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Research article

Fixed point results for a new contraction mapping with integral and fractional applications

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Abstract: The purpose of this manuscript is to present some fixed point results for a Λ-Ćirić mapping in the setting of non-triangular metric spaces. Also, two numerical examples are given to support the theoretical study. Finally, under suitable conditions, the existence and uniqueness of a solution to a general Fredholm integral equation, a Riemann-Liouville fractional differential equation and a Caputo non-linear fractional differential equation are discussed as applications.

Keywords: fixed point technique; non-triangular metric space; Fredholm integral equation; fractional differential equation
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1. Introduction

Fixed point theory is an active field of research with wide range of applications in numerous directions. It is interested with the results about stating that due to some conditions a self mapping T on a set Λ possesses one or more fixed points. Fixed point theory began almost immediately after the classical analysis started its quick development. The main next growth was investigated by the need to establish existence results for problems dealing with integral and differential equations. Hence, the fixed point theory began a pure analytical theory.

The Banach contraction principle [1] is one of the most nice theorems in fixed point theory. Due to its application in variant fields such as biology, physics, computer science, chemistry and several branches of mathematics, this classical famous theorem has been improved, extended and generalized in nonlinear analysis.

In mathematics, the Fredholm integral equation is an integral equation whose solution is due to Fredholm operators and the study of Fredholm kernels. Several types of numerical and analytical methods and numerical methods were used to solve this problem. One of useful techniques to solve such equations is the usage of fixed point method, see [2–6]. Fractional differential equations appear in many fields such as physics, mechanics, chemistry, economics, engineering and biological sciences, etc.; see for example [7–12]. The theory of fractional differential (evolution) equations is a useful branch of mathematics by which variant physical phenomena in several fields of engineering and science can be studied. In the recent years, there has been a remarkable development in partial and ordinary differential equations using fractional derivatives. Many authors studied the existence and uniqueness of positive solutions for (nonlinear) fractional differential equation boundary value problems, see [13, 14]. Among them, new existence results in Banach spaces by using the fractional derivatives and fixed point theorems have been presented, see [15, 16].

In last years, many generalizations of standard metric spaces related to generalizing the Banach contraction theorem have been investigated. Most of known fixed point achievements in literature are given by taking into account the triangle inequality or other its generalizations (b-metric [17], partial metric [18], dislocated metric [19], G-metric [20], extended b-metric [21], controlled metric [22], double controlled metric [23], etc). Many works appeared in order to make weaker the triangle inequality. Amon them, Jleli and Samet [24] introduced a new metric setting, called as a JS-metric involving the power of sequences, covering some generalized metrics, as the b-metric, the dislocated metric and the modular metric. Recently, Khojasteh and Khandani [25] initiated the notion of non-triangular metric spaces in order to weaken the condition suggested in [24]. Their concept is based on the fact that if the limit of a convergent sequence exists, it is unique. Some fixed point results using manageable functions are presented in [26].

When the triangle inequality is omitted, the results becomes more difficult to establish. However, several real applications suffer from the lack of the triangle inequality, and so the related results will be more interesting and nice. Our paper goes with this point of view. In particular, we deal with a non-triangulat metric space. We will prove some related fixed point results involving Ćirić [27] and Wardowski [28] contraction mappings. Some concrete examples are provided. At the end, by applying our obtained results we ensure the existence of solutions of a Fredholm integral equation, a Riemann-Liouville fractional differential equation and a Caputo non-linear fractional differential equation.

2. Background and material

This part is devoted to present the initial characteristics of contraction mappings and some previously defined ideas with references.

Definition 2.1. [25] Let ∇ be a non-empty set and $\ell : \nabla^2 \to [0, \infty)$ be a mapping satisfying, for all $\vartheta, \theta, \varsigma \in \nabla$,

(n_1) $\ell(\vartheta, \vartheta) = 0$;
(n_2) $\ell(\vartheta, \theta) = \ell(\theta, \vartheta)$;
(n_3) for a sequence $\{\vartheta_i\}_{i\in\mathbb{N}} \in \nabla$ with $\lim_{i\to\infty} \ell(\vartheta_i, \vartheta) = 0$ and $\lim_{i\to\infty} \ell(\vartheta_i, \theta) = 0$, we have $\vartheta = \theta$.

Then ℓ is called a non-triangular metric and the pair (∇, ℓ) is called a non-triangular metric space (NTMS, for short).

The definition of a Ćirić contraction mapping is stated as follows:

Definition 2.2. [27] A self mapping ζ on a metric space (∇, ℓ) is called a Ćirić contraction if there exists $\hbar \in (0, \frac{1}{2})$ so that the inequality below holds:

$$\ell(\zeta\vartheta, \zeta\theta) \leq \hbar \left(\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right), \ \forall \vartheta, \theta \in \nabla.$$

In 2012, Wardowski [28] generalized the Banach contraction mapping [1] and introduced some different forms for contraction mappings. His definition is stated as follows:

Definition 2.3. Assume that $\Lambda : \mathbb{R}_+ \to \mathbb{R}$ is a function justifying

(Λ_i) Λ is strictly increasing;
(Λ_{ii}) for each sequence $\{\vartheta_i\}_{i\in\mathbb{N}}$ of positive numbers with $\lim_{i\to\infty} \vartheta_i = 0 \Leftrightarrow \lim_{i\to\infty} \Lambda(\vartheta_i) = -\infty$;
(Λ_{iii}) there is a constant $\mu \in (0, 1)$ so that $\lim_{i\to 0^+} (\vartheta_i)^\mu \Lambda(\vartheta_i) = 0$.

Let Σ be the family of all functions $\Lambda : \mathbb{R}_+ \to \mathbb{R}$ fulfilling (i)-(iii). A function $\zeta : \nabla \to \nabla$ is called a $\Lambda-$contraction if the following inequality

$$\ell(\zeta\vartheta, \zeta\theta) > 0 \Rightarrow \alpha + \Lambda\left(\ell(\zeta\vartheta, \zeta\theta)\right) \leq \Lambda\left(\ell(\vartheta, \theta)\right). \tag{2.1}$$

holds for all $\vartheta, \theta \in \nabla$ and $\Lambda \in \Sigma$.

Based on the inequality (2.1), the same author presented some various contractions as follows: For all $\vartheta, \theta \in \Lambda$ with $\zeta\vartheta \neq \zeta\theta$,

$$(i)\ \Lambda_1(\vartheta) = \ln(\vartheta), \qquad \frac{\ell(\zeta\vartheta,\zeta\theta)}{\ell(\vartheta,\theta)} \leq e^{-\alpha},$$

$$(ii)\ \Lambda_2(\vartheta) = \ln(\vartheta) + \vartheta, \qquad \ell(\zeta\vartheta, \zeta\theta)e^{\ell(\zeta\vartheta,\zeta\theta)} \leq \ell(\vartheta, \theta)e^{\ell(\vartheta,\theta)-\alpha},$$

$$(iii)\ \Lambda_3(\vartheta) = \frac{-1}{\sqrt{\vartheta}}, \qquad \ell(\zeta\vartheta, \zeta\theta)\left(1 + \ell\sqrt{\ell(\vartheta, \theta)}\right)^2 \leq \ell(\vartheta, \theta),$$

$$(iv)\ \Lambda_4(\vartheta) = \ln(\vartheta^2 + \vartheta), \qquad \ell(\zeta\vartheta, \zeta\theta)(1 + \ell(\zeta\vartheta, \zeta\theta))) \leq e^{-\alpha}\ell(\vartheta, \theta)(1 + \ell(\vartheta, \theta)),$$

where $\{\Lambda_i : i = 1, 2, 3, 4\} \in \Sigma$.

