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Research article

Transcritical bifurcation in a multiparametric nonlinear system

Osmin Ferrer[1,*], **José Guerra**[1] **and Alberto Reyes**[2]

[1] Departament of Mathematics, University of Sucre, Cra. 28 # 5-267, Puerta Roja, Sincelejo, Sucre, Colombia

[2] Departament of Mathematics, University of Atlántico, Carrera 30 # 8-49, Puerto Colombia, Barranquilla, Atlántico, Colombia

* **Correspondence:** Email: osmin.ferrer@unisucre.edu.co; Tel: +57604795255.

Abstract: In this paper we study a multiparametric nonlinear system with a transcritical bifurcation in a region of points of $\mathbb{R}^3$. The parametric regions that constitute the boundaries where important qualitative changes occur in the dynamics of the system are determined. The equilibrium points in each of the regions are also established and classified. Finally, the stability of the equilibrium points at infinity of the system obtained from the Poincare compactification is classified, and the global phase portrait of the system is made.

Keywords: dynamic system; singularity; Poincare compactification; stability; bifurcation; global phase portrait
Mathematics Subject Classification: 42C15, 46C05, 46C20

1. Introduction

Dynamical systems have their origin in mathematics and physics. They are attributed to Henri Poincaré (1854–1912), who developed the foundations of chaos theory, the pioneer of dynamical systems [1]. A dynamical system is considered as a mathematical model, to study deterministic or random iterative processes. Dynamical systems have proven to be a powerful tool, for example, for analyzing the stability of cryptocurrencies and the modeling of Covid-19 disease as evidenced in [2, 3]. In the present paper a multiparametric nonlinear system is studied with the objective of establishing tools to analyze dynamic models applicable to different contexts. All the results of the present study are part of the master thesis of the second author. Similar ideas have been developed independently in [4–7].

The elements necessary to carry out the qualitative study of the nonlinear multiparameter system are presented in Section 2. A definition of autonomous system is given there (Definition 2.1); through

Definition 2.2 it is established what it is and how an equilibrium point is determined for an autonomous system. The types of equilibrium points or singularities are specified according to the structure of the eigenvalues (Theorems 2.1–2.4). The types of sectors presented by autonomous systems are defined (Definition 2.3), Poincaré's definition of the index of a Jordan curve (Definition 2.4), the index of a critical point with respect to a vector field (Definition 2.5) and the Poincaré compactification (Definition 2.6).

The study of the linear multiparametric system begins in Section 3. Since the system under study has three parameters, we proceed to partition the space $\mathbb{R}^3$ into parametric regions, which determine the boundaries with respect to which qualitative changes occur in the dynamics of the system. In Proposition 3.1, we show that the eight parametric regions form a partition of $\mathbb{R}^3$. Then, in Section 4, we determine the number of equilibrium points or singularities existing for the system under study in each of the parametric regions, with the objective of analyzing the appearance or destruction of equilibrium points when varying the parameters of the system from one region to another (Proposition 4.1).

In Section 5, with the objective of analyzing the changes that occur in the stability of the equilibrium points of the system when varying the parameters from one region to another, its stability is classified in each of the parametric regions taking into account the structure of the eigenvalues of the Jacobian matrix associated to the system in each case (Propositions 5.1–5.3). In Section 6, starting from the stability of equilibrium points studied in Section 5 and the creation or destruction of equilibrium points from one parametric region to another (Section 4), we analyze whether or not the system presents bifurcations and the values of the parameters where the bifurcations occur, called bifurcation points. In Section 7 with the objective of obtaining a global behavior of the system, both in the finite plane and near infinity, we determine and classify singularities near infinity from the Poincaré compactification and the theorems used to classify singularities in the finite plane are determined and classified. In Section 8, the global phase portraits for the system are shown, where the behavior of the trajectories both in the finite plane and near infinity can be visualized.

2. Preliminares

Definition 2.1. *[8] A plane autonomous system is a system of two differential equations of the form*

$$\begin{aligned} \dot{x} &= F(x,y) \\ \dot{y} &= G(x,y), \end{aligned} \tag{2.1}$$

where F and G are continuous functions with continuous first order partial derivatives in the entire plane.

Definition 2.2. *[8] The point (x_0, y_0) such that $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$ is called a critical point of the system.*

Theorem 2.1. *[9] Let $(0,0)$ be an isolated singularity of the vector field $X(x,y) = (ax+by+F(x,y), cx+dy+G(x,y))$, where F and G are analytic in a neighborhood of the origin and have series expansions starting with terms of degree two in x and y. We say that $(0,0)$ is a nondegenerate singularity if $ad-bc \neq 0$. Let λ_1 and λ_2 be the eigenvalues of $DX(0,0)$. Then:*
1) If λ_1, λ_2 are real and $\lambda_1\, \lambda_2 < 0$, then $(0,0)$ is saddle point whose separatrices tend to $(0,0)$ in the directions given by the eigenvectors associated with λ_1 and λ_2.

2) If λ_1, λ_2 are real and $\lambda_1\ \lambda_2 > 0$, then $(0,0)$ is a node. If $\lambda_1 > 0$ ($\lambda_1 < 0$) then is a source (sink).
3) If $\lambda_1 = \alpha+\beta i$ and $\lambda_2 = \alpha-\beta i$ with $\alpha,\beta \neq 0$ then $(0,0)$ is a focus. If $\alpha > 0(\alpha < 0)$ then it is a repulsor (attractor).
4) If $\lambda_1 = \beta i$ and $\lambda_2 = -\beta i$, then $(0,0)$ is a linear center, topologically a focus or a center.

Definition 2.3. *[10] A sector that is topologically equivalent to the sector shown in Figure 1 (a) is called a hyperbolic sector. A sector that is topologically equivalent to the sector shown in Figure 1 (b), is called a parabolic sector; and a sector that is topologically equivalent to the one shown in Figure 1 (c) is called an elliptic sector.*

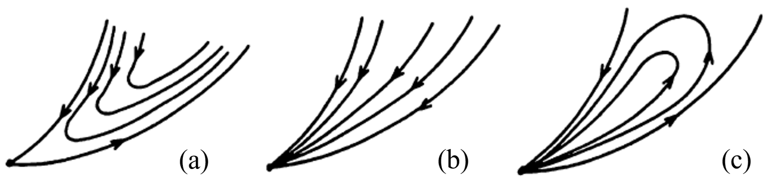


Figure 1. (a) A hyperbolic sector, (b) A parabolic sector, and (c) An elliptic sector.

The following definition is essential to define the index of a point of a vector field **f**.

Definition 2.4. *[10] The index $I_f(C)$ of a Jordan curve C relative to a vector field $\boldsymbol{f} \in C^1(\mathbb{R}^2)$, where $\boldsymbol{f}$ has no critical point on C, is defined as the integer*

$$I_f(C) = \frac{\Delta\Theta}{2\pi}.$$

Where $\Delta\Theta$ is the total change in the angle Θ that the vector $\boldsymbol{f}= (P,Q)^T$ makes with respect to the x-axis, i.e., $\Delta\Theta$ the change in

$$\Theta(x,y) = \arctan\frac{Q(x,y)}{P(x,y)},$$

as the point (x,y) traverses C exactly once in the positive direction.

