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Research article

Iterative solutions via some variants of extragradient approximants in Hilbert spaces

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Abstract: This paper provides iterative solutions, via some variants of the extragradient approximants, associated with the pseudomonotone equilibrium problem (EP) and the fixed point problem (FPP) for a finite family of η-demimetric operators in Hilbert spaces. The classical extragradient algorithm is embedded with the inertial extrapolation technique, the parallel hybrid projection technique and the Halpern iterative methods for the variants. The analysis of the approximants is performed under suitable set of constraints and supported with an appropriate numerical experiment for the viability of the approximants.

Keywords: inertial extrapolation technique; parallel hybrid projection technique equilibrium problem; fixed point problem; strong convergence
Mathematics Subject Classification: 47H05, 47H10, 47J25, 49M30, 54H25

1. Introduction

Mathematical modelling provides a systematic formalism for the understanding of the corresponding real-world problem. Moreover, adequate mathematical tools for the analysis of the translated real-world problem are at our disposal. Fixed point theory (FPT), an important branch of nonlinear functional analysis, is prominent for modelling a variety of real-world problems. It is worth mentioning that the real-world phenomenon can be translated into well known existential as well as computational FPP.

The EP theory provides an other systematic formalism for modelling the real-world problems with possible applications in optimization theory, variational inequality theory and game theory [7, 10, 13, 17–19, 22, 25, 28, 31, 32]. In 1994, Blum and Oettli [13] proposed the (monotone-) EP in Hilbert spaces. Since then various classical iterative algorithms are employed to compute the optimal solution of the (monotone-) EP and the FPP. It is remarked that the convergence characteristic and the speed of convergence are the principal attributes of an iterative algorithm. All the classical iterative algorithms from FPT or EP theory have a common shortcoming that the convergence characteristic occurs with respect to the weak topology. In order to enforce the strong convergence characteristic, one has to assume stronger assumptions on the domain and/or constraints. Moreover, strong convergence characteristic of an iterative algorithm is often more desirable than weak convergence characteristic in an infinite dimensional framework.

The efficiency of an iterative algorithm can be improved by employing the inertial extrapolation technique [29]. This technique has successfully been combined with the different classical iterative algorithms; see e.g., [2–6, 8, 9, 14–16, 23, 27]. On the other hand, the parallel architecture of the algorithm helps to reduce the computational cost.

In 2006, Tada and Takahashi [33] suggested a hybrid framework for the analysis of monotone EP and FPP in Hilbert spaces. On the other hand, the iterative algorithm proposed in [33] fails for the case of pseudomonotone EP. In order to address this issue, Anh [1] suggested a hybrid extragradient method, based on the seminal work of Korpelevich [24], to address the pseudomonotone EP together with the FPP. Inspired by the work of Anh [1], Hieu et al. [22] suggested a parallel hybrid extragradient framework to address the pseudomonotone EP together with the FPP associated with nonexpansive operators.

Inspired and motivated by the ongoing research, it is natural to study the pseudomonotone EP together with the FPP associated with the class of an η-demimetric operators. We therefore, suggest some variants of the classical Mann iterative algorithm [26] and the Halpern iterative algorithm [20] in Hilbert spaces. We formulate these variants endowed with the inertial extrapolation technique and parallel hybrid architecture for speedy strong convergence results in Hilbert spaces.

The rest of the paper is organized as follows. We present some relevant preliminary concepts and useful results regarding the pseudomonotone EP and FPP in Section 2. Section 3 comprises strong convergence results of the proposed variants of the parallel hybrid extragradient algorithm as well as Halpern iterative algorithm under suitable set of constraints. In Section 4, we provide detailed numerical results for the demonstration of the main results in Section 3 as well as the viability of the proposed variants with respect to various real-world applications.

2. Preliminaries

Throughout this section, the triplet $(\mathcal{H}, <\cdot,\cdot>, \|\cdot\|)$ denotes the real Hilbert space, the inner product and the induced norm, respectively. The symbolic representation of the weak and strong convergence characteristic are $\rightharpoonup$ and $\rightarrow$, respectively. Recall that a Hilbert space satisfies the Opial's condition, i.e., for a sequence $(p_k) \subset \mathcal{H}$ with $p_k \rightharpoonup \nu$ then the inequality $\liminf_{k\to\infty} \|p_k - \nu\| < \liminf_{k\to\infty} \|p_k - \mu\|$ holds for all $\mu \in \mathcal{H}$ with $\nu \neq \mu$. Moreover, $\mathcal{H}$ satisfies the the Kadec-Klee property, i.e., if $p_k \rightharpoonup \nu$ and $\|p_k\| \to \|\nu\|$ as $k \to \infty$, then $\|p_k - \nu\| \to 0$ as $k \to \infty$.

For a nonempty closed and convex subset $K \subseteq \mathcal{H}$, the metric projection operator $\Pi_K^{\mathcal{H}} : \mathcal{H} \to K$ is defined as $\Pi_K^{\mathcal{H}}(\mu) = argmin_{\nu\in K}\|\mu - \nu\|$. If $T : \mathcal{H} \to \mathcal{H}$ is an operator then $Fix(T) = \{\nu \in \mathcal{H} | \nu = T\nu\}$ represents the set of fixed points of the operator T. Recall that the operator T is called η-demimetric (see [35]) where $\eta \in (-\infty, 1)$, if $Fix(T) \neq \emptyset$ and

$$\langle \mu - \nu, \mu - T\mu \rangle \geq \frac{1}{2}(1-\eta)\|\mu - T\mu\|^2, \ \ \forall \ \mu \in \mathcal{H} \text{ and } \nu \in Fix(T).$$

The above definition is equivalently represented as

$$\|T\mu - \nu\|^2 \leq \|\mu - \nu\|^2 + \eta\|\mu - T\mu\|^2, \ \ \forall \ \mu \in \mathcal{H} \text{ and } \nu \in Fix(T),$$

Recall also that a bifunction $g : K \times K \to \mathbb{R} \cup \{+\infty\}$ is coined as (i) monotone if $g(\mu,\nu) + g(\nu,\mu) \leq 0, \text{ for all } \mu,\nu \in K$; and (ii) strongly pseudomonotone if $g(\mu,\nu) \geq 0 \Rightarrow g(\nu,\mu) \leq -\alpha\|\mu - \nu\|^2, \text{ for all } \mu,\nu \in K$, where $\alpha > 0$. It is worth mentioning that the monotonicity of a bifunction implies the pseudo-monotonicity, but the converse is not true. Recall the EP associated with the bifunction g is to find $\mu \in K$ such that $g(\mu,\nu) \geq 0$ for all $\nu \in K$. The set of solutions of the equilibrium problem is denoted by $EP(g)$.

