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Research article

Some fractional integral inequalities via h-Godunova-Levin preinvex function

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Abstract: In recent years, integral inequalities are investigated due to their extensive applications in several domains. The aim of the paper is to investigate certain new fractional integral inequalities which include Hermite-Hadamard inequality and different forms of trapezoid type inequalities related to Hermite-Hadamard inequality for h-Godunova-Levin preinvex function. Moreover, we compare our obtained results with the existing work in the literature and are represented by corollaries.

Keywords: fractional inequalities; h-Godunova-Levin convex and preinvex function; Hadamard inequality
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1. Introduction

Fractional integrals inequalities have immense applications in the fields of mathematical analysis and approximation theory [1–11]. It is generalization and extension of classical integral inequalities, which have great contribution in fractional theory and the field of analysis. The demand of fractional operators has increased because of its applications in various subfield of analysis, specially in modification of fractional inequalities and advance fractional calculus. There are many integral inequalities which have been generalized in the form of fractional version of these inequalities by applying Riemann-Lioville fractional operator. The theory of convexity has also been widely discussed in [12–23] due to its many applications in several fields. The significance of convexity has obtained remarkable recognition and researchers are continuously working to enlarge its frame. The numerous applications of convex function have great achievement in the field of fractional analysis due to the modifications of well known fractional inequalities [9, 10]. Convexity relates to the theory of inequalities and control optimization. Many well known inequalities are analyzed and reported in the field of mathematics by using convexity theory [24–35]. One of the inequalities is Hermite-Hadamard inequality, which is defined as

$$\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\aleph(x)dx \leq \frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2},$$

for convex function [36–39], $\aleph : J \to \mathbb{R}, \kappa_1, \kappa_2 \in J, \kappa_1 < \kappa_2, \kappa_1, \kappa_2 \in \mathbb{R}, J \subseteq \mathbb{R}$, which is playing remarkable role in the field of inequalities [40, 41].

Recently, the concept of convexity has been modified to s-Godunova-Levin type of convexity by Dragomir [42]. Moreover, s-Godunova-Levin type convexity has been studied in the literature [43]. The h-convexity was introduced by Varošanec in [44]. Ohud Almutari, introduced h-Godunova-Levin convexity and preinvexity [45] by combining the concepts of Dragomir and Varošanec. In the present work, we consider h-Godunova-Levin convex and preinvex functions to obtain new fractional version of Hermite-Hadamard and Trapezoid type inequalities.

Definition 1.1. *[37, 46] For $t \in [0, 1]$, and $\forall \kappa_1, \kappa_2 \in J$, then the convex function $\aleph : J \to \mathbb{R}$ is defined as follows*

$$\aleph\Big(t\kappa_1 + (1-t)\kappa_2\Big) \leq t\aleph(\kappa_1) + (1-t)\aleph(\kappa_2). \tag{1.1}$$

Definition 1.2. *[47] Let a real bifunction $\aleph : J \times J \to \mathbb{R}$, then the invex set $J \subseteq \mathbb{R}$ with respect to bifunction is defined as follows*

$$\kappa_2 + \lambda\aleph(\kappa_1, \kappa_2) \in J,$$

where $\kappa_1, \kappa_2 \in J, \lambda \in [0, 1]$.

Definition 1.3. *[47] Let $\kappa_1, \kappa_2 \in J$ and $\lambda \in [0, 1]$, then the preinvex function $\aleph : J \to \mathbb{R}$ is defined as*

$$\aleph(\kappa_2 + \lambda\zeta(\kappa_1, \kappa_2)) \leq \lambda\aleph(\kappa_1) + (1-\lambda)\aleph(\kappa_2),$$

where J is an invex set with respect to ζ.

Definition 1.4. *[48] Let the function $\aleph : J \subseteq \mathbb{R} \to \mathbb{R}$ be positive, then it is called the Godunova-Levin if*

$$\aleph(t\kappa_1 + (1-t)\kappa_2) \leq \frac{\aleph(\kappa_1)}{t} + \frac{\aleph(\kappa_2)}{1-t},$$

for all $\kappa_1, \kappa_2 \in J, t \in (0, 1)$.

