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Research article

A study on the fractal-fractional tobacco smoking model

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Abstract: In this article, we consider a fractal-fractional tobacco mathematical model with generalized kernels of Mittag-Leffler functions for qualitative and numerical studies. From qualitative point of view, our study includes; existence criteria, uniqueness of solution and Hyers-Ulam stability. For the numerical aspect, we utilize Lagrange's interpolation polynomial and obtain a numerical scheme which is further illustrated simulations. Lastly, a comparative analysis is presented for different fractal and fractional orders. The numerical results are divided into four figures based on different fractal and fractional orders. We have found that the fractional and fractal orders have a significant impact on the dynamical behaviour of the model.

Keywords: fractal-fractional derivative; existence; tobacco model; stability; simulation; interpolation
Mathematics Subject Classification: 26A33, 34A08, 35R11

1. Introduction

Cigarette smoking is one of the basic reasons of causing high mortality risks. Andrew et al. [1] discussed that one out of five deaths in the United States is due to the tobacco smoking, 28% deaths are due to lung cancer. Cigarette smoking are causing 37% of vascular diseases and 26% are other respiratory problems. In comparison, the male smokers are very high than women smokers. Three million smokers die and half of them before the age of 70 years. Wald et al. [2] investigated that there are one billion smokers worldwide and one third of them belong to China. In several reports it has been noticed that the smokers all over the world mistakenly believe that it has fruitful psychological effects and can help to relieve the feeling of stress [3]. Lloyd et al. [3] believed that the assumption that smoking reduces stress is wrong, and that better awareness and health education are needed to alert smokers about this belief, as it raises stress many times. Cohen and Lichtenstein [4] supported Lloyd et al.'s views and explained that nicotine is not only addictive, but it also increases stress rather than alleviating it. Regular smokers need to maintain their nicotine levels in order to avoid irritability and tension if their plasma nicotine levels drop.

According to recent study, smoking promotes a variety of fatal ailments. It can harm the lungs' air sacs (alveoli) and induce a variety of deadly disorders. It is a major cause of cancer in the bladder, cervix, liver, larynx, oesophagus, colon, and rectum, as well as the neck, tongue, tonsils, soft palate, stomach, pancreas, lung, trachea, bronchus, ureter, kidney, and blood. Tobacco use causes heart problems such as coronary artery and stroke, blood clots in the legs and skin, blood vessel damage and narrowing, reduced blood supply to the brain, and other cardiovascular disorders. It impairs immune system function, causes diabetes and rheumatoid arthritis, and can harm a tiny region near the retina, for detail see Mokdad et al. [5].

Mathematical modeling is one of the best techniques used for the dynamical problems. Researchers used different mathematical tools to produce more realistic analysis. Recently, several scientists worked on the modeling of tobacco smoking, its adverse effects on human body and suggested different optimal control strategies. For example, Zeb et al. [6] presented a new mathematical model for the dynamical study of tobacco smoking and its optimization. Sharomi and Gumel et al. [7] developed a numerical analysis of the tobacco smoking based on their new model and predicted its impact on the population. Alkhudhari et al. [8] presented stability results for their smoking model and given their numerical simulations. For more details on the smoking models and their analytical and numerical studies, we refer the readers to the work in [9–21].

Fractional order modeling have attracted researchers in almost all the fields of science and engineering. Several notions have been developed and were applied to dynamical systems. Although, each version the definition has its own set of limitations. For example, The Riemann-Liouville derivative fails to deliver the traditional case of the derivative of the constant being zero when explaining the importance of the initial condition. Later on, Caputo developed a new definition known as "Caputo's fractional derivative" satisfies the classical example and shown the significance of the initial condition, but due to the singular kernel it had also several limitations. In order to overcome the singularity issue, in 2015, Caputo and Fabrizio proposed a new definition that eliminates all of the aforementioned requirements [22]. Many mathematicians and physicists worked on this operator and found that the non-singular kernel is non-local and the anti-derivative is simply the average of the function and its integral. For the modification of this operator, Atangana and Baleanu [23] proposed a novel fractional differential operator in 2016 to solve all of these restrictions with generalized Mittag-Leffler function as its kernel. The derivatives based on the Mittag-Leffler kernel were applied to a large number of physical problems and were analyzed for the comparative results [24–27]. One can also consider for further details of modelling anthrax in animals [28], Hepatitis C [29], memristor-based circuit [30], Lorenz-Stenflo hyperchaotic system [31], dynamics of environmental persistence of infections [32], Langevin equation [33], genetic regulatory networks [34], Mump virus [35], Zika virus [36,37], mosaic disease [38], Computer viruses [39], thermostat control [40,41], pantograph equation [42,43], canine distemper virus [44], Q-fever [45], hybrid equation of p-Laplacian operators [46], co-dynamics of COVID-19 and diabetes [47], chemical structure of glucose [48], Navier systems [49], and so on [50–56].

It has been observed that the concept of a local operator of differentiation is not very useful for modelling complicated real-world problems including those physical events that exhibit fractal tendencies. In mathematics, a fractal derivative is a nonstandard type of derivative in which the variable is scaled according to t^a. This derivative was developed to simulate physical issues in which conventional physical principles, such as Darcy's, Fourier's, and Fick's laws, are no longer valid. These issues are assumed to be based on Euclidean geometry and are incompatible with non-integral fractal dimension media. Fractal features can be found in a range of real-world issues, including porous medium, aquifers, turbulence, and a variety of other media. Statistical fractal features can be found in a variety of real-world circumstances. Although there are a number of difficulties in this subject, including measuring fractal dimension, which is influenced by a variety of mechanical constraints, as well as scanning aggregate data for numerical and experimental noise and limits. Because appraised fractal dimensions for statistically self-similar singularities may have practical applications in a variety of fields such as electrochemical processes, physics, diagnostic imagining, neuroscience, image analysis, acoustics, physiology, and Riemann zeta zeros. Due to this importance, this field has attracted several researchers and is rapidly growing [57–66].

By defining mathematical models and the refinement of numerical approaches, there is a need to use new mathematical operators with high computational capabilities to model processes. As a result, Atangana [67] used fractal derivatives to introduce a new type of hybrid operators and introduced fractional-fractal derivatives into the world of modeling in 2017. In fact, to define these advanced operators, he used two arguments to represent the order of the operator and the dimension of the operator, which he called the fractional order and the fractional dimension of the fractional-fractional derivatives, respectively [67]. Atangana divided these derivatives into three different categories and, with the help of different integral kernels, extracted the numerical algorithms associated with them. In 2021, Arfan et al. [68] designed a prey-predation structure for the four-compartmental fractal-fractional model of syn-ecosymbiosis and examined some conditions for species survival in an ecological system. Abdulwasaa et al. [69] conducted a case study with these fractal-fractional operators in which they examined the dynamics of new cases and the number of deaths from the COVID-19 epidemic over a specific period of time in India. Shah et al. [70] conducted the same study on a new model in Pakistan. Khan et al. [71] simulated and evaluated models of smoking at the incidence rate under the Caputo fractal-fractional derivative operator. Arif et al. [72] utilized the same fractal-fractional operators in engineering to analyze MHD stress fluid in a single channel. For more, see [73,74].

The fractal-fractional derivative is applied to a cigarette smoking model in this article. In fact, our key contribution is that we are the first to use newly developed operators to describe tobacco smoking dynamics, and we examine their correctness and effectiveness using a numerical approach. The applicability of these hybrid fractal-fractional operators is demonstrated by varying fractal dimensions and fractional orders, as well as by comparing the results to integer-order simulation. The fractal-fractional derivative model is critical in expressing the fundamental and auxiliary features in this direction. One of them is that the model can have an endless number of fractional order solutions in the assumed domain of derivatives. When the population of a class rises or decreases, all of the solutions behave similarly. For orders closer to the integer case, the findings converge to basic solutions.

2. Preliminaries

We here recall several required definitions and lemmas.

Definition 2.1. *Let $\psi \in \mathcal{L}^1([a,b]),\mathcal{R})$ and $\forall\,\alpha > 0$, the α-th fractional integral of ψ in Riemann-Liouviille sense (RL) is defined to be:*

$$I^{\alpha}_{a+}\psi(\theta) = \frac{1}{\Gamma(\alpha)}\int_a^{\theta}(\theta - s)^{\alpha-1}\psi(s)ds,$$

s.t. the R.H.S exists.

Definition 2.2. *For ψ given above, the α-th RL-derivative is defined to be*

$$D^{\alpha}_{a+}\psi(\theta) = \frac{1}{\Gamma(\alpha)}(\frac{d}{d\theta})^n\int_a^{\theta}(\theta - s)^{\alpha-1}\psi(s)ds,$$

where $n = [\alpha] + 1$.

Definition 2.3. *[67, 75] Let $\psi(\theta)$ be continues and fractional differentiable on (a, b) with order ν. Then the fractional-fractal RL-derivatives of $\psi(\theta)$ of dimension ω equipped with generalized Mittag-Liffler type kernel is*

$${}^{FFM}_{\ a}\mathcal{D}^{\nu,\omega}_{\theta}\psi(\theta) = \frac{\mathbb{AB}(\nu)}{1-\nu}\frac{d}{d\theta^{\omega}}\int_a^{\theta}E_{\nu}[\frac{-\nu}{1-\nu}(\theta - s)^{\nu}]\psi(s)ds, \tag{2.1}$$

where, $AB(\nu) = 1 - \nu + \frac{\nu}{\Gamma\nu}$, $0 < \nu, \omega \leq 1$.

Definition 2.4. *[67, 75] Let ψ be the same map mentioned above. The fractional-fractal integral for ψ of order $0 < \nu \leq 1$ via the Mittag-Leffler-type kernel is defined to be*

$${}^{FFM}_{\ 0}I^{\nu,\omega}_{\theta}\psi(\theta) = \frac{\nu\omega}{AB(\nu)\Gamma(\nu)}\int_0^{\theta}s^{\omega-1}\psi(s)(\theta - s)^{\nu-1}ds + \frac{\omega(1-\nu)\theta^{\omega-1}}{AB(\nu)}\psi(\theta),$$

where, $AB(\nu) = 1 - \nu + \frac{\nu}{\Gamma(\nu)}$.

3. Analysis of fractal-fractional tobacco model

Our present model was considered by Awan et al. [76] for a smoking model in ordinary differential settings of first order. We will consider the same model with same parameters by changing derivatives. In fact, we substitute derivatives with new fractal-fractional derivatives with Mittag-Leffler type kernels. Our fractal-fractional tobacco smoking model (TSM) is given below

$$\begin{cases} {}_{0}^{FFM}D_{\theta}^{\nu,\omega}\mathcal{P}(\theta) = \lambda - \lambda\mathcal{P} - \mu\sqrt{\mathcal{P}\mathcal{S}} + \kappa\mathcal{R}, \\ {}_{0}^{FFM}D_{\theta}^{\nu,\omega}\mathcal{S}(\theta) = -(\lambda + y)\mathcal{S} + \mu\sqrt{\mathcal{P}\mathcal{S}} + \alpha\mathcal{R}, \\ {}_{0}^{FFM}D_{\theta}^{\nu,\omega}\mathcal{R}(\theta) = -(\lambda + \alpha)\mathcal{R} + y(1-\sigma)\mathcal{S} - \kappa\mathcal{R}, \\ {}_{0}^{FFM}D_{\theta}^{\nu,\omega}\mathcal{Q}(\theta) = -\lambda\mathcal{Q} + \sigma y\mathcal{S}, \end{cases} \tag{3.1}$$

via initial conditions

$$\mathcal{P}(0) = \mathcal{P}^0, \ \ \mathcal{S}(0) = \mathcal{S}^0, \ \ \mathcal{R}(0) = \mathcal{R}^0, \ \ \mathcal{Q}(0) = \mathcal{Q}^0,$$

where the variables and parameters are described as follows: $\mathcal{P}$ as potential smokers or susceptible persons, $\mathcal{S}$ as smokers persons that are in the infected group, $\mathcal{R}$ as temporary quitters, and $\mathcal{Q}$ as permanent quitters. The parameter κ shows a rate for which the persons belonging to $\mathcal{R}$ move into the class $\mathcal{P}$ again for the sake of severe cravings to smoke. Also, $y(1-\sigma)$ and $y\sigma$ are two rates for which smokers knock off smoking temporarily, and knock off smoking at all, respectively. Moreover, $0 < \sigma < 1$, the rate λ denotes the natural death. The parameter α specifies the transmission rate of $\mathcal{S}$ and $\mathcal{R}$ who regress back to smoking. The contact rate is denoted by μ between two classes $\mathcal{P}$ and $\mathcal{S}$. And the operator ${}_{0}^{FFM}D_{\theta}^{\nu,\omega}$ representss the fractal-fractional differential operator for the orders ν, $\omega \in (0, 1]$ showing the fractaional order and fractal dimension.

We will prove the existence of a solution to the fractal fractional tobacco smoking model (3.1) by employing consecutive iterative techniques. We use the integral given in Definition 2.4 from [67] on the suggested tobacco smoking model (3.1), and we get

$$\begin{aligned} \mathcal{P}(\theta) - \mathcal{P}(0) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^{\theta} s^{\omega-1}(\theta-s)^{\nu-1}[\lambda - \lambda\mathcal{P} - \mu\sqrt{\mathcal{P}\mathcal{S}} + \kappa\mathcal{R}]ds \\ &\quad + \frac{\omega(1-\nu)\theta^{\omega-1}}{\mathbb{AB}(\nu)}[\lambda - \lambda\mathcal{P} - \mu\sqrt{\mathcal{P}\mathcal{S}} + \kappa\mathcal{R}], \\ \mathcal{S}(\theta) - \mathcal{S}(0) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^{\theta} s^{\omega-1}(\theta-s)^{\nu-1}[-(\lambda + y)\mathcal{S} + \mu\sqrt{\mathcal{P}\mathcal{S}} + \alpha\mathcal{R}]ds \\ &\quad + \frac{\omega(1-\nu)\theta^{\omega-1}}{\mathbb{AB}(\nu)}[-(\lambda + y)\mathcal{S} + \mu\sqrt{\mathcal{P}\mathcal{S}} + \alpha\mathcal{R}], \\ \mathcal{R}(\theta) - \mathcal{R}(0) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^{\theta} s^{\omega-1}(\theta-s)^{\nu-1}[-(\lambda + \alpha)\mathcal{R} + y(1-\sigma)\mathcal{S} - \kappa\mathcal{R}]ds \\ &\quad + \frac{\omega(1-\nu)\theta^{\omega-1}}{\mathbb{AB}(\nu)}[-(\lambda + \alpha)\mathcal{R} + y(1-\sigma)\mathcal{S} - \kappa\mathcal{R}], \\ \mathcal{Q}(\theta) - \mathcal{Q}(0) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^{\theta} s^{\omega-1}(\theta-s)^{\nu-1}[-\lambda\mathcal{Q} + \sigma y\mathcal{S}]ds \\ &\quad + \frac{\omega(1-\nu)\theta^{\omega-1}}{\mathbb{AB}(\nu)}[-\lambda\mathcal{Q} + \sigma y\mathcal{S}]. \end{aligned} \tag{3.2}$$

Consider the functions $\mathcal{V}_i$ for $i = 1, 2, 3, 4$, given below:

$$\begin{cases} \mathcal{V}_1(\theta, \mathcal{P}) = \lambda - \lambda\mathcal{P} - \mu\sqrt{\mathcal{P}\mathcal{S}} + \kappa\mathcal{R}, \\ \mathcal{V}_2(\theta, \mathcal{S}) = -(\lambda + y)\mathcal{S} + \mu\sqrt{\mathcal{P}\mathcal{S}} + \alpha\mathcal{R}, \\ \mathcal{V}_3(\theta, \mathcal{R}) = -(\lambda + \alpha)\mathcal{R} + y(1 - \sigma)\mathcal{S} - \kappa\mathcal{R}, \\ \mathcal{V}_4(\theta, \mathcal{Q}) = -\lambda\mathcal{Q} + \sigma y\mathcal{S}. \end{cases} \tag{3.3}$$

4. Existence criteria

In this section, we make the following assumption in order to prove the existence property theorem. $(\mathcal{H}^*)$ All functions $\mathcal{P}, \mathcal{P}^*, \mathcal{S}, \mathcal{S}^*, \mathcal{R}, \mathcal{R}^*, \mathcal{Q}, \mathcal{Q}^* \in L[0, 1]$ are continuous so that $\|\mathcal{P}\| \leq a_1, \|\mathcal{S}\| \leq a_2, \|\mathcal{R}\| \leq a_3, \|\mathcal{Q}\| \leq a_4$ for some positive constants $a_1,\ a_2,\ a_3,\ a_4 > 0$.

For proving our results, we shall utilize the assumption $\mathcal{H}^*$.

Theorem 4.1. *The $\mathcal{V}_i$ as the kernal functions for $i \in \mathcal{N}_1^4$ satisfy Lipschitz condition with constants $\Phi_i > 0$ provided that $(\mathcal{H}^*)$ is fulfilled, and Φ_i's are determined in the proof.*

Proof. We first check $\mathcal{V}_1(\theta, \mathcal{P})$. Applying $\mathcal{P}(\theta)$ and $\mathcal{P}^*(\theta)$, we estimate

$$\begin{aligned} \|\mathcal{V}_1(\theta, \mathcal{P}) - \mathcal{V}_1(\theta, \mathcal{P}^*)\| &= \|(\lambda - \lambda\mathcal{P} - \mu\sqrt{\mathcal{P}\mathcal{S}} + \kappa\mathcal{R}) \\ &- (\lambda - \lambda\mathcal{P}^* - \mu\sqrt{\mathcal{P}^*\mathcal{S}} + \kappa\mathcal{R})\| \\ &\leq \| - \lambda(\mathcal{P} - \mathcal{P}^*)\| + \| - \mu\sqrt{\mathcal{S}}(\sqrt{\mathcal{P}} - \sqrt{\mathcal{P}^*})\| \\ &\leq \lambda\|\mathcal{P} - \mathcal{P}^*\| + \mu\sqrt{\mathcal{S}}\|(\mathcal{P} - \mathcal{P}^*)\| \\ &\leq (\lambda + \mu\sqrt{a_2})\|\mathcal{P} - \mathcal{P}^*\| \leq \Phi_1\|\mathcal{P} - \mathcal{P}^*\|, \end{aligned} \tag{4.1}$$

where $\Phi_1 = \lambda + \mu\sqrt{a_2} > 0$. For the $\mathcal{V}_2(\theta, \mathcal{S})$, we

$$\begin{aligned} \|\mathcal{V}_2(\theta, \mathcal{S}) - \mathcal{V}_2(\theta, \mathcal{S}^*)\| &= \| - (\lambda + y)\mathcal{S} + \mu\sqrt{\mathcal{P}\mathcal{S}} + \alpha\mathcal{R} \\ &- (-(\lambda + y)\mathcal{S}^* + \mu\sqrt{\mathcal{P}\mathcal{S}^*} + \alpha\mathcal{R}\| \\ &\leq (\lambda + y)\|\mathcal{S} - \mathcal{S}^*\| + \mu\|\sqrt{\mathcal{P}}(\sqrt{\mathcal{S}} - \sqrt{\mathcal{S}^*})\| \\ &\leq (\lambda + y)\|\mathcal{S} - \mathcal{S}^*\| + \mu\sqrt{\mathcal{P}}\|\mathcal{S} - \mathcal{S}^*\| \\ &\leq (\lambda + y + \mu\sqrt{a_1})\|\mathcal{S} - \mathcal{S}^*\| \leq \Phi_2\|\mathcal{S} - \mathcal{S}^*\|, \end{aligned} \tag{4.2}$$

where $\Phi_2 = \lambda + y + \mu\sqrt{a_1} > 0$. The function $\mathcal{V}_3(\theta, \mathcal{R})$ implies that

$$\begin{aligned} \|\mathcal{V}_3(\theta, \mathcal{R}) - \mathcal{V}_3(\theta, \mathcal{R}^*)\| &= \| - (\lambda + \alpha)\mathcal{R} + y(1 - \sigma)\mathcal{S} - \kappa\mathcal{R}) \\ &- (-(\lambda + \alpha)\mathcal{R}^* + y(1 - \sigma)\mathcal{S} - \kappa\mathcal{R}^*)\| \end{aligned} \tag{4.3}$$

$$\leq \quad (\lambda + \alpha + \kappa)\|\mathcal{R} - \mathcal{R}^*\| \leq \Phi_3\|\mathcal{R} - \mathcal{R}^*\|,$$

where $\Phi_3 = \lambda + \alpha + \kappa > 0$. For $\mathcal{V}_4(\theta, \mathcal{Q})$, we have

$$\begin{aligned} \|\mathcal{V}_4(\theta, \mathcal{Q}) - \mathcal{V}_4(\theta, \mathcal{Q}^*)\| &= \| - \lambda\mathcal{Q} + \sigma y\mathcal{S} - (-\lambda\mathcal{Q}^* + \sigma y\mathcal{S})\| \\ &\leq \lambda\|\mathcal{Q} - \mathcal{Q}^*\| \leq \Phi_4\|\mathcal{Q} - \mathcal{Q} * \|, \end{aligned} \tag{4.4}$$

where $\Phi_4 = \lambda > 0$. Thus, from (4.1) to (4.4), we have that the $\mathcal{V}_i$'s for $i = 1, 2, 3, 4$, satisfy the Lipschitz condition. □

Let's assume:

$$\begin{cases} \mathcal{P}(\theta) - \mathcal{P}(0) = \dfrac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \displaystyle\int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_1(s, \mathcal{P}) ds + \dfrac{\omega(1-\nu)}{\mathbb{AB}(\nu)} \theta^{\omega-1} \mathcal{V}_1(\theta, \mathcal{P}), \\ \mathcal{S}(\theta) - \mathcal{S}(0) = \dfrac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \displaystyle\int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_2(s, \mathcal{S}) ds + \dfrac{\omega(1-\nu)}{\mathbb{AB}(\nu)} \theta^{\omega-1} \mathcal{V}_2(\theta, \mathcal{S}), \\ \mathcal{R}(\theta) - \mathcal{R}(0) = \dfrac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \displaystyle\int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_3(s, \mathcal{R}) ds + \dfrac{\omega(1-\nu)}{\mathbb{AB}(\nu)} s^{\omega-1} \mathcal{V}_3(\theta, \mathcal{R}), \\ \mathcal{Q}(\theta) - \mathcal{Q}(0) = \dfrac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \displaystyle\int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_4(s, \mathcal{Q}) ds + \dfrac{\omega(1-\nu)}{\mathbb{AB}(\nu)} \theta^{\omega-1} \mathcal{V}_4(\theta, \mathcal{Q}). \end{cases}$$

Now, we define the following recursive formulas for the model (3.1):

$$\mathcal{P}_n(\theta) - \mathcal{P}(0) = \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_1(s, \mathcal{P}_{n-1}(s)) ds + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)} \theta^{\omega-1} \mathcal{V}_1(\theta, \mathcal{P}_{n-1}(\theta)),$$

$$\mathcal{S}_n(\theta) - \mathcal{S}(0) = \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_1(s, \mathcal{S}_{n-1}(s)) ds + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)} \theta^{\omega-1} \mathcal{V}_1(\theta, \mathcal{S}_{n-1}(\theta)),$$

$$\mathcal{R}_n(\theta) - \mathcal{R}(0) = \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_1(s, \mathcal{R}_{n-1}(s)) ds + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)} \theta^{\omega-1} \mathcal{V}_1(\theta, \mathcal{R}_{n-1}(\theta)),$$

$$\mathcal{Q}_n(\theta) - \mathcal{Q}(0) = \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} \mathcal{V}_1(s, \mathcal{Q}_{n-1}(s)) ds + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)} \theta^{\omega-1} \mathcal{V}_1(\theta, \mathcal{Q}_{n-1}(\theta)).$$

Next, we consider the differences as follows:

$$\Delta\mathcal{P}_{n+1}(\theta) = \mathcal{P}_{n+1}(\theta) - \mathcal{P}_n(\theta),$$

where

$$\Delta\mathcal{P}_{n+1}(\theta) \quad = \quad \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)} \int_0^\theta (\theta - s)^{\nu-1} s^{\omega-1} [\mathcal{V}_1(s, \mathcal{P}_n(s)) - \mathcal{V}_1(s, \mathcal{P}_{n-1}(s))] ds$$

$$+ \quad \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}[\mathcal{V}_1(\theta,\mathcal{P}_n(\theta)) - \mathcal{V}_1(\theta,\mathcal{P}_{n-1}(\theta))], \tag{4.5}$$

and

$$\Delta\mathcal{S}_{n+1}(\theta) = \mathcal{S}_{n+1}(\theta) - \mathcal{S}_n(\theta),$$

where

$$\begin{aligned}\Delta\mathcal{S}n+1(\theta) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}[\mathcal{V}_2(s,\mathcal{S}_n(s)) - \mathcal{V}_2(s,\mathcal{S}_{n-1}(s))]ds \\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}[\mathcal{V}_2(\theta,\mathcal{S}_n(\theta)) - \mathcal{V}_2(\theta,\mathcal{S}_{n-1}(\theta))],\end{aligned} \tag{4.6}$$

and

$$\Delta\mathcal{R}_{n+1}(\theta) = \mathcal{R}_{n+1}(\theta) - \mathcal{R}_n(\theta),$$

where

$$\begin{aligned}\Delta\mathcal{R}_{n+1}(\theta) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}[\mathcal{V}_3(s,\mathcal{R}_n(s)) - \mathcal{V}_3(s,\mathcal{R}_{n-1}(s))]ds \\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}[\mathcal{V}_3(\theta,\mathcal{R}_n(\theta) - \mathcal{V}_3(\theta,\mathcal{R}_{n-1}(\theta)))],\end{aligned} \tag{4.7}$$

and

$$\Delta\mathcal{Q}_{n+1}(\theta) = \mathcal{Q}_{n+1}(\theta) - \mathcal{Q}_n(\theta),$$

where

$$\begin{aligned}\Delta\mathcal{Q}_{n+1}(\theta) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}[\mathcal{V}_4(s,\mathcal{Q}_n(s)) - \mathcal{V}_1(s,\mathcal{Q}_{n-1}(s))]ds \\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}[\mathcal{V}_4(\theta,\mathcal{Q}_n(\theta)) - \mathcal{V}_4(\theta,\mathcal{Q}_{n-1}(\theta))].\end{aligned} \tag{4.8}$$

Now, by making use of the norm on (4.5)–(4.8), we get

$$\|\Delta_{\mathcal{P}}n+1(\theta)\| = \|\mathcal{P}n+1(\theta) - \mathcal{P}_n(\theta)\|,$$

$$\|\Delta\mathcal{S}_{n+1}(\theta)\| = \|\mathcal{S}_{n+1}(\theta) - \mathcal{S}_n(\theta)\|,$$

$$\|\Delta\mathcal{R}_{n+1}(\theta)\| = \|\mathcal{R}_{n+1}(\theta) - \mathcal{R}_n(\theta)\|,$$

$$\|\Delta\mathcal{Q}_{n+1}(\theta)\| = \|\mathcal{Q}_{n+1}(\theta) - \mathcal{Q}_n(\theta)\|.$$

In other words,

$$\|\Delta\mathcal{P}_{n+1}(\theta)\| = \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\|[\mathcal{V}_1(s,\mathcal{P}_n(s)) - \mathcal{V}_1(s,\mathcal{P}_{n-1}(s))]\|ds$$

$$+ \quad \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\|[\mathcal{V}_1(\theta,\mathcal{P}_n(\theta)) - \mathcal{V}_1(\theta,\mathcal{P}_{n-1}(\theta))]\|,$$

$$\begin{aligned}
\|\Delta\mathcal{S}_{n+1}(\theta)\| &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\|[\mathcal{V}_2(s,\mathcal{P}_n(s)) - \mathcal{V}_2(s,\mathcal{S}_{n-1}(s))]\|ds \\
&+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\|[\mathcal{V}_2(\theta,\mathcal{S}_n(\theta)) - \mathcal{V}_2(\theta,\mathcal{S}_{n-1}(\theta))]\|,
\end{aligned}$$

$$\begin{aligned}
\|\Delta\mathcal{R}_{n+1}(\theta)\| &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\|[\mathcal{V}_3(s,\mathcal{R}_n(s)) - \mathcal{V}_3(s,\mathcal{R}_{n-1}(s))]\|ds \\
&+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\|[\mathcal{V}_3(\theta,\mathcal{R}_n(\theta)) - \mathcal{V}_3(\theta,\mathcal{R}_{n-1}(\theta))]\|,
\end{aligned}$$

$$\begin{aligned}
\|\Delta\mathcal{Q}_{n+1}(\theta)\| &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\|[\mathcal{V}_4(s,\mathcal{Q}_n(s)0) - \mathcal{V}_4(s,\mathcal{Q}_{n-1}(s))]\|ds \\
&+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\|[\mathcal{V}_4(\theta,\mathcal{Q}_n(\theta)) - \mathcal{V}_4(\theta,\mathcal{Q}_{n-1}(\theta))]\|.
\end{aligned}$$

Theorem 4.2. *There is at least one solution for the fractal-fractional tobacco Smoking model* (3.1) *if*

$$\Phi = \max\{\Phi_i\} < 1, \quad i = 1, \dots, 4. \tag{4.9}$$

Proof. We define four functions as follows:

$$\begin{cases}
\mathcal{M}1_n(\theta) = \mathcal{P}_{n+1}(\theta) - \mathcal{P}(\theta), \\
\mathcal{M}2_n(\theta) = \mathcal{S}_{n+1}(\theta) - \mathcal{S}(\theta), \\
\mathcal{M}3_n(\theta) = \mathcal{R}_{n+1}(\theta) - \mathcal{R}(\theta), \\
\mathcal{M}4_n(\theta) = \mathcal{Q}_{n+1}(\theta) - \mathcal{Q}(\theta).
\end{cases} \tag{4.10}$$

Then, we find that

$$\begin{aligned}
\|\mathcal{M}1_n(\theta)\| &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\|[\mathcal{V}_1(s,\mathcal{P}_n(s)) - \mathcal{V}_1(s,\mathcal{S}(s))]\|ds \\
&+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\|[\mathcal{V}_1(\theta,\mathcal{P}_n(\theta)) - \mathcal{V}_1(\theta,\mathcal{P}(\theta))]\| \\
&\le \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\Phi_1\|\mathcal{P}_n - \mathcal{P}\|ds \\
&+ \frac{\omega(1-\omega)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\Phi_1\|\mathcal{P}_n - \mathcal{P}\|
\end{aligned} \tag{4.11}$$

$$\leq \left[\frac{\nu\omega\Gamma(\omega)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\right]\Phi_1\|\mathcal{P}_n - \mathcal{P}\|$$

$$\leq \left[\frac{\nu\omega\Gamma(\omega)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\right]\Phi_1\|\mathcal{P}_n - \mathcal{P}\|$$

$$\leq \left[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\right]^n\Phi_1^n\|\mathcal{P}_n - \mathcal{P}\|.$$

In which for $\Phi_1 < 1$ and as $n \to \infty$, we have $\mathcal{P}_n \to \mathcal{P}$. So $\mathcal{M}1_n \to 0$ as $n \to \infty$.
Similarly,

$$\|\mathcal{M}2_n(\theta)\| \leq \left[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\right]^n\Phi_2^n\|\mathcal{P}_n - \mathcal{P}\|, \tag{4.12}$$

$$\|\mathcal{M}3_n(\theta)\| \leq \left[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\right]^n\Phi_3^n\|\mathcal{R}_n - \mathcal{R}\|, \tag{4.13}$$

and

$$\|\mathcal{M}4_n(\theta)\| \leq \left[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\right]^n\Phi_4^n\|\mathcal{Q}_n - \mathcal{Q}\|. \tag{4.14}$$

By (4.11)–(4.14), when $n \to \infty$, then $\mathcal{M}i_n(\theta) \to 0,\ i \in \mathbb{N}_2^4$, for $\Phi_i < 1, (i = 2, \ldots, 4)$. Ultimately, the tobacco Smoking system (3.1) has a solution. □

4.1. Unique solution

For our suggested tobacco smoking model (3.1), we follow the analysis of the uniqueness property.

Theorem 4.3. *The fractal-fractional tobacco smoking model* (3.1) *possesses one solution exactly if* ($\mathcal{H}^*$) *is satisfied and the following holds:*

$$\left[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\right]\Phi_i \leq 1,\ i \in \mathbb{N}_1^4. \tag{4.15}$$

Proof. We assume that the conclusion of the theorem is not valid. In other words, another solution exists for the supposed tobacco smoking (TS) model (3.1) in the fractal-fractional settings. Hence, $\mathcal{P}^*(\theta), \mathcal{S}^*(\theta), \mathcal{R}^*(\theta), \mathcal{Q}^*(\theta)$ is another solution with $\mathcal{P}^*(0) = \mathcal{P}^0, \mathcal{S}^*(0) = \mathcal{S}^0, \mathcal{R}^*(0) = \mathcal{R}^0, \mathcal{Q}^*(0) = \mathcal{Q}^0$ such that

$$\begin{aligned}\mathcal{P}^*(\theta) - \mathcal{P}^*(0) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\mathcal{V}_1(s, \mathcal{P}^*(s))ds,\\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\mathcal{V}_1(\theta, \mathcal{P}^*(\theta)).\end{aligned} \tag{4.16}$$

And similarly,

$$\mathcal{S}^*(\theta) - \mathcal{S}^*(0) = \frac{\nu\omega}{AB(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\mathcal{V}_2(s, \mathcal{S}^*(s))ds$$

$$+ \quad \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\mathcal{V}_2(\theta,\mathcal{S}^*(\theta)), \tag{4.17}$$

$$\begin{aligned}\mathcal{R}^*(\theta)-\mathcal{R}^*(0) &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma\nu}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\mathcal{V}_3(s,\mathcal{R}^*(s))ds,\\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\mathcal{V}_3(\theta,\mathcal{R}^*(\theta)),\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\mathcal{Q}^*(\theta)-\mathcal{Q}^*(0) &= \frac{\nu\omega}{AB(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\mathcal{V}_4(s,\mathcal{Q}^*(s))ds,\\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\mathcal{V}_4(\theta,\mathcal{Q}^*(\theta).\end{aligned} \tag{4.19}$$

Now, we write

$$\begin{aligned}\|\mathcal{P}-\mathcal{P}^*\| &= \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\|\mathcal{V}_1(s,\mathcal{P})-\mathcal{V}_1(s,\mathcal{P}^*(s))\|ds\\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\|\mathcal{V}_1(\theta,\mathcal{P})-\mathcal{V}_1(\theta,\mathcal{P}^*)\|\\ &\leq \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta(\theta-s)^{\nu-1}s^{\omega-1}\Phi_1\|\mathcal{P}-\mathcal{P}^*\|\\ &+ \frac{\omega(1-\nu)}{\mathcal{AB}(\nu)}\theta^{\omega-1}\Phi_1\|\mathcal{P}-\mathcal{P}^*\|\\ &\leq [\frac{\nu\omega\Gamma(\omega)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)}+\frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_1\|\mathcal{P}-\mathcal{P}^*\|,\end{aligned}$$

and so

$$[1-[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)}+\frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_1]\|\mathcal{P}-\mathcal{P}^*\|\leq 0. \tag{4.20}$$

The above inequality (4.20) is true if $\|\mathcal{P}-\mathcal{P}^*\|=0$, and accordingly, $\mathcal{P}=\mathcal{P}^*$.
Similarly, from

$$\|\mathcal{S}-\mathcal{S}^*\|\leq[\frac{\nu\Gamma(\omega+1)}{\mathcal{AB}(\nu)\Gamma(\nu+\omega)}+\frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_2\|\mathcal{S}-\mathcal{S}^*\|, \tag{4.21}$$

we arrive at

$$[1-[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)}+\frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_2]\|\mathcal{S}-\mathcal{S}^*\|\leq 0.$$

This implies, $\|\mathcal{S}-\mathcal{S}^*\|=0$ and $\mathcal{S}=\mathcal{S}^*$.
Also,

$$[1-[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)}+\frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_3]\|\mathcal{R}-\mathcal{R}^*\|\leq 0.$$

This inequality is true, if $\|\mathcal{R}-\mathcal{R}^*\|=0$, and accordingly, $\mathcal{R}=\mathcal{R}^*$. In similar manner, the inequality

$$[1-[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)}+\frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_4]\|\mathcal{Q}-\mathcal{Q}^*\|\leq 0.$$

is valid if $\|\mathcal{Q}-\mathcal{Q}^*\|=0$, which gives $\mathcal{Q}=\mathcal{Q}^*$. Therefore the tobacco smoking model (3.1) contains a solution uniquely. □

5. HU-Stability

In this section, we introduce the HU-stability and investigate it for solutions of the system (3.1).

Definition 5.1. *The fractal-fractional tobacco smoking system* (3.1) *is termed as HU-stable if* $\exists\,\eta_i>0, i\in\mathbb{N}_1^4$ *provided that* $\forall\,\xi_i>0, i\in\mathbb{N}_1^4$ *and for each* $(\mathcal{P}^*,\mathcal{S}^*,\mathcal{R}^*,\mathcal{Q}^*)$ *satisfying*

$$\begin{cases}\left|{}_0^{FFM}D_\theta^{\nu,\omega}\mathcal{P}^*(\theta)-\mathcal{V}_1(\theta,\mathcal{P}^*(\theta))\right|\leq\xi_1,\\ \left|{}_0^{FFM}D_\theta^{\nu,\omega}\mathcal{S}^*(\theta)-\mathcal{V}_1(\theta,\mathcal{S}^*(\theta))\right|\leq\xi_2,\\ \left|{}_0^{FFM}D_\theta^{\nu,\omega}\mathcal{R}^*(\theta)-\mathcal{V}_1(\theta,\mathcal{R}^*(\theta))\right|\leq\xi_3,\\ \left|{}_0^{FFM}D_\theta^{\nu,\omega}\mathcal{Q}^*(\theta)-\mathcal{V}_1(\theta,\mathcal{Q}^*(\theta))\right|\leq\xi_4,\end{cases}\tag{5.1}$$

there exists $(\mathcal{P},\mathcal{S},\mathcal{R},\mathcal{Q})$ satisfying the TSM system (3.1) and further we have

$$\begin{cases}\|\mathcal{P}-\mathcal{P}^*\|\leq\eta_1\xi_1,\\ \|\mathcal{S}-\mathcal{S}^*\|\leq\eta_2\xi_2,\\ \|\mathcal{R}-\mathcal{R}^*\|\leq\eta_3\xi_3,\\ \|\mathcal{Q}-\mathcal{Q}^*\|\leq\eta_4\xi_4,\end{cases}$$

where $\mathcal{V}_i$, $i\in\mathbb{N}_1^4$ are introduced in (3.3).

Remark 5.2. *Consider that the function* $\mathcal{P}^*$ *is a solution of the first inequality* (5.1) *iff a continuous map* h_1 *exists (depending on* $\mathcal{P}^*$*) so that (a)* $|h_1(\theta)|<\xi_1$*, and*

$$(b)\ {}_0^{FFM}D_\theta^{\nu,\omega}\mathcal{P}^*(\theta)=\mathcal{V}_1(\theta,\mathcal{P}^*)+h_1(\theta).$$

Similarly, one can indicate such a definition for each of solutions to inequalities (5.1) *by finding* h_i *for* $i\in\mathbb{N}_2^4$.

Theorem 5.3. *Let the hypothesis* $(\mathcal{H}^*)$ *be true. Then the fractal-fractional tobacco smoking model* (3.1) *is HU-stable if*

$$[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)}+\frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_i\leq 1, i\in\mathbb{N}_1^4.$$

Proof. Let $\xi_1 > 0$ and the function $\mathcal{P}^*$ be arbitrary so that

$$\left|{}_0^{FFM}D_\theta^{\nu,\omega}\mathcal{P}^*(\theta) - \mathcal{V}_1(\theta, \mathcal{P}^*(\theta))\right| \leq \xi_1.$$

In view of Remark 5.2, we have a function like h_1 with $|h_1(\theta)| < \xi_1$ which satisfies

$${}_0^{FFM}D_\theta^{\nu,\omega}\mathcal{P}^*(\theta) = \mathcal{V}_1(\theta, \mathcal{P}^*) + h_1(\theta).$$

Accordingly, we get

$$\begin{aligned}\mathcal{P}^*(\theta) &= \mathcal{P}^0 + \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta (\theta - s)^{\nu-1}s^{\omega-1}\mathcal{V}_1(s, \mathcal{P}^*(s))ds + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\mathcal{V}_1(\theta, \mathcal{P}^*(\theta))\\ &+ \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta (\theta - s)^{\nu-1}s^{\omega-1}h_1(s)ds + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}h_1(\theta).\end{aligned}$$

Consider $\mathcal{P}$ as the unique solution of the fractal-fractional tobacco smoking model (3.1). Then, it becomes

$$\mathcal{P}(\theta) = \mathcal{P}^0 + \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta (\theta - s)^{\nu-1}s^{\omega-1}\mathcal{V}_1(s, \mathcal{P}(s))ds + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\mathcal{V}_1(\theta, \mathcal{P}(\theta)).$$

Hence,

$$\begin{aligned}\left\|\mathcal{P}^*(\theta) - \mathcal{P}(\theta)\right\| &\leq \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta (\theta - s)^{\nu-1}s^{\omega-1}\left|\mathcal{V}_1(s, \mathcal{P}^*(s)) - \mathcal{V}_1(s, \mathcal{P}(s))\right|ds\\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu_)}\theta^{\omega-1}\left|\mathcal{V}_1(\theta, \mathcal{P}^*(\theta)) - \mathcal{V}_1(\theta, \mathcal{P}(\theta))\right|\\ &+ \frac{\nu\omega}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta (\theta - s)^{\nu-1}s^{\omega-1}\left|h_1(s)\right|ds\\ &+ \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}\theta^{\omega-1}\left|h_1(\theta)\right|\\ &\leq [\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]\Phi_1\left|\mathcal{P}^*(\theta) - \mathcal{P}(\theta)\right|\\ &+ [\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu_)}]\xi_1.\end{aligned}$$

In consequence,

$$\left\|\mathcal{P}^* - \mathcal{P}\right\| \leq \frac{[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]}{1 - [\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\omega)}{\mathbb{AB}(\nu)}]\Phi_1}\xi_1. \tag{5.2}$$

If, we take

$$\eta_1 := \frac{[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\nu)}{\mathbb{AB}(\nu)}]}{1-[\frac{\nu\Gamma(\omega+1)}{\mathbb{AB}(\nu)\Gamma(\nu+\omega)} + \frac{\omega(1-\omega)}{\mathbb{AB}(\nu)}]\Phi_1},$$

then $\left\|\mathcal{P}^* - \mathcal{P}\right\| \leq \eta_1\xi_1$. Similarly, we have

$$\|\mathcal{S}^* - \mathcal{S}\| \leq \eta_2\xi_2,\ \ \|\mathcal{R}^* - \mathcal{R}\| \leq \eta_3\xi_3,\ \ \|\mathcal{Q}^* - \mathcal{Q}\| \leq \eta_4\xi_4.$$

Thus, the fractal-fractional tobacco smoking model (3.1) is HU-stable which ends the argument. □

6. Numerical algorithm

In this section, we describe the numerical scheme in relation to the fractal-fractional tobacco smoking model (3.1). For this, we have taken help from the technique regarding the two-step Lagrange polynomials. For the numerical scheme, consider the linear general differential equation ${}^{FFM}{}_0D_\theta^{\nu,\omega}\varphi(\theta) = \mathcal{V}(\theta,\varphi(\theta))$, where $\varphi(0) = \varphi_0$ is the initial value. The latter equation can be rewritten with respect to the Atangana-Baleanu fractal-fractional derivative as

$${}^{AB}{}_0D_\theta^{\nu,\omega}\varphi(\theta) = \omega\theta^{\omega-1}\mathcal{V}(\theta,\varphi(\theta)) = \mathcal{Y}(\theta,\varphi(\theta)).$$

With the help of the fractal-fractional integral operator having kernel of the generalized Mittag-Leffler type, we get

$$\varphi(\theta) = \varphi(0) + \frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{Y}(\theta,\varphi(\theta)) + \frac{\nu}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^\theta \varpi^{\omega-1}(\theta-\varpi)^{\nu-1}\mathcal{Y}(\varpi,\varphi(\varpi))d\varpi.$$

Replacing θ by θ_{n+1}, we have

$$\varphi^{n+1} = \varphi(0) + \frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{Y}(\theta_n,\varphi(\theta_n)) + \frac{\nu}{\mathbb{AB}(\nu)\Gamma(\nu)}\int_0^{\theta_{n+1}} \varpi^{\omega-1}(\theta_{n+1}-\varpi)^{\nu-1}\mathcal{Y}(\varpi,\varphi(\varpi))d\varpi. \tag{6.1}$$