Assumption 2.1. *[12, 13] Let $g : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying the following assumptions:*
(A1) g is pseudomonotone, i.e., $g(\mu,\nu) \geq 0 \Rightarrow g(\mu,\nu) \leq 0$, for all $\mu,\nu \in K$;
(A2) g is Lipschitz-type continuous, i.e., there exist two nonnegative constants d_1, d_2 such that

$$g(\mu,\nu) + g(\nu,\xi) \geq g(\mu,\xi) - d_1\|\mu - \nu\|^2 - d_2\|\nu - \xi\|^2, \textit{ for all } \mu,\nu,\xi \in K;$$

(A3) g is weakly continuous on $K \times K$ imply that, if $\mu,\nu \in K$ and (p_k), (q_k) are two sequences in K such that $p_k \rightharpoonup \mu$ and $q_k \rightharpoonup \nu$ respectively, then $f(p_k,q_k) \to f(\mu,\nu)$;
(A4) For each fixed $\mu \in K$, $g(\mu,\cdot)$ is convex and subdifferentiable on K.

In view of the Assumption 2.1, $EP(g)$ associated with the bifunction g is weakly closed and convex.

Let $g_i : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a finite family of bifunctions satisfying Assumption 2.1. Then for all $i \in \{1, 2, \cdots, M\}$, we can compute the same Lipschitz coefficients (d_1, d_2) for the family of bifunctions g_i by employing the condition (A2) as

$$g_i(\mu,\xi) - g_i(\mu,\nu) - g_i(\nu,\xi) \leq d_{1,i}\|\mu - \nu\|^2 + d_{2,i}\|\nu - \xi\|^2 \leq d_1\|\mu - \nu\|^2 + d_2\|\nu - \xi\|^2,$$

where $d_1 = \max_{1\leq i\leq M}\{d_{1,i}\}$ and $d_2 = \max_{1\leq i\leq M}\{d_{2,i}\}$. Therefore, $g_i(\mu,\nu) + g_i(\nu,\xi) \geq g_i(\mu,\xi) - d_1\|\mu - \nu\|^2 - d_2\|\nu - \xi\|^2$. In addition, we assume $T_j : \mathcal{H} \to \mathcal{H}$ to be a finite family of η-demimetric operators such that $\Gamma := \left(\bigcap_{i=1}^M EP(g_i)\right) \cap \left(\bigcap_{j=1}^N Fix(T_j)\right) \neq \emptyset$. Then we are interested in the following problem:

$$\hat{p} \in \Gamma. \tag{2.1}$$

Lemma 2.2. *[11] Let $\mu, \nu \in \mathcal{H}$ and $\beta \in \mathbb{R}$ then*

(1) $\|\mu + \nu\|^2 \leq \|\mu\|^2 + 2\langle \nu, \mu + \nu\rangle$*;*

(2) $\|\mu - \nu\|^2 = \|\mu\|^2 - \|\nu\|^2 - 2\langle \mu - \nu, \nu\rangle$*;*

(3) $\|\beta\mu + (1-\beta)\nu\|^2 = \beta\|\mu\|^2 + (1-\beta)\|\nu\|^2 - \beta(1-\beta)\|\mu - \nu\|^2$.

Lemma 2.3. *[35] Let $T : K \to \mathcal{H}$ be an η-demimetric operator defined on a nonempty, closed and convex subset K of a Hilbert space $\mathcal{H}$ with $\eta \in (-\infty, 1)$. Then $Fix(T)$ is closed and convex.*

Lemma 2.4. *[36] Let $T : K \to \mathcal{H}$ be an η-demimetric operator defined on a nonempty, closed and convex subset K of a Hilbert space $\mathcal{H}$ with $\eta \in (-\infty, 1)$. Then the operator $L = (1-\gamma)Id + \gamma T$ is quasi-nonexpansive provided that $Fix(T) \neq \emptyset$ and $0 < \gamma < 1 - \eta$.*

Lemma 2.5. *[11] Let $T : K \to K$ be a nonexpansive operator defined on a nonempty closed convex subset K of a real Hilbert space $\mathcal{H}$ and let (p_k) be a sequence in K. If $p_k \rightharpoonup x$ and if $(Id - T)p_k \to 0$, then $x \in Fix(T)$.*

Lemma 2.6. *[37] Let $h : K \to \mathbb{R}$ be a convex and subdifferentiable function on nonempty closed and convex subset K of a real Hilbert space $\mathcal{H}$. Then, p_* solves the $\min\{h(q) : q \in K\}$, if and only if $0 \in \partial h(p_*) + N_K(p_*)$, where $\partial h(\cdot)$ denotes the subdifferential of h and $N_K(\bar{p})$ is the normal cone of K at $\bar{p}$.*

3. Algorithm and convergence analysis

Our main iterative algorithm of this section has the following architecture:

Algorithm 1 Parallel Hybrid Inertial Extragradient Algorithm (Alg.1)

Initialization: Choose arbitrarily, $p_0, p_1 \in \mathcal{H}$, $K \subseteq \mathcal{H}$ and $C_1 = \mathcal{H}$. Set $k \geq 1$, $\{\alpha_1, \cdots, \alpha_N\} \subset (0,1)$ such that $\sum_{j=1}^{N} \alpha_j = 1$, $0 < \mu < min(\frac{1}{2d_1}, \frac{1}{2d_2})$, $\xi_k \subset [0,1)$ and $\gamma_k \in (0, \infty)$.

Iterative Steps: Given $p_k \in \mathcal{H}$, calculate e_k, $\bar{v}_k$ and w_k as follows:

Step 1. Compute

$$\begin{cases} e_k = p_k + \xi_k(p_k - p_{k-1}); \\ u_{i,k} = \arg\min\{\mu g_i(e_k, v) + \frac{1}{2}\|e_k - v\|^2 : v \in K\}, \quad i = 1, 2, \cdots, M; \\ v_{i,k} = \arg\min\{\mu g_i(u_{i,k}, v) + \frac{1}{2}\|e_k - v\|^2 : v \in K\}, \quad i = 1, 2, \cdots, M; \\ i_k = \arg\max\{\|v_{i,k} - p_k\| : i = 1, 2, \cdots, M\}, \bar{v}_k = v_{i_k,k}; \\ w_k = \sum_{j=1}^{N} \alpha_j((1-\gamma_k)Id + \gamma_k T_j)\bar{v}_k; \end{cases}$$

If $w_k = \bar{v}_k = e_k = p_k$ then terminate and p_k solves the problem 2.1. Else

Step 2. Compute

$$\begin{aligned} C_{k+1} &= \{z \in C_k : \|w_k - z\|^2 \leq \|p_k - z\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - z, p_k - p_{k-1}\rangle\}, \\ p_{k+1} &= \Pi_{C_{k+1}}^{\mathcal{H}} p_1, \ \forall\, k \geq 1. \end{aligned}$$

Set $k =: k + 1$ and return to **Step 1.**

Theorem 3.1. *Let the following conditions:*

(C1) $\sum_{k=1}^{\infty} \xi_k \|p_k - p_{k-1}\| < \infty$;
(C2) $0 < a^* \le \gamma_k \le min\{1-\eta_1, \cdots, 1-\eta_N\}$,

hold. Then Algorithm 1 solves the problem 2.1.

