AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 13845–13855.
DOI: 10.3934/math.2022764
Received: 18 December 2021
Revised: 19 April 2022
Accepted: 10 May 2022
Published: 24 May 2022

Research article

A new family of differential and integral equations of hybrid polynomials via factorization method

Raziya Sabri[1], Mohammad Shadab[2], Kotakkaran Sooppy Nisar[3,*] and Shahadat Ali[1]

[1] Department of Natural and Applied Sciences, School of Science and Technology, Glocal University, Saharanpur 247121, India

[2] Department of Mathematics, School of Basic and Applied Sciences, Lingaya's Vidyapeeth (Deemed to be University), Faridabad-121002, Haryana, India

[3] Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia

* **Correspondence:** Email: n.sooppy@psau.edu.sa

Abstract: In this study, we investigate differential and integral equations of some hybrid families of truncated exponential-based Sheffer polynomials. We also derive some integro-differential equation and new recurrence relations for the truncated exponential based Sheffer polynomials by using the factorization method. We also discuss some special cases as illustrative examples.

Keywords: truncated exponential based Sheffer polynomials; factorization methods; differential equations; integral equations; integro-differential equation
Mathematics Subject Classification: 05A15, 33C45, 33E20, 44A45

1. Introduction and preliminaries

The family of Sheffer polynomial sequences was introduced by I. M. Sheffer. The Sheffer sequence is the one of the main classes of the polynomial sequence [20]. The Sheffer sequence arise in many fields of applied mathematics and other branches of mathematics. With reference to application of umbral composition of polynomial sequence, the set of Sheffer sequences is closed.

In physical problems, the class of 2-variable hybrid polynomials gives importance of its formation as solution of different important differential equations again and again encountered in [14, 15].

The pervasive investigation of Appell and Sheffer sequences along with the structure of modern umbral calculus collected by the Roman [19]. Roman [19] elaborated the following generating function of Sheffer sequence.

$$\frac{e^{xf^{-1}(t)}}{g(f^{-1}(t))} = \sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}, \tag{1.1}$$

for all x in $\mathbb{C}$, wherever $f^{-1}(t)$ is the inverse function of $f(t)$ and $f(t)$ is differently fixed by 2-power series.

Let $f(t)$ and $g(t)$ be a delta and an invertible series respectively, are given by

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, f_1 \neq 0, f_0 = 0, \tag{1.2}$$

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, g_0 \neq 0. \tag{1.3}$$

If $g(t) = 1$, then the Sheffer sequence is the associated Sheffer sequence and for the Appell sequence $g(t) = t$ [2]. These polynomials are appeared in number theory, physics, and various other branches of mathematics. The class of the Sheffer sequence has some valuable sequences as Hermite, Bessel, Laguerre polynomials etc. According to the physical applications, the Sheffer sequences are very useful.

By the series [6, p.597 (14)], a noteworthy interest of a group of truncated polynomials is presented. These polynomials [3, 18] come out in a vast variety of physical problems as optics, quantum mechanics problems and furthermore play a job in the assessment of the integral including result of special functions. 2-variables special polynomials are significant according to the applications.

This is the series of truncated exponential polynomials $e_n(x)$. [1]

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{1.4}$$

This is the first $(n + 1)$ terms of Maclaurin series for e^x. In many problems of quantum mechanics and optics, these polynomials appears. In the beginning Dattoli [6] gave the systematic research of these polynomials.

Generally, these polynomials have the properties which may be acquired by definition (1.4). Truncated exponential polynomials can be addressed by the generating function [6, p. 596 (4)],

$$\frac{e^{xt}}{(1-t)} = \sum_{n=0}^{\infty} e_n(x)t^n. \tag{1.5}$$

Now, we see the 2-variable higher order truncated polynomials are specified by the series [7, p. 174 (29)]

$$e_n^{(r)}(x, y) = \sum_{k=0}^{[\frac{n}{r}]} \frac{y^k x^{(n-rk)}}{(n-rk)!}, \tag{1.6}$$

the generating function of 2-variable truncated exponential polynomial (2VTP) [7, p.174 (30)]

$$\frac{e^{xt}}{(1-yt^r)} = \sum_{n=0}^{\infty} e_n^{(r)}(x,y)\frac{t^n}{n!}. \tag{1.7}$$