Definition 1.5. *[45] Let $h : (0, 1) \to \mathbb{R}$ be a function and let the function $\aleph : J \to \mathbb{R}$ be non-negative, then it is said to be h-Godunova-Levin, for all $\kappa_1, \kappa_2 \in J$ and $t \in (0, 1)$ if*

$$\aleph(t\kappa_1 + (1-t)\kappa_2) \leq \frac{\aleph(\kappa_1)}{h(t)} + \frac{\aleph(\kappa_2)}{h(1-t)}.$$

Definition 1.6. *[45] Let $\aleph : J \to \mathbb{R}$ be a function, then it is said to be h-Godunova-Levin preinvex with respect to ζ if,for all $\kappa_1, \kappa_2 \in J, t \in (0,1)$,*

$$\aleph(\kappa_1 + t\zeta(\kappa_2, \kappa_1)) \leq \frac{\aleph(\kappa_1)}{h(1-t)} + \frac{\aleph(\kappa_2)}{h(t)},$$

holds.

Definition 1.7. *[49] The integral form of gamma function is defined for $\Re(t) > 0$, as follows*

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

Definition 1.8. *[50] Consider $\aleph \in L^1[u, v]$. The Riemann-Liouville integrals $J^\eta_{u^+}\aleph$ and $J^\eta_{v^-}\aleph$ of order $\eta > 0$ with $u \geq 0$ are defined as:*

$$J^\eta_{u^+}\aleph(x) = \frac{1}{\Gamma(\eta)} \int_u^x (x-t)^{\eta-1} \aleph(t) dt, \qquad x > u,$$

and

$$J^\eta_{v^-}\aleph(x) = \frac{1}{\Gamma(\eta)} \int_x^v (t-x)^{\eta-1} \aleph(t) dt, \qquad x < v.$$

The goals of this paper is to investigate Hermite-Hadamard and Trapezoid type inequalities by employing fractional integrals.

The paper consists of two sections. In section 2, we modify the Hermite-Hadamard Inequalities by utilizing h-Godunova-Levin convex function. In Section 3, we establish Trapezoid type Inequalities related to Hermite-Hadamard Inequality for h-Godunova-Levin preinvex function.

2. Hermite Hadamard type inequalities associated with h-Godunova-Levin convex function

In this section, we discuss the Hermite-Hadamard inequalities for h-Godunova-Levin convex function by using Riemann-Liouville fractional operator.

Theorem 2.1. *Let $\aleph : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a h-Godunova-Levin convex function where $0 < \kappa_1 < \kappa_2$ and $\aleph \in L_1[\kappa_1, \kappa_2]$ such that the function $h : (0,1) \to \mathbb{R}$ is positive and $h(\vartheta) \neq 0$, then for fractional integral, we have*

$$\frac{h(\frac{1}{2})}{2}\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \leq \frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2) + J^\eta_{\kappa_2^-}\aleph(\kappa_1)\Big]$$
$$\leq \frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}\eta \int_0^1 [\frac{1}{h(\vartheta)} + \frac{1}{h(1-\vartheta)}]\vartheta^{\eta-1} d\vartheta.$$

Proof. By the definition of h-Godunova-Levin convex function on the interval $[\kappa_1, \kappa_2]$, let $x, y \in [\kappa_1, \kappa_2]$, we have

$$\aleph(\delta x + (1-\delta)y) \leq \frac{\aleph(x)}{h(\delta)} + \frac{\aleph(y)}{h(1-\delta)},$$

where by taking

$$x = \vartheta\kappa_1 + (1-\vartheta)\kappa_2, y = (1-\vartheta)\kappa_1 + \vartheta\kappa_2$$
$$\delta = \frac{1}{2},$$

leads to

$$\aleph(\frac{\kappa_1+\kappa_2}{2}) \le \frac{1}{h(\frac{1}{2})}[\aleph(\vartheta\kappa_1 + (1-\vartheta)\kappa_2) + \aleph((1-\vartheta)\kappa_1 + \vartheta\kappa_2)].$$

