6} > 1$$

and by Corollary 3.13, B_φ^* is not contractive.

3.2. H-Toeplitz operators with coanalytic symbols

In this subsection, we consider the properties of H-Toeplitz operators B_φ and B_φ^* with coanalytic, or antianalytic symbols. First, we study the contractivity and expansivity of B_φ and B_φ^* with $\varphi = b\bar{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Next, we extend the symbol φ of the form $\varphi(z) = \sum_{i=1}^{\infty} b_i \bar{z}^i$ with $b_i \in \mathbb{C}$.

Theorem 3.15. *Let $\varphi(z) = b\bar{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then B_φ is contractive if and only if $|b| \le 1$.*

Proof. For any $k \in L_a^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi k(z) &= PM_\varphi \left[\sum_{n=0}^{\infty} \left(\frac{\sqrt{n+1}}{\sqrt{2n+1}} c_{2n} z^n + \frac{\sqrt{n+2}}{\sqrt{2n+2}} c_{2n+1} \bar{z}^{n+1} \right) \right] \\ &= b \sum_{n=N}^{\infty} \frac{n-N+1}{\sqrt{2n+1}\sqrt{n+1}} c_{2n} z^{n-N} \\ &= b \sum_{n=0}^{\infty} \frac{n+1}{\sqrt{2n+2N+1}\sqrt{n+N+1}} c_{2n+2N} z^n. \end{aligned}$$

Thus

$$||B_\varphi k(z)||^2 = |b|^2 \sum_{n=0}^{\infty} \frac{n+1}{(2n+2N+1)(n+N+1)} |c_{2n+2N}|^2.$$

Hence B_φ is contractive if and only if

$$|b|^2 \sum_{n=0}^{\infty} \frac{n+1}{(2n+2N+1)(n+N+1)} |c_{2n+2N}|^2 \leq \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2.$$

If we compare the coefficients of c_{2n+2N}, we have

$$\frac{|b|^2(n+1)}{(2n+2N+1)(n+N+1)} |c_{2n+2N}|^2 \leq \frac{1}{2n+2N+1} |c_{2n+2N}|^2$$

for any $n \geq 0$; thus,

$$|b|^2 \leq \frac{n+N+1}{n+1}$$

for any $n \geq 0$. Since $\frac{n+N+1}{n+1}$ is decreasing for n, B_φ is contractive if and only if

$$|b|^2 \leq \min_{n\geq 0} \frac{n+N+1}{n+1} = \lim_{n\to\infty} \frac{n+N+1}{n+1} = 1.$$

This completes the proof. □

Corollary 3.16. *Let $\varphi(z) = b\overline{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then B_φ is neither expansive nor isometric.*

Proof. From the proof of Theorem 3.15, B_φ is expansive if and only if

$$|b|^2 \sum_{n=0}^{\infty} \frac{n+1}{(2n+2N+1)(n+N+1)} |c_{2n+2N}|^2 \geq \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2.$$

Since c_i's $(0 \leq i < 2N)$ are arbitrary, we put $c_i \neq 0$ if i is odd and $c_i = 0$ if i is even; then, $0 \geq \frac{1}{i+1}$; it is a contradiction. □

In the next result, we get a sufficient condition for the contractivity of H-Toeplitz operators B_φ with symbols $\varphi(z) = \sum_{i=1}^{\infty} b_i \overline{z}^i$, where $b_i \in \mathbb{C}$ on $L_a^2(\mathbb{D})$.

Theorem 3.17. *Let $\varphi(z) = \sum_{i=1}^{\infty} b_i \overline{z}^i$ and $b_i \in \mathbb{C}$. If B_φ is contractive then*

$$\sum_{i=1}^{s} (s-i+1)|b_i|^2 \leq s+1$$

for any $s \in \mathbb{N}$.

Proof. For any $k \in L_a^2(\mathbb{D})$,

$$B_\varphi k(z) = P\left(\sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} b_i c_{2n} z^n \overline{z}^i \right) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{2n+1}} \cdot \frac{n-i+1}{\sqrt{n+1}} b_i c_{2n} z^{n-i}. \tag{3.11}$$

Then on comparing the coefficient of z^m, by the equation (3.11), we have that

$$\sum_{n=m+1}^{\infty} \frac{m+1}{\sqrt{2n+1}\sqrt{n+1}} b_{n-m} c_{2n}.$$

We set $c_\ell \neq 0$ if $\ell = 2s$ and $c_\ell = 0$ if $\ell \neq 2s$ for some $s \in \mathbb{N}$. Thus B_φ on $L^2_a(\mathbb{D})$ is contractive then

$$\sum_{i=1}^{s} \frac{(s-i+1)|b_i|^2}{(2s+1)(s+1)} |c_{2s}|^2 \leq \frac{1}{2s+1} |c_{2s}|^2.$$