There are the integral representation of truncated exponential

$$e_n^{(r)}(x) = \frac{1}{n!}\int_0^{+\infty} e^{-\xi}(x+\xi)^n d\xi,$$

where

$$n! = \int_0^{\infty} e^{-\xi}\xi^n d\xi.$$

2. Truncated exponential based Sheffer polynomials

In 2014, Khan et al. introduced the two variables truncated-exponential-Sheffer polynomials (2VTESP) and derived with the frame of reference of the operational method

$$\frac{exp(xf^{-1}(t))}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)} = \sum_{n=0}^{\infty} {}_{e^{(r)}}s_n(x,y)\frac{t^n}{n!}.$$

The generating function of Truncated exponential-Sheffer polynomial (2VTESP) ${}_{e^{(r)}}s_n(x,y)$

$$\frac{1}{g(f^{-1}(t))}e^{(xf^{-1}(t))}C_0(-y(f^{-1}(t))^r) = \sum_{n=0}^{\infty} {}_{e^{(r)}}s_n(x,y)\frac{t^n}{n!}, \tag{2.1}$$

$$\frac{1}{g(f^{-1}(t))}e^{(xf^{-1}(t))}e^{(D_y^{-1}(f^{-1}(t))^r)} = \sum_{n=0}^{\infty} {}_{e^{(r)}}s_n(x,y)\frac{t^n}{n!}, \tag{2.2}$$

where $C_0(ax)$ is Tricomi function of order zero [1].

Since $C_n(x) = \sum_{n=0}^{\infty}\frac{(-1^r)(x)^r}{r!(n+r)!}$, and where D_y^{-1} is defined as the inverse derivative operator $D_y = \frac{\partial}{\partial y}$, and is specified by $D_y^{-1}\{f(y)\} = \int_0^y f(\theta)d\theta$.

The two variables truncated exponential-Sheffer polynomials (2VTESP) ${}_{e^r}s_n(x,y)$ are as stipulated by series

$${}_{e^{(r)}}S_n(x,y) = n!\sum_{k=0}^{[\frac{n}{r}]}\frac{S_{n-rk}y^k}{(n-rk)!}. \tag{2.3}$$

In 2002, He and Ricci exploited the idea to derive the differential equations by the factorization method for the Appell polynomials in [16] (see also, [17]). Recently, differential equations for confluent type Sheffer polynomials are investigated by many authors [9, 10, 14, 15, 21, 23]. This motivates to investigate the differential equations for the truncated exponential-Sheffer polynomials by apply the factorization technique. We retrace a few initially allied to the factorization technique.

The technique of factorization method is very useful to derive the differential and integro-differential equations of hybrid class of polynomials.

Now, let us suppose that $\{P_n(x)\}_{k=0}^{\infty}$ be a sequence of polynomials like as the $\deg(P_n(x)) = n, (n \in \mathbb{N}_0 := 0, 1, 2, 3, ...)$. The differential operators $\mathbf{Q}_n^-$ and $\mathbf{Q}_n^+$ holding the properties

$$\mathbf{Q}_n^- P_n(x) = P_{n-1}(x), \tag{2.4}$$

$$\mathbf{Q}_n^+ P_n(x) = P_{n+1}(x). \tag{2.5}$$

$\mathbf{Q}_n^-$ known as the derivative operator and $\mathbf{Q}_n^+$ the multiplicative operator. The principle of monomiality [4, 22] and also related operational rules are applied in [5] to expand new families of cospectral which leads to non-trivial generalized polynomials.

The differential equation for the class of hybrid polynomials

$$(\mathbf{Q}_{n+1}^- \mathbf{Q}_n^+) P_n(x) = P_n(x), \tag{2.6}$$

will be derived by the application of the operators $\mathbf{Q}_n^-$ and $\mathbf{Q}_\mathbf{n}^+$. By applying this method for obtaining the differential equation through the Eq (2.6) is called factorization method [8]. The factorization method has the main purpose to present derivative operator $\mathbf{Q}_n^-$ and multiplicative operator $\mathbf{Q}_n^+$ as the Eq (2.6) holds.