Multiplying both sides by $\vartheta^{\eta-1}$, and integrating with respect ϑ on the interval $[0,1]$, we obtain

$$h(\frac{1}{2})\aleph(\frac{\kappa_1+\kappa_2}{2})\int_0^1 \vartheta^{\eta-1}d\vartheta \le \int_0^1 \vartheta^{\eta-1}\aleph(\vartheta\kappa_1 + (1-\vartheta)\kappa_2)d\vartheta + \int_0^1 \vartheta^{\eta-1}\aleph((1-\vartheta)\kappa_1 + \vartheta\kappa_2)d\vartheta.$$

By substituting $w = \vartheta\kappa_1 + (1-\vartheta)\kappa_2$ and $t = \vartheta\kappa_1 + (1-\vartheta)\kappa_2$,

$$h(\frac{1}{2})\aleph(\frac{\kappa_1+\kappa_2}{2})\frac{1}{\eta} \le \int_{\kappa_1}^{\kappa_2} (\frac{\kappa_2 - w}{\kappa_2-\kappa_1})^{\eta-1}\aleph(w)\frac{dw}{\kappa_2-\kappa_1} + \int_{\kappa_1}^{\kappa_2} (\frac{t-\kappa_1}{\kappa_2-\kappa_1})^{\eta-1}\aleph(w)\frac{dt}{\kappa_2-\kappa_1}.$$

$$\frac{h(\frac{1}{2})}{2}\aleph(\frac{\kappa_1+\kappa_2}{2}) \le \frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2) + J^\eta_{\kappa_2^-}\aleph(\kappa_1)\Big]. \tag{2.1}$$

Now, for second part of inequality, we consider h-Godunova-Levin convexity of $\aleph$

$$\aleph(\vartheta\kappa_1 + (1-\vartheta)\kappa_2) \le \frac{\aleph(\kappa_1)}{h(\vartheta)} + \frac{\aleph(\kappa_2)}{h(1-\vartheta)},$$

and

$$\aleph((1-\vartheta)\kappa_1 + \vartheta\kappa_2) \le \frac{\aleph(\kappa_1)}{h(1-\vartheta)} + \frac{\aleph(\kappa_2)}{h(\vartheta)}.$$

Adding these two inequalities gives

$$\aleph(\vartheta\kappa_1 + (1-\vartheta)\kappa_2) + \aleph((1-\vartheta)\kappa_1 + \vartheta\kappa_2) \le (\aleph(\kappa_1) + \aleph(\kappa_2))[\frac{1}{h(\vartheta)} + \frac{1}{h(1-\vartheta)}].$$

Multiplying both sides by $\vartheta^{\eta-1}$ and integrating the resulting inequality on $[0,1]$ with respect to ϑ, we get

$$\int_0^1 \vartheta^{\eta-1}\aleph(\vartheta\kappa_1 + (1-\vartheta)\kappa_2)d\vartheta + \int_0^1 \vartheta^{\eta-1}\aleph((1-\vartheta)\kappa_1 + \vartheta\kappa_2)d\vartheta$$
$$\le (\aleph(\kappa_1) + \aleph(\kappa_2))\int_0^1 [\frac{1}{h(\vartheta)} + \frac{1}{h(1-\vartheta)}]\vartheta^{\eta-1}d\vartheta.$$

On solving, we obtain

$$\frac{\Gamma(\eta+1)}{(\kappa_2-\kappa_1)^\eta}[J^\eta_{\kappa_1^+}\aleph(\kappa_2) + J^\eta_{\kappa_2^-}\aleph(\kappa_1)] \le \eta[\aleph(\kappa_1) + \aleph(\kappa_2)]\int_0^1 \Big[\frac{1}{h(\vartheta)} + \frac{1}{h(1-\vartheta)}\Big]\vartheta^{\eta-1}d\vartheta. \tag{2.2}$$

Combining (2.1) and (2.2), we reach to the required inequality. □

Corollary 2.1. *Choosing $h(\vartheta) = \vartheta^s$ in Theorem 2.1, we obtain Hermite-Hadamard type inequality for s-Godunova Levin function.*

$$\frac{(\frac{1}{2})^s}{2}\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \le \frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{\kappa_2^-}\aleph(\kappa_1)\Big]$$
$$\le \frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}\eta\int_0^1\Big[\frac{1}{(\vartheta)^s}+\frac{1}{(1-\vartheta)^s}\Big]\vartheta^{\eta-1}d\vartheta.$$

Corollary 2.2. *Applying Theorem 2.1 for $h(\vartheta) = 1$, we obtain the following Hermite-Hadamard type inequality for P function given by*

$$\frac{1}{2}\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \le \frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{\kappa_2^-}\aleph(\kappa_1)\Big]$$
$$\le [\aleph(\kappa_1)+\aleph(\kappa_2)].$$

Corollary 2.3. *Applying Theorem 2.1 for $h(\vartheta) = \frac{1}{\vartheta}$, we get Theorem 2 given in [23] by:*

$$\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \le \frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{\kappa_2^-}\aleph(\kappa_1)\Big]$$
$$\le \frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}.$$

Corollary 2.4. *Applying Theorem 2.1 for $h(\vartheta) = \vartheta$, we obtain the following inequality by*

$$\frac{1}{4}\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \le \frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{\kappa_2^-}\aleph(\kappa_1)\Big]$$
$$\le \frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}\eta\int_0^1\frac{\vartheta^{\eta-2}}{1-\vartheta}d\vartheta.$$

Corollary 2.5. *(1) Applying Theorem 2.1 for $h(\vartheta) = \frac{1}{\vartheta^s}$, we obtain the following inequality for S-convex function by*

$$2^{s-1}\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \le \frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{\kappa_2^-}\aleph(\kappa_1)\Big]$$
$$\le \frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}\eta\int_0^1[\vartheta^s+(1-\vartheta)^s]\vartheta^{\eta-1}d\vartheta.$$

(2) Here choosing $\eta = 1$, we obtain Theorem 2.1 proved by Dragomir [51].

$$2^{s-1}\aleph\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\aleph(\vartheta)d\vartheta \le \frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{s+1}.$$

3. Trapezoid type inequalities related to Hermite-Hadamard Inequality for h-Godunova-Levin preinvex function

In this section, Trapezoid type Inequalities related to Hermite-Hadamard inequality by utilizing fractional integral can be obtained with the help of following Lemma which has been proved by [52].

Lemma 3.1. *Consider a function* $\aleph : J = [\kappa_1, \kappa_1 + \zeta(\kappa_2, \kappa_1)] \to \mathbb{R}$ *with* $\kappa_1, \kappa_2 \in \mathbb{R}$, $\aleph \in L_1[\kappa_1, \kappa_1 + \zeta(\kappa_2, \kappa_1)]$ *be a differentiable function where* $J = [\kappa_1, \kappa_1 + \zeta(\kappa_2, \kappa_1)]$ *is taken to be an open invex set with respect to* $\zeta : J \times J \to \mathbb{R}$ *with* $\zeta(\kappa_2, \kappa_1) > 0$ *for* $\kappa_1, \kappa_2 \in J$*. Then for fractional integrals , the following inequality holds*

$$\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{\Gamma(\eta + 1)}{2\zeta^{\eta}(\kappa_2, \kappa_1)}\Big[J^{\eta}_{\kappa_1^+}\aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1)) + J^{\eta}_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-}\aleph(\kappa_1)\Big]$$
$$= \frac{\zeta(\kappa_2, \kappa_1)}{2}\int_0^1 [\vartheta^{\eta} - (1-\vartheta)^{\eta}]\aleph'(\kappa_1 + \vartheta\zeta(\kappa_2, \kappa_1))d\vartheta.$$

By using the Lemma 3.1, we have the following theorem.

Theorem 3.1. *Suppose a mapping* $\aleph : J = [\kappa_1, \kappa_1 + \zeta(\kappa_2, \kappa_1)] \to (0, \infty)$ *with* $J \in \mathbb{R}$*, be a differentiable function on* J*, and let* $|\aleph'|$ *be a h-Godunova-Levin preinvex function on* J*, then the following inequality holds for fractional integrals*

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{\Gamma(\eta + 1)}{2\zeta^{\eta}(\kappa_2, \kappa_1)}\Big[J^{\eta}_{\kappa_1^+}\aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1)) + J^{\eta}_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-}\aleph(\kappa_1)\Big]\right|$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2}[|\aleph'(\kappa_1)| + |\aleph'(\kappa_2)|]\int_0^1 \left|\vartheta^{\eta} - (1-\vartheta)^{\eta}\right|\frac{1}{h(\vartheta)}d\vartheta.$$

Proof.

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{\Gamma(\eta + 1)}{2\zeta^{\eta}(\kappa_2, \kappa_1)}\Big[J^{\eta}_{\kappa_1^+}\aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1)) + J^{\eta}_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-}\aleph(\kappa_1)\Big]\right|$$
$$= \left|\frac{\zeta(\kappa_2, \kappa_1)}{2}\int_0^1 [\vartheta^{\eta} - (1-\vartheta)^{\eta}]\aleph'(\kappa_1 + \vartheta\zeta(\kappa_2, \kappa_1))d\vartheta\right|$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2}\int_0^1 \left|\vartheta^{\eta} - (1-\vartheta)^{\eta}\right|\left|\aleph'(\kappa_1 + \vartheta\zeta(\kappa_2, \kappa_1))\right|d\vartheta$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2}\int_0^1 \left|\vartheta^{\eta} - (1-\vartheta)^{\eta}\right|\left|\frac{\aleph'(\kappa_1)}{h(\vartheta)} + \frac{\aleph'(\kappa_2)}{h(1-\vartheta)}\right|d\vartheta$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2}\int_0^1 \left|\vartheta^{\eta} - (1-\vartheta)^{\eta}\right|\Big[\frac{\left|\aleph'(\kappa_1)\right|}{h(\vartheta)} + \frac{\aleph'(\kappa_2)}{h(1-\vartheta)}\Big]d\vartheta$$
$$= \frac{\zeta(\kappa_2, \kappa_1)}{2}[|\aleph'(\kappa_1)| + |\aleph'(\kappa_2)|]\int_0^1 |\vartheta^{\eta} - (1-\vartheta)^{\eta}|\frac{1}{h(\vartheta)}d\vartheta,$$

as required. □

Corollary 3.1. *If in theorem 3.1,* $\eta = 1$*, we obtain Theorem (2) of [45]*

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{1}{\zeta(\kappa_2, \kappa_1)}\int_{\kappa_1}^{\kappa_1+\zeta(\kappa_2,\kappa_1)} \aleph(\vartheta)d\vartheta\right|$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2}[|\aleph'(\kappa_1)| + |\aleph'(\kappa_2)|]\int_0^1 \frac{\left|1 - 2\vartheta\right|}{h(\vartheta)}d\vartheta.$$

Corollary 3.2. *If $|\aleph'|$ is h-Godunova Levin convex i.e $\zeta(\kappa_2,\kappa_1) = \kappa_2 - \kappa_1$, then by Theorem 3.1, we obtain following inequality*

$$\begin{aligned}&\Big|\frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}-\frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{(\kappa_2)^-}\aleph(\kappa_1)\Big]\Big|\\&\le\frac{\kappa_2-\kappa_1}{2}[|\aleph'(\kappa_1)|+|\aleph'(\kappa_2)|]\int_0^1\Big|\vartheta^\eta-(1-\vartheta)^\eta\Big|\frac{1}{h(\vartheta)}d\vartheta.\end{aligned}$$

Corollary 3.3. *If $h(\vartheta)=\frac{1}{\vartheta}$, then $|\aleph'|$ is convex, then it leads to the inequality given in [23]:*

$$\begin{aligned}&\Big|\frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}-\frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{(\kappa_2)^-}\aleph(\kappa_1)\Big]\Big|\\&\le\frac{\kappa_2-\kappa_1}{2}\frac{1}{\eta+1}(1-\frac{1}{2^\eta})[|\aleph'(\kappa_1)|+|\aleph'(\kappa_2)|].\end{aligned}$$

Theorem 3.2. *Suppose that $\aleph : J = [\kappa_1, \kappa_1+\zeta(\kappa_2,\kappa_1)] \to (0,\infty)$ with $J \in \mathbb{R}$, be a differentiable real valued function on J. Also, suppose that $|\aleph'|^q$ is a h-Godunova-Levin preinvex function on J with $p > 1$ and $q = \frac{p}{p-1}$, then for fractional integral , we have*

$$\begin{aligned}&\Big|\frac{\aleph(\kappa_1)+\aleph(\kappa_1+\zeta(\kappa_2,\kappa_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta(\kappa_2,\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_1+\zeta(\kappa_2,\kappa_1))+J^\eta_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-}\aleph(\kappa_1)\Big|\\&\le\frac{\zeta(\kappa_2,\kappa_1)}{2}(|\aleph'(\kappa_1)|^q+|\aleph'(\kappa_2)|^q)^{\frac{1}{q}}\Big(\int_0^1\Big|\vartheta^\eta-(1-\vartheta)^\eta\Big|^pd\vartheta\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{1}{h(\vartheta)}d\vartheta\Big)^{\frac{1}{q}}.\end{aligned}$$

Proof. Using Lemma 3.1, we have

$$\begin{aligned}&\Big|\frac{\aleph(\kappa_1)+\aleph(\kappa_1+\zeta(\kappa_2,\kappa_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta(\kappa_2,\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\zeta(\kappa_2,\kappa_1)+J_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-}\aleph(\kappa_1)\Big|\\&=\Big|\frac{\zeta(\kappa_2,\kappa_1)}{2}\int_0^1[\vartheta^\eta-(1-\vartheta)^\eta]\aleph'(\kappa_1+\vartheta\zeta(\kappa_2,\kappa_1))d\vartheta.\Big|\\&\le\frac{\zeta(\kappa_2,\kappa_1)}{2}\int_0^1\Big|\vartheta^\eta-(1-\vartheta)^\eta\Big|\Big|\aleph'(\kappa_1+\vartheta\zeta(\kappa_2,\kappa_1))\Big|d\vartheta.\end{aligned}$$

Using Hölder's integral inequality, we have

$$\le\frac{\zeta(\kappa_2,\kappa_1)}{2}\Big(\int_0^1\Big|\vartheta^\eta-(1-\vartheta)^\eta\Big|^pd\vartheta\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|\aleph'(\kappa_1+\vartheta\zeta(\kappa_2,\kappa_1))\Big|^qd\vartheta\Big)^{\frac{1}{q}},\tag{3.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$.
Now, since $|\aleph'|^q$ is a h-Godunova-Levin preinvex, we have

$$\begin{aligned}&\int_0^1|\aleph'(\kappa_1+\vartheta\zeta(\kappa_2,\kappa_1))|^qd\vartheta\le\int_0^1\Big(\frac{|\aleph'(\kappa_1)|^q}{h(\vartheta)}+\frac{|\aleph'(\kappa_2)|^q}{h(1-\vartheta)}\Big)d\vartheta\\&\le(|\aleph'(\kappa_1)|^q+|\aleph'(\kappa_2)|^q)\int_0^1\frac{1}{h(\vartheta)}d\vartheta.\end{aligned}\tag{3.2}$$

Using (3.2) in (3.1), we get the required result. □

Corollary 3.4. *(1) If in theorem 3.2, $\eta = 1$, we obtain Theorem 3 given in [45] as:*