Therefore, $\sum_{i=1}^{s}(s-i+1)|b_i|^2 \leq s+1$. This completes the proof. □

On the other hand, we have that

Corollary 3.18. *Let $\varphi(z) = \sum_{i=1}^{\infty} b_i \bar{z}^i$ and $b_i \in \mathbb{C}$. Then B_φ is not expansive.*

Proof. Using the equation (3.11), we set $c_i = 0$ if i is even and $c_i \neq 0$ if i is odd; then, $B_\varphi k(z) = 0$. Thus, B_φ on $L^2_a(\mathbb{D})$ is not expansive. □

The following theorem is purposed to find the necessary and sufficient conditions for the contractivity of the adjoint H-Toeplitz operator B^*_φ with coanalytic symbols φ.

Theorem 3.19. *Let $\varphi(z) = b\bar{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then B^*_φ is contractive if and only if $|b| \leq 1$.*

Proof. For any $k \in L^2_a(\mathbb{D})$,

$$B^*_\varphi k(z) = K^* P_{harm}\left(\bar{b} z^N \sum_{n=0}^{\infty} c_n z^n\right) = \bar{b} \sum_{n=0}^{\infty} \frac{\sqrt{2n+2N+1}}{\sqrt{n+N+1}} c_n z^{2n+2N};$$

then,

$$\|B^*_\varphi k(z)\|^2 = |b|^2 \sum_{n=0}^{\infty} \frac{1}{n+N+1} |c_n|^2.$$

Thus, B^*_φ on $L^2_a(\mathbb{D})$ is contractive if and only if

$$|b|^2 \sum_{n=0}^{\infty} \frac{1}{n+N+1} |c_n|^2 \leq \sum_{n=0}^{\infty} \frac{1}{n+1} |c_n|^2.$$

Since $\frac{n+N+1}{n+1}$ is decreaing, B^*_φ on $L^2_a(\mathbb{D})$ is contractive if and only if

$$|b|^2 \leq \min_{n \geq 0} \frac{n+N+1}{n+1} = 1.$$

This completes the proof. □

From Theorem 3.19, we get the following corollary and example.

Corollary 3.20. *Let $\varphi(z) = b\bar{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then, B^*_φ is expansive if and only if $|b|^2 \geq N+1$.*

Proof. From the proof of Theorem 3.19, B_φ^* is expansive if and only if

$$|b|^2 \sum_{n=0}^{\infty} \frac{1}{n+N+1}|c_n|^2 \geq \sum_{n=0}^{\infty} \frac{1}{n+1}|c_n|^2$$

or equivalently,

$$|b|^2 \geq \frac{n+N+1}{n+1}$$

for any $n \geq 0$. Hence B_φ^* is expansive if and only if

$$|b|^2 \geq \max_{n\geq 0} \frac{n+N+1}{n+1} = N+1.$$

□

Example 3.21. *Let* $\varphi(z) = \frac{3}{2}\overline{z}^2$*. By direct calculations,*

$$\|B_\varphi^* k(z)\|^2 = \frac{9}{4} \sum_{n=0}^{\infty} \frac{1}{n+3}|c_n|^2$$

and

$$\|k(z)\|^2 = \sum_{n=0}^{\infty} \frac{1}{n+1}|c_n|^2.$$

Since c_i*'s are arbitrary, we set* $c_0 \neq 0$ *and* $c_i = 0$ *for* $i > 0$*; then,* $\|B_\varphi^* k(z)\|^2 = \frac{3}{4}|c_0|^2$ *and* $\|k(z)\|^2 = |c_0|^2$*. Thus,* $\|B_\varphi^* k(z)\|^2 < \|k(z)\|^2$*. Set* $c_1 \neq 0$ *and* $c_i = 0$ *for* $i \neq 1$*; then,* $\|B_\varphi^* k(z)\|^2 = \frac{9}{16}|c_1|^2$ *and* $\|k(z)\|^2 = \frac{1}{2}|c_1|^2$*. Thus,* $\|B_\varphi^* k(z)\|^2 > \|k(z)\|^2$*. Hence,* B_φ^* *is neither contractive nor expansive.*

In view of Corollaries 3.9 and 3.10, we have the following result.

Corollary 3.22. *Let* $\varphi(z) = b\overline{z}^N$ *for* $N \in \mathbb{N}$ *and* $b \in \mathbb{C}$*. Then* B_φ^* *is neither self-adjoint nor normal.*

In the next theorem, we have the necessary and sufficient condition for the contractivity and expansivity of adjoint H-Toeplitz operators B_φ^* with symbols $\varphi(z) = \sum_{i=1}^{\infty} b_i \overline{z}^i$ where $b_i \in \mathbb{C}$ on $L_a^2(\mathbb{D})$.

Theorem 3.23. *Let* $\varphi(z) = \sum_{i=1}^{\infty} b_i \overline{z}^i$ *and* $b_i \in \mathbb{C}$*.*
(i) B_φ^* *is contractive if and only if*

$$\sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m+1}|b_i|^2|c_{m-i}|^2 \leq \sum_{j=0}^{\infty} \frac{1}{j+1}|c_j|^2.$$

(ii) B_φ^* *is expansive if and only if*

$$\sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m+1}|b_i|^2|c_{m-i}|^2 \geq \sum_{j=0}^{\infty} \frac{1}{j+1}|c_j|^2.$$

Proof. For any $k \in L_a^2(\mathbb{D})$,

$$B_\varphi^* k(z) = \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{2n+2i+1}}{\sqrt{n+i+1}} \overline{b_i} c_n z^{2n+2i}.$$

Then B_φ^* is contractive if and only if

$$\|B_\varphi^* k(z)\|^2 = \sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m+1} |b_i|^2 |c_{m-i}|^2 \leq \sum_{j=0}^{\infty} \frac{1}{j+1} |c_j|^2.$$

Similarly, B_φ^* is expansive if and only if

$$\|B_\varphi^* k(z)\|^2 = \sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{1}{m+1} |b_i|^2 |c_{m-i}|^2 \geq \sum_{j=0}^{\infty} \frac{1}{j+1} |c_j|^2.$$

This completes the proof. □

Corollary 3.24. *Let $\varphi(z) = b_1 \overline{z} + b_2 \overline{z}^2$ and $b_1, b_2 \in \mathbb{C}$. Then B_φ^* is contractive if and only if*

$$\frac{1}{s+2} |b_1|^2 + \frac{1}{s+3} |b_2|^2 \leq \frac{1}{s+1}$$

for any nonnegative integer s.

3.3. H-Toeplitz operators with harmonic symbols

Finally, we study the properties of H-Toeplitz operators B_φ with harmonic symbols of the form $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{i=1}^{\infty} b_i \overline{z}^i$ with $a_i, b_i \in \mathbb{C}$. Specifically, we focus on the necessary and sufficient conditions of contractivity and expansivity for B_φ and B_φ^*, respectively.

Theorem 3.25. *Let $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{i=1}^{\infty} b_i \overline{z}^i$ and $a_i, b_i \in \mathbb{C}$.*
(i) *If B_φ is contractive then*

$$\sum_{i=1}^{\infty} \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^{s} \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \leq 1$$

and

$$\sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2} |a_i|^2 \leq \frac{1}{s+2}$$

for any nonnegative integer s.
(ii) *If B_φ is expansive then*

$$\sum_{i=1}^{\infty} \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^{s} \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \geq 1$$

and

$$\sum_{i=s+1}^{\infty} \frac{i-s}{(i+1)^2} |a_i|^2 \geq \frac{1}{s+2}$$

for any nonnegative integer s.

Proof. By a similar argument as in the proof of Theorems 3.3 and 3.17, for any $k \in L_a^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi k(z) = &\sum_{i=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} a_i c_{2n} z^{n+i} + \sum_{i=0}^{\infty}\sum_{n=0}^{i-1} \frac{\sqrt{n+2}}{\sqrt{2n+2}} \cdot \frac{i-n}{i+1} a_i c_{2n+1} z^{i-n-1} \\ &+ \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{1}{\sqrt{2n+1}} \cdot \frac{n-i+1}{\sqrt{n+1}} b_i c_{2n} z^{n-i} \end{aligned}$$

for any $c_j \in \mathbb{C}$ $(j = 0, 1, 2, \cdots)$. Set $c_\ell \neq 0$ for some ℓ and $c_j = 0$ for any $j \neq \ell$. Then we consider the following two cases:
Case 1: If $\ell = 2s$ for any nonnegative integer s and $c_{2s} \neq 0$ then

$$B_\varphi k(z) = \sum_{i=1}^{\infty} \frac{\sqrt{s+1}}{\sqrt{2s+1}} a_i c_{2s} z^{s+i} + \sum_{i=1}^{s} \frac{s-i+1}{\sqrt{2s+1}\sqrt{s+1}} b_i c_{2s} z^{s-i}.$$

If B_φ on $L_a^2(\mathbb{D})$ is contractive then

$$\sum_{i=1}^{\infty} \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^{s} \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \leq 1.$$

Similarly, if B_φ on $L_a^2(\mathbb{D})$ is expansive then

$$\sum_{i=1}^{\infty} \frac{s+1}{(2s+1)(s+i+1)} |a_i|^2 + \sum_{i=1}^{s} \frac{s-i+1}{(2s+1)(s+1)} |b_i|^2 \geq 1.$$