Definition 2.5. *[10] Let $\boldsymbol{f} \in C^1(E)$ where E is an open subset of $\mathbb{R}^2$ and let bf $\boldsymbol{x_0} \in E$ be an isolated critical point of $\boldsymbol{f}$. Let C be a Jordan curve contained in E and containing $\boldsymbol{x_0}$ and no other critical point of $\boldsymbol{f}$ on its interior. Then the index of the critical point $\boldsymbol{x_0}$ with respect to $\boldsymbol{f}$*

$$I_f(\boldsymbol{x}_0) = I_f(C).$$

The following theorem is fundamental to analyze the behavior around singularities when the matrix $A = DX(p)$ has an eigenvalue equal to zero, the determinant is equal to zero, the trace is nonzero and the matrix is different from the zero matrix.

Theorem 2.2. *[9] Let $(0,0)$ be an isolated singularity of the system:*

$$\begin{aligned} \dot{x} &= X(x,y) \\ \dot{y} &= y + Y(x,y), \end{aligned} \tag{2.2}$$

where X and Y are analytic in a neighborhood of the origin and have expansions starting with second degree terms in x and y. Let $y = f(x)$ *be the solution of the equation* $y + Y(x, y) = 0$ *in a neighborhood of* $(0, 0)$*, and suppose that the serial expression of the function* $g(x) = X(x, f(x))$ *has the form* $g(x) = a_m x^m + ...$ *where* $m \geq 2$*,* $a_m \neq 0$*.*

Then:

1) If m is odd and $a_m > 0$*, then* $(0, 0)$ *is topologically a node.*

2) If m is odd and $a_m < 0$*, then* $(0, 0)$ *is topologically a saddle, where two of its separatrices tend to* $(0, 0)$ *in the* 0 *and* π *directions and the other two in the* $\frac{\pi}{2}$ *and* $\frac{3\pi}{2}$ *directions.*

3) If m is even, then $(0, 0)$ *is a saddle-node, i.e., a singularity whose neighborhood is the union of a parabolic sector and two hyperbolic sectors, two of its separatrices tend to* $(0, 0)$ *in the* $\frac{\pi}{2}$ *and* $\frac{3\pi}{2}$ *directions, and the other two in the* 0 *or* π *directions according to* $a_m < 0$ *or* $a_m > 0$*.*

Now, we will make considerations about two theorems necessary to analyze the behavior of a singularity when the matrix, $A = DX(p)$, has two eigenvalues equal to zero, the determinant and the trace are equal to zero, but the corresponding matrix is different from the zero matrix.

Theorem 2.3. *[11] Let* $(0, 0)$ *be an isolated singularity of the system:*

$$\begin{aligned} \dot{x} &= y + X(x, y) \\ \dot{y} &= Y(x, y), \end{aligned} \tag{2.3}$$

where X and Y are analytic in a neighborhood of the origin and have expansions starting with second degree terms in x and y. Let $y = F(x) = a_2 x^2 + a_3 x^3 + ...$ *be a solution of the equation* $y + Y(x, y) = 0$ *in a neighborhood of* $(0, 0)$*, and suppose they have the following series expansion of the function* $f(x) = Y(x, F(x)) = ax^{\alpha}(1 + ...)$ *and* $\phi(x) = (\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y})(x, F(x)) = bx^{\beta}(1 + ...)$ *where* $a \neq 0$*,* $\alpha \geq 2$ *and* $\beta \geq 1$*.*

Then:

1) If α *is even and:*

a) $\alpha > 2\beta + 1$*, then the origin is a saddle-node(index* 0*), (Figure* 2 *(a)).*

b) If $\alpha < 2\beta + 1$*, or,* $\phi(x) \equiv 0$*, then the origin is a singularity whose neighborhood is the union of two hyperbolic sectors(index* 0*), (Figure* 2 *(b)).*

2) If α *is odd and* $a > 0$*, then the origin is a saddle (Index* -1*), (Figure* 2 *(c)).*

3) If alpha is odd, $a < 0$*, and:*

a) If ($\alpha > 2\beta + 1$*, and,* β *even, or,* $\alpha = 2\beta + 1$*, and,* β *is even) and* $b^2 + 4a(\beta + 1) \geq 0$*, then the origin is a node (Index + 1), (Figure* 2 *(d)). The node is stable if* $b < 0$*, or unstable if* $b > 0$*.*

b) If ($\alpha > 2\beta + 1$*, and,* β *odd, or, (*$\alpha = 2\beta + 1$*), and,* β *odd) and* $b^2 + 4a(\beta + 1) \geq 0$*, then the origin is the union of a hyperbolic sector and an elliptic sector (Index + 1), (Figure* 2 *(e)).*

c) If $\alpha = 2\beta + 1$ *and* $b^2 + 4a(\beta + 1) < 0$*, or, (*$\alpha < 2\beta + 1$*, or,* $\phi \equiv 0$*) then the origin is a focus, or, a center (Index + 1).*

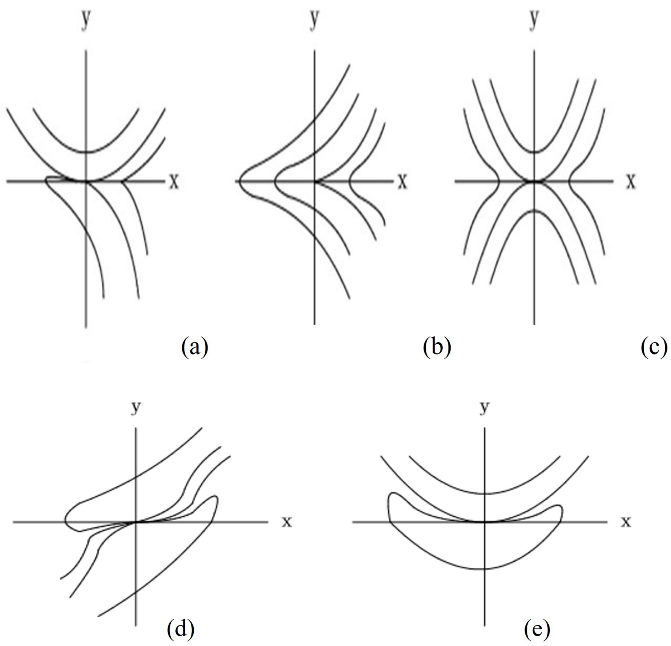


Figure 2. Local behavior near a singularity.

Given the polynomial differential system

$$\begin{aligned}\dot{\mathbf{x}} &= P(x, y)\\ \dot{\mathbf{y}} &= Q(x, y).\end{aligned} \tag{2.4}$$

According to [10], it can be transformed to the "normal" form

$$\begin{aligned}\dot{\mathbf{x}} &= y\\ \dot{\mathbf{y}} &= a_k x^k[1 + h(x)] + b_n x^n y[1 + g(x)] + y^2[R(x, y)],\end{aligned} \tag{2.5}$$

where $h(x)$, $g(x)$, and, $R(x, y)$ are analytic in a neighborhood of the origin, $h(0) = g(0) = 0$, $k \geq 2$, $a_k \neq 0$ and $n \geq 1$.

From this transformation the following theorem is established.