According to two-step Lagrange polynomials, we have

$$\begin{aligned}\mathcal{V}(x,\varphi(\theta)) &= \frac{(x-\theta_{\ell-1})\mathcal{V}(\theta_\ell,\varphi(\theta_\ell))}{\theta_\ell-\theta_{\ell-1}} - \frac{(x-\theta_\ell)\mathcal{V}(\theta_{\ell-1},\varphi(\theta_{\ell-1}))}{\theta_\ell-\theta_{\ell-1}}\\ &= \frac{\mathcal{V}(\theta_\ell,\varphi(\theta_\ell)(x-\theta_{\ell-1})}{\theta_\ell-\theta_{\ell-1}} - \frac{\mathcal{V}(\theta_{\ell-1},\varphi(\theta_{\ell-1}))(x-\theta_\ell)}{\theta_\ell-\theta_{\ell-1}}\\ &= \frac{\mathcal{V}(\theta_\ell,\varphi_\ell)(x-\theta_{\ell-1})}{h} - \frac{\mathcal{V}(\theta_{\ell-1},\varphi_{\ell-1})(x-\theta_\ell)}{h}.\end{aligned}$$

In this case, if we use the aforesaid Lagrange polynomial to (6.1), we obtain

$$\varphi^{n+1} = \varphi(0) + \frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{V}(\theta_n,\varphi(\theta_n))$$

$$+\frac{\nu}{\mathbb{AB}(\nu)\Gamma(\nu)}\sum_{\ell=1}^{n}\Big[\frac{\mathcal{V}(\theta_\ell,\varphi(\theta_\ell))}{h}\int_{\theta_\ell}^{\theta_{\ell+1}}(\varpi-\theta_{\ell-1})(\theta_{n+1}-\varpi)^{\nu-1}d\varpi$$

$$-\frac{\mathcal{V}(\theta_{\ell-1},\varphi(\theta_{\ell-1}))}{h}\int_{\theta_\ell}^{\theta_{n+1}}(\varpi-\theta_\ell)(\theta_{n+1}-\varpi)^{\nu-1}d\varpi\Big].$$

Further, we solve the above integral equation and obtain

$$\varphi^{n+1}=\varphi_0+\frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{V}(\theta_n,\varphi(\theta_n))$$

$$+\frac{\nu h^\nu}{\mathbb{AB}(\nu_1)\Gamma(\nu+2)}\sum_{\ell=1}^{n}\Big[\mathcal{V}(\theta_\ell,\varphi(\theta_\ell))\Big((n+1-\ell)^\nu(n-\ell+2+\nu)-(n-\ell)^\nu(n-\ell+2+2\nu)\Big)$$

$$-\mathcal{V}(\theta_{\ell-1},\varphi_{\ell-1})\Big((n+1-\ell)^{\nu+1}-(n-\ell+1+\nu)(n-\ell)^\nu\Big)\Big].$$

Inserting the value of $\mathcal{Y}(t,\varphi(\theta))$, it becomes

$$\varphi^{n+1}=\varphi_0+\omega\theta_n^{\omega-1}\frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{V}(\theta_n,\varphi(\theta_n))$$

$$+\frac{\omega h^\nu}{\mathbb{AB}(\nu)\Gamma(\nu+2)}\sum_{\ell=1}^{n}\Big[\theta_\ell^{\omega-1}\mathcal{V}(\theta_\ell,\varphi(\theta_\ell))\Big((n+1-\ell)^\nu(n-\ell+2+\nu)-(n-\ell)^\nu(n-\ell+2+2\nu)\Big)$$

$$-\theta_{\ell-1}^{\omega-1}\mathcal{V}(\theta_{\ell-1},\varphi_{\ell-1})\Big((n+1-\ell)^{\nu+1}-(n-\ell+1+\nu)(n-\ell)^\nu\Big)\Big].$$

Thus, by assuming

$$\psi_1(n,\ell):=(n+1-\ell)^\nu(n-\ell+2+\nu)-(n-\ell)^\nu(n-\ell+2+2\nu),$$

$$\psi_2(n,\ell):=(n+1-\ell)^{\nu+1}-(n-\ell+1+\nu)(n-\ell)^\nu,$$

the numerical scheme for the integral system (3.2) is obtained as

$$\mathcal{P}(\theta_{n+1})=\mathcal{P}(0)+\omega\theta_n^{\omega-1}\frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{V}_1(\theta_n,\mathcal{P}(\theta_n))+\frac{\omega h^\nu}{\mathbb{AB}(\nu)\Gamma(\nu+2)}$$
$$\times\sum_{\ell=1}^{n}\Big[\theta_\ell^{\omega-1}\mathcal{V}_1(\theta_\ell,\mathcal{P}(\theta_\ell))\psi_1(n,\ell)-\theta_{\ell-1}^{\omega-1}\mathcal{V}_1(\theta_{\ell-1},\mathcal{P}(\theta_{\ell-1}))\psi_2(n,\ell)\Big],$$

$$\mathcal{S}(\theta_{n+1})=\mathcal{S}(0)+\omega\theta_n^{\omega-1}\frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{V}_2(\theta_n,\mathcal{S}(\theta_n))+\frac{\omega h^\nu}{\mathbb{AB}(\nu)\Gamma(\nu+2)}$$
$$\times\sum_{\ell=1}^{n}\Big[\theta_\ell^{\omega-1}\mathcal{V}_2(\theta_\ell,\mathcal{S}(\theta_\ell))\psi_1(n,\ell)-\theta_{\ell-1}^{\omega-1}\mathcal{V}_2(\theta_{\ell-1},\mathcal{S}(\theta_{\ell-1}))\psi_2(n,\ell)\Big],$$

$$\mathcal{R}(\theta_{n+1})=\mathcal{R}(0)+\omega\theta_n^{\omega-1}\frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{V}_3(\theta_n,\mathcal{R}(\theta_n))+\frac{\omega h^\nu}{\mathbb{AB}(\nu)\Gamma(\nu+2)}$$

$$\times \sum_{\ell=1}^{n}\Big[\theta_{\ell}^{\omega-1}\mathcal{V}_3(\theta_\ell,\mathcal{R}(\theta_\ell))\psi_1(n,\ell)-\theta_{\ell-1}^{\omega-1}\mathcal{V}_3(\theta_{\ell-1},\mathcal{S}(\theta_{\ell-1}))\psi_2(n,\ell)\Big],$$

$$\mathcal{Q}(\theta_{n+1}) = \mathcal{Q}(0) + \omega\theta_n^{\omega-1}\frac{1-\nu}{\mathbb{AB}(\nu)}\mathcal{V}_4(\theta_n,\mathcal{Q}(\theta_n)) + \frac{\omega h^{\nu}}{\mathbb{AB}(\nu)\Gamma(\nu+2)}$$
$$\times \sum_{\ell=1}^{n}\Big[\theta_{\ell}^{\omega-1}\mathcal{V}_4(\theta_\ell,\mathcal{Q}(\theta_\ell))\psi_1(n,\ell)-\theta_{\ell-1}^{\omega-1}\mathcal{V}_4(\theta_{\ell-1},\mathcal{Q}(\theta_{\ell-1}))\psi_2(n,\ell)\Big].$$

6.1. Numerical simulation

The numerical values used are: $\nu = 0.001$, $\lambda = 0.001$, $\xi = 0.005$, $\gamma = 0.1$, $a = 0.002$, $\sigma = 0.02$, $\alpha = 0.02$ with the initial values $\mathcal{P}(1) = 6$, $\mathcal{S}(1) = 10$, $\mathcal{R}(0) = 0$, $\mathcal{Q}(0) = 40$.

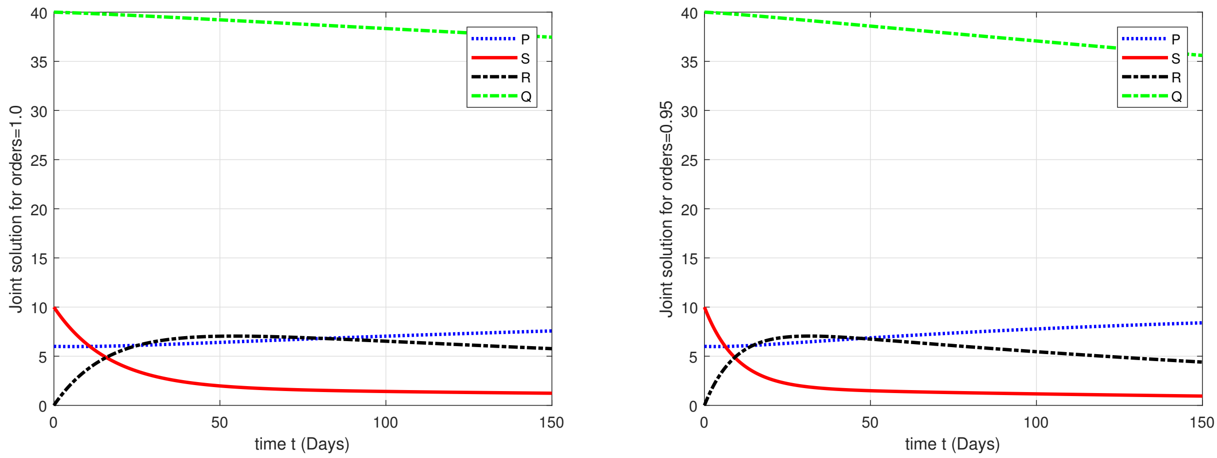

Figure 1. Joint solution of the model (3.1) for the integer order 1.0, and 0.95 (from left to right).

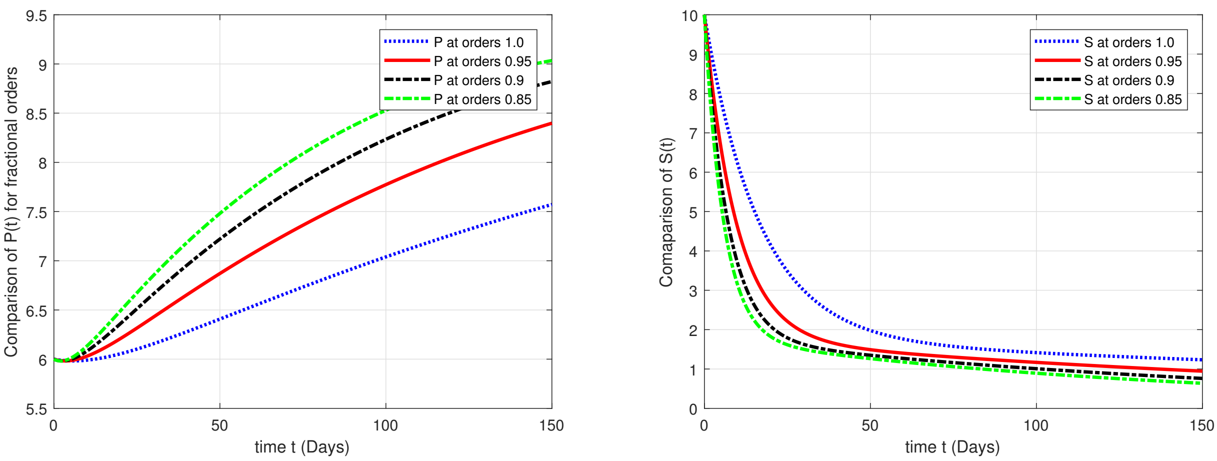

Figure 2. Potential smokers $\mathcal{P}(t)$ and $\mathcal{S}(t)$ for the fractal-fractional orders $\nu = \omega = 1.0, 0.95, 0.90, 0.85$.

The first Figure 1 represents the numerical solution of the integer order 1.0 and fractal-fractional orders equal to 0.95. This graphical representation is analyzed in comparison to the fractal-fractional orders $\nu = \omega = 1.0,\ 0.95,\ 0.90,\ 0.85$ for all the classes. Figure 2 represents a numerical comparison of $\mathcal{P}$ and $\mathcal{S}$ for the fractal-fractional orders $\nu = \omega = 1.0,\ 0.95,\ 0.90,\ 0.85$.

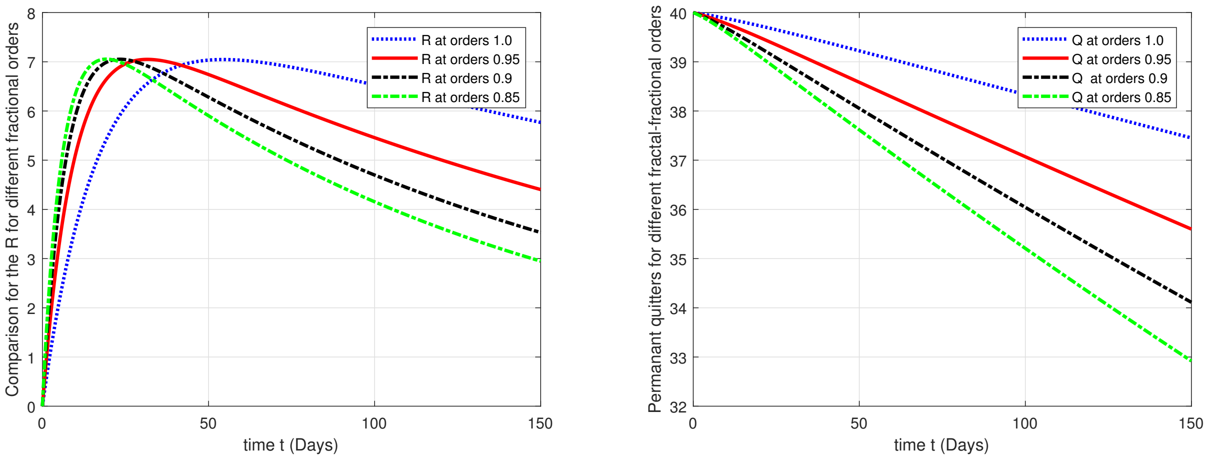

Figure 3. Temporary quitters $\mathcal{R}(t)$ and $\mathcal{Q}(t)$ for the fractal-fractional orders $\nu = \omega = 1.0,\ 0.95,\ 0.90,\ 0.85$.

The potential smokers are increased with the passage of time and this portion is based on the growth in the population of the temporary quitters $\mathcal{R}(t)$ in Figure 3 and permanent quitters $\mathcal{Q}(t)$ in Figure 3. Ultimately, the smokers are reduced with the passage of time as shown in Figure 3. We presume that this model is representing a better situation of the population.

7. Conclusions

A fractal-fractional tobacco mathematical model was investigated in this article. For demonstrating theoretical existence results, we used an iterative approach. The Ulam-Hyers criterion for stable solutions was examined. To create and investigate numerical prediction of the fractal-fractional tobacco smoking model, an interpolation method based on Lagrange's polynomials was applied. The models depict a more favourable population condition, with fewer smokers and more potential smokers. This is because of the population's quitting classes. Four figures detail the numerical results. In the first figure, we have compared the integer order joint solution with the fractional order for $\nu = \omega = 0.95$. We discovered that the fractional order has a significant impact on the model's dynamics. Fractional derivatives with non-singular kernels have been frequently used to solve dynamical problems in recent years. As previously stated, the researchers feel that this model extension strategy is one of the most effective ways to collect additional relevant data. In this scenario, the fractal-fractional derivative model is crucial in conveying the basic and supplementary properties. One of these is that the model can have an endless number of fractional order solutions in the assumed domain of the derivatives, i.e., $(0, 1)$. All of the solutions respond similarly to the classical case when the population of a class increases or decreases. The results converge to the traditional solutions for orders approaching the integer case. We encourage readers to investigate further aspects of the model, as well as other models, for a variety of research purposes, including optimal proximity points, asymptotic stabilities, local and global stabilities with bifurcation, and the stochastic and fuzzy extensions of the models. For scholars, these themes present fresh perspectives.

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Conflict of interest

The authors declare no conflict of interest.

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