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{1}{\zeta(\kappa_2, \kappa_1)} \int_{\kappa_1}^{\kappa_1+\zeta(\kappa_2,\kappa_1)} \aleph(\vartheta) d\vartheta\right|$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2(p+1)^{\frac{1}{p}}} (|\aleph'(\kappa_1)|^q + |\aleph'(\kappa_2)|^q)^{\frac{1}{q}} \Big(\int_0^1 \frac{1}{h(\vartheta)} d\vartheta\Big)^{\frac{1}{q}}.$$

(2)Here by taking $\zeta(\kappa_2, \kappa_1) = \kappa_2 - \kappa_1$ and $h(\vartheta) = \frac{1}{\vartheta^s}$, we obtain Theorem 2.1 proved by Mudassar [53] is given below:

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_2)}{2} - \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \aleph(\vartheta) d\vartheta\right|$$
$$\leq \frac{(\kappa_2 - \kappa_1)}{2(p+1)^{\frac{1}{p}}} \Big(\frac{|\aleph'(\kappa_1)|^q + |\aleph'(\kappa_2)|^q}{s+1}\Big)^{\frac{1}{q}}.$$

Corollary 3.5. *If in theorem 3.2, $h(\vartheta) = \vartheta^s$ i.e if $\aleph$ is s-Godunova-Levin, then we obtain Theorem 3.2 presented by Noor [55] as given by;*

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{1}{\zeta(\kappa_2, \kappa_1)} \int_{\kappa_1}^{\kappa_1+\zeta(\kappa_2,\kappa_1)} \aleph(\vartheta) d\vartheta\right|$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2(p+1)^{\frac{1}{p}}} \Big[\frac{|\aleph'(\kappa_1)|^{\frac{p}{p-1}} + |\aleph'(\kappa_2)|^{\frac{p}{p-1}}}{1-s}\Big]^{\frac{p-1}{p}}.$$

Theorem 3.3. *By the assumptions of theorem 3.2, the following inequality holds, which is related to Hermite-Hadamard inequality*

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\kappa_2, \kappa_1)} \Big[J^\eta_{\kappa_1^+} \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1)) + J^\eta_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-} \aleph(\kappa_1)\Big]\right|$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2^{\frac{1}{q}}(\eta+1)^{1-\frac{1}{q}}} (|\aleph'(\kappa_1)|^q + |\aleph'(\kappa_2)|^q)^{\frac{1}{q}} (1 - \frac{1}{2^\eta})^{1-\frac{1}{q}}$$
$$\Big[\int_0^1 \frac{\big|\vartheta^\eta - (1-\vartheta)^\eta\big|}{h(\vartheta)} d\vartheta\Big]^{\frac{1}{q}}.$$

Proof. Using lemma 3.1, we have

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\kappa_2, \kappa_1)} \Big[J^\eta_{\kappa_1^+} \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1)) + J^\eta_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-} \aleph(\kappa_1)\Big]\right|$$
$$= \left|\frac{\zeta(\kappa_2, \kappa_1)}{2} \int_0^1 [\vartheta^\eta - (1-\vartheta)^\eta] \aleph'(\kappa_1 + \vartheta\zeta(\kappa_2, \kappa_1)) d\vartheta.\right|$$
$$\leq \frac{\zeta(\kappa_2, \kappa_1)}{2} \int_0^1 \big|\vartheta^\eta - (1-\vartheta)^\eta\big| \big|\aleph'(\kappa_1 + \vartheta\zeta(\kappa_2, \kappa_1))\big| d\vartheta.$$

Applying power-mean inequality, we get

$$\left|\frac{\aleph(\kappa_1) + \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\kappa_2, \kappa_1)} \Big[J^\eta_{\kappa_1^+} \aleph(\kappa_1 + \zeta(\kappa_2, \kappa_1)) + J^\eta_{(\kappa_1+\zeta(\kappa_2,\kappa_1))^-} \aleph(\kappa_1)\Big]\right|$$

$$\leq \frac{\zeta(\kappa_2,\kappa_1)}{2}\Big(\int_0^1 \big|\vartheta^\eta-(1-\vartheta)^\eta\big|d\vartheta\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \big|\vartheta^\eta-(1-\vartheta)^\eta\big|\big|\aleph'(\kappa_1+\vartheta\zeta(\kappa_2,\kappa_1))\big|^q d\vartheta\Big)^{\frac{1}{q}}.$$