Theorem 2.4. *[10] Let $k = 2m + 1$, with $m \geq 1$ in (2.5) and be $\lambda = b_n^2 + 4(m + 1)a_k$.*
Then, if $a_k > 0$, the origin is a (topologically) saddle. If $a_k < 0$ the origin is (1) a focus or a center if $b_n = 0$ and also if $b_n \neq 0$ and $n > m$ or if $n = m$ and $\lambda < 0$, (2) a node if $b_n \neq 0$ n is an even number and $n < m$ and also if $b_n \neq 0$, n is an even number , $n = m$ and $\lambda \geq 0$ and (3) a critic point with an eliptical sector if $b_n \neq 0$, n is an odd number and $n < m$ and also if $b_n \neq 0$, n is an odd number, $n = m$ and $\lambda \geq 0$.

Definition 2.6. *[12] Let $X = P\dfrac{\partial}{\partial x_1} + Q\dfrac{\partial}{\partial x_2}$ be a polynomial vector field(the functions P and Q are polynomials of arbitrary degree in the variables x_1 and x_2 and $p(X)$ the Poincaré compactification of the vector field X in $\mathbb{R}^2$. The expression for $p(X)$ in the local chart (U_1, ϕ_1) is given by:*

$$\dot{u} = v^d \Big[-u\,P\Big(\frac{1}{v}, \frac{u}{v}\Big) + Q\Big(\frac{1}{v}, \frac{u}{v}\Big)\Big]$$
$$\dot{v} = -v^{d+1}\,P\Big(\frac{1}{v}, \frac{u}{v}\Big).$$

The expression for (U_2, ϕ_2) *is*

$$\dot{u} = v^d \Big[P\Big(\frac{u}{v}, \frac{1}{v}\Big) - u\,Q\Big(\frac{u}{v}, \frac{1}{v}\Big)\Big]$$
$$\dot{v} = -v^{d+1}\,Q\Big(\frac{u}{v}, \frac{1}{v}\Big).$$

The Figure 3, taken from [12], shows a representation of the Poincaré compactification.

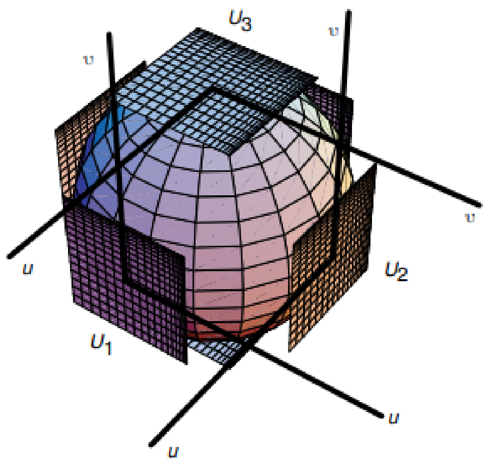


Figure 3. The local charts (U_k, ϕ_k) para $k = 1, 2, 3$ of the Poincaré sphere.

The definitions specified in this section are essential to understand the terminology used in the study, and the theorems are fundamental to classify the stability of the critical points that exist in each of the parametric regions defined in Section 3, which are established in order to determine the regions where the parameters cause qualitative changes in the dynamics of the system called bifurcations (Section 6).

3. Parametric regions

In this section we divide the space $\mathbb{R}^3$ into parametric regions which are useful for the development of the study.

Let us consider the following regions defined in $\mathbb{R}^3$:

$B_1 = \{(a, b, c) \mid b^2 + 4c < 0\}$
$B_2 = \{(a, b, c) \mid b^2 + 4c > 0,\ a = 0,\ c \neq 0\}$
$B_3 = \{(a, b, c) \mid b^2 + 4c = 0,\ a = 0,\ b \neq 0\}$
$B_4 = \{(a, b, c) \mid b^2 + 4c = 0, a \neq 0,\ b = 0,\ c = 0\}$
$B_5 = \{(a, b, c) \mid b^2 + 4c = 0,\ ab \neq 0\}$
$B_6 = \{(a, b, c) \mid b^2 + 4c > 0, ab \neq 0, c = 0\}$

$B_7 = \{(a,b,c) \mid\ b^2 + 4c > 0,\ a \neq 0, c \neq 0\}$
$B_8 = \{(a,b,c) \mid a = 0,\ c = 0\} = \{(a,b,c) | b^2 + 4c = 0, a = c = 0\} \cup \{(a,b,c) | b^2 + 4c > 0, a = c = 0\}$.

Proposition 3.1. *The family* $\{B_k\}_{k=1}^{8}$ *forms a partition of* $\mathbb{R}^3$.

Proof. In fact, we have to
$\mathbb{R}^3 = \{(a,b,c) | b^2 + 4c < 0, b^2 + 4c = 0 \quad and \quad b^2 + 4c > 0\}$
$\mathbb{R}^3 = \{(a,b,c) | b^2 + 4c < 0\} \cup \{(a,b,c) | b^2 + 4c = 0\} \cup \{(a,b,c) | b^2 + 4c > 0\}$
$\mathbb{R}^3 = \{(a,b,c) | b^2 + 4c < 0\} \cup \{(a,b,c) \mid b^2 + 4c = 0,\ a = 0,\ b \neq 0\} \cup \{(a,b,c) \mid b^2 + 4c = 0,$ $a \neq 0,\ b = 0,\ c = 0\} \cup \{(a,b,c) \mid b^2 + 4c = 0,\ ab \neq 0\} \cup \{(a,b,c) | b^2 + 4c = 0, a = c = 0\} \cup$ $\{(a,b,c) | b^2 + 4c > 0, a = c = 0\} \cup \{(a,b,c) \mid b^2 + 4c > 0,\ a = 0,\ c \neq 0\} \cup \{(a,b,c) \mid b^2 + 4c > 0, ab \neq$ $0, c = 0\} \cup \{(a,b,c) \mid\ b^2 + 4c > 0,\ a \neq 0, c \neq 0\}$
$\mathbb{R}^3 = B_1 \cup B_3 \cup B_4 \cup B_5 \cup B_8 \cup B_2 \cup B_6 \cup B_7 = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6 \cup B_7 \cup B_8$
$\mathbb{R}^3 = \bigcup_{k=1}^{8} B_k.$
Moreover, $B_k \cap B_j = \emptyset$ for $j \neq k$, where $j,k \in \{1,2,...,8\}$.
Hence, $\{B_k\}_{k=1}^{8}$ is a partition of $\mathbb{R}^3$. □

4. Existence of critical points in parametric regions

In this section we analyze the existence of critical points in each of the parametric regions defined previously for the system

$$\begin{aligned} \dot{\mathbf{x}} &= y \\ \dot{\mathbf{y}} &= (3ax + b)\,y + (-a^2x^2 - abx + c)\,x. \end{aligned} \tag{4.1}$$

Proposition 4.1. *Given the system* (4.1) *then it follows that:*

1) *If* $(a,b,c) \in \bigcup_{k=1}^{4} B_k$, *then* $(0,0)$ *is the only critical point.*
2) *If* $(a,b,c) \in \bigcup_{k=5}^{6} B_k$, *then the system has two critical points.*
3) *If* $(a,b,c) \in B_7$, *then the system has three critical points.*
4) *If* $(a,b,c) \in B_8$, *then the system has infinite critical points.*