3. Recurrence relation and differential equation

Now, we obtain some recurrence relations for the truncated exponential based Sheffer polynomials (2VTESP) ${}_{e^r}s_n(x, y)$.

Theorem 3.1. *The following recurrence relation for the truncated exponential-Sheffer polynomials (2VTESP)* ${}_{e^{(r)}}s_n(x, y)$ *holds true*

$$\begin{aligned} {}_{e^{(r)}}s_{n+1}(x, y) = &\Bigg((x + \alpha_0)\, {}_{e^r}s_n(x, y) + \sum_{k=1}^{n} \binom{n}{k} \alpha_k\, {}_{e^r}S_{n-k}(x, y) \\ &+ \frac{n!}{(n-r+1)!}\, r D_y^{-1}\, {}_{e^{(r)}}S_{n-r+1}(x, y) \Bigg) \frac{1}{f'(t)}, \end{aligned} \tag{3.1}$$

coefficients $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *can be express by*

$$\frac{g'(t)}{g(t)} = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k. \tag{3.2}$$

Proof. Differentiate the Eq (2.2) with respect to t, we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{e^{(r)}}s_{n+1}(x, y) \frac{t^n}{n!} = &\frac{1}{g(f^{-1}(t))} exp(D_y^{-1}(f^{-1}(t))^r + x f^{-1}(t)) \\ &\times \left(r D_y^{-1} (f^{-1}(t))^{r-1} + x + \frac{g'(f^{-1}(t))}{g(f^{-1}(t))}) \right) \frac{1}{f'(f^{-1}(t))}. \end{aligned} \tag{3.3}$$

Using the Eqs (3.2) and (2.2) in the Eq (3.3), and then in the left hand side we use the Cauchy product rulers follows

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}s_{n+1}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\alpha_{k\,e^r}s_{n-k}(x,y)\right.$$
$$\left. + x_{e^{(r)}}s_n(x,y) + \frac{n!}{(n-r+1)!}rD_y^{-1}{}_{e^{(r)}}S_{n-r+1}(x,y)\right)\frac{t^n}{n!}\frac{1}{f'(f^{-1}(t))}.$$

On comparing the coefficients of equal power of t, on the both sides of the above equation and after that solving the equation with $t = f^{-1}(t)$, then we get Eq (3.1). □

Now, we derive shift operators for truncated exponential-Sheffer polynomial (2VTESP).

Theorem 3.2. *The shift operators for the truncated exponential-Sheffer polynomials (2VTESP) ${}_{e^{(r)}}s_n(x,y)$ are as follows*

$$ {}_x£_n^- = \frac{1}{n}D_x, \tag{3.4}$$

$$ {}_y£_n^- = \frac{1}{n}D_yD_x^{-(r-1)}, \tag{3.5}$$

$$ {}_x£_n^+ = \left((x+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k + rD_y^{-1}D_x^{r-1}\right)\frac{1}{f'(t)}, \tag{3.6}$$

$$ {}_y£_n^+ = \left((x+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^kD_x^{-k(r-1)} + rD_y^{r-2}D_x^{-(r-1)^2}\right)\frac{1}{f'(t)}, \tag{3.7}$$

$$\textit{where} \qquad D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y} \textit{ and } D_x^{-1} = \int_0^x f(\xi)d\xi.$$

Proof. First, the generating function of truncated exponential-Sheffer polynomials (2VTESP) will be differentiated and then comparing the coefficients of equal powers of t, we get

$$\frac{\partial}{\partial x}\left({}_{e_n^{(r)}}S_n(x,y)\right) = n_{e^{(r)}}S_{n-1}(x,y). \tag{3.8}$$

So that

$$\frac{1}{n}\frac{\partial}{\partial x}\left({}_{e^{(r)}}s_n(x,y)\right) = {}_{e^{(r)}}s_{n-1}(x,y).$$

Consequently,

$$ {}_x£_n^-[{}_{e^{(r)}}s_n(x,y)] = \frac{1}{n}D_x\{{}_{e^{(r)}}s_n(x,y)\} = {}_{e^{(r)}}s_{n-1}(x,y), \tag{3.9}$$

which proves (3.4).