The following result is crucial for the strong convergence result of the Algorithm 1.

Lemma 3.2. *[1, 30] Suppose that $v_* \in EP(g_i)$, and p_k, e_k, $u_{i,k}$, $v_{i,k}$, $i \in \{1, 2, \cdots, M\}$ are defined in Step 1 of the Algorithm 1. Then we have*

$$\|v_{i,k} - v_*\|^2 \le \|e_k - v_*\|^2 - (1 - 2\mu d_1)\|u_{i,k} - e_k\|^2 - (1 - 2\mu d_2)\|u_{i,k} - v_{i,k}\|^2.$$

Proof of Theorem 3.1.
Step 1. The Algorithm 1 is stable.
Observe the following representation of the set C_{k+1}:

$$C_{k+1} = \{z \in C_k : \langle w_k - p_k, z\rangle \le \frac{1}{2}(\|w_k\|^2 - \|p_k\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - z, p_k - p_{k-1}\rangle)\}.$$

This infers that C_{k+1} is closed and convex for all $k \ge 1$. It is well-known that $EP(g_i)$ and $Fix(T_j)$ (from the Assumption 2.1 and Lemma 2.3, respectively) are closed and convex. Hence Γ is nonempty, closed and convex. For any $p_* \in \Gamma$, it follows from Algorithm 1 that

$$\begin{aligned} \|e_k - p_*\|^2 &= \|p_k - p_* + \xi_k(p_k - p_{k-1})\|^2 \\ &\le \|p_k - p_*\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - p_*, p_k - p_{k-1}\rangle. \end{aligned} \tag{3.1}$$

From (3.1) and recalling Lemma 2.4, we obtain

$$\begin{aligned} \|w_k - p_*\| &= \Big\|\sum_{j=1}^{N} \alpha_j((1-\gamma_k)Id + \gamma_k T_j)\bar{v}_k - p_*\Big\| \le \sum_{j=1}^{N} \alpha_j\|((1-\gamma_k)Id + \gamma_k T_j)\bar{v}_k - p_*\| \\ &\le \sum_{j=1}^{N} \alpha_j\|\bar{v}_k - p_*\| = \|\bar{v}_k - p_*\|. \end{aligned}$$

Now recalling Lemma 3.2, the above estimate implies that

$$\begin{aligned} \|w_k - p_*\|^2 &\le \|\bar{v}_k - p_*\|^2 \\ &\le \|p_k - p_*\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - p_*, p_k - p_{k-1}\rangle. \end{aligned} \tag{3.2}$$

The above estimate (3.2) infers that $\Gamma \subset C_{k+1}$. It is now clear from these facts that the Algorithm 1 is well-defined.
Step 2. The limit $\lim_{k\to\infty} \|p_k - p_1\|$ exists.
From $p_{k+1} = \Pi_{C_{k+1}}^{\mathcal{H}} p_1$, we have $\langle p_{k+1} - p_1, p_{k+1} - v\rangle \le 0$ for each $v \in C_{k+1}$. In particular, we have $\langle p_{k+1} - p_1, p_{k+1} - p_*\rangle \le 0$ for each $p_* \in \Gamma$. This proves that the sequence $(\|p_k - p_1\|)$ is bounded. However, from $p_k = \Pi_{C_k}^{\mathcal{H}_1} p_1$ and $p_{k+1} = \Pi_{C_{k+1}}^{\mathcal{H}_1} p_1 \in C_{k+1}$, we have that

$$\|p_k - p_1\| \le \|p_{k+1} - p_1\|.$$

This infers that $(\|p_k - p_1\|)$ is nondecreasing and hence

$$\lim_{k\to\infty} \|p_k - p_1\| \quad \text{exists.} \tag{3.3}$$

Step 3. $\tilde{p}_* \in \Gamma$.
Compute

$$\begin{aligned} \|p_{k+1} - p_k\|^2 &= \|p_{k+1} - p_1 + p_1 - p_k\|^2 \\ &= \|p_{k+1} - p_1\|^2 + \|p_k - p_1\|^2 - 2\langle p_k - p_1, p_{k+1} - p_1\rangle \\ &= \|p_{k+1} - p_1\|^2 + \|p_k - p_1\|^2 - 2\langle p_k - p_1, p_{k+1} - p_k + p_k - p_1\rangle \\ &= \|p_{k+1} - p_1\|^2 - \|p_k - p_1\|^2 - 2\langle p_k - p_1, p_{k+1} - p_k\rangle \\ &\leq \|p_{k+1} - p_1\|^2 - \|p_k - p_1\|^2. \end{aligned}$$

Utilizing (3.3), the above estimate infers that

$$\lim_{k\to\infty} \|p_{k+1} - p_k\| = 0. \tag{3.4}$$

Recalling the definition of (e_k) and the condition (C1), we have

$$\lim_{k\to\infty} \|e_k - p_k\| = \lim_{k\to\infty} \xi_k \|p_k - p_{k-1}\| = 0. \tag{3.5}$$

Recalling (3.4) and (3.5), the following relation

$$\|e_k - p_{k+1}\| \leq \|e_k - p_k\| + \|p_k - p_{k+1}\|,$$

infers that

$$\lim_{k\to\infty} \|e_k - p_{k+1}\| = 0. \tag{3.6}$$

Note that $p_{k+1} \in C_{k+1}$, therefore the following relation

$$\|w_k - p_{k+1}\| \leq \|p_k - p_{k+1}\| + 2\xi_k \|p_k - p_{k-1}\| + 2\xi_k \langle p_k - p_{k+1}, p_k - p_{k-1}\rangle,$$

infers, on employing (3.4) and the condition (C1), that

$$\lim_{k\to\infty} \|w_k - p_{k+1}\| = 0. \tag{3.7}$$

Again, recalling (3.4) and (3.7), the following relation

$$\|w_k - p_k\| \leq \|w_k - p_{k+1}\| + \|p_{k+1} - p_k\|$$

infers that

$$\lim_{k\to\infty} \|w_k - p_k\| = 0. \tag{3.8}$$

In view of the condition (C2), observe the variant of (3.2)

$$\begin{aligned} &(1 - 2\mu d_1)\|u_{i_k,k} - e_k\|^2 - (1 - 2\mu d_2)\|u_{i_k,k} - v_{i_k,k}\|^2 \\ &\quad \leq (\|p_k - p_*\| + \|w_k - p_*\|)\|p_k - w_k\| + \xi_k^2 \|p_k - p_{k-1}\|^2 + 2\xi_k \|p_k - p_*\| \|p_k - p_{k-1}\|. \end{aligned}$$

Recalling (3.8) and condition (C1), we get

$$(1 - 2\mu d_1) \lim_{k\to\infty} \|u_{i_k,k} - e_k\|^2 - (1 - 2\mu d_2) \lim_{k\to\infty} \|u_{i_k,k} - v_{i_k,k}\|^2 = 0. \tag{3.9}$$

The above estimate (3.9) implies that

$$\lim_{k\to\infty} \|u_{i_k,k} - e_k\|^2 = \lim_{k\to\infty} \|u_{i_k,k} - v_{i_k,k}\|^2 = 0. \tag{3.10}$$

Reasoning as above, recalling (3.5), (3.8) and (3.10), we have

- $\|\bar{v}_k - e_k\| \leq \|\bar{v}_k - u_{i_k,k}\| + \|u_{i_k,k} - e_k\| \to 0$;
- $\|\bar{v}_k - p_k\| \leq \|\bar{v}_k - e_k\| + \|e_k - p_k\| \to 0$;
- $\|w_k - e_k\| \leq \|w_k - p_k\| + \|p_k - e_k\| \to 0$;
- $\|w_k - \bar{v}_k\| \leq \|w_k - e_k\| + \|e_k - \bar{v}_k\| \to 0$.

In view of the estimate $\lim_{k\to\infty} \|w_k - \bar{v}_k\| = 0$, we have

$$\lim_{k\to\infty} \|T_j\bar{v}_k - \bar{v}_k\| = 0, \ \ \forall \ \ j = \{1, 2, \cdots, N\}. \tag{3.11}$$

Next, we show that $\tilde{p}_* \in \bigcap_{i=1}^{M} EP(g_i)$.
Observe that

$$u_{i,k} = \arg\min\{\mu g_i(e_k, \nu) + \frac{1}{2}\|e_k - \nu\|^2 : \nu \in K\}.$$

Recalling Lemma 2.6, we get

$$0 \in \partial_2\{\mu g_i(e_k, \nu) + \frac{1}{2}\|e_k - \nu\|^2\}(u_{i,k}) + N_K(u_{i,k}).$$

This implies the existence of $\tilde{x} \in \partial_2 g_i(e_k, u_{i,k})$ and $\tilde{x}_* \in N_K(u_{i,k})$ such that

$$\mu\tilde{x} + e_k - u_{i,k} + \tilde{x}_*. \tag{3.12}$$

Since $\tilde{x}_* \in N_K(u_{i,k})$ and $\langle \tilde{x}_*, \nu - u_{i,k}\rangle \leq 0$ for all $\nu \in K$. Therefore recalling (3.12), we have

$$\mu\langle \tilde{x}, \nu - u_{i,k}\rangle \geq \langle u_{i,k} - e_k, \nu - u_{i,k}\rangle, \ \ \forall \ \ \nu \in K. \tag{3.13}$$

Since $\tilde{x} \in \partial_2 g_i(e_k, u_{i,k})$,

$$g_i(e_k, \nu) - g_i(e_k, u_{i,k}) \geq \langle p, \nu - u_{i,k}\rangle, \ \ \forall \ \ \nu \in K. \tag{3.14}$$

Therefore recalling (3.13) and (3.14), we obtain

$$\mu(g_i(e_k, \nu) - g_i(e_k, u_{i,k})) \geq \langle u_{i,k} - e_k, \nu - u_{i,k}\rangle, \ \ \forall \ \ \nu \in K. \tag{3.15}$$

Observe from the fact that (p_k) is bounded then $p_{k_t} \rightharpoonup \tilde{p}_* \in \mathcal{H}$ as $t \to \infty$ for a subsequence (p_{k_t}) of (p_k). This also infers that $\bar{w}_{k_t} \rightharpoonup \tilde{p}_*$, $\bar{v}_{k_t} \rightharpoonup \tilde{p}_*$ and $b_{k_t} \rightharpoonup \tilde{p}_*$ as $t \to \infty$. Since $e_k \rightharpoonup \tilde{p}_*$ and $\|e_k - u_{i,k}\| \to 0$ as $k \to \infty$, this implies $u_{i,k} \rightharpoonup \tilde{p}_*$. Recalling the assumption (A3) and (3.15), we deduce that $g_i(\tilde{p}_*, \nu) \geq 0$ for all $\nu \in K$ and $i \in \{1, 2, \cdots, M\}$. Therefore, $\tilde{p}_* \in \bigcap_{i=1}^{M} EP(g_i)$. Moreover, recall that $\bar{v}_{k_t} \rightharpoonup \tilde{p}_*$ as $t \to \infty$ and (3.11) we have $\tilde{p}_* \in \bigcap_{j=1}^{N} Fix(T_j)$. Hence $\tilde{p}_* \in \Gamma$.
Step 4. $p_k \to p_* = \Pi_{\Gamma}^{\mathcal{H}} p_1$.
Since $p_* = \Pi_{\Gamma}^{\mathcal{H}} p_1$ and $\tilde{p}_* \in \Gamma$, therefore we have $p_{k+1} = \Pi_{C_{k+1}}^{\mathcal{H}} p_1$ and $p_* \in \Gamma \subset C_{k+1}$. This implies that

$$\|p_{k+1} - p_1\| \leq \|p_* - p_1\|.$$

By recalling the weak lower semicontinuity of the norm, we have

$$\|p_1 - p_*\| \leq \|p_1 - \tilde{p}_*\| \ \ \leq \ \ \liminf_{t\to\infty} \|p_1 - p_{k_t}\| \leq \limsup_{t\to\infty} \|p_1 - p_{k_t}\| \leq \|p_1 - p_*\|.$$

Recalling the uniqueness of the metric projection operator yields that $\tilde{p}_* = p_* = \Pi_{\Gamma}^{\mathcal{H}} p_1$. Also $\lim_{t\to\infty} \|p_{k_t} - p_1\| = \|p_* - p_1\| = \|\tilde{p}_* - p_1\|$. Moreover, recalling the Kadec-Klee property of $\mathcal{H}$ with the fact that $p_{k_t} - p_1 \rightharpoonup \tilde{p}_* - p_1$, we have $p_{k_t} - p_1 \to \tilde{p}_* - p_1$ and hence $p_{k_t} \to \tilde{p}_*$. This completes the proof. ■

Corollary 3.3. *Let $K \subseteq \mathcal{H}$ be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. For all $i \in \{1, 2, \cdots, M\}$, let $g_i : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a finite family of bifunctions satisfying Assumption 2.1. Assume that $\Gamma := \bigcap_{i=1}^{M} EP(g_i) \neq \emptyset$, such that*

$$\begin{cases} e_k = p_k + \xi_k(p_k - p_{k-1}); \\ u_{i,k} = \arg\min\{\mu g_i(e_k, v) + \frac{1}{2}\|e_k - v\|^2 : v \in K\}, \quad i = 1, 2, \cdots, M; \\ v_{i,k} = \arg\min\{\mu g_i(u_{i,k}, v) + \frac{1}{2}\|e_k - v\|^2 : v \in K\}, \quad i = 1, 2, \cdots, M; \\ i_k = \arg\max\{\|v_{i,k} - p_k\| : i = 1, 2, \cdots, M\}, \bar{v}_k = v_{i_k,k}; \\ C_{k+1} = \{z \in C_k : \|\bar{v}_k - z\|^2 \leq \|p_k - z\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - z, p_k - p_{k-1}\rangle\}; \\ p_{k+1} = \Pi^{\mathcal{H}}_{C_{k+1}} p_1, \forall\, k \geq 1. \end{cases} \tag{3.16}$$

Assume that the condition (C1) holds, then the sequence (p_k) generated by (3.16) strongly converges to a point in Γ.

We now propose an other variant of the hybrid iterative algorithm embedded with the Halpern iterative algorithm [20].

Algorithm 2 Parallel Hybrid Inertial Halpern-Extragradient Algorithm (Alg.2)

Initialization: Choose arbitrarily $q, p_0, p_1 \in \mathcal{H}$, $K \subseteq \mathcal{H}$ and $C_1 = \mathcal{H}$. Set $k \geq 1$, $\{\alpha_1, \cdots, \alpha_N\}, \beta_k \subset (0, 1)$ such that $\sum_{j=1}^{N} \alpha_j = 1$, $0 < \mu < min(\frac{1}{2d_1}, \frac{1}{2d_2})$, $\xi_k \subset [0, 1)$ and $\gamma_k \in (0, \infty)$.

Iterative Steps: Given $p_k \in \mathcal{H}$, calculate e_k, $\bar{v}_k$ and w_k as follows:

Step 1. Compute

$$\begin{cases} e_k = p_k + \xi_k(p_k - p_{k-1}); \\ u_{i,k} = \arg\min\{\mu g_i(e_k, v) + \frac{1}{2}\|e_k - v\|^2 : v \in K\}, \quad i = 1, 2, \cdots, M; \\ v_{i,k} = \arg\min\{\mu g_i(u_{i,k}, v) + \frac{1}{2}\|e_k - v\|^2 : v \in K\}, \quad i = 1, 2, \cdots, M; \\ i_k = \arg\max\{\|v_{i,k} - p_k\| : i = 1, 2, \cdots, M\}, \bar{v}_k = v_{i_k,k}; \\ w_k = \sum_{j=1}^{N} \alpha_j((1 - \gamma_k)Id + \gamma_k T_j)\bar{v}_k; \\ t_{l,k} = \beta_k q + (1 - \beta_k)w_k; \\ l_k = \arg\max\{\|t_{j,k} - p_k\| : j = 1, 2, \cdots, P\}, \bar{t}_k = t_{l_k,k}. \end{cases}$$

If $\bar{t}_k = w_k = \bar{v}_k = e_k = p_k$ then terminate and p_k solves the problem 2.1. Else

Step 2. Compute

$$\begin{aligned} C_{k+1} &= \{z \in C_k : \|\bar{t}_k - z\|^2 \leq \beta_k\|q - z\|^2 + (1 - \beta_k)(\|p_k - z\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 \\ &\quad + 2\xi_k\langle p_k - z, p_k - p_{k-1}\rangle)\}; \\ p_{k+1} &= \Pi^{\mathcal{H}}_{C_{k+1}} p_1, \ \forall\, k \geq 1. \end{aligned}$$

Set $k =: k + 1$ and go back to **Step 1.**

Remark 3.4. *Note for the Algorithm 2 that the claim p_k is a common solution of the EP and FPP provided that $p_{k+1} = p_k$, in general is not true. So intrinsically a stopping criterion is implemented for $k > k_{max}$ for some chosen sufficiently large number k_{max}.*

Theorem 3.5. *Let $\Gamma \neq \emptyset$ and the following conditions:*

(C1) $\sum_{k=1}^{\infty} \xi_k\|p_k - p_{k-1}\| < \infty$;

(C2) $0 < a^* \leq \gamma_k \leq min\{1 - \eta_1, \cdots, 1 - \eta_N\}$ *and* $\lim_{k\to\infty} \beta_k = 0$, *hold. Then the Algorithm 2 solves the problem 2.1.*

Proof. Observe that the set C_{k+1} can be expressed in the following form:

$$C_{k+1} = \{z \in C_k : \|\bar{t}_k - z\|^2 \leq \beta_k\|q - z\|^2 + (1-\beta_k)(\|p_k - z\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - z, p_k - p_{k-1}\rangle)\}.$$

Recalling the proof of Theorem 3.1, we deduce that the sets Γ and C_{k+1} are closed and convex satisfying $\Gamma \subset C_{k+1}$ for all $k \geq 0$. Further, (p_k) is bounded and

$$\lim_{k\to\infty} \|p_{k+1} - p_k\| = 0. \tag{3.17}$$

Since $p_{k+1} = \Pi^{\mathcal{H}}_{C_{k+1}}(q) \in C_{k+1}$, we have

$$\|\bar{t}_k - p_{k+1}\|^2 \leq \beta_k\|q - p_{k+1}\|^2 + (1-\beta_k)(\|p_k - p_{k+1}\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - p_{k+1}, p_k - p_{k-1}\rangle).$$

Recalling the estimate (3.17) and the conditions (C1) and (C2), we obtain

$$\lim_{k\to\infty} \|\bar{t}_k - p_{k+1}\| = 0.$$

Reasoning as above, we get

$$\lim_{k\to\infty} \|\bar{t}_k - p_k\| = 0.$$

The rest of the proof of Theorem 3.5 follows from the proof of Theorem 3.1 and is therefore omitted. ■

The following remark elaborate how to align condition (C1) in a computer-assisted iterative algorithm.

Remark 3.6. *We remark here that the condition* (C1) *can easily be aligned in a computer-assisted iterative algorithm since the value of* $\|p_k - p_{k-1}\|$ *is quantified before choosing* ξ_k *such that* $0 \leq \xi_k \leq \widehat{\xi_k}$ *with*

$$\widehat{\xi_k} = \begin{cases} \min\{\frac{\sigma_k}{\|p_k - p_{k-1}\|}, \xi\} & if \ p_k \neq p_{k-1}; \\ \xi & otherwise. \end{cases}$$

Here $\{\sigma_k\}$ *denotes a sequence of positives* $\sum_{k=1}^{\infty} \sigma_k < \infty$ *and* $\xi \in [0,1)$.

As a direct application of Theorem 3.1, we have the following variant of the problem 2.1, namely the generalized split variational inequality problem associated with a finite family of single-valued monotone and hemicontinuous operators $A_j : K \to \mathcal{H}$ defined on a nonempty closed convex subset K of a real Hilbert space $\mathcal{H}$ for each $j \in \{1, 2, \cdots, N\}$. The set $VI(K, A)$ represents all the solutions of the following variational inequality problem $\langle A\mu, \nu - \mu\rangle \geq 0 \ \forall \ \nu \in C$.

Theorem 3.7. *Assume that* $\Gamma = \bigcap_{i=1}^{M} VI(C, A_i) \cap \bigcap_{j=1}^{N} Fix(T_j) \neq \emptyset$ *and the conditions (C1)–(C4) hold. Then the sequence* (p_k)

$$\begin{cases} e_k = p_k + \xi_k(p_k - p_{k-1}); \\ u_{i,k} = \Pi_K(e_k - \mu A_i(e_k)), \quad i = 1, 2, \cdots, M; \\ v_{i,k} = \Pi_K(e_k - \mu A_i(u_{i,k})), \quad i = 1, 2, \cdots, M; \\ i_k = \arg\max\{\|v_{i,k} - p_k\| : i = 1, 2, \cdots, M\}, \bar{v}_k = v_{i_k,k}; \\ w_k = \sum_{j=1}^{N} \alpha_j((1-\gamma_k)Id + \gamma_k T_j)\bar{v}_k; \\ C_{k+1} = \{z \in C_k : \|w_k - z\|^2 \leq \|p_k - z\|^2 + \xi_k^2\|p_k - p_{k-1}\|^2 + 2\xi_k\langle p_k - z, p_k - p_{k-1}\rangle\}, \\ p_{k+1} = \Pi^{\mathcal{H}}_{C_{k+1}} p_1, \ \forall \ k \geq 1, \end{cases} \tag{3.18}$$

generated by (3.18) solves the problem 2.1.

Proof. Observe that, if we set $g_i(\bar{\mu}, \bar{\nu}) = \langle A_i(\bar{\mu}), \bar{\nu} - \bar{\mu}\rangle$ for all $\bar{\mu}, \bar{\nu} \in K$, then each A_i being L-Lipschitz continuous infers that g_i is Lipschitz-type continuous with $d_1 = d_2 = \frac{L}{2}$. Moreover, the pseudo-monotonicity of A_i ensures the pseudo-monotonicity of g_i. Recalling the assumptions (A3)–(A4) and the Algorithm 1, note that

$$\begin{aligned} u_{i,k} &= \arg\min\{\mu\langle A_i(p_k), \nu - p_k\rangle + \frac{1}{2}\|p_k - \nu\|^2 : \nu \in K\}; \\ v_{i,k} &= \arg\min\{\mu\langle A_i(u_{i,k}), \nu - u_{i,k}\rangle + \frac{1}{2}\|p_k - \nu\|^2 : \nu \in K\}, \end{aligned}$$

can be transformed into

$$\begin{aligned} u_{i,k} &= \arg\min\{\frac{1}{2}\|\nu - (p_k - \mu A_i(p_k)\|^2 : \nu \in K\} = \Pi_K(p_k - \mu A_i(p_k)); \\ v_{i,k} &= \arg\min\{\frac{1}{2}\|\nu - (p_k - \mu A_i(u_{i,k})\|^2 : \nu \in K\} = \Pi_K(p_k - \mu A_i(u_{i,k})). \end{aligned}$$

Hence recalling $g_i(\bar{\mu}, \bar{\nu}) = \langle A_i(\bar{\mu}), \bar{\nu} - \bar{\mu}\rangle$ for all $\bar{\mu}, \bar{\nu} \in K$ and for all $i \in \{1, 2, \cdots, M\}$ in Theorem 3.1, we have the desired result. ■

4. Numerical experiment and results

This section provides the effective viability of the algorithm via a suitable numerical experiment.

Example 4.1. *Let $\mathcal{H} = \mathbb{R}$ be the set of all real numbers with the inner product defined by $\langle p, q\rangle = pq$, for all $p, q \in \mathbb{R}$ and the induced usual norm $|\cdot|$. For each $i = \{1, 2, \cdots, M\}$, let the family of pseudomonotone bifunctions $g_i(p, q) : K \times K \to \mathbb{R}$ on $K = [0, 1] \subset \mathcal{H}$, is defined by $g_i(p, q) = S_i(p)(q - p)$, where*

$$S_i(p) = \begin{cases} 0, & 0 \le p \le \lambda_i; \\ \sin(p - \lambda_i) + \exp(p - \lambda_i) - 1, & \lambda_i \le p \le 1. \end{cases}$$

where $0 < \lambda_1 < \lambda_2 < ... < \lambda_M < 1$. Note that $EP(g_i) = [0, \lambda_i]$ if and only if $0 \le p \le \lambda_i$ and $q \in [0, 1]$. Consequently, $\bigcap_{i=1}^{M} EP(g_i) = [0, \lambda_1]$. For each $j \in \{1, 2, \cdots, N\}$, let the family of operators $T_j : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_j(p) = \begin{cases} -\frac{3p}{j}, & p \in [0, \infty); \\ p, & p \in (-\infty, 0). \end{cases}$$

Clearly, T_j defines a finite family of η-demimetric operators with $\bigcap_{j=1}^{N} Fix(T_j) = \{0\}$. Hence $\Gamma = (\bigcap_{i=1}^{M} EP(g_i)) \cap (\bigcap_{j=1}^{N} Fix(T_j)) = 0$. In order to compute the numerical values of the Algorithm 1, we choose $\xi = 0.5$, $\alpha_k = \frac{1}{100k+1}$, $\mu = \frac{1}{7}$, $\lambda_i = \frac{i}{(M+1)}$, $M = 2 \times 10^5$ and $N = 3 \times 10^5$. Since

$$\begin{cases} \min\{\frac{1}{k^2\|p_k - p_{k-1}\|}, 0.5\} & if \quad p_k \neq p_{k-1}; \\ 0.5 & otherwise, \end{cases}$$

Observe that the expression

$$u_{i,k} = \arg\min\{\mu S_i(e_k)(\nu - e_k) + \frac{1}{2}(y - p_k)^2, \ \forall \ \nu \in [0, 1]\},$$

in the Algorithm 1 is equivalent to the following relation $u_{i,k} = e_k - \mu S_i(e_k)$, *for all* $i \in \{1, 2, \cdots, M\}$. *Similarly* $v_{i,k} = e_k - \mu S_i(ui, k)$, *for all* $i \in \{1, 2, \cdots, M\}$. *Hence, we can compute the intermediate approximation* $\bar{v}_k$ *which is farthest from* e_k *among* $v_{i,k}$, *for all* $i \in \{1, 2, \cdots, M\}$. *Generally, at the* k^{th} *step if* $E_k = \|p_k - p_{k-1}\| = 0$ *then* $p_k \in \Gamma$ *implies that* p_k *is the required solution of the problem. The terminating criteria is set as* $E_k < 10^{-6}$. *The values of the Algorithm 1 and its variant are listed in the following table (see Table 1):*

Table 1. Numerical values of Algorithm 1.

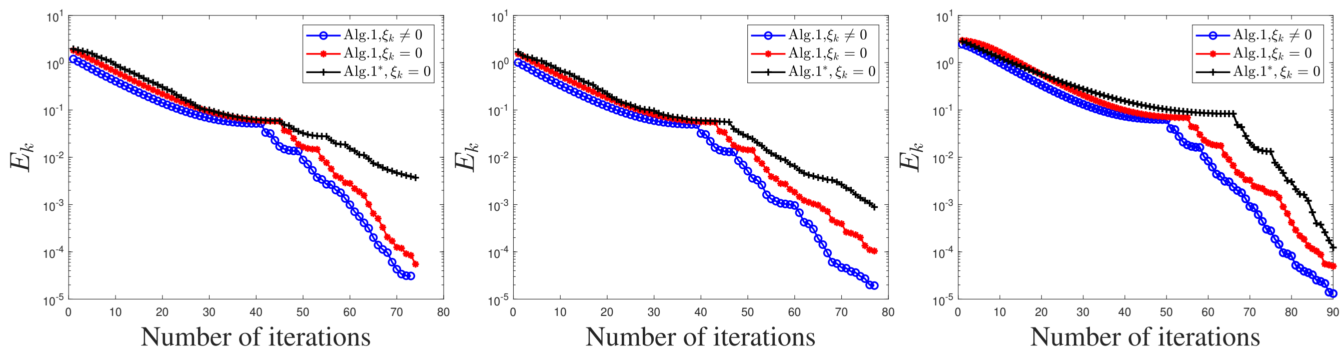
	No. of Iter.		CPU-Time (Sec)	
N0.	Alg.1,$\xi_k = 0$	Alg.1, $\xi_k \neq 0$	Alg.1, $\xi_k = 0$	Alg.1, $\xi_k \neq 0$
Choice 1. $p_0 = (5)$, $p_1 = (2)$	87	75	0.088153	0.073646
Choice 2. $p_0 = (4.3)$, $p_1 = (1.7)$	88	79	0.072250	0.068662
Choice 3. $p_0 = (-7)$, $p_1 = (3)$	99	92	0.062979	0.051163

The values of the non-inertial and non-parallel variant of the Algorithm 1 referred as Alg.1^* *are listed in the following table (see Table 2):*

Table 2. Numerical values of Algorithm Alg.1^*.

No. of Choices	No. of Iter.	CPU-Time (Sec)
Choice 1. $p_0 = (5)$, $p_1 = (2)$	111	0.091439
Choice 2. $p_0 = (4.3)$, $p_1 = (1.7)$	106	0.089872
Choice 3. $p_0 = (-7)$, $p_1 = (3)$	104	0.081547

The error plotting E_k *against the Algorithm 1 and its variants for each choices in Tables 1 and 2 are illustrated in Figure 1.*



(a) Choice 1 **(b)** Choice 2 **(c)** Choice 3

Figure 1. Comparison between Algorithm 1 and its variants in view of Example 4.1.

Example 4.2. *Let $\mathcal{H} = \mathbb{R}^n$ with the induced norm $\|p\| = \sqrt{\sum_{i=1}^{n} |p_i|^2}$ and the inner product $\langle p, q\rangle = \sum_{i=1}^{n} p_i q_i$, for all $p = (p_1, p_2, \cdots, p_n) \in \mathbb{R}^n$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbb{R}^n$. The set K is given by $K = \{p \in \mathbb{R}^n_+ : |p_k| \leq 1\}$, where $k = \{1, 2, \cdots, n\}$. Consider the following problem:*

$$\text{find } p_* \in \Gamma := \bigcap_{i=1}^{M} EP(g_i) \cap \bigcap_{j=1}^{N} Fix(T_j),$$

where $g_i : K \times K \to \mathbb{R}$ is defined by:

$$g_i(p, q) = \sum_{k=1}^{n} S_{i,k}(q_k^2 - p_k^2), \ \forall \ i \in \{1, 2, \cdots, M\},$$

where $S_{i,k} \in (0, 1)$ is randomly generated for all $i = \{1, 2, \cdots, M\}$ and $k = \{1, 2, \cdots, n\}$. For each $j \in \{1, 2, \cdots, N\}$, let the family of operators $T_j : \mathcal{H} \to \mathcal{H}$ be defined by

$$T_j(p) = \begin{cases} -\frac{4p}{j}, & p \in [0, \infty); \\ p, & p \in (-\infty, 0). \end{cases}$$

for all $p \in \mathcal{H}$. It is easy to observe that $\Gamma = \bigcap_{i=1}^{M} EP(g_i) \cap \bigcap_{j=1}^{N} Fix(T_j) = 0$. The values of the Algorithm 1 and its non-inertial variant are listed in the following table (see Table 3):

Table 3. Numerical values of Algorithm 1.

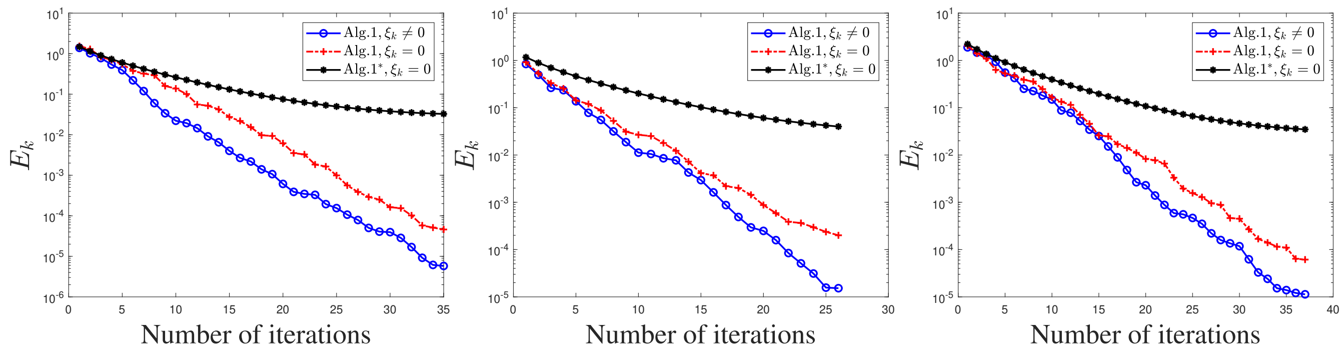
	No. of Iter.		CPU-Time (Sec)	
N0.	Alg.1,$\xi_k = 0$	Alg.1, $\xi_k \neq 0$	Alg.1, $\xi_k = 0$	Alg.1, $\xi_k \neq 0$
Choice 1. $p_0 = (5)$, $p_1 = (2)$, $n = 5$	46	35	0.061975	0.054920
Choice 2. $p_0 = (1)$, $p_1 = (1.5)$,$n = 10$	38	27	0.056624	0.040587
Choice 3. $p_0 = (-8)$, $p_1 = (3)$, $n = 30$	50	37	0.055844	0.041246

The values of the non-inertial and non-parallel variant of the Algorithm 1 referred as Alg.1^* are listed in the following table (see Table 4):

Table 4. Numerical values of Algorithm Alg.1^*.