Proof. The proof of this proposition is based on the number of zeros possessed by the function:

$$g(x) = a^2x^2 + abx - c = \left(ax + \frac{b}{2}\right)^2 - \frac{b^2 + 4c}{4}.$$

1) If $(a,b,c) \in \bigcup_{k=1}^{4} B_k$ then $(a,b,c) \in B_1$ o $(a,b,c) \in B_2$ or $(a,b,c) \in B_3$ o $(a,b,c) \in B_4$.
   Analyzing each of the cases, we obtain the following:
   a. If $(a,b,c) \in B_1$, we have that $b^2 + 4c < 0$.
      Since $g(x) = \left(\left(ax + \frac{b}{2}\right)^2 - \frac{b^2 + 4c}{4}\right) > 0$ then the system has a single critical point corresponding to $(0,0)$.

b. If $(a,b,c) \in B_2$, then $b^2+4c>0$, $a=0$ *and* $c \neq 0$. Also, $g(x)=(0)^2x^2+(0)bx-c=-c$. But $c \neq 0$, therefore, $g(x) \neq 0$. Thus, the only critical point is $(0,0)$.
c. If $(a,b,c) \in B_3$, then $b^2+4c=0$, $a=0$ and $b \neq 0$; $g(x)=-c$. Since $b \neq 0$ then $c \neq 0$, therefore, $g(x) \neq 0$. Thus, the only critical point of the system is $(0,0)$.
d. If $(a,b,c) \in B_4$, then $b^2+4c=0$, $a \neq 0, b=0$ and $c=0$; $g(x)=a^2x^2$. Since $g(x)=0$ if $x=0$ and $g(x)>0$ for $x \neq 0$ then $(0,0)$ is the only critical point.

In conclusion, it follows that if $(a,b,c) \in \bigcup_{k=1}^{4} B_k$, then $(0,0)$ is the only critical point.

2) If $(a,b,c) \in \bigcup_{k=5}^{6} B_k$ then $(a,b,c) \in B_5$ or $(a,b,c) \in B_6$.

a. If $(a,b,c) \in B_5$, then $b^2+4c=0$ and $ab \neq 0$. Since $b^2+4c=0$, then $g(x)=\left(ax+\frac{b}{2}\right)^2$. Then, $g(x)=0$ whenever $x=-\frac{b}{2a}$. Therefore, the system (4.1) has two critical points which are $(0,0)$ and $\left(-\frac{b}{2a},0\right)$.
b. If $(a,b,c) \in B_6$, then $b^2+4c>0$, $ab \neq 0$ and $c=0$; $g(x)=a^2x^2+abx=ax(ax+b)$. Then, $g(x)$ has two zeros. Thus, the system (4.1) has two critical points which are $(0,0)$ and $\left(-\frac{b}{a},0\right)$.

In conclusion, if $(a,b,c) \in \bigcup_{k=5}^{6} B_k$, then the system has two critical points.

3) If $(a,b,c) \in B_7$, then $b^2+4c>0$, $a \neq 0$ and $c \neq 0$. In addition, $g(x)$ can be written as

$$g(x)=\left(ax+\frac{b}{2}-\frac{\sqrt{b^2+4c}}{2}\right)\left(ax+\frac{b}{2}+\frac{\sqrt{b^2+4c}}{2}\right)$$

Therefore, $g(x)=0$ whenever $x=\frac{-b-\sqrt{b^2+4c}}{2a}$ ór $x=\frac{-b+\sqrt{b^2+4c}}{2a}$ with $a \neq 0$ by hypothesis. Also, $g(x)$ has two zeros since $b^2+4c>0$.

As well, the system (4.1) has three critical points, which are $(0,0)$, $\left(\frac{-b+\sqrt{b^2+4c}}{2a},0\right)$ and $\left(\frac{-b-\sqrt{b^2+4c}}{2a},0\right)$.

4) If $(a,b,c) \in B_8$ then $a=0$ and $c=0$, then the system (4.1) reduces to:

$$\begin{aligned} \dot{\mathbf{x}} &= y \\ \dot{\mathbf{y}} &= by. \end{aligned} \tag{4.2}$$

Thus, the system has infinite critical points. □

In the next section we study the stability of the critical points of the system (4.1) in each of the parametric regions.

5. Stability in parametric regions

5.1. Stability in $\bigcup_{k=1}^{4} B_k$

Proposition 5.1. *Given the system* (4.1) *with* $(a,b,c) \in \mathbb{R}^3$, *then:*

1) *If* $(a,b,c) \in B_1$ *and* $b > 0$*, then* $(0,0)$ *is a repulsor focus.*
2) *If* $(a,b,c) \in B_1$ *and* $b < 0$*, then* $(0,0)$ *is an attractor focus.*
3) *If* $(a,b,c) \in B_1$*, and* $b = 0$*, then* $(0,0)$ *is a center.*
4) *If* $(a,b,c) \in B_2$*, then* $(0,0)$ *is a repulsor node if* $b > 0$ *and attractor if* $b < 0$*.*
5) *If* $(a,b,c) \in B_2$ *and* $c > 0$*, then* $(0,0)$ *is a saddle point.*
6) *If* $(a,b,c) \in B_3$*, then* $(0,0)$ *is a repulsor node if* $b > 0$ *and attractor if* $b < 0$*.*
7) *If* $(a,b,c) \in B_4$*, then* $(0,0)$ *consists of the union of a hyperbolic sector and an elliptic sector.*

Proof. For the system (4.1) the resulting Jacobian matrix evaluated at the critical point $(0,0)$ corresponds to:

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & b \end{bmatrix}.$$

The eigenvalues corresponding to the critical point $(0,0)$ are:

$$\lambda_1 = \frac{b + \sqrt{b^2+4c}}{2} \quad \& \quad \lambda_2 = \frac{b - \sqrt{b^2+4c}}{2}.$$

1) If $(a,b,c) \in B_1$, then $b^2 + 4c < 0$, this implies that $\lambda_1 = \dfrac{b + i\sqrt{-(b^2+4c)}}{2}$ & $\lambda_2 = \dfrac{b - i\sqrt{-(b^2+4c)}}{2}$. Therefore, if $b > 0$, $(0,0)$ is a repulsor focus.
2) If $(a,b,c) \in B_1$, then $b^2 + 4c < 0$, this implies that $\lambda_1 = \dfrac{b + i\sqrt{-(b^2+4c)}}{2}$ & $\lambda_2 = \dfrac{b - i\sqrt{-(b^2+4c)}}{2}$.
   Then, if $b < 0$, $(0,0)$ is an attractor focus.
3) If $(a,b,c) \in B_1$, then $b^2 + 4c < 0$. Now, if $b = 0$ we have that, $\lambda_1 = i\sqrt{-c}$ and $\lambda_2 = -i\sqrt{-c}$. Therefore, $(0,0)$ is a center.
4) If $(a,b,c) \in B_2$, then $b^2 + 4c > 0, a = 0$ and $c \neq 0$. Note that $\lambda_1\lambda_2 = -c$. Now, if $b > 0$ and $c < 0$ then $\lambda_1\lambda_2 > 0$. Since $b > 0$ then $\lambda_1 > 0$ and consequently, $\lambda_2 > 0$. Thus, it is concluded that $(0,0)$ is a repulsor node. On the other hand, if $b < 0$ and $c < 0$ then $\lambda_1\lambda_2 > 0$ and $\lambda_2 < 0$, consequently, $\lambda_1 < 0$. Therefore, $(0,0)$ is an attractor node.
5) If $(a,b,c) \in B_2$, then $b^2 + 4c > 0, a = 0$ and $c \neq 0$. Now, if $c > 0$ then $\lambda_1\lambda_2 < 0$. Then, $(0,0)$ is a saddle point.
6) If $(a,b,c) \in B_3$, then $b^2 + 4c = 0, a = 0$ and $b \neq 0$. Therefore, $\lambda_1 = \lambda_2 = \dfrac{b}{2}$. Now, if $b > 0$ then $\lambda_1 > 0$ and $\lambda_2 > 0$, therefore, $(0,0)$ is a repulsor node. On the other hand, if $b < 0$ then $\lambda_1 < 0$ and $\lambda_2 < 0$, therefore, $(0,0)$ is an attractor node.
7) If $(a,b,c) \in B_4$, then $b^2 + 4c = 0$, $a \neq 0, b = 0$ and $c = 0$. The only critical point for this system is $(0,0)$. The eigenvalues associated with the Jacobian matrix $Df(0,0)$ are $\lambda_1 = 0$

and $\lambda_2 = 0$, theorem 2.4 is applied for the characterization of the critical point. Taking the data of the associated system, we obtain that: $k = 2m + 1$, $m = 1$, $a_k = -a^2$ and $n = m = 1$. Thus, $\lambda = b_n^2 + 4(m+1)a_k$ and substituting we observe that $\lambda = (3a)^2 + 4(1+1)(-a^2)$, this is, $\lambda = 9a^2 - 8a^2$ and thus $\lambda = a^2 > 0$. Therefore, analyzing the hypotheses of the theorem we see that the number (3) is satisfied, with $b_n \neq 0$, n odd, $n = m$ and $\lambda \geq 0$. Therefore, $(0, 0)$ is a critical point with an elliptic sector.

□

5.2. *Stability in* $\bigcup_{k=5}^{6} B_k$

Proposition 5.2. *Given the system* (4.1) *then*

1) *If* $(a, b, c) \in B_5$, $a \neq 0$ *and* $b > 0$ *then* $(0, 0)$ *is a repulsor node and* $\left(-\frac{b}{2a}, 0\right)$ *is a saddle-node point.*
2) *If* $(a, b, c) \in B_5$, $a \neq 0$ *and* $b < 0$ *then* $(0, 0)$ *is an attractor node and* $\left(-\frac{b}{2a}, 0\right)$ *is a saddle-node point.*
3) *If* $(a, b, c) \in B_6$, $a \neq 0$, $b > 0$ *and* $c = 0$ *then* $(0, 0)$ *is a saddle-node point and* $\left(-\frac{b}{a}, 0\right)$ *is an attractor node.*
4) *If* $(a, b, c) \in B_6$, $a \neq 0$, $b < 0$ *and* $c = 0$ *then* $(0, 0)$ *is a saddle-node point and* $\left(-\frac{b}{a}, 0\right)$ *is a repulsor node.*

Proof. 1) If $(a, b, c) \in B_5$, then $b^2 + 4c = 0$ and $ab \neq 0$. If we consider and $a \neq 0$ then the eigenvalues of the Jacobian evaluated at $(0, 0)$ are $\lambda_1 = \lambda_2 = \frac{b}{2}$. Then, if $b > 0$ then $(0, 0)$ is a repulsor node.

The eigenvalues of $Df\left(-\frac{b}{2a}, 0\right)$ are $\lambda_1 = 0$ and $\lambda_2 = -\frac{b}{2}$. Since one of the eigenvalues of the system is zero we proceed to apply the theorem 2.2. Doing the change of variables $u = x - x_0$. and $v = y - y_0$ we obtain that $u = x + \frac{b}{2a}$ and $v = y$. In addition, $x = u - \frac{b}{2a}$ and $v = y$.

Calculating $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ we obtain the system:

$$\begin{aligned} \dot{\mathbf{u}} &= v \\ \dot{\mathbf{v}} &= 3auv - \tfrac{1}{2}bv - a^2u^3 - \tfrac{1}{2}abu^2. \end{aligned}$$

We find the critical points of this system. Making $\dot{\mathbf{u}} = 0$ we obtain that $v = 0$. Substituting this expression in $3auv - \frac{1}{2}bv - a^2u^3 - \frac{1}{2}abu^2 = 0$ we obtain that $u = -\frac{b}{2a}$ or $u = 0$. Therefore, the only critical points are $(0, 0)$ and $\left(-\frac{b}{2a}, 0\right)$. The critical point of interest in this case is $(0, 0)$. The Jacobian of this new system is

$$Df(u, v) = \begin{bmatrix} 0 & 1 \\ 3av - 3a^2u^2 - 2abu & 3au - \frac{1}{2}b \end{bmatrix}.$$

Evaluating the critical point $(0,0)$ we get

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2}b \end{bmatrix}.$$

The eigenvalues of $Df(0,0)$ are $\lambda_1 = 0$ and $\lambda_2 = -\dfrac{1}{2}b$. Now, by applying the theorem 2.2 we obtain that: $X(u,v) = v$.
Let v be the solution of $3auv - \frac{1}{2}bv - a^2u^3 - \frac{1}{2}abu^2 = 0$. Therefore, clearing we obtain that:

$$v = \frac{a^2u^3 - \frac{1}{2}abu^2}{3au - \frac{1}{2}b}.$$

$$g(u) = U(u, f(u)) = \frac{a^2u^3 - \frac{1}{2}abu^2}{3au - \frac{1}{2}b}$$

Doing the Taylor's series development, we obtain that $g(u) = au^2 + ...$ Since $m = 2$ is even then we apply the item 3 of the theorem 2.2 and we obtain that $(0,0)$ is a saddle-node point. Consequently, $\left(-\dfrac{b}{2a}, 0\right)$ is a saddle-node point of the original system.

1) If $(a,b,c) \in B_5$, then $b^2+4c = 0$ and $ab \neq 0$. If we take $b^2+4c = 0$ and $a \neq 0$ then the eigenvalues of the Jacobian evaluated at $(0,0)$ are $\lambda_1 = \lambda_2 = \frac{b}{2}$. Then, if $b < 0$ then $(0,0)$ is an attractor node. Also, $\left(-\dfrac{b}{2a}, 0\right)$ is saddle point for the previous item.
2) If $(a,b,c) \in B_6$, then $ab \neq 0$ and $c = 0$. Taking $a \neq 0, b > 0$ and $c = 0$ we have that the critical points of the system are $(0,0)$ and $\left(-\dfrac{b}{a}, 0\right)$. The eigenvalues of $Df\left(-\dfrac{b}{a}, 0\right)$ are

$$\lambda_1 = \lambda_2 = -b.$$

Therefore, si $b > 0$, then $\left(-\dfrac{b}{a}, 0\right)$ is an attractor node.
For the critical point $(0,0)$ the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = b$, this is, we must use a criterion for the analysis of semi-hyperbolic critical points [12].
On the system (4.1) with $c = 0$, let's perform the change of variables $u = bx - y$, which gives rise to the topologically equivalent system

$$\begin{aligned} \dot{\mathbf{u}} &= -\frac{3a}{b}(u+y)y + \frac{a}{b}(u+y)^2 + \frac{a^2}{b^3}(u+y)^3 \\ \dot{\mathbf{y}} &= by + \frac{3a}{b}(u+y)y + \frac{a}{b}(u+y)^2 + \frac{a^2}{b^3}(u+y)^3 \end{aligned}$$

Now identifying the elements of the theorem,we have $\lambda = b$,
$A(u,y) = -\frac{3a}{b}(u+y)y + \frac{a}{b}(u+y)^2 + \frac{a^2}{b^3}(u+y)^3$
$B(u,y) = \frac{3a}{b}(u+y)y + \frac{a}{b}(u+y)^2 + \frac{a^2}{b^3}(u+y)^3 (deg(B(u,y)) \geq 2)$.

If we calculate the solution of the equation $\lambda y + B(u,y) = 0$ in Taylor's series expansion we have $y = f(u) = -\frac{a}{b^2}u^2 + O(u^2)$, $g(u) = Div(A,B)|_{(u,f(u))} = \frac{a}{b}u^2 + O(u^2)$ then $m = 2$ and $a_m = a/b$.
So, under the above conditions, it follows that $(0,0)$ is a critical saddle-node point.

3) If $(a, b, c) \in B_6$, then $ab \neq 0$ and $c = 0$. Taking $a \neq 0, b < 0$ and $c = 0$, we have that the critical points of the system are $(0, 0)$ and $\left(-\frac{b}{a}, 0\right)$. Using a procedure analogous to the previous item, we obtain that if $b < 0$ then $\left(-\frac{b}{a}, 0\right)$ is a repulsor node and $(0, 0)$ is a saddle point.

□

5.3. *Stability in B_7*

Proposition 5.3. *Given the system* (4.1) *then*

1) *If* $(a, b, c) \in B_7$, $b^2 + 4c > 0$, $a \neq 0$, $b > 0$ *and* $c > 0$ *then* $(0, 0)$ *is a saddle point,* $\left(\frac{-b + \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is a repulsor node and* $\left(\frac{-b - \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is an attractor node.*
2) *If* $(a, b, c) \in B_7$, $b^2 + 4c > 0$, $a \neq 0$, $b > 0$ *and* $c < 0$, *then* $(0, 0)$ *is a repulsor node,* $\left(\frac{-b + \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is a saddle point and* $\left(\frac{-b - \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is an attractor node.*
3) *If* $(a, b, c) \in B_7$, $b^2 + 4c > 0$, $a \neq 0$, $b < 0$ *and* $c > 0$, *then* $(0, 0)$ *is a saddle point,* $\left(\frac{-b + \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is a repulsor node and* $\left(\frac{-b - \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is an attractor node.*
4) *If* $(a, b, c) \in B_7$, $b^2 + 4c > 0$, $a \neq 0$, $b < 0$ *and* $c < 0$, *then* $(0, 0)$ *is an attractor node,* $\left(\frac{-b + \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is a repulsor node and* $\left(\frac{-b - \sqrt{b^2 + 4c}}{2a}, 0\right)$ *is a saddle point.*
5) *If* $(a, b, c) \in B_7$, $a \neq 0$, $c > 0$ *and* $b = 0$ *then* $(0, 0)$ *is a saddle point,* $\left(\frac{\sqrt{c}}{a}, 0\right)$ *is a repulsor node and* $\left(-\frac{\sqrt{c}}{a}, 0\right)$ *is an attractor node.*

Proof. For the system (4.1) the resulting Jacobian matrix evaluated at the critical point $P_0 = (0, 0)$ corresponds to:

$$Df(P_0) = \begin{bmatrix} 0 & 1 \\ c & b \end{bmatrix}.$$

The eigenvalues of the matrix $Df(P_0)$ are

$$\lambda_1 = \frac{b + \sqrt{b^2 + 4c}}{2} \quad \& \quad \lambda_2 = \frac{b - \sqrt{b^2 + 4c}}{2}.$$

Let $P_1 = \left(\frac{-b + \sqrt{b^2 + 4c}}{2a}, 0\right)$, then the eigenvalues of the matrix $Df(P_1)$ are:

$$\lambda_1 = \sqrt{b^2 + 4c} \quad \& \quad \lambda_2 = \frac{-b + \sqrt{b^2 + 4c}}{2}.$$

Also, if we take $P_2 = \left(\frac{-b - \sqrt{b^2 + 4c}}{2a}, 0\right)$, the eigenvalues of the matrix $Df(P_2)$ would be:

$$\lambda_1 = \frac{-b - \sqrt{b^2 + 4c}}{2} \quad \& \quad \lambda_2 = -\sqrt{b^2 + 4c}.$$

Now:

1) If $(a, b, c) \in B_7$, then $b^2 + 4c > 0$, $a \neq 0$ and $c \neq 0$. Considering $b^2 + 4c > 0$, $a \neq 0$, $c > 0$ and $b > 0$. For the critical point P_0 we have that $\lambda_1 \lambda_2 = -c$. Since $c > 0$, then $\lambda_1 \lambda_2 < 0$, i.e., one eigenvalue is positive and the other negative, therefore, P_0 is a saddle point. For the critical point P_1, we have that $\lambda_1 > 0$ and $\lambda_2 > 0$, so P_1 is a repulsor node. For the critical point P_2, $\lambda_1 < 0$ and $\lambda_2 < 0$, therefore, P_2 is an attractor node.
2) If $(a, b, c) \in B_7$, then $b^2 + 4c > 0, a \neq 0$ and $c \neq 0$. Let $a \neq 0$, $b > 0$ and $c < 0$, then $\lambda_1 \lambda_2 = -c > 0$ and $\lambda_1 > 0$, therefore, P_0 is a repulsor node. For the critical point P_1, $\lambda_2 < 0$ and $\lambda_1 > 0$, therefore, P_1 is a saddle point. For the critical point P_2 we have that $\lambda_1 < 0$ and $\lambda_2 < 0$, then P_2 is an attractor node.
3) If $(a, b, c) \in B_7$, then $b^2 + 4c > 0, a \neq 0$ and $c \neq 0$. Let's take $a \neq 0$, $c > 0$ and $b < 0$, then for the critical point P_0 we have that, $\lambda_1 \lambda_2 = -c < 0$, therefore, P_0 is a saddle point. For the critical point P_1, since $\lambda_1 > 0$ and $\lambda_2 > 0$, then P_1 is a repulsor node. For the critical point P_2, it happens that $\lambda_2 < 0$ and $\lambda_1 < 0$ then P_2 is an attractor node.
4) If $(a, b, c) \in B_7$, then $b^2 + 4c > 0, a \neq 0$ and $c \neq 0$. When $a \neq 0$, $c < 0$ and $b < 0$, for the critical point P_0, $\lambda_1 \lambda_2 > 0$ and $\lambda_1 < 0$, then P_0 is an attractor node. For the critical point P_1, we have that $\lambda_1 > 0$ and $\lambda_2 > 0$, then, P_1 is a repulsor node. For the critical point P_2, we have that $\lambda_2 < 0$ and Therefore, $\lambda_1 > 0$. Then, P_2 is a saddle point.
5) If $(a, b, c) \in B_7$, then $b^2 + 4c > 0, a \neq 0$ and $c \neq 0$. If we consider $a \neq 0$, $c > 0$ and $b = 0$, then the critical points of the system are reduced to $(0, 0)$, $\left(\sqrt{\frac{c}{a^2}}, 0\right)$ and $\left(-\sqrt{\frac{c}{a^2}}, 0\right)$.

   For the critical point $(0, 0)$,the eigenvalues of the matrix $Df(0, 0)$ are $\lambda_{1,2} = \pm\sqrt{c}$, i.e., $(0, 0)$ is a saddle point.

   For the critical point $\left(\sqrt{\frac{c}{a^2}}, 0\right)$, the eigenvalues of $Df\left(\sqrt{\frac{c}{a^2}}, 0\right)$ are

   $$\lambda_1 = 2\sqrt{c} \quad \& \quad \lambda_2 = \sqrt{c}.$$

   Then, the critical point $\left(\sqrt{\frac{c}{a^2}}, 0\right)$ is a repulsor node.

   For the critical point $\left(-\sqrt{\frac{c}{a^2}}, 0\right)$,the eigenvalues of the matrix $Df\left(-\sqrt{\frac{c}{a^2}}, 0\right)$ are

   $$\lambda_1 = -\sqrt{c} \quad \& \quad \lambda_2 = 2\sqrt{c}$$

   Then the point is an attractor node.

□

In the following section we study the bifurcations of the system (4.1).

6. Bifurcation analysis

Let us consider in B_7 the following subsets:

$$W_1 = \{(a,b,c) \mid \; b^2 + 4c > 0, \; a \neq 0, b > 0, c > 0\}$$

$$W_2 = \{(a,b,c) \mid \; b^2 + 4c > 0, \; a \neq 0, b < 0, c < 0\}.$$

Proposition 6.1. *The system undergoes a transcritical bifurcation for each element in the region B_4.*

Proof. Sean $P_0 = (0,0)$ and $P_2 = \left(\frac{-b - \sqrt{b^2+4c}}{2a}, 0\right)$ then:
a) If $(a,b,c) \in W_2$, for the Proposition 5.3, P_0 is an attractor node and P_2 is a saddle point (Figure 4).
b) If $(a,b,c) \in B_4$, by Proposition 5.1, P_0 and P_2 collide at a non-hyperbolic critical point consisting of the junction of a hyperbolic sector and an elliptic sector (Figure 5).
c) When $(a,b,c) \in W_1$, for the Proposition 5.3, P_0 and P_2 exchange their stability, i.e., P_0 is a saddle point and P_2 is an attractor node (Figure 6). Therefore, the system undergoes a transcritical bifurcation for all elements of B_4.

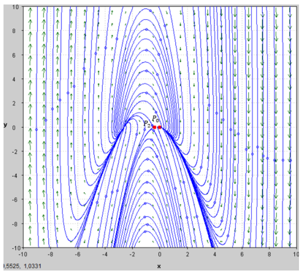


Figure 4. P_0 is an attractor node and P_2 is a saddle point in W_2.

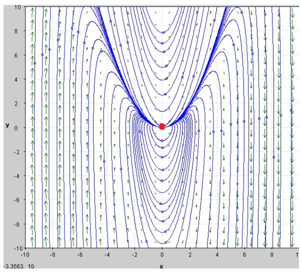


Figure 5. P_0 and P_2 collide at a non-hyperbolic critical point en B_4.

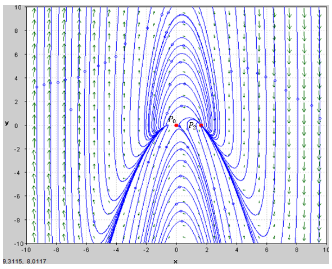


Figure 6. P_0 is a saddle point and P_2 is an attractor node in W_1

□

7. Singularities in infinity

System of differential equations at infinity on the chart U_1 of the system (4.1) is given by

$$\begin{aligned} \dot{\mathbf{u}} &= -u^2 v^2 + 3auv + buv^2 - a^2 - abv + cv^2 \\ \dot{\mathbf{v}} &= -uv^3. \end{aligned} \tag{7.1}$$

Similarly, we obtain that:

1) If $a \neq 0$ then the system has no critical points.
2) If $a = 0$ and $b^2 + 4c < 0$, then the system has no critical points.
3) If $a = 0$, $c = 0$ and $b \neq 0$ then $(b, 0)$ is an attractor node (repulsor) if $b > 0$ $(b < 0)$ and $(0, 0)$ is a topological chair or node.
4) If $a = 0$, $b^2 + 4c = 0$ and $b \neq 0$, then $\left(\frac{b}{2}, 0\right)$ is a saddle-node point.
5) If $a = 0$, $b = 0$ and $c = 0$, then $(0, 0)$ is a repulsor node for $u < 0$ and attractor for $u > 0$.
6) If $a = 0$, $b = 0$ and $c > 0$, then $(\sqrt{c}, 0)$ is an atract node and $(-\sqrt{c}, 0)$ is a repulsor node.
7) If $b^2 + 4c > 0$, $a = 0$, $b > 0$ and $c > 0$, then $\left(\frac{b+\sqrt{b^2+4c}}{2}, 0\right)$ is an attractor node and $\left(\frac{b-\sqrt{b^2+4c}}{2}, 0\right)$ is a repulsor node.
8) If $b^2 + 4c > 0$, $a = 0$, $b < 0$ and $c > 0$, then $\left(\frac{b+\sqrt{b^2+4c}}{2}, 0\right)$ is an attractor node and $\left(\frac{b-\sqrt{b^2+4c}}{2}, 0\right)$ is a repulsor node.
9) If $b^2 + 4c > 0$, $a = 0$, $b < 0$ and $c < 0$, then $\left(\frac{b+\sqrt{b^2+4c}}{2}, 0\right)$ is a saddle point and $\left(\frac{b-\sqrt{b^2+4c}}{2}, 0\right)$ is a repulsor node.
10) If $b^2 + 4c > 0$, $a = 0$, $b > 0$ and $c < 0$, then $\left(\frac{b+\sqrt{b^2+4c}}{2}, 0\right)$ is an attractor node and $\left(\frac{b-\sqrt{b^2+4c}}{2}, 0\right)$ is a saddle point.

System of differential equations at infinity on the chart U_2 of the system (4.1) is given by

$$\begin{aligned} \dot{\mathbf{u}} &= v^2 - 3au^2v - buv^2 + a^2u^4 + abu^3v - cu^2v^2 \\ \dot{\mathbf{v}} &= -3auv^2 - bv^3 + a^2u^3v + abu^2v^2 - cuv^3. \end{aligned} \tag{7.2}$$

Similarly, we obtain that:

1) If $a \neq 0$, then $(0, 0)$ is a critical point with elliptical domain.
2) If $a = 0$ and $b^2 + 4c < 0$, then the system has no critical points.
3) If $a = 0$, $c = 0$ and $b \neq 0$ then $\left(\frac{1}{b}, 0\right)$ is an attractor node (repulsor) if $b > 0$ $(b < 0)$.
4) If $a = 0$, $b^2 + 4c = 0$ and $b \neq 0$, then $\left(-\frac{b}{2c}, 0\right)$ is a saddle-node point.
5) If $a = 0$, $b = 0$ and $c = 0$, then the system has no critical points.