Since $|\aleph'|^q$ is a h-Godunova-Levin preinvex, we get

$$\begin{aligned}\int_0^1 \big|\vartheta^\eta-(1-\vartheta)^\eta\big|\big|\aleph'(\kappa_1+\vartheta\zeta(\kappa_2,\kappa_1))\big|^q d\vartheta &\leq \int_0^1 \big|\vartheta^\eta-(1-\vartheta)^\eta\big|\Big(\frac{|\aleph'(\kappa_1)|^q}{h(\vartheta)}+\frac{|\aleph'(\kappa_2)|^q}{h(1-\vartheta)}\Big)d\vartheta\\ &\leq \int_0^1 \frac{\big|\vartheta^\eta-(1-\vartheta)^\eta\big|}{h(\vartheta)}(|\aleph'(\kappa_1|^q+|\aleph'(\kappa_2)|^q)d\vartheta.\end{aligned}$$

Now by basic calculus, we have

$$\int_0^1 \big|\vartheta^\eta-(1-\vartheta)^\eta\big|d\vartheta=\frac{2}{(\eta+1)}\Big(1-\frac{1}{2^\eta}\Big).$$

□

Corollary 3.6. *If $\eta=1$, we obtain inequality reported by Ohud Almutairi and Adem Kiliçman in [45] as follows;*

Corollary 3.7. *If $\zeta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$, $h(\vartheta)=\frac{1}{\vartheta}$, $q=1$, and $\eta=1$, we have*

$$\left|\frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\aleph(x)dx\right|\leq\frac{\kappa_2-\kappa_1}{8}(|\aleph'(\kappa_1)|+|\aleph'(\kappa_2)|),$$

which is proposed by Dragomir and Agarwal [54].

Corollary 3.8. *If $\eta=1$, $h(\vartheta)=\vartheta^s$, we obtain Theorem 3.3 presented by Noor [55] as follows;*

$$\begin{aligned}&\left|\frac{\aleph(\kappa_1)+\aleph(\kappa_1+\zeta(\kappa_2,\kappa_1))}{2}-\frac{1}{\zeta(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\zeta(\kappa_2,\kappa_1)}\aleph(\vartheta)d\vartheta\right|\\ &\leq\frac{\zeta(\kappa_2,\kappa_1)}{4}\Big[[|\aleph'(\kappa_1)|^q+|\aleph'(\kappa_2)|^q]\Big[\frac{2^{s+1}-2s}{(s-2)(s-1)}\Big]\Big]^{\frac{1}{q}}.\end{aligned}$$

Corollary 3.9. *If $\zeta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$, $h(\vartheta)=\frac{1}{\vartheta}$ and $q=1$, we have*

$$\left|\frac{\aleph(\kappa_1)+\aleph(\kappa_2)}{2}-\frac{\Gamma(\eta+1)}{2(\kappa_2-\kappa_1)^\eta}\Big[J^\eta_{\kappa_1^+}\aleph(\kappa_2)+J^\eta_{(\kappa_2)^-}\aleph(\kappa_1)\Big]\right|\leq\frac{\kappa_2-\kappa_1}{2(\eta+1)}(1-\frac{1}{2^\eta})[|\aleph'(\kappa_1)|+|\aleph'(\kappa_2)|],$$

which is Theorem 3 given in [23].

4. Conclusions

In the present study, we developed new version of Hermite-Hadamard fractional integral inequality and trapezoid type inequalities by utilizing Riemann-Liouville fractional integral operator for h-Godunova-Levin convex function and h-Godunova-Levin preinvex function, and we have authenticated our results by drawing corollaries which are well known results in literature [23,45,54,55]. Numerous fractional version of different well known inequalities can be developed for h-Godunova-Levin convex and preinvex functions which provide the theoretical achievement of extensive work in the field of fractional inequalities.

Conflict of interest

The authors declare no conflict of interest.

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