Next, differentiate the generating function (2.2) and then comparing the coefficients of equal powers of t, we get

$$\frac{\partial}{\partial y}\{_{e^{(r)}}s_n(x,y)\} = \frac{n!}{(n-r)!}\{_{e^{(r)}}s_{n-r}(x,y)\}. \tag{3.10}$$

From the Eq (3.8), it can be observed that

$$\frac{\partial}{\partial y}\{_{e^{(r)}}s_n(x,y)\} = n\frac{\partial^{r-1}}{\partial_x^{r-1}}\,_{e^{(r)}}s_{n-1}(x,y).$$

Thus

$$_y£_n^-[_{e^{(r)}}s_n(x,y)] = \frac{1}{n!}D_yD_x^{-(r-1)}\{_{e^{(r)}}s_n(x,y)\} = {}_{e^{(r)}}s_{n-1}(x,y), \tag{3.11}$$

which proves (3.5).

Next, to obtain the raising operator

$$_{e^{(r)}}s_{n-k}(x,y) = \left(_x£_{n-k+1}^-\,_x£_{n-k+2}^-........_x£_n^-\right)\,_{e^{(r)}}s_n(x,y),$$

which on using Eq (3.9), gives

$$_{e^{(r)}}s_{n-k}(x,y) = \frac{(n-k)!}{n!}D_x^k\,_{e^{(r)}}s_n(x,y). \tag{3.12}$$

Now, by using the Eq (3.12) in Eq (3.1)

$$_x£_n^+\{_{e^{(r)}}s_n(x,y)\} = {}_{e^{(r)}}s_{n+1}(x,y),$$

can be given by

$$_x£_n^+ = \left((x+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k + rD_y^{-1}D_x^{(r-1)}\right)\frac{1}{f'(t)},$$

which proves (3.6).

Now, for $_y£^+$ obtain the raising operator, considered the relation

$$_{e^{(r)}}s_{n-k}(x,y) = \left(_y£_{n-k+1}^-\,_y£_{n-k+2}^-........_y£_{n-k+1}^-\right)\,_{e^{(r)}}s_{n-k}(x,y),$$

which on using Eq (3.11), gives

$$_{e^{(r)}}s_{n-k}(x,y) = \frac{(n-k)!}{n!}D_y^kD_x^{k(1-r)}\{_{e^{(r)}}s_n(x,y)\}. \tag{3.13}$$

Using above equation in (3.1), we obtain

$$_y£_n^+\{_{e_n^y}S_n(x,y)\} = {}_{e^{(r)}}s_{n+1}(x,y),$$

$$_y£_n^+ = \left((x+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^kD_x^{-k(r-2)} + rD_y^{(r-2)}D^{-(r-1)^2}\right)\frac{1}{f'(t)},$$

which proves Eq (3.7). □

Theorem 3.3. *The differential equation for truncated exponential-Sheffer polynomials (2VTESP)* ${}_{e^{(r)}}s_n(x,y)$ *satisfy the following equation*

$$\left(\{(x+\alpha_0)D_x+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{k+1}+ryD_x^r\}\frac{1}{f'(t)}-n\right){}_{e^{(r)}}s_n(x,y)=0. \tag{3.14}$$

Proof. Considering the factorization method for the derivation of differential equation

$${}_x\pounds_{n+1}^{-}\,{}_x\pounds_n^{+}\{{}_{e^{(r)}}s_n(x,y)\}={}_{e^{(r)}}s_n(x,y),$$