6) If $a = 0, b = 0$ and $c > 0$, then $\left(\frac{1}{\sqrt{c}}, 0\right)$ is an attractor node and $\left(-\frac{1}{\sqrt{c}}, 0\right)$ is a repulsor node.

7) If $b^2 + 4c > 0$, $a = 0$, $b > 0$ and $c > 0$, then $\left(\frac{b + \sqrt{b^2 + 4c}}{2c}, 0\right)$ is an attractor node and $\left(\frac{b - \sqrt{b^2 + 4c}}{2c}, 0\right)$ is a repulsor node.

8) If $b^2 + 4c > 0$, $a = 0$, $b < 0$ and $c > 0$, then $\left(\frac{-b + \sqrt{b^2 + 4c}}{2c}, 0\right)$ is an attractor node and $\left(\frac{-b - \sqrt{b^2 + 4c}}{2c}, 0\right)$ is a repulsor node.

9) If $b^2 + 4c > 0$, $a = 0$, $b < 0$ and $c < 0$, then $\left(\frac{-b + \sqrt{b^2 + 4c}}{2c}, 0\right)$ is a saddle point and $\left(\frac{-b - \sqrt{b^2 + 4c}}{2c}, 0\right)$ is a repulsor node.

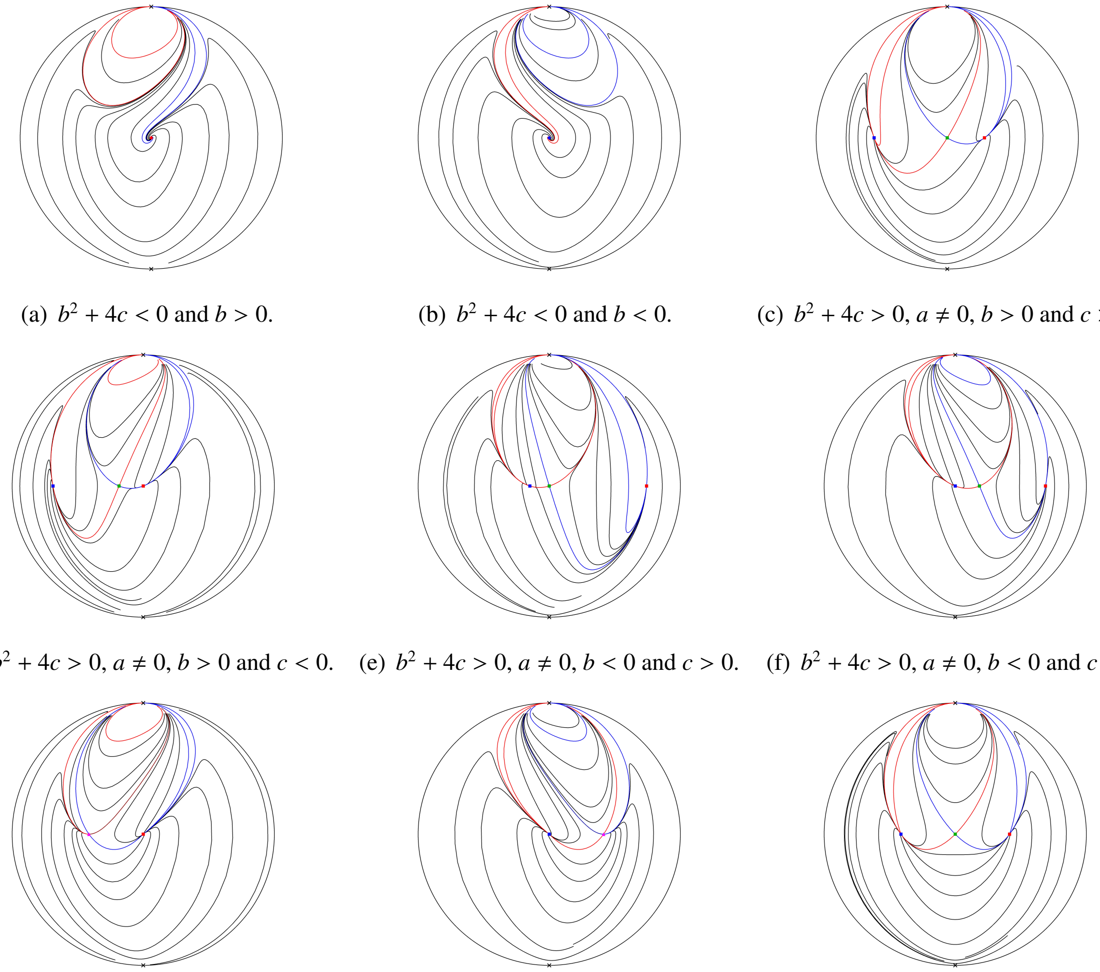
10) If $b^2 + 4c > 0$, $a = 0$, $b > 0$ and $c < 0$, then $\left(\frac{-b + \sqrt{b^2 + 4c}}{2c}, 0\right)$ is an attractor node and $\left(\frac{-b - \sqrt{b^2 + 4c}}{2c}, 0\right)$ is a saddle point.

8. Global phase portraits

HThis section shows the overall phase portrait of the system (4.1) on the Poincaré disk. The red dot represents a repulsor node, the blue one an attractor node, the green one a saddle point and the pink one a saddle-node point (in Figure 7).

9. Conclusions

In this article, we have performed a qualitative analysis of a multiparametric nonlinear system, which arises from the foliation of exercise 10 in [[13], 1.3.3] for the case $n = 1$ and $k = 1$. The system studied in this work presents critical points in the parametric regions and a transcritical bifurcation for a region of points $\mathbb{R}^3$. In addition, the system analyzed in this article is potentially applicable to perform various studies such as in [14]. Also, according to [13] the analyzed system allows studies associated with the theory of heat and mass transfer, nonlinear mechanics, elasticity, hydrodynamics, theory of nonlinear oscillations, theory of combustion, chemical engineering science, etc. As an added value of this research, the scientific and academic community is provided with a methodology to study bifurcations of multiparametric nonlinear systems, which begins with the definition of the parametric regions that determine the stability of the system and ends with the obtaining of the region of points where the bifurcation occurs.



(a) $b^2 + 4c < 0$ and $b > 0$. (b) $b^2 + 4c < 0$ and $b < 0$. (c) $b^2 + 4c > 0$, $a \neq 0$, $b > 0$ and $c > 0$.

(d) $b^2 + 4c > 0$, $a \neq 0$, $b > 0$ and $c < 0$. (e) $b^2 + 4c > 0$, $a \neq 0$, $b < 0$ and $c > 0$. (f) $b^2 + 4c > 0$, $a \neq 0$, $b < 0$ and $c < 0$.

(g) $b^2 + 4c = 0$, $a \neq 0$ and $b > 0$. (h) $b^2 + 4c = 0$, $a \neq 0$ and $b < 0$. (i) $a \neq 0$, $c > 0$ and $b = 0$.

Figure 7. Global phase portraits for system (4.1).

Conflict of interest

The authors declare that they have no conflict of interest in this work.

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