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Research article

Min-max differential game with partial differential equation

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Abstract: In this paper, we are concerned with a min-max differential game with Cauchy initial value problem (CIVP) as the state trajectory for the differential game, we studied the analytical solution and the approximate solution by using Picard method (PM) of this problem. We obtained the equivalent integral equation to the CIVP. Also, we suggested a method for solving this problem. The existence, uniqueness of the solution and the uniform convergence are discussed for the two methods.

Keywords: Cauchy initial value problem; min-max differential game; uniform convergence; existence; uniqueness; Picard method
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1. Introduction

Most physical phenomena use partial differential equations (PDEs) for example, fluid dynamics, electricity, magnetism, mechanics, optics, or heat flow. It's true that simplifications can be made that reduce the equations in question to ordinary differential equations, but, nevertheless, the complete description of these systems resides in the general area of PDEs [1,2]. In [3] Hemeda, A. A. introduced an integral iterative method (IIM) as a modification for PM to solve nonlinear integro-differential and systems of nonlinear integro-differential equations. Hemeda, A. A. and Eladdad, E. E. used IIM with the new iterative method (NIM) in [4] for solving linear and nonlinear Fokker-Planck equations. Hemeda, A. A. used local integral iterative method (LