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Research article

On the exact solutions of nonlinear extended Fisher-Kolmogorov equation by using the He's variational approach

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Abstract: In this article, we investigate existence and the exact solutions of the extended Fisher-Kolmogorov (EFK) equation. This equation is used in the population growth dynamics and wave propagation. The fourth-order term in this model describes the phase transitions near critical points which are also known as Lipschitz points. He's variational method is adopted to construct the soliton solutions as well as the periodic wave solutions successfully for the extended (higher-order) EFK equation. This approach is simple and has the greatest advantages because it can reduce the order of our equation and make the equation more simple. So, the results that are obtained by this approach are very simple and straightforward. The graphics behavior of these solutions are also sketched in 3D, 2D, and corresponding contour representations by the different choices of parameters.

Keywords: soliton solutions; extended Fisher-Kolmogorov equation; He's variational methods; semi-inverse method; variational principle
Mathematics Subject Classification: 35C08, 37K05, 49K20

1. Introduction

In 1937 fisher's equation is proposed named by Ronald fisher, which is a reaction-diffusion equation in the inhomogeneous form of the partial differential equation. This equation is used in the population growth dynamics and wave propagation. Fisher proposed the advantages in population dynamics of wave spatial spread of an advantageous allele [1]. In the same year, in 1937 Fisher introduced a more general reaction-diffusion model by the contribution with Kolmogorov, Petrovsky, and Piskunov and proposed a new model named as KPP equation which is used in population genetics [2]. The population genetics is a subfield of genetics that deals with genetic differences within and between populations and is a part of evolutionary biology. Such equations are accruing in ecology, plasma physics, physiology, and phase transition problems.

The standard form of Fisher-Kolmogorov FK equation is,

$$\phi_t - \Delta\phi + \phi^3 - \phi = 0, \qquad \Omega \times [0, T]. \tag{1.1}$$

This is classical Fisher-Kolmogorov equation [3–5]. In above Eq (1.1) by adding a stabilizing fourth-order derivative term to the Fisher-Kolmogorov equation obtained the extended Fisher-Kolmogorov EFK equation for the real valued function P defined on $\Omega \times [0, T]$ as [6–10],

$$\phi_t + \gamma\Delta^2\phi - \Delta\phi + \phi^3 - \phi = 0, \qquad \Omega \times [0, T], \tag{1.2}$$

where Ω is bounded in domain R^2, γ is positive constant, the Laplace operator is $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and biharmonic operator is taken as $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$, So, Eq (1.2) takes the form,

$$\frac{\partial\phi}{\partial t} + \gamma\frac{\partial^4\phi}{\partial x^4} + 2\gamma\frac{\partial^4\phi}{\partial x^2\partial y^2} + \gamma\frac{\partial^4\phi}{\partial y^4} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} + \phi^3 - \phi = 0. \tag{1.3}$$

In the Eq (1.3) the fourth-order term describes the phase transitions near critical points which are also known as Lipschitz points. The phase transitions or phase changes are the physical process of transition one state medium to another medium with different values of parameters near the critical point. Commonly the term is used to refer to changes among the basic states of matter, solid, liquid, and gas, as well as plasma in rare cases [11].

Now, a day the interesting field of research is to find the exact solitary wave solutions, so, we will consider the EFK equation. So there are many different approaches to find the exact solutions of nonlinear PDEs such as, (G'/G)-expansion method [12, 13], new MEDA technique [14, 15], Riccati equation mapping method [16, 17], generalized exponential rational functional method [18, 19], ϕ^6-model expansion method [20, 21], the tanh-coth method [22, 23], the Lie symmetry method [24, 25], etc. But in this article, we use He's variational technique for more detail see [26–29]. This is an easily applicable and powerful approach to finding the solitons solution via an energy method or semi-inverse principle [30–33].

2. Existence results

In this section, we aim at solving the fourth order PDE Eq (1.2). Since PDE Eq (1.2) carries the first order derivative with respect to time and the fourth ordered partial derivatives w.r.t. the space variable x

so, classically $\phi \in C^4[a,b]$, $\phi \in C^1[0,T]$ by simple integrating Eq (1.2) can be reduced to the following integral equation in operator form,

$$\Phi = \phi_0(x) + \int_0^t \left(\phi(x,s) - \phi^3(x,s) + \Delta\phi(x,s) - \gamma\Delta^2\phi(x,s)\right) ds, \tag{2.1}$$

clearly, $|\Delta\phi| \le k_d$ and $|\Delta^2\phi| \le k_b$ are bounded values and we are going to apply the schauder fixed point theorem [34]. For this, we shall choose the solution in the ball

$$B_r(\Theta) = \{\phi, \phi \in C[0,\rho], \|\phi\| \le r\}, \tag{2.2}$$

now, the following conditions for Schauder theorem, will be verified

- Self mapping, $\Phi : B_r(\Theta) \to B_r(\Theta)$,
- $\Phi(B)$ is relatively compact.

Firstly, taking the norm of Eq (1.2),

$$\begin{aligned} \|\Phi\| &\le \|\phi_0\| + \int_0^t \left(\|\phi\| - \|\phi^3\| + \|\Delta\phi\| - \gamma\|\Delta^2\phi\|\right) ds, && (2.3)\\ \|\Phi\| &\le c + \rho(r + r^3 + k_d + \gamma k_b), && (2.4)\\ \|\Phi\| &\le r && (2.5)\\ \rho &\le \frac{r-c}{(r + r^3 + k_d + \gamma k_b)}. && (2.6) \end{aligned}$$

Now, for relatively compactness condition,

$$\begin{aligned} \|\Phi_i(t) - \Phi(t^*) \quad \| &\le \int_t^{t^*} \left(\|\phi_i\| - \|\phi_i^3\| + \|\Delta\phi_i\| - \gamma\|\Delta^2\phi_i\|\right) ds, && (2.7)\\ &\le (r + r^3 + k_d + \gamma k_b)|t - t^*|, && (2.8) \end{aligned}$$

$\Rightarrow$ are equi-continuous. So, by Arzela-Ascoli theorem theorem, there exists a uniformly convergent subsequence Φ_{ij} of ϕ_i, so $\Phi(B_r(\Theta))$ is relatively compact and by Schauder fixed point theorem there exists at least one fixed point of Eq (1.2) and hence the solutions exists.

3. Variational principle

To obtained the variational principle order, we use the transformation $\phi(x,y,t) = u(\rho)$ where $\rho = \nu x + \nu y - ct + \rho_0$. So, apply this transformation on Eq (1.3) to change PDE into ODE as,

$$-cu' + 4\gamma\nu^4 u'''' - 2\nu^2 u'' + u^3 - u = 0, \tag{3.1}$$

where $' = \frac{d}{d\rho}$.

Integrating the above Eq (3.1) once with respect to ρ and reduce in the form,

$$-cu + 4\gamma\nu^4 u''' - 2\nu^2 u' + \frac{u^4}{4} - \frac{u^2}{2} = 0, \tag{3.2}$$

here integrating constant is neglected. Again integrated the above relation and takes form as,

$$6\gamma\nu^4 u'' - 4\nu^2 u + \frac{u^5}{10} - \frac{u^3}{3} - cu^2 = 0. \tag{3.3}$$

Now, by using the semi-inverse method [30–33], we obtained the variational formulation of Eq (3.3) as follows,

$$J(\rho) = \int \left(-3\gamma\nu^4 (u')^2 - 2\nu^2 u^2 + \frac{u^6}{60} - \frac{u^4}{12} - \frac{1}{3}cu^3\right) d\rho. \tag{3.4}$$

So, by using variational principle we reduced the order of Eq (5.2) which is in more simplest form. In this next section section we use He's variational methods to find the solitons and solitary wave solutions.

4. Solitons solutions

The solitons solution of Eq (3.3) is assumed in the following form,

$$u(\rho) = \delta \sec h^2(\kappa\rho), \tag{4.1}$$

where δ and κ are arbitrary constants that we will determined later. So, substitute the Eq (4.1) into Eq (5.2), then it will gives,

$$\begin{aligned} J(\delta,\kappa) &= \int_0^\infty \left(-3\gamma\nu^4 (u')^2 - 2\nu^2 u^2 + \frac{u^6}{60} - \frac{u^4}{12} - \frac{1}{3}cu^3\right) d\rho, \qquad (4.2)\\ &= \int_0^\infty \Big(-12\gamma\nu^4\delta^2\kappa^2 \sec h^4(\kappa\rho)\tanh^2(\kappa\rho) - 2\nu^2\delta^2 \sec h^4(\kappa\rho) \\ &\quad + \frac{\delta^6 \sec h^{12}(\kappa\rho)}{60} - \frac{\delta^4 \sec h^8(\kappa\rho)}{12} - \frac{1}{3}c\delta^3 \sec h^6(\kappa\rho)\Big) d\rho, \\ &= -\frac{8}{5}\gamma\nu^2\delta^2\kappa - \frac{2\,\nu^2\delta^2}{\kappa} + \frac{2}{225}\frac{\delta^6}{\kappa} - \frac{1}{18}\frac{\delta^4}{\kappa} - \frac{1}{12}\frac{c\delta^3\pi}{\kappa}. \qquad (4.3) \end{aligned}$$

By He's variational methods there are

$$\frac{\partial J}{\partial \delta} = 0, \tag{4.4}$$

$$\frac{\partial J}{\partial \kappa} = 0. \tag{4.5}$$

So, above Eq (4.3) give the results as follows,

$$-\frac{16}{5}\gamma\nu^2\delta\kappa - \frac{8}{3}\frac{\nu^2\delta}{\kappa} + \frac{128}{3465}\frac{\delta^5}{\kappa} - \frac{16}{105}\frac{\delta^3}{\kappa} - \frac{8}{15}\frac{c\delta^2}{\kappa} = 0, \tag{4.6}$$

$$-\frac{8}{5}\gamma\nu^2\delta^2 + \frac{4}{3}\frac{\nu^2\delta^2}{\kappa^2} - \frac{64}{10395}\frac{\delta^6}{\kappa^2} + \frac{4}{105}\frac{\delta^4}{\kappa^2} + \frac{8}{45}\frac{c\delta^3}{\kappa^2} = 0, \tag{4.7}$$

$$\delta = \frac{1}{2}\sqrt{d} + \frac{1}{2}\sqrt{\frac{1155c}{\sqrt[32]{d}} - b - a + \frac{99}{16}}, \tag{4.8}$$

$$\kappa = \frac{i}{2\sqrt{42}\sqrt{\gamma\nu}} \times \sqrt{-70\nu^2 - \frac{7}{2}c\sqrt{d} - \frac{7}{2}c\sqrt{\frac{1155c}{\sqrt[32]{d} - b - a + \frac{99}{32}}} - \frac{1}{2}\sqrt{d}\sqrt{\frac{1155c}{\sqrt[32]{d}} - b - a + \frac{99}{16}} - \frac{1155c}{\sqrt[28]{d}}}, \tag{4.9}$$

where,

$a = \frac{99(297-17920\nu^2)}{32\ 2^{2/3}\sqrt[3]{-h+2305195200c^2-9484231680\nu^2-52396146}}$,

$b = \frac{\sqrt[3]{-h+2305195200c^2-9484231680\nu^2-52396146}}{192\sqrt[3]{2}}$,

where

$h = \sqrt{(-2305195200c^2 + 9484231680\nu^2 + 52396146)^2 - 4(88209 - 5322240\nu^2)^3}$,

$d = a + b + \frac{99}{32}$.

So, the solitons solutions are obtained as,

$$\phi(x, y, t) = \left(\frac{1}{2}\sqrt{d} + \frac{1}{2}\sqrt{\frac{1155c}{\sqrt[32]{d}} - b - a + \frac{99}{16}}\right)\sec h^2(\kappa(\nu x + \nu y - ct + \rho_0)) \tag{4.10}$$

here we use κ from the Eq (4.9).

5. Solitary wave solutions

In this section we investigates the periodic wave solutions by using the He's variational method of Eq (1.2) for more detail see [29, 35]. So, we assume the periodic wave solution in the form of

$$u(\rho) = \delta\cos(\kappa\rho), \tag{5.1}$$

substituting Eq (5.1) in Eq (5.2) and result as in the periodic form is,

$$J(\rho) = \int_0^{\frac{\pi}{2}} \left(-3\gamma\nu^4(u')^2 - 2\nu^2u^2 + \frac{u^6}{60} - \frac{u^4}{12} - \frac{1}{3}cu^3\right)d\rho \tag{5.2}$$

$$= \int_0^{\frac{\pi}{2}} \left(-3\gamma\nu^4\rho^2\kappa^2\sin^2(\kappa\rho) - 2\nu^2\delta^2\cos^2(\kappa\rho) + \frac{\delta^6}{60}\cos^6(\kappa\rho) - \frac{\delta^4}{12}\cos^4(\kappa\rho) - \frac{1}{3}c\rho^3\cos^3(\kappa\rho)\right)d\rho \tag{5.3}$$

$$= \frac{1}{\kappa}\int_0^{\frac{\pi}{2}} \left(-3\gamma\nu^4\rho^2\kappa^2\sin^2(\theta) - 2\nu^2\delta^2\cos^2(\theta) + \frac{\delta^6}{60}\cos^6(\theta) - \frac{\delta^4}{12}\cos^4(\theta) - \frac{1}{3}c\rho^3\cos^3(\theta)\right)d\theta. \tag{5.4}$$

By He's variational methods the above Eq (5.4) give the results as follows,

$$\frac{\partial J}{\partial \delta} = -6\gamma\nu^2\delta\kappa - 4\nu^2\delta + \frac{1}{64}\delta^5\pi - \frac{1}{16}\delta^3\pi - \frac{2}{3}c\delta^2 = 0, \tag{5.5}$$

$$\frac{\partial J}{\partial \kappa} = -6\gamma\nu^2\delta^2\kappa - 2\nu^2\delta^2 + \frac{1}{384}\delta^6\pi - \frac{1}{64}\delta^4\pi - \frac{2}{9}c\delta^3 = 0, \tag{5.6}$$

So, by the help of *mathematica* solve the above equations and get the values of δ and κ as follows,

$$\delta = \frac{1}{2}\sqrt{-\frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5}} - \frac{1}{2}\sqrt{-\frac{1024c}{15\pi\sqrt{\frac{36\sqrt[3]{2}(9\pi-1280\nu^2)}{5a}}} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5} + \frac{a}{45\sqrt[3]{2}\pi} + \frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} + \frac{24}{5}}, \tag{5.7}$$

$$\begin{aligned}\kappa = \frac{1}{2880\gamma\nu^2}\Bigg(&-768\nu^2 + 32c\sqrt{-\frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5}}\\
&+32c\sqrt{-\frac{1024c}{15\pi\sqrt{-\frac{36\sqrt[3]{2}(9\pi-1280\nu^2)}{5a} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{25}}} + \frac{a}{45\sqrt[3]{2}\pi} + \frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} + \frac{24}{5}}\\
&-\frac{3\pi}{2}\sqrt{-\frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5}}\\
&\sqrt{-\frac{1024c}{15\pi\sqrt{-\frac{36\sqrt[3]{2}(9\pi-1280\nu^2)}{5a} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5}}} + \frac{a}{45\sqrt[3]{2}\pi} + \frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} + \frac{24}{5}}\\
&+\frac{256c}{5\sqrt{-\frac{36\sqrt[3]{2}(9\pi-1280\nu^2)}{5a} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5}}} - \frac{27\pi}{5}\Bigg),\end{aligned} \tag{5.8}$$

where $a = \sqrt[3]{\sqrt{(-106168320\pi c^2 + 134369280\pi^2\nu^2 + 314928\pi^3)^2 - 4\,(2916\pi^2 - 414720\pi\nu^2)^3}}$ $- \sqrt[3]{106168320\pi c^2 + 134369280\pi^2\nu^2 + 314928\pi^3}$,
So, the periodic wave or solitary wave solution of Eq (3.3),

$$u(\rho) = \left(\frac{1}{2}\sqrt{-\frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5}} - \frac{1}{2}\sqrt{-\frac{1024c}{15\pi\sqrt{\frac{36\sqrt[3]{2}(9\pi-1280\nu^2)}{5a}}} - \frac{a}{45\sqrt[3]{2}\pi} + \frac{12}{5} + \frac{a}{45\sqrt[3]{2}\pi} + \frac{36\sqrt[3]{2}\,(9\pi - 1280\nu^2)}{5a} + \frac{24}{5}}\right)\cos(\kappa\rho), \tag{5.9}$$

here we use the above value of κ and get the approximate solution of Eq (1.2) in the view of Eq (3.1) as above. So, the exact solitary wave solution of EFK equation is,

$$\phi(x,y,t)=\left(\frac{1}{2}\sqrt{-\frac{36\sqrt[3]{2}\,(9\pi-1280\nu^2)}{5a}-\frac{a}{45\sqrt[3]{2}\pi}+\frac{12}{5}}\right.$$
$$-\frac{1}{2}\sqrt{-\frac{1024c}{15\pi\sqrt{\frac{36\sqrt[3]{2}(9\pi-1280\nu^2)}{5a}}}-\frac{a}{45\sqrt[3]{2}\pi}+\frac{12}{5}}$$
$$\left.+\frac{a}{45\sqrt[3]{2}\pi}+\frac{36\sqrt[3]{2}\,(9\pi-1280\nu^2)}{5a}+\frac{24}{5}\right)\cos(\kappa(\nu x+\nu y-ct+\rho_0)). \tag{5.10}$$

Remark 5.1. On the same procedure we can obtain other solitons solutions by choosing the $u(\rho)=\delta\tanh(\kappa\rho)$, $u(\rho)=\delta\operatorname{csc}h(\kappa\rho)$, $u(\rho)=\delta\coth(\kappa\rho)$ and for the mixed functions as well. Similarly for the periodic functions $u(\rho)=\delta\tan(\kappa\rho)$, $u(\rho)=\delta\sin(\kappa\rho)$, $u(\rho)=\delta\cot(\kappa\rho)$ and for mixed periodic functions as well.

6. Graphical behavior

In this section, we discussed the graphical behavior of the solutions by the different choices of parameters. The solitary wave solution of Eqs (4.10) and (5.10) are described physically by choosing the two sets of parameters. The solutions are useful to study the physical process of the transition from one state medium to another medium and in the population growth dynamics and wave propagation. The 2D and 3D and their corresponding contour plots of the solutions Eq (4.10), $\phi(x,y,t)$ are shown in Figures (1) and (2) which shows the soliton solutions and also for Eq (5.10), $\phi(x,y,t)$ are shown in Figures (3) and (4) shows the periodic solutions by the different choices of parameters for the the EFK model. Hence, the physical description of our results may fruitful tool for investigating the further results for nonlinear wave problems in applied science.

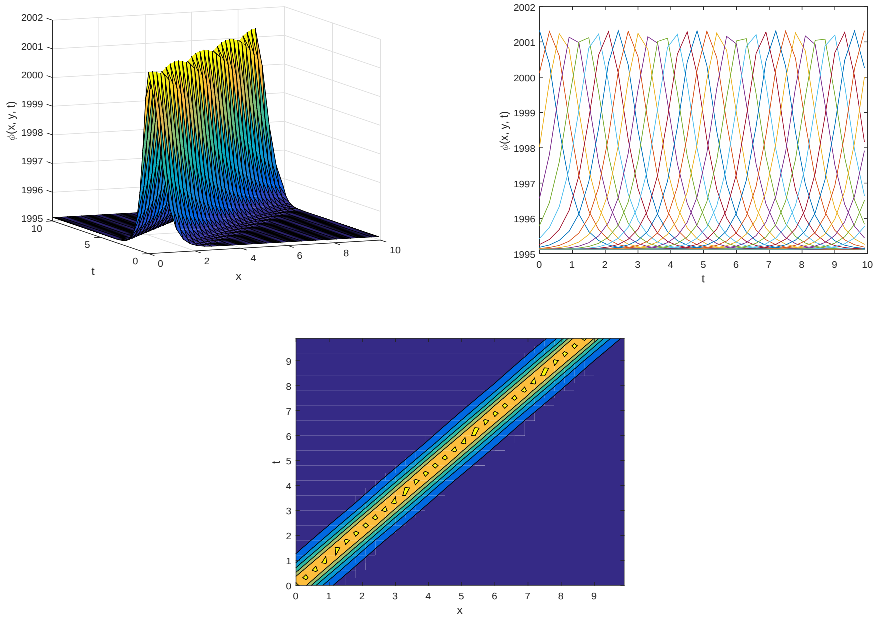

Figure 1. The plots of solutions $\phi(x, y, t)$ for different values of parameters as $\nu = 0.0595$, $c = 1.195$ and $\gamma = 3.9$.

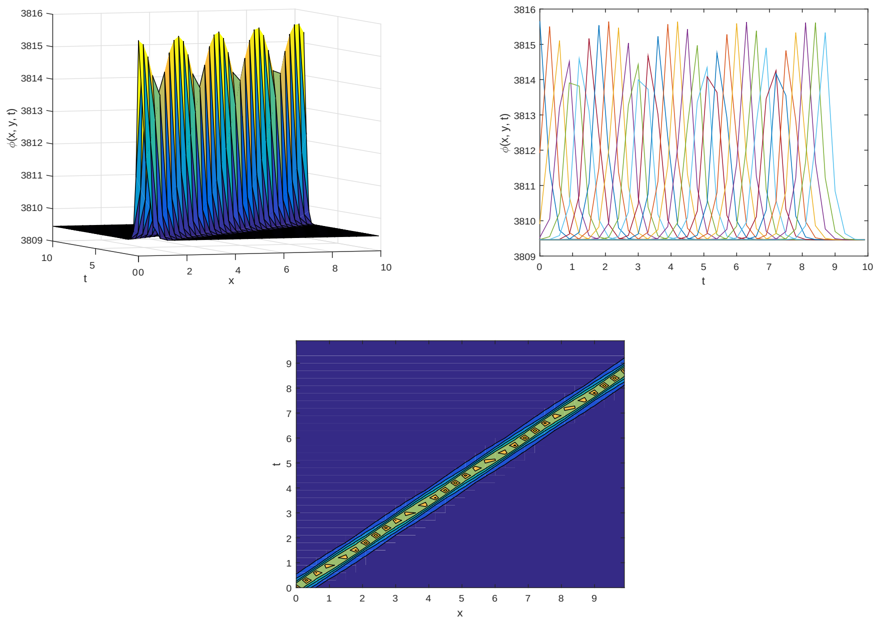

Figure 2. The plots of solutions $\phi(x, y, t)$ for different values of parameters as $\nu = 2.2$, $c = 2.52$ and $\gamma = 1.5$.