It should be noted that, the inequality (2.1) yields that the mapping ℓ is contractive. Hence, every $\Lambda-$contraction is continuous.

Rashwan and Hasanen [29] added a new function to the family Σ but the shape of the contraction under this function is not known until now. This function takes the form $\Lambda(\vartheta) = \frac{-1}{\sqrt[r]{\vartheta}}$, where $r > 1$ and $\vartheta > 0$.

Now, we merge the results of Ćirić and Wardowski to obtain the following contraction mapping in the context of NTMSs.

Definition 2.4. Let (∇, ℓ) be an NTMS. We say that $\zeta : \nabla \to \nabla$ is a $\Lambda-$Ćirić mapping if there exists $\hbar \in (0,1)$ such that for each $\vartheta, \theta \in \nabla$ with $\varrho > 0$, we have

$$\varrho + \Lambda\left(\ell(\zeta\vartheta, \zeta\theta)\right) \leq \Lambda\left(\hbar\left[\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right]\right). \tag{2.2}$$

Clearly, the inequality (2.2) reduces to (2.1), if we take $\Lambda(\vartheta) = \ln(\vartheta)$.

Definition 2.5. [25] Let (∇, ℓ) be an NTMS. A mapping $\zeta : \nabla \to \nabla$ is called asymptotically regular if for all $\{\vartheta_i\} \subset \nabla$, $\lim_{i\to\infty} \ell\left(\zeta\vartheta_i, \zeta\vartheta_{i+1}\right) = 0$.

Definition 2.6. [25] Let (∇, ℓ) be an NTMS. A sequence $\{\vartheta_i\} \subset \nabla$ is said to be

- convergent to the point $\vartheta \in \nabla$ if $\lim_{i\to\infty} \ell\left(\vartheta_i, \vartheta\right) = 0$,
- a Cauchy sequence if $\lim_{i,j\to\infty} \ell\left(\vartheta_i, \vartheta_j\right) = 0$.

If every Cauchy sequence in ∇ converges to some element $\vartheta \in \nabla$, then the NTMS (∇, ℓ) is called complete.

3. Novel results

This part is devoted to discuss the existence and uniqueness of an FP for the $\Lambda-$Ćirić mapping under asymptotic regularity in the setting of an NTMS.

Theorem 3.1. *Let (∇, ℓ) be a complete NTMS. Then a $\Lambda-$Ćirić mapping $\zeta : \nabla \to \nabla$ owns a unique FP.*

Proof. Let ϑ_0 be an arbitrary point in ∇ and define a sequence $\{\vartheta_i\}_{i\geq 1}$ by $\vartheta_{i+1} = \zeta\vartheta_i$ and $\vartheta_i = \zeta^i\vartheta_0$ for all $i \in \mathbb{N}$.

Based on the definition of ζ, we get

$$\varrho + \Lambda\left(\ell(\zeta\vartheta, \zeta\theta)\right) \leq \Lambda\left(\hbar\left[\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right]\right), \text{ for some } \varrho > 0.$$

It follows that, for some $\hbar \in (0, \frac{1}{2})$,

$$\Lambda\left(\ell(\zeta\vartheta, \zeta\theta)\right) < \Lambda\left(\hbar\left[\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right]\right).$$

From (Λ_i), we can write

$$\ell(\zeta\vartheta, \zeta\theta) < \varpi\left[\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right], \ \forall \vartheta, \theta \in \nabla. \tag{3.1}$$

Putting $\vartheta = \vartheta_{i-1}$ and $\theta = \vartheta_i$ in (3.1), we have

$$\begin{aligned} \ell(\vartheta_i, \vartheta_{i+1}) &= \ell(\zeta\vartheta_{i-1}, \zeta\vartheta_i) < \varpi\left[\ell(\vartheta_{i-1}, \zeta\vartheta_{i-1}) + \ell(\vartheta_i, \zeta\vartheta_i)\right] \\ &= \varpi\left[\ell(\vartheta_{i-1}, \vartheta_i) + \ell(\vartheta_i, \vartheta_{i+1})\right], \end{aligned}$$

which implies that

$$\ell(\vartheta_i, \vartheta_{i+1}) < \rho\ell(\vartheta_{i-1}, \vartheta_i), \text{ for all } i \in \mathbb{N}, \tag{3.2}$$

where $\rho = \frac{\varpi}{1-\varpi} < 1$. Effecting the function Λ on both sides of (3.1), we obtain

$$\Lambda\left(\ell(\vartheta_i, \vartheta_{i+1})\right) < \Lambda\left(\rho\ell(\vartheta_{i-1}, \vartheta_i)\right).$$

Hence,

$$\Lambda\left(\ell(\vartheta_i,\vartheta_{i+1})\right) \leq \Lambda\left(\rho\ell(\vartheta_{i-1},\vartheta_i)\right) - \varrho,\ \text{for some } \varrho > 0.$$

Similarly, one can write

$$\Lambda\left(\ell(\vartheta_{i-1},\vartheta_i)\right) \leq \Lambda\left(\rho^2\ell(\vartheta_{i-2},\vartheta_{i-1})\right) - 2\varrho,\ \text{for some } \varrho > 0.$$

Following the same scenario, for some $\varrho > 0$, we find that

$$\Lambda\left(\ell(\zeta\vartheta_{i-1},\zeta\vartheta_i)\right) = \Lambda\left(\ell(\vartheta_i,\vartheta_{i+1})\right) \leq \Lambda\left(\rho^i\ell(\vartheta_0,\vartheta_1)\right) - i\varrho. \tag{3.3}$$

Letting $i \to \infty$ in (3.3), we conclude that

$$\lim_{i\to\infty}\Lambda\left(\ell(\zeta\vartheta_{i-1},\zeta\vartheta_i)\right) = -\infty.$$

Using (Λ_{ii}), we have

$$\lim_{i\to\infty}\ell(\zeta\vartheta_{i-1},\zeta\vartheta_i) = 0. \tag{3.4}$$

This shows that the sequence $\{\zeta\vartheta_i\}$ is asymptotically regular and hence the sequence $\{\vartheta_{i+1}\}$ or $\{\vartheta_i\}$.

Now, we prove that $\{\zeta\vartheta_i\}$ is a Cauchy sequence. Indeed, for $i, j \in \mathbb{N}$ with $j \geq i$, putting $\vartheta = \vartheta_i$ and $\theta = \vartheta_j$ in (2.2), we have

$$\ell(\zeta\vartheta_i,\zeta\vartheta_j) < \varpi\left[\ell(\vartheta_i,\zeta\vartheta_i) + \ell(\vartheta_j,\zeta\vartheta_j)\right] = \varpi\left[\ell(\zeta\vartheta_{i-1},\zeta\vartheta_i) + \ell(\zeta\vartheta_{j-1},\zeta\vartheta_j)\right]. \tag{3.5}$$

Taking the limit as $i \to \infty$ in (3.5) and using (3.4), we have

$$\ell(\zeta\vartheta_i,\zeta\vartheta_j) \to 0 \text{ as } i \to \infty.$$

This proves that the sequence $\{\zeta\vartheta_i\}$ is Cauchy in ∇. Since (∇,ℓ) is complete, there is a point $\vartheta^* \in \nabla$ such that

$$\lim_{i\to\infty}\zeta\vartheta_i = \vartheta^*. \tag{3.6}$$

In order to obtain an FP of ζ, choosing $\vartheta = \vartheta_i$ and $\theta = \vartheta^*$ in (2.2), we get

$$\varrho + \Lambda\left(\ell(\zeta\vartheta_i,\zeta\vartheta^*)\right) \leq \Lambda\left(\hbar\left[\ell(\vartheta_i,\zeta\vartheta_i) + \ell(\vartheta^*,\zeta\vartheta^*)\right]\right),\ \text{for some } \varrho > 0.$$

Hence,

$$\ell(\zeta\vartheta_i,\zeta\vartheta^*) \leq \hbar\left[\ell(\vartheta_i,\zeta\vartheta_i) + \ell(\vartheta^*,\zeta\vartheta^*)\right].$$

As $i \to \infty$ in the above inequality and using (3.6), we can write

$$(1-\hbar)\ell(\vartheta^*,\zeta\vartheta^*) \leq 0,$$

since $\hbar < 1$. Then the above inequality holds only if $\ell(\vartheta^*,\zeta\vartheta^*) = 0$, that is, $\vartheta^* = \zeta\vartheta^*$. Hence, ϑ^* is an FP of ζ. For the uniqueness, let $\widehat{\vartheta} \in \nabla$ be another distinct FP of ζ, i.e.,

$$\vartheta^* = \zeta\vartheta^* \text{ and } \widehat{\vartheta} = \zeta\widehat{\vartheta}.$$

Selecting $\vartheta = \vartheta^*$ and $\theta = \widehat{\vartheta}$ in (2.2), we have

$$\varrho + \Lambda\left(\ell(\zeta\vartheta^*,\zeta\widehat{\vartheta})\right) \leq \Lambda\left(\hbar\left[\ell(\vartheta^*,\zeta\vartheta^*) + \ell(\widehat{\vartheta},\zeta\widehat{\vartheta})\right]\right),\ \text{for some } \varrho > 0.$$

Hence,

$$\ell(\vartheta^*,\widehat{\vartheta}) = \ell(\zeta\vartheta^*,\zeta\widehat{\vartheta}) \leq \hbar\left[\ell(\vartheta^*,\zeta\vartheta^*) + \ell(\widehat{\vartheta},\zeta\widehat{\vartheta})\right] = 0.$$

This implies that $\vartheta^* = \widehat{\vartheta}$. This completes the proof. $\square$

The following results follow immediately from Theorem 3.1.

Corollary 3.2. *Let* (∇, ℓ) *be a complete NTMS and* ζ *be a self-mapping such that*

$$\ell(\zeta\vartheta, \zeta\theta) < \hbar \min\{\ell(\vartheta, \zeta\vartheta), \ell(\theta, \zeta\theta)\},$$

for all $\vartheta, \theta \in \nabla$, *where* $\hbar \in (0, \frac{1}{2})$. *Then* ζ *owns a unique FP in* ∇.

Proof. For each $\vartheta, \theta \in \nabla$, it's easy to see that

$$\ell(\zeta\vartheta, \zeta\theta) < \hbar \min\{\ell(\vartheta, \zeta\vartheta), \ell(\theta, \zeta\theta)\} \leq \hbar\left[\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right].$$

Since Λ is monotonically increasing, we obtain

$$\Lambda\left(\ell(\zeta\vartheta, \zeta\theta)\right) < \Lambda\left(\hbar\left[\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right]\right).$$

For some constant $\varrho > 0$, we can write

$$\varrho + \Lambda\left(\ell(\zeta\vartheta, \zeta\theta)\right) \leq \Lambda\left(\hbar\left[\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right]\right).$$

This implies that ζ is a $\Lambda-$Ćirić mapping. Applying Theorem 3.1, we can find a unique FP of ζ. □

Corollary 3.3. *Let* (∇, ℓ) *be a complete NTMS and* ζ *be a self-mapping such that*

$$\ell(\zeta\vartheta, \zeta\theta) \leq \hbar\left(\frac{\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)}{1 + \ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)}\right),$$

for all $\vartheta, \theta \in \nabla$, *where* $\hbar \in (0, \frac{1}{2})$. *Then* ζ *owns a unique FP in* ∇.

Proof. Consider

$$\ell(\zeta\vartheta, \zeta\theta) \leq \hbar\left(\frac{\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)}{1 + \ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)}\right) \leq \hbar\left(\ell(\vartheta, \zeta\vartheta) + \ell(\theta, \zeta\theta)\right).$$

Then ζ is a $\Lambda-$Ćirić mapping. Hence, the desired result is obtained. □

To support our studies, we present the examples below.

Example 3.4. Let $\nabla = [0, \infty)$ and $\ell : \nabla^2 \to [0, \infty)$ be a metric described as

$$\ell(\vartheta, \theta) = \begin{cases} \frac{\vartheta+\theta}{\vartheta+\theta+1}, & \text{if } \vartheta \neq \theta, \vartheta \neq 0, \theta \neq 0, \\ 0, & \text{if } \vartheta = \theta, \\ \frac{\vartheta}{2}, & \text{if } \theta = 0, \\ \frac{\theta}{2}, & \text{if } \vartheta = 0. \end{cases}$$

Clearly, (∇, ℓ) is an NTMS [25]. It is not a metric space. Define a self-mapping ζ by $\zeta\vartheta = \frac{1}{17}\vartheta$. Consider for all $\vartheta, \theta \in \nabla$,

$$\Lambda\left(\ell(\zeta\vartheta, \zeta\theta)\right) = \Lambda\left(\ell\left(\frac{\vartheta}{17}, \frac{\theta}{17}\right)\right) = \Lambda\left(\frac{\vartheta + \theta}{\vartheta + \theta + 17}\right).$$

Also,

$$\begin{aligned}\Lambda\left(\hbar\left[\ell(\vartheta,\zeta\vartheta)+\ell(\theta,\zeta\theta)\right]\right) &= \Lambda\left(\frac{1}{3}\left[\ell\left(\vartheta,\frac{\vartheta}{17}\right)+\ell\left(\theta,\frac{\theta}{17}\right)\right]\right)\\ &= \Lambda\left(6\left[\frac{\vartheta}{18\vartheta+17}+\frac{\theta}{18\theta+17}\right]\right)\\ &\geq \Lambda\left(3\left(\frac{\vartheta+\theta}{\vartheta+\theta+17}\right)\right),\end{aligned}$$

where $\hbar=\frac{1}{3}$. Let $\Lambda\in\Sigma$ be a function defined by $\Lambda(\vartheta)=\ln(\vartheta)$, for $\vartheta>0$. Then

$$\begin{aligned}\Lambda\left(\ell(\zeta\vartheta,\zeta\theta)\right)-\Lambda\left(\hbar\left[\ell(\vartheta,\zeta\vartheta)+\ell(\theta,\zeta\theta)\right]\right) &\leq \ln\left(\frac{\vartheta+\theta}{\vartheta+\theta+17}\right)-\ln\left(3\left(\frac{\vartheta+\theta}{\vartheta+\theta+17}\right)\right)\\ &= \ln\left(\frac{\left(\frac{\vartheta+\theta}{\vartheta+\theta+17}\right)}{3\left(\frac{\vartheta+\theta}{\vartheta+\theta+17}\right)}\right)=\ln(\frac{1}{3})=-\ln(3).\end{aligned}$$

Therefore, ζ a $\Lambda-$Ćirić mapping with $\varrho=\ln(3)>0$. According to Theorem 3.1, ζ has 0 as a unique FP.