Putting the values of the shift operators from Eqs (3.5) and (3.7), we get

$$\left(\{(x+\alpha_0)D_x+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{k+1}+rD_y^{-1}D_x^{(r-1)}D_x\}\frac{1}{f'(t)}-n\right){}_{e^{(r)}}s_n(x,y)=0,$$

or

$$\left(\{(x+\alpha_0)D_x+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{k+1}+rD_y^{-1}D_x^{r}\}\frac{1}{f'(t)}-n\right){}_{e^{(r)}}s_n(x,y)=0,$$

which proves (3.14). □

4. Integro-differential equation for truncated exponential-Sheffer polynomials

Theorem 4.1. *The following integro-differential equation of truncated exponential-Sheffer polynomials holds true*

$$\left(\{(x+\alpha_0)D_x+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^{k}D_xD_x^{-k(r-1)}+rD_y^{(r-2)}D_x^{-(r-1)^2+1}\}\frac{1}{f'(t)}-nD_x\right){}_{e^{(r)}}s_n(x,y)=0. \tag{4.1}$$

Proof. By using the factorization method, and the pair of shift operators ${}_x\pounds_{n+1}^{-}$ and ${}_y\pounds_n^{+}$, we have assertion

$${}_x\pounds_{n+1}^{-}\,{}_y\pounds_n^{+}\{{}_{e^{(r)}}s_n(x,y)\}={}_{e^{(r)}}s_n(x,y).$$

Putting the value of shift operator in above equation, we get (4.1). □

Corollary 4.1. *By differentiating the equation of integro-differential equation n -times with respect to y, then we find a partial differential equation of truncated exponential-Sheffer polynomials*

$$\left(\{(x+\alpha_0)D_xD_y^n+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^{n+k}D_x^{-k(r-1)+1}+rD_y^{(r-2)+n}D_x^{(r-1)^2+1}\}\frac{1}{f'(t)}-nD_xD_y^n\right){}_{e^{(r)}}s_n(x,y)=0. \tag{4.2}$$

5. Integral equation for truncated exponential based Sheffer polynomials

Integral equations appear in engineering and logical applications. The applications of integral equations can be observed in different directions of scientific fields like quantum physics, water waves diffraction problems etc. [11, 13].

Theorem 5.1. *The following homogeneous Volterra integral equation of the truncated exponential-Sheffer polynomials holds true*

$$\begin{aligned}\phi(x) = \quad & -\frac{\alpha_1}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2(x) + \mathbb{P}_1\right) \\ & -\frac{\alpha_0}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_2\frac{x^2}{2!} + \mathbb{P}x + n\mathbb{R}_{n-1}\right) \\ & -\frac{1}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_2\frac{x^3}{2!} + \mathbb{P}x^2 + n\mathbb{R}n-1x\right) \\ & +\frac{nf'(t)}{ry}\left(\mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n\right) \\ & -\frac{1}{ry}\int_0^x\left(\alpha_1\frac{(x-\xi)^{r-3}}{(r-3)!} + (x+\alpha_0)\frac{(x-\xi)^{r-2}}{(r-2)!} - n\frac{(x-\xi)^{r-1}}{(r-1)!}\right)\phi(\xi)d\xi. \end{aligned} \tag{5.1}$$

Proof. Consider the differential equation of truncated exponential-Sheffer polynomial for $k = 1$ in the following form

$$\left([\{ryD_x^r + (x+\alpha_0)D_x + \alpha_1 D_x^2\}\frac{1}{f'(t)} - n]\frac{1}{ry}\right)_{e^{(r)}}s_n(x,y) = 0. \tag{5.2}$$

The generating function (2.2) with $y = 0$ can be presented

$$\frac{1}{g(f^{-1}(t))}e^{(xf^{-1}(t))} = \sum_{n=0}^{\infty} {}_{e^{(r)}}s_n(x,0)\frac{t^n}{n!} = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}.$$

By using $A(t) = \frac{1}{g(f^{-1}(t))}$ [12, p.923] with Substituting the series expression of exponential in left hand side and then apply Cauchy product rule, we have

$$s_n(x) = \sum_{k=0}^{n}\binom{n}{k}s_{n-k}x^k.$$

The initial condition can be obtained as

$${}_{e^{(r)}}s_n(x,0) = s_n(x) = \sum_{k=0}^{n}\binom{n}{k}s_{n-k}x^k.$$

Letting $s_n(x) = \mathbb{R}_n$. Then the derivative of $s_n(x)$ with respect to x

$$\frac{d}{dx}{}_{e^{(r)}}s_n(x,0) = ns_{n-1}(x) = n\sum_{k=0}^{n}\binom{n-1}{k}s_{n-1-k}x^k = n\mathbb{R}_{n-1},$$

$$\frac{d^2}{dx^2}{}_{e^{(r)}}s_n(x,0) = n(n-1)s_{n-2}(x) = \prod_{k=0}^{1}(n-k)\mathbb{R}_{n-2} = \mathbb{P}_1.$$