Case 2: If $\ell = 2s+1$ for any nonnegative integer s and $c_{2s+1} \neq 0$, then it follows from Case 2 of Theorem 3.3. This completes the proof. □

Theorem 3.26. *Let* $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{i=1}^{\infty} b_i \overline{z}^i$ *and* $a_i, b_i \in \mathbb{C}$.
(i) *If* B_φ^* *is contractive, then*

$$\sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=1}^{\infty} \frac{1}{s+i+1} |b_i|^2 \leq \frac{1}{s+1},$$

for any nonnegative integer s.
(ii) *If* B_φ^* *is expansive, then*

$$\sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2} |a_i|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2} |a_i|^2 + \sum_{i=1}^{\infty} \frac{1}{s+i+1} |b_i|^2 \geq \frac{1}{s+1},$$

for any nonnegative integer s.

Proof. By a similar argument as in the proof of Theorems 3.11 and 3.23, for any $k \in L_a^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi^* k(z) = &\sum_{n=0}^{i-1}\sum_{i=1}^{\infty} \frac{i-n+1}{i+1} \cdot \frac{\sqrt{2i-2n}}{\sqrt{i-n+1}} \overline{a}_i c_n z^{2i-2n-1} \\ &+ \sum_{n=i}^{\infty}\sum_{i=0}^{\infty} \frac{n-i+1}{n+1} \cdot \frac{\sqrt{2n-2i+1}}{\sqrt{n-i+1}} \overline{a}_i c_n z^{2n-2i} + \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} \frac{\sqrt{2n+2i+1}}{\sqrt{n+i+1}} \overline{b}_i c_n z^{2n+2i}. \end{aligned}$$

for any $c_j \in \mathbb{C}$ ($j = 0, 1, 2, \cdots$). Set $c_s \neq 0$ for some s and $c_j = 0$ for any $j \neq s$. Then

$$B_\varphi^* k(z) = \sum_{i=s+1}^{\infty} \frac{\sqrt{i-s+1}\,\sqrt{2i-2s}}{i+1} \overline{a_i} c_s z^{2i-2s-1} + \sum_{i=0}^{s} \frac{\sqrt{s-i+1}\,\sqrt{2s-2i+1}}{s+1} \overline{a_i} c_s z^{2s-2i} + \sum_{i=1}^{\infty} \frac{\sqrt{2s+2i+1}}{\sqrt{s+i+1}} \overline{b_i} c_s z^{2s+2i},$$

for any nonnegative integer s. If B_φ^* on $L_a^2(\mathbb{D})$ is contractive, then

$$\sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2}|a_i|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2}|a_i|^2 + \sum_{i=1}^{\infty} \frac{1}{s+i+1}|b_i|^2 \leq \frac{1}{s+1}.$$

Similarly, if B_φ^* on $L_a^2(\mathbb{D})$ is expansive, then

$$\sum_{i=s+1}^{\infty} \frac{i-s+1}{(i+1)^2}|a_i|^2 + \sum_{i=0}^{s} \frac{s-i+1}{(s+1)^2}|a_i|^2 + \sum_{i=1}^{\infty} \frac{1}{s+i+1}|b_i|^2 \geq \frac{1}{s+1},$$

for any nonnegative integer s. This completes the proof. □

The following results are immediate from Theorem 3.26.

Corollary 3.27. *Let $\varphi(z) = a_1 z + a_2 z^2 + b_1 \overline{z} + b_2 \overline{z}^2$ and $a_i, b_i \in \mathbb{C}$ where $i = 1, 2$. Then, B_φ^* is contractive*

$$\Longrightarrow \begin{cases} \frac{s}{(s+1)^2}|a_1|^2 + \frac{s+1}{3^{s+1}}|a_2|^2 + \frac{1}{s+2}|b_1|^2 + \frac{1}{s+3}|b_2|^2 \leq \frac{1}{s+1} & \text{if } s = 0, 1, \\ \frac{s}{(s+1)^2}|a_1|^2 + \frac{s-1}{(s+1)^2}|a_2|^2 + \frac{1}{s+2}|b_1|^2 + \frac{1}{s+3}|b_2|^2 \leq \frac{1}{s+1} & \text{if } s \geq 2. \end{cases}$$

Corollary 3.28. *Let $\varphi(z) = a_1 z + b_1 \overline{z}$ and $a_1, b_1 \in \mathbb{C}$. Then, B_φ^* is contractive; then,*

$$\frac{s}{(s+1)^2}|a_1|^2 + \frac{1}{s+2}|b_1|^2 \leq \frac{1}{s+1},$$

for any $s \in \mathbb{N}$.

4. Conclusions

We characterized the necessary or sufficient conditions for the contractive and expansive H-Toeplitz operators B_φ with various symbols φ on the Bergman space $L_a^2(\mathbb{D})$. By these results, we expect to provide the properties of these operators on the Bergman space.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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