Example 3.5. Let $\nabla=\left\{\frac{1}{2^{j-4}}:j\in\mathbb{N}\right\}\cup\{0\}$ under the metric defined in Example 3.4. Then the pair (∇,ℓ) is an NTMS. Define a nonlinear mapping $\zeta:\nabla\to\nabla$ by

$$\zeta\vartheta=\begin{cases}\{\frac{1}{2^{2j}}\}, & \text{if}\quad \vartheta\in\left\{\frac{1}{2^{2j-4}};j\in\mathbb{N}\right\},\\ 0, & \text{if}\qquad \vartheta=0.\end{cases}$$

To prove that ζ is a $\Lambda-$Ćirić mapping, we discuss the following cases:

(i) If $\vartheta=\frac{1}{2^{2j-4}}$ and $\theta=\frac{1}{2^{2m-4}}$, for $m>j\geq1$, then one can write

$$\begin{aligned}\Lambda(\ell(\zeta\vartheta,\zeta\theta)) &= \Lambda\left(\ell\left(\zeta\left(\frac{1}{2^{2j-4}}\right),\zeta\left(\frac{1}{2^{2m-4}}\right)\right)\right)=\Lambda\left(\ell\left(\frac{1}{2^{2j}},\frac{1}{2^{2m}}\right)\right)\\ &= \Lambda\left(\frac{\frac{1}{2^{2j}}+\frac{1}{2^{2m}}}{1+\frac{1}{2^{2j}}+\frac{1}{2^{2m}}}\right)=\Lambda\left(\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right).\end{aligned}\tag{3.7}$$

Also,

$$\begin{aligned}\Lambda\left(\hbar\left[\ell(\vartheta,\zeta\vartheta)+\ell(\theta,\zeta\theta)\right]\right) &= \Lambda\left(\frac{1}{3}\left[\ell\left(\frac{1}{2^{2j-4}},\zeta\left(\frac{1}{2^{2j-4}}\right)\right)+\ell\left(\frac{1}{2^{2m-4}},\zeta\left(\frac{1}{2^{2m-4}}\right)\right)\right]\right)\\ &= \Lambda\left(\frac{1}{3}\left[\ell\left(\frac{1}{2^{2j-4}},\frac{1}{2^{2j}}\right)+\ell\left(\frac{1}{2^{2m-4}},\frac{1}{2^{2m}}\right)\right]\right)\\ &= \Lambda\left(\frac{1}{3}\left[\frac{16}{16+2^{2j}}+\frac{16}{16+2^{2m}}\right]\right)\\ &= \Lambda\left(\frac{16}{3}\left[\frac{32+2^{2m}+2^{2j}}{256+16\left(2^{2m}+2^{2j}\right)+2^{2m+2j}}\right]\right)\end{aligned}$$

$$\geq \quad \Lambda\left(\frac{3}{2}\left[\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right]\right). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}
&\Lambda(\ell(\zeta\vartheta,\zeta\theta)) - \Lambda\left(\hbar\left[\ell(\vartheta,\zeta\vartheta)+\ell(\theta,\zeta\theta)\right]\right)\\
&\leq \Lambda\left(\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right) - \Lambda\left(\frac{3}{2}\left[\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right]\right)\\
&= \ln\left(\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right) - \ln\left(\frac{3}{2}\left[\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right]\right)\\
&= ln\left(\frac{\left(\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right)}{\frac{3}{2}\left(\frac{2^{2m}+2^{2j}}{2^{2m}+2^{2j}+2^{2m+2j}}\right)}\right) = \ln\left(\frac{2}{3}\right) = -\ln\left(\frac{3}{2}\right) \cong -0.4055.
\end{aligned}$$

(ii) If $\vartheta = \frac{1}{2^{2j-4}}$ and $\theta = 0$, then we have

$$\begin{aligned}
\Lambda(\ell(\zeta\vartheta,\zeta\theta)) &= \Lambda\left(\ell\left(\zeta\left(\frac{1}{2^{2j-4}}\right),\zeta(0)\right)\right) = \Lambda\left(\ell\left(\frac{1}{2^{2j}},0\right)\right)\\
&= \Lambda\left(\frac{\frac{1}{2^{2j}}}{1+\frac{1}{2^{2j}}}\right) = \Lambda\left(\frac{1}{1+2^{2j}}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\Lambda\left(\hbar\left[\ell(\vartheta,\zeta\vartheta)+\ell(\theta,\zeta\theta)\right]\right) &= \Lambda\left(\frac{1}{3}\left[\ell\left(\frac{1}{2^{2j-4}},\zeta\left(\frac{1}{2^{2j-4}}\right)\right)+\ell(0,\zeta 0)\right]\right)\\
&= \Lambda\left(\frac{1}{3}\left[\ell\left(\frac{1}{2^{2j-4}},\frac{1}{2^{2j}}\right)\right]\right) = \Lambda\left(\frac{1}{3}\left[\frac{16}{16+2^{2j}}\right]\right)\\
&= \Lambda\left(\frac{16}{3}\left[\frac{1}{16+2^{2j}}\right]\right) \geq \Lambda\left(\frac{3}{2}\left[\frac{1}{1+2^{2j}}\right]\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Lambda(\ell(\zeta\vartheta,\zeta\theta)) - \Lambda\left(\hbar\left[\ell(\vartheta,\zeta\vartheta)+\ell(\theta,\zeta\theta)\right]\right) &\leq \Lambda\left(\frac{1}{1+2^{2j}}\right) - \Lambda\left(\frac{3}{2}\left[\frac{1}{1+2^{2j}}\right]\right)\\
&= \ln\left(\frac{1}{1+2^{2j}}\right) - \ln\left(\frac{3}{2}\left[\frac{1}{1+2^{2j}}\right]\right)\\
&= \ln\left(\frac{\left(\frac{1}{1+2^{2j}}\right)}{\frac{3}{2}\left[\frac{1}{1+2^{2j}}\right]}\right) = \ln\left(\frac{2}{3}\right) = -\ln\left(\frac{3}{2}\right) \cong -0.4055.
\end{aligned}$$

(iii) If $\vartheta = 0$ and $\theta = \frac{1}{2^{2m-4}}$, then the proof follows immediately from Cases (i) and (ii).