Since $\mathbb{R}_n = {}_{e^{(r)}}A_n(x) = s_n(x)$, we have

$$D_x^{r-2} = \{{}_{e^{(r)}}s_n(x,0)\} = n(n-1)(n-2)......(n-r+3)\mathbb{R}_{n-r+2} = \mathbb{P}_{r-3} = \prod_{k=0}^{r-3}(n-k)\mathbb{R}_{n-r+2},$$
$$D_x^{r-1}\{{}_{e^{(r)}}s_n(x,0)\} = n(n-1)(n-2)........(n-r+1)\mathbb{R}_{n-r+1} = \mathbb{P}_{r-2} = \prod_{k=0}^{r-2}(n-k)\mathbb{R}_{n-r+1}.$$

Consider the equation

$$D_x^r\{{}_{e^{(r)}}s_n(x,y)\} = \phi(x). \tag{5.3}$$

On integrating the Eq (5.3) with initial conditions, we get

$$D_x^{r-1}\{{}_{e^{(r)}}s_n(x,y)\} = \int_0^x \phi(\xi)d\xi + \mathbb{P}_{r-2},$$
$$D_x^{r-2}\{{}_{e^{(r)}}s_n(x,y)\} = \int_0^x \phi(\xi)d^2\xi + \mathbb{P}_{r-3}.$$

$$D_x^2\{{}_{e^{(r)}}s_n(x,y)\} = \int_0^x \phi(\xi)d\xi^{r-2} + \mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2 x + \mathbb{P}_1,$$
$$D_x\{{}_{e^{(r)}}s_n(x,y)\} = \int_0^x \phi(\xi)d\xi^{r-1} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + ... + \mathbb{P}_1 x + ... + n\mathbb{R}n-1,$$
$${}_{e^{(r)}}s_n(x,y) = \int_0^x \phi(\xi)d\xi^r + \mathbb{P}r-2\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n, \tag{5.4}$$

where

$$\mathbb{P}_{r-s} = \prod_{k=0}^{r-s}(n-k)\mathbb{R}_{n-r+(s-1)},\ s = r-1, r-2, ...3, 2.$$

Using the expression (5.4) in Eq (5.2), we obtain

$$\phi(x) = -\frac{(x+\alpha_0)}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-1} + \mathbb{P}_{r-2}\frac{x^{r-2}}{(r-2)!} + \mathbb{P}_{r-3}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_2\frac{x^2}{2!} + \mathbb{P}_1\frac{x}{1!} + n\mathbb{R}_{n-1}\right)$$
$$-\frac{\alpha_1}{ry}\left(\int_0^x \phi(\xi)d\xi^{r-2} + \mathbb{P}_{r-2}\frac{x^{r-3}}{(r-3)!} + \mathbb{P}_{r-3}\frac{x^{r-4}}{(r-4)!} + ... + \mathbb{P}_2 x + \mathbb{P}_1\right)$$
$$+\frac{nf'(t)}{ry}\left(\int_0^x \phi(\xi)d\xi + \mathbb{P}_{r-2}\frac{x^{r-1}}{(r-1)!} + \mathbb{P}_{r-3}\frac{x^{r-2}}{(r-2)!} + ... + \mathbb{P}_1\frac{x^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n\right),$$

which proves assertion (5.1). □

6. Conclusions

We derive differential equation, shift operators, and integral equation for the truncated-exponential based Sheffer polynomials by the factorization method in the present investigation. The development of these type techniques may be useful in different scientific areas. The truncated exponential-Sheffer polynomials (2VTESP) and for their relatives can be taken in further investigations of mathematical and engineering sciences.

Conflict of interest

The authors declare that there are no conflicts of interest.

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