No. of Choices	No. of Iter.	CPU-Time (Sec)
Choice 1. $p_0 = (5)$, $p_1 = (2)$, $n = 5$	81	0.072992
Choice 2. $p_0 = (1)$, $p_1 = (1.5)$,$n = 10$	75	0.065654
Choice 3. $p_0 = (-8)$, $p_1 = (3)$, $n = 30$	79	0.068238

The error plotting $E_k \leq 10^{-6}$ against the Algorithm 1 and its variants for each choices in Tables 3 and 4 are illustrated in Figure 2.



(a) Choice 1 **(b)** Choice 2 **(c)** Choice 3

Figure 2. Comparison between Algorithm 1 and its variants in view of Example 4.2.

Example 4.3. *Let $L^2([0,1]) = \mathcal{H}$ with induced norm $\|p\| = (\int_0^1 |p(s)|^2 ds)^{\frac{1}{2}}$ and the inner product $\langle p, q\rangle = \int_0^1 p(s)q(s)ds$, for all $p, q \in L^2([0,1])$ and $s \in [0,1]$. The feasible set K is given by: $K = \{p \in L^2([0,1]) : \|p\| \leq 1\}$. Consider the following problem:*

$$\text{find } \bar{p} \in \Gamma := \bigcap_{i=1}^{M} EP(g_i) \cap \bigcap_{j=1}^{N} Fix(T_j),$$

where $g_i(p,q)$ is defined as $\langle S_i p, q - p\rangle$ with the operator $S_i : L^2([0,1]) \to L^2([0,1])$ given by

$$S_i(p(s)) = \max\left\{0, \frac{p(s)}{i}\right\}, \ \forall\, i \in \{1, 2, \cdots, M\}, \ s \in [0,1].$$

Since each g_i is monotone and hence pseudomonotone on C. For each $j \in \{1, 2, \cdots, N\}$, let the family of operators $T_j : \mathcal{H} \to \mathcal{H}$ be defined by

$$T_j(p) = \Pi_C(p) = \begin{cases} \frac{p}{\|p\|}, & \|p\| > 1; \\ p, & \|p\| \leq 1. \end{cases}$$

Then T_j is a finite family of η-demimetric operators. It is easy to observe that $\Gamma = \bigcap_{i=1}^{M} EP(g_i) \cap \bigcap_{j=1}^{N} Fix(T_j) = 0$. Choose $M = 50$ and $N = 100$. The values of the Algorithm 1 and its non-inertial variant have been computed for different choices of p_0 and p_1 in the following table (see Table 5):

Table 5. Numerical values of Algorithm 1.

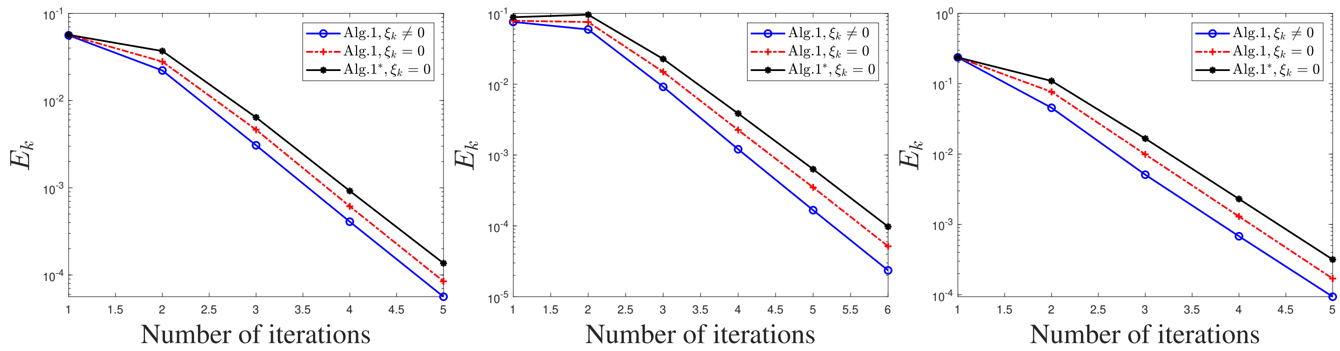
	No. of Iter.		CPU-Time (Sec)	
N0.	Alg.1,$\xi_k = 0$	Alg.1, $\xi_k \neq 0$	Alg.1, $\xi_k = 0$	Alg.1, $\xi_k \neq 0$
Choice 1. $p_0 = exp(3s) \times \sin(s)$, $p_1 = 3s^2 - s$	10	5	1.698210	0.981216
Choice 2. $p_0 = \frac{1}{1+s}$, $p_1 = \frac{s^2}{10}$	14	6	2.884623	1.717623
Choice 3. $p_0 = \frac{\cos(3s)}{7}$, $p_1 = s$	16	5	2.014687	1.354564

The values of the non-inertial and non-parallel variant of the Algorithm 1 referred as Alg.1^* have been computed for different choices of p_0 and p_1 in the following table (see Table 6):

Table 6. Numerical values of Algorithm Alg.1*.

No. of Choices	No. of Iter.	CPU-Time (Sec)
Choice 1. $p_0 = exp(3s) \times \sin(s)$, $p_1 = 3s^2 - s$	23	2.65176
Choice 2. $p_0 = \frac{1}{1+s}$, $p_1 = \frac{s^2}{10}$	27	3.102587
Choice 3. $p_0 = \frac{\cos(3s)}{7}$, $p_1 = s$	26	2.903349

The error plotting $E_k =< 10^{-4}$ against the Algorithm 1 and its variants for each choices in Tables 5 and 6 are illustrated in Figure 3.



(a) Choice 1 **(b)** Choice 2 **(c)** Choice 3

Figure 3. Comparison between Algorithm 1 and its variants in view of Example 4.3.

We can see from Tables 1–6 and Figures 1–3 that the Algorithm 1 out performs its variants with respect to the reduction in the error, time consumption and the number of iterations required for the convergence towards the common solution.

5. Conclusions

In this paper, we have constructed some variants of the classical extragradient algorithm that are embedded with the inertial extrapolation and hybrid projection techniques. We have shown that the algorithm strongly converges towards the common solution of the problem 2.1. A useful instance of the main result, that is, Theorem 3.1, as well as an appropriate example for the viability of the algorithm, have also been incorporated. It is worth mentioning that the problem 2.1 is a natural mathematical model for various real-world problems. As a consequence, our theoretical framework constitutes an important topic of future research.

Conflict of interest

The authors declare that they have no competing interests.

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