Based on the above cases, we conclude that ζ is a $\Lambda-$Ćirić mapping with $\varrho = 0.4055$ and $\Lambda(\vartheta) = \ln(\vartheta)$ for $\vartheta > 0$. So by Theorem 3.1, 0 is the unique FP of ζ.

4. A general Fredholm integral equation

In this part, we apply Theorem 3.1 to discuss the existence and uniqueness of a unique solution to a general Fredholm integral equation. This solution is equivalent to find a unique fixed point of the mapping ζ.

Consider the following problem:

$$\vartheta(\tau)\varphi(\tau) = e^{-\varrho}\nu(\tau) + e^{-\varrho}\int_{\delta_0}^{\delta_1} \varpi(\tau, s)\varphi(s)ds,\ \forall\tau, s \in [\delta_0, \delta_1],\ \varrho > 0, \tag{4.1}$$

where $\varphi(\tau) \in C[\delta_0, \delta_1]$ is a continuous function, $\varpi : [\delta_0, \delta_1] \times [\delta_1, \delta_0] \to \mathbb{R}$ is a square integrable function and $\nu : [\delta_0, \delta_1] \to \mathbb{R}$ is a known function.

Assume that $\nabla = (C[\delta_0, \delta_1], \mathbb{R})$ is the set of real continuous functions on $[\delta_0, \delta_1]$ endowed with

$$\ell(\vartheta, \theta) = \max_{\tau\in[\delta_1,\delta_0]} \{|\vartheta(\tau) - \theta(\tau)|\},\ \text{for each } \vartheta, \theta \in \nabla. \tag{4.2}$$

Problem (4.1) will be considered under the following assumptions:

(p_1) there exist functions $\vartheta_1(\tau)$ and $\vartheta_2(\tau)$ in $C[\delta_0, \delta_1]$ so that $\vartheta_2(\tau) \geq \vartheta_1(\tau)$ for each $\vartheta_1(\tau), \vartheta_2(\tau) > 0$;

(p_2) for some $\varphi_1(\tau), \varphi_2(\tau) \in \nabla$ and for all $\tau \in [\delta_0, \delta_1]$, we have

$$\nu_2(\tau) + \int_{\delta_0}^{\delta_1} \varpi_2(\tau, s)\varphi_2(s)ds = (\varphi_1(\tau) - \varphi_2(\tau))\, e^{\varrho}\vartheta_2(\tau),$$

for any function $\nu_2(\tau) \in \mathbb{R}$ and $\varrho > 0$.

Now, we can state and prove our main theorem in this section.

Theorem 4.1. *Under the hypotheses (p_1) and (p_3), Eq (4.1) has a unique solution in ∇.*

Proof. Define the mapping $\zeta : \nabla \to \nabla$ by

$$\zeta(\varphi_1)(\tau) = \frac{\nu_1(\tau) + \int_{\delta_0}^{\delta_1} \varpi_1(\tau, s)\varphi_1(s)ds}{e^{\varrho}\vartheta_1(\tau)}, \tag{4.3}$$

for all $\varphi_1(\tau) \in \nabla$ and $\tau \in [\delta_0, \delta_1]$. Using (4.3), we get

$$|\zeta(\varphi_2)(\tau) - \zeta(\varphi_1)(\tau)| = \left| \frac{\nu_2(\tau) + \int_{\delta_0}^{\delta_1} \varpi_2(\tau, s)\varphi_2(s)ds}{e^{\varrho}\vartheta_2(\tau)} - \frac{\nu_1(\tau) + \int_{\delta_0}^{\delta_1} \varpi_1(\tau, s)\varphi_1(s)ds}{e^{\varrho}\vartheta_1(\tau)} \right|.$$

From assumption (p_1), $\vartheta_2(\tau) \geq \vartheta_1(\tau)$, i.e., $\frac{1}{\vartheta_1(\tau)} \geq \frac{1}{\vartheta_2(\tau)}$, we obtain $\frac{1}{\vartheta_1(\tau)} > \frac{1}{2\vartheta_2(\tau)}$. Applying this fact in the above equation, we can write

$$\begin{aligned} &|\zeta(\varphi_2)(\tau) - \zeta(\varphi_1)(\tau)| \\ &\leq \left| \frac{\nu_2(\tau) + \int_{\delta_0}^{\delta_1} \varpi_2(\tau, s)\varphi_2(s)ds}{e^{\varrho}\vartheta_2(\tau)} - \frac{\nu_1(\tau) + \int_{\delta_0}^{\delta_1} \varpi_1(\tau, s)\varphi_1(s)ds}{2e^{\varrho}\vartheta_2(\tau)} \right| \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&= \frac{1}{2}\left|\frac{2\nu_2(\tau)+2\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds}{e^{\varrho}\vartheta_2(\tau)}-\frac{\nu_1(\tau)+\int_{\delta_0}^{\delta_1}\varpi_1(\tau,s)\varphi_1(s)ds}{e^{\varrho}\vartheta_2(\tau)}\right| \\
&= \frac{1}{2}\left|\frac{\nu_2(\tau)+\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds}{e^{\varrho}\vartheta_2(\tau)}+\frac{\nu_2(\tau)+\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds}{e^{\varrho}\vartheta_2(\tau)}-\frac{\nu_1(\tau)+\int_{\delta_0}^{\delta_1}\varpi_1(\tau,s)\varphi_1(s)ds}{e^{\varrho}\vartheta_2(\tau)}\right|.
\end{aligned}
$$

Applying the assumption (p_2) in (4.5), one can write

$$
\begin{aligned}
&|\zeta(\varphi_2)(\tau)-\zeta(\varphi_1)(\tau)| \\
&\leq \frac{1}{2}\left|\frac{\nu_2(\tau)+\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds}{e^{\varrho}\vartheta_2(\tau)}+\frac{(\varphi_1(\tau)-\varphi_2(\tau))\,e^{2\varrho}\vartheta_2(\tau)}{e^{\varrho}\vartheta_2(\tau)}-\frac{\nu_1(\tau)+\int_{\delta_0}^{\delta_1}\varpi_1(\tau,s)\varphi_1(s)ds}{e^{\varrho}\vartheta_2(\tau)}\right| \\
&= \frac{1}{2}\left|\left(\frac{e^{-\varrho}\left(\nu_2(\tau)+\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds\right)}{\vartheta_2(\tau)}-\varphi_2(\tau)e^{\varrho}\right)\right. \\
&\qquad\left.-\left(\frac{e^{-\varrho}\left(\nu_1(\tau)+\int_{\delta_0}^{\delta_1}\varpi_1(\tau,s)\varphi_1(s)ds\right)}{\vartheta_2(\tau)}-\varphi_1(\tau)e^{\varrho}\right)\right| \\
&\leq \frac{1}{2}\left|\left(\frac{e^{-\varrho}\left(\nu_2(\tau)+\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds\right)}{\vartheta_2(\tau)}-\varphi_2(\tau)e^{-\varrho}\right)\right. \\
&\qquad\left.-\left(\frac{e^{-\varrho}\left(\nu_1(\tau)+\int_{\delta_0}^{\delta_1}\varpi_1(\tau,s)\varphi_1(s)ds\right)}{\vartheta_2(\tau)}-\varphi_1(\tau)e^{-\varrho}\right)\right| \\
&\leq \frac{e^{-\varrho}}{2}\left[\left|\left(\frac{\nu_2(\tau)+\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds}{\vartheta_2(\tau)}-\varphi_2(\tau)\right)\right|+\left|\left(\frac{\nu_1(\tau)+\int_{\delta_0}^{\delta_1}\varpi_1(\tau,s)\varphi_1(s)ds}{\vartheta_2(\tau)}-\varphi_1(\tau)\right)\right|\right].
\end{aligned}
$$

Since $\frac{1}{\vartheta_1(\tau)} \geq \frac{1}{\vartheta_2(\tau)}$, we have

$$
\begin{aligned}
|\zeta(\varphi_2)(\tau)-\zeta(\varphi_1)(\tau)| &\leq \frac{e^{-\varrho}}{2}\left(\left|\frac{\nu_1(\tau)+\int_{\delta_0}^{\delta_1}\varpi_1(\tau,s)\varphi_1(s)ds}{\vartheta_1(\tau)}-\varphi_1(\tau)\right|\right. \\
&\qquad\left.+\left|\frac{\nu_2(\tau)+\int_{\delta_0}^{\delta_1}\varpi_2(\tau,s)\varphi_2(s)ds}{\vartheta_2(\tau)}-\varphi_2(\tau)\right|\right) \\
&= \frac{e^{-\varrho}}{2}\left(|\zeta(\varphi_1)(\tau)-\varphi_1(\tau)|+|\zeta(\varphi_2)(\tau)-\varphi_2(\tau)|\right).
\end{aligned}
$$

Using (4.2), we obtain

$$
e^{\varrho}\ell(\zeta(\varphi_2),\zeta(\varphi_1)) \leq \frac{1}{2}\left[\ell(\zeta(\varphi_1),\varphi_1)+\ell(\zeta(\varphi_1),\varphi_1)\right].
$$

Taking $\hbar = \frac{1}{2}$ and applying ln to both sides, we get

$$\varrho + \ln\left[\ell\left(\zeta\left(\varphi_2\right), \zeta\left(\varphi_1\right)\right)\right] \leq \ln\left[\hbar\left(\ell\left(\zeta\left(\varphi_1\right), \varphi_1\right) + \ell\left(\zeta\left(\varphi_1\right), \varphi_1\right)\right)\right], \text{ for some } \varrho > 0.$$

This implies that ζ is a Λ−Ćirić mapping with $\Lambda(\vartheta) = \ln(\vartheta) \in \Sigma$. By Theorem 3.1, there exists a unique FP of a mapping ζ, that is, the unique solution of the integral equation (4.1). □

5. Riemann-Liouville fractional order operator

In this part, we discuss the existence and uniqueness solution of a Riemann-Liouville fractional order operator by Theorem 3.1. This operator is defined as follows: Assume that $\varphi(\tau)$ is a class of functions, which have $c + 1$ continuous derivatives for all $\tau \in [0, \delta]$, then the Riemann-Liouville fractional derivative of the function $\varphi(\tau)$ with the order ε is described as

$$\begin{aligned} {}_{\delta}D_{\tau}^{\varepsilon}\varphi(\tau) &= \frac{1}{\Gamma(-\varepsilon + c + 1)} \frac{d}{d(\tau)^c} \int_{\delta}^{\tau} (\tau - \vartheta)^{c-\varepsilon} \varphi(\vartheta) d\vartheta \\ &= \sum_{k=0}^{c} \frac{\varphi^{(k)}(\delta)(\tau - \delta)^{k-\varepsilon}}{\Gamma(k - \varepsilon + 1)} + \frac{1}{\Gamma(-\varepsilon + c + 1)} \int_{\delta}^{\tau} (\tau - \vartheta)^{c-\varepsilon} \varphi^{(c+1)}(\vartheta) d\vartheta, \end{aligned} \tag{5.1}$$

where $c < \varepsilon \leq c + 1$ and $\tau \in [0, \delta]$.

Assume that $\nabla = (C[0, \delta], \mathbb{R})$ is the set of real continuous functions on $[0, \delta]$ equipped with

$$\ell(\vartheta, \theta) = \max_{\tau \in [0,\delta]} |\vartheta(\tau) - \theta(\tau)|, \text{ for all } \vartheta, \theta \in \nabla.$$

The existence solution of the integral operator (5.1) will be discussed under the postulate below:

(IO) there exist functions $\varphi_1(\tau)$ and $\varphi_2(\tau)$ in ∇ so that for each $\tau \in [0, \delta]$, we have $\varphi_1(\tau) \geq \varphi_2(\tau)$ and

$$\varphi_1^{(s)}(\tau) - \varphi_2^{(s)}(\tau) \leq e^{-\varrho} \left(\frac{\varphi_1^{(s)}(\tau) + \varphi_2^{(s)}(\tau)}{3} \right),$$

for all $s \in (0, \infty)$ and for some $\varrho > 0$. Here $\varphi_1^{(s)}(\tau)$ refers to the s^{th}-order derivative of the function φ_1.

Theorem 5.1. *Riemann-Liouville fractional derivative operator (5.1) has a unique solution in ∇ provided that the postulate (IO) holds.*

Proof. Define an operator $\zeta : \nabla \to \nabla$ by

$$\zeta\varphi(\tau) = \sum_{k=0}^{c} \frac{\varphi^{(k)}(\delta)(\tau - \delta)^{k-\varepsilon}}{\Gamma(k - \varepsilon + 1)} + \frac{1}{\Gamma(c - \varepsilon + 1)} \int_{\delta}^{\tau} (\tau - \vartheta)^{c-\varepsilon} \varphi^{(c+1)}(\vartheta) d\vartheta, \tag{5.2}$$

for all $\varphi(\tau) \in \nabla$ and $\tau \in [0, \delta]$. Then the unique solution of the integral operator (5.1) is equivalent to find a unique FP of the operator (5.2). For each $\tau \in [0, \delta]$ and $c < \varepsilon \leq c + 1$, we have

$$|\zeta\varphi_1(\tau) - \varphi_1(\tau)| + |\zeta\varphi_2(\tau) - \varphi_2(\tau)|$$

$$= \left|\sum_{k=0}^{c}\frac{\varphi_1^{(k)}(\delta)(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\varphi_1^{(c+1)}(\vartheta)d\vartheta-\varphi_1(\vartheta)\right|$$
$$+\left|\sum_{k=0}^{c}\frac{\varphi_2^{(k)}(\delta)(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\varphi_2^{(c+1)}(\vartheta)d\vartheta-\varphi_2(\vartheta)\right|. \tag{5.3}$$

Consider

$$\begin{aligned}|\zeta\varphi_1(\tau)-\zeta\varphi_2(\tau)| &= \left|\sum_{k=0}^{c}\frac{\varphi_1^{(k)}(\delta)(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\varphi_1^{(c+1)}(\vartheta)d\vartheta\right.\\
&\quad\left.-\sum_{k=0}^{c}\frac{\varphi_2^{(k)}(\delta)(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}-\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\varphi_2^{(c+1)}(\vartheta)d\vartheta\right|\\
&= \left|\sum_{k=0}^{c}\frac{(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}\left[\varphi_1^{(k)}(\delta)-\varphi_2^{(k)}(\delta)\right]\right.\\
&\quad\left.+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\left[\varphi_1^{(c+1)}(\vartheta)-\varphi_2^{(c+1)}(\vartheta)\right]d\vartheta\right|.\end{aligned}$$

Applying the condition (IO) and using (5.3), one can write

$$\begin{aligned}&|\zeta\varphi_1(\tau)-\zeta\varphi_2(\tau)|\\
\le& \left|\sum_{k=0}^{c}\frac{(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}\left[\varphi_1^{(k)}(\delta)-\varphi_2^{(k)}(\delta)\right]\right.\\
&\left.+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\left[\varphi_1^{(c+1)}(\vartheta)-\varphi_2^{(c+1)}(\vartheta)\right]d\vartheta+\varphi_2(\vartheta)-\varphi_1(\vartheta)\right|\\
=& \left|\sum_{k=0}^{c}\frac{(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}\left[\varphi_1^{(k)}(\delta)-\varphi_2^{(k)}(\delta)\right]\right.\\
&\left.+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\left[\varphi_1^{(c+1)}(\vartheta)-\varphi_2^{(c+1)}(\vartheta)\right]d\vartheta-[\varphi_1(\vartheta)-\varphi_2(\vartheta)]\right|\\
\le& \frac{e^{-\varrho}}{3}\left|\sum_{k=0}^{c}\frac{(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}\left[\varphi_1^{(k)}(\delta)+\varphi_2^{(k)}(\delta)\right]\right.\\
&\left.+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\left[\varphi_1^{(c+1)}(\vartheta)+\varphi_2^{(c+1)}(\vartheta)\right]d\vartheta-[\varphi_1(\vartheta)+\varphi_2(\vartheta)]\right|\\
=& \frac{e^{-\varrho}}{3}\left(\left|\sum_{k=0}^{c}\frac{\varphi_1^{(k)}(\delta)(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\varphi_1^{(c+1)}(\vartheta)d\vartheta-\varphi_1(\vartheta)\right.\right.\\
&\left.\left.+\sum_{k=0}^{c}\frac{\varphi_2^{(k)}(\delta)(\tau-\delta)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}+\frac{1}{\Gamma(-\varepsilon+c+1)}\int_{\delta}^{\tau}(\tau-\vartheta)^{c-\varepsilon}\varphi_2^{(c+1)}(\vartheta)d\vartheta-\varphi_2(\vartheta)\right|\right)\\
=& \frac{e^{-\varrho}}{3}\left(|\zeta\varphi_1(\tau)-\varphi_1(\tau)|+|\zeta\varphi_2(\tau)-\varphi_2(\tau)|\right).\end{aligned}$$

By (4.2), we have

$$e^{\varrho}\ell\left(\zeta\left(\varphi_2\right),\zeta\left(\varphi_1\right)\right)\le\frac{1}{3}\left[\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)+\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)\right].$$

Putting $\hbar = \frac{1}{3}$ and applying ln to both sides, we get

$$\varrho + \ln\left[\ell\left(\zeta\left(\varphi_2\right), \zeta\left(\varphi_1\right)\right)\right] \leq \ln\left[\hbar\left(\ell\left(\zeta\left(\varphi_1\right), \varphi_1\right) + \ell\left(\zeta\left(\varphi_1\right), \varphi_1\right)\right)\right], \text{ for some } \varrho > 0,$$

or equivalently

$$\varrho + \Lambda\left(\ell\left(\zeta\left(\varphi_2\right), \zeta\left(\varphi_1\right)\right)\right) \leq \Lambda\left(\hbar\left(\ell\left(\zeta\left(\varphi_1\right), \varphi_1\right) + \ell\left(\zeta\left(\varphi_1\right), \varphi_1\right)\right)\right).$$

This implies that ζ is a Λ−Ćirić mapping with $\Lambda(\vartheta) = \ln(\vartheta) \in \Sigma$. Based on Theorem 3.1, the mapping ζ has a unique FP, which is the unique solution of the integral operator (5.1). □

6. Caputo non-linear fractional differential equation

There is no doubt that non-linear fractional differential equations play a great role in many applications such as mathematical modeling, engineering, physics, and many real-world problems. So, the goal of this part is to study the existence of a solution to non-linear fractional differential equation of Caputo type by Theorem 3.1. El-Hady and Agrekci [30] studied the stability problem of some fractional differential equations with Caputo derivatve in the sense of Hyers-Ulam and Hyers-Ulam-Rassias based on some fixed point techniques.

Caputo's formula for derivatives is presented as follows:

$$^{C}D^{\varepsilon}\varphi(\tau) = \Xi(\tau, \varphi(\tau)), \tag{6.1}$$

with boundary conditions

$$\varphi(0) = 0,\ \varphi(1) = \int_0^{\delta} \varphi(\vartheta)d\vartheta,\ \delta \in (0, 1),$$

where$^{C}D^{\varepsilon}$ represents the Caputo fractional derivative with order ε. Moreover, for a continuous function $\varphi : [0, \infty) \to \mathbb{R}$, the Caputo fractional derivative with order ε is described as

$$^{C}D^{\varepsilon}\varphi(\tau) = \frac{1}{\Gamma(c - \varepsilon)} \int_0^{\delta_1} (\delta_1 - \vartheta)^{c-\varepsilon-1}\varphi^{(c)}(\vartheta)d\vartheta,\ c - 1 < \varepsilon \leq c.$$

Let $\nabla = C[0, 1]$ be the set of all real-valued continuous functions on $[0, 1]$. Define $\ell : \nabla^2 \to [0, \infty)$ by

$$\ell(\vartheta, \theta) = \max_{\tau\in[0,1]} |\vartheta(\tau) - \theta(\tau)|, \text{ for each } \vartheta, \theta \in \nabla. \tag{6.2}$$

Now, we consider the following hypotheses:

(h_1) there exist continuous functions $\Xi_1, \Xi_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\Xi(\tau, \varphi_2(\tau)) - \Xi(\tau, \varphi_1(\tau)) \leq \frac{\Xi(\tau, \varphi_2(\tau)) + \Xi(\tau, \varphi_1(\tau))}{4e^{\varrho}},$$

for all $\tau \in [0, 1]$ and $\varphi_1, \varphi_2 \in \nabla$;

(h_2) there are $\varphi_1(\tau), \varphi_2(\tau) \geq 0$ with $\varphi_1(\tau) \leq \varphi_2(\tau)$ such that

$$\varphi_2(\tau) - \varphi_1(\tau) \leq \frac{\varphi_2(\tau) + \varphi_1(\tau)}{4e^{\varrho}},$$

for all $\varphi_1(\tau), \varphi_2(\tau) \in \nabla$.

We are ready to state and prove the main theorem in this section.

Theorem 6.1. *Under hypotheses* (h_1) *and* (h_2), *the boundary value problem (6.1) has a unique solution in* ∇.

Proof. Let $\zeta : \nabla \to \nabla$ be a mapping defined by

$$\begin{aligned}\zeta\varphi(\tau) &= \frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi(\vartheta))d\vartheta - \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^1 (1-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi(\vartheta))d\vartheta \\ &\quad + \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^\delta\int_0^\vartheta (\vartheta-\phi)^{\varepsilon-1}\Xi(\phi,\varphi(\phi))d\phi d\vartheta, \end{aligned} \tag{6.3}$$

for $\tau \in [0,1]$. The function $\varphi \in \nabla$ is a unique solution of the problem (6.1) if and only if $\varphi = \zeta\varphi$, i.e., φ is a unique FP of ζ. To achieve that, we shall prove that the ζ is a $\Lambda-$Ćirić mapping. Consider,

$$\begin{aligned} & |\zeta\varphi_1(\tau)-\varphi_1(\tau)| + |\zeta\varphi_2(\tau)-\varphi_2(\tau)| \\ = & \left|\frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_1(\vartheta))d\vartheta - \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^1 (1-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_1(\vartheta))d\vartheta \right. \\ & \left. + \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^\delta\int_0^\vartheta (\vartheta-\phi)^{\varepsilon-1}\Xi(\phi,\varphi_1(\phi))d\phi d\vartheta - \varphi_1(\tau)\right| \\ & + \left|\frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_2(\vartheta))d\vartheta - \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^1 (1-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_2(\vartheta))d\vartheta \right. \\ & \left. + \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^\delta\int_0^\vartheta (\vartheta-\phi)^{\varepsilon-1}\Xi(\phi,\varphi_2(\phi))d\phi d\vartheta - \varphi_1(\tau)\right| \\ = & \left|\frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\vartheta)^{\varepsilon-1}\left[\Xi(\vartheta,\varphi_1(\vartheta)) + \Xi(\vartheta,\varphi_2(\vartheta))\right]d\vartheta \right. \\ & \left. - \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^1 (1-\vartheta)^{\varepsilon-1}\left[\Xi(\vartheta,\varphi_1(\vartheta)) + \Xi(\vartheta,\varphi_2(\vartheta))\right]d\vartheta - \left[\varphi_1(\tau)+\varphi_2(\vartheta)\right]\right|. \end{aligned} \tag{6.4}$$

Also, we obtain

$$\begin{aligned} & |\zeta\varphi_2(\tau)-\zeta\varphi_1(\tau)| \\ = & \left|\frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_2(\vartheta))d\vartheta - \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^1 (1-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_2(\vartheta))d\vartheta \right. \\ & + \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^\delta\int_0^\vartheta (\vartheta-\phi)^{\varepsilon-1}\Xi(\phi,\varphi_2(\phi))d\phi d\vartheta \\ & - \frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_1(\vartheta))d\vartheta + \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^1 (1-\vartheta)^{\varepsilon-1}\Xi(\vartheta,\varphi_1(\vartheta))d\vartheta \\ & \left. - \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^\delta\int_0^\vartheta (\vartheta-\phi)^{\varepsilon-1}\Xi(\phi,\varphi_1(\phi))d\phi d\vartheta\right| \\ \le & \left|\frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\vartheta)^{\varepsilon-1}\left[\Xi(\vartheta,\varphi_2(\vartheta)) - \Xi(\vartheta,\varphi_1(\vartheta))\right]d\vartheta \right. \\ & - \frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^1 (1-\vartheta)^{\varepsilon-1}\left[\Xi(\vartheta,\varphi_2(\vartheta)) - \Xi(\vartheta,\varphi_1(\vartheta))\right]d\vartheta - \left[\varphi_2(\vartheta)-\varphi_1(\vartheta)\right] \end{aligned}$$

$$+\frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^{\delta}\int_0^{\vartheta}(\vartheta-\phi)^{\varepsilon-1}\left[\Xi(\phi,\varphi_2(\phi))-\Xi(\phi,\varphi_1(\phi))\right]d\phi d\vartheta\Bigg|.$$

Applying the assumptions (h_1), (h_2) and using (6.3), we get

$$\begin{aligned}
& |\zeta\varphi_2(\tau)-\zeta\varphi_1(\tau)| \\
\leq\ & \frac{e^{-\varrho}}{4}\Bigg(\Bigg|\frac{1}{\Gamma(\varepsilon)}\int_0^{\tau}(\tau-\vartheta)^{\varepsilon-1}\left[\Xi(\vartheta,\varphi_2(\vartheta))+\Xi(\vartheta,\varphi_1(\vartheta))\right]d\vartheta \\
& -\frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^{1}(1-\vartheta)^{\varepsilon-1}\left[\Xi(\vartheta,\varphi_2(\vartheta))+\Xi(\vartheta,\varphi_1(\vartheta))\right]d\vartheta-\left[\varphi_2(\vartheta)+\varphi_1(\vartheta)\right] \\
& +\frac{2\tau}{(2-\delta^2)\,\Gamma(\varepsilon)}\int_0^{\delta}\int_0^{\vartheta}(\vartheta-\phi)^{\varepsilon-1}\left[\Xi(\phi,\varphi_2(\phi))+\Xi(\phi,\varphi_1(\phi))\right]d\phi d\vartheta\Bigg|\Bigg) \\
=\ & \frac{e^{-\varrho}}{4}\left(|\zeta\varphi_1(\tau)-\varphi_1(\tau)|+|\zeta\varphi_2(\tau)-\varphi_2(\tau)|\right).
\end{aligned}$$

Using (6.2), we can write

$$e^{\varrho}\ell\left(\zeta\left(\varphi_2\right),\zeta\left(\varphi_1\right)\right)\leq\frac{1}{4}\left[\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)+\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)\right].$$

Letting $\hbar=\frac{1}{4}$ and applying ln to both sides, we obtain

$$\varrho+\ln\left[\ell\left(\zeta\left(\varphi_2\right),\zeta\left(\varphi_1\right)\right)\right]\leq\ln\left[\hbar\left(\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)+\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)\right)\right],\ \text{for some }\varrho>0,$$

or equivalently

$$\varrho+\Lambda\left(\ell\left(\zeta\left(\varphi_2\right),\zeta\left(\varphi_1\right)\right)\right)\leq\Lambda\left(\hbar\left(\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)+\ell\left(\zeta\left(\varphi_1\right),\varphi_1\right)\right)\right).$$

Thus, the mapping ζ is a $\Lambda-$Ćirić mapping with $\Lambda(\vartheta)=\ln(\vartheta)\in\Sigma$. According to Theorem 3.1, ζ has a unique FP. This completes the proof. □

Open Question: It is clear that the space of interest lacks many assumptions and situations through which we can address many applications. Among these situations, how can we study Hyers-Ulam-Rassias and Hyers-Ulam stability (see [31–33]) of some fractional differential equation with Caputo derivative?

7. Conclusions

In this work, we presented some fixed point results involving $\Lambda-$Ćirić mappings in the setting of non-triangular metric spaces. We illustrated the obtained results by some concrete examples and some applications. We solved a general Fredholm integral equation, a Riemann-Liouville fractional differential equation and a Caputo non-linear fractional differential equation. As perspectives, it would be interesting to extend the paper and give related applications to non classical metric spaces, like fuzzy bipolar metric spaces [34].

Conflict of interest

The authors declare that they have no conflict of interest.

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