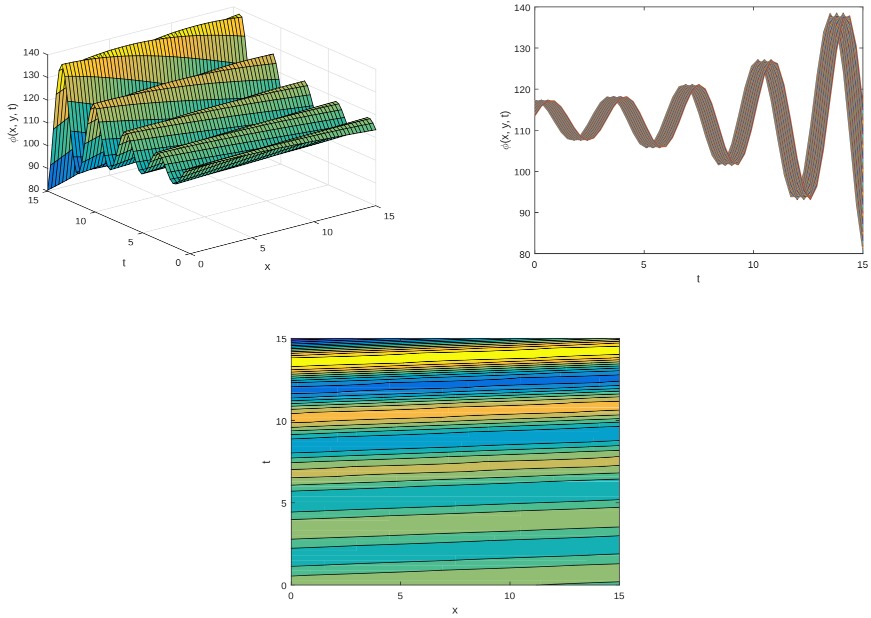

Figure 3. The plots of solutions $\phi(x, y, t)$ for different values of parameters as $\nu = 1.2$, $c = 1.052$ and $\gamma = 3.5$.

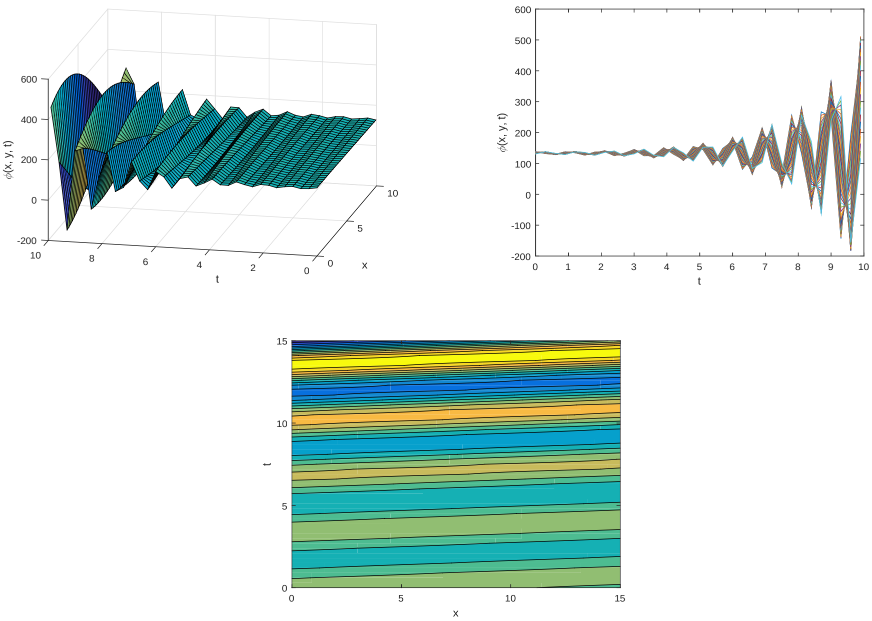

Figure 4. The plots of solutions $\phi(x, y, t)$ for different values of parameters as $\nu = 0.0095$, $c = 1.195$ and $\gamma = 1.9$.

7. Conclusions

In this research, the extended Fisher-Kolmogorov (EFK) equation is investigated for the solitons and periodic wave solutions. This equation is used in the population growth dynamics and wave propagation. The fourth-order term in this model describes the phase transitions near critical points which are also known as Lipschitz points. The advantages in population dynamics of wave spatial spread of an advantageous allele. The existence of solutions of the EFK equation is successfully found by using the Schauder theorem. He's variational method is adopted to construct the singular soliton solutions as well as the periodic wave solutions successfully. This method gives us the advantage to reduce the order of the equation and make it simple for the calculation as compared to the other techniques. The graphics of solutions are also sketched in 3D and 2D and corresponding contour representations. The state variable has spatial dynamical behavior with two spatial independent variables which leads to the important physical phenomena as compared to the one-dimensional case.

Conflict of interest

The authors declare no conflict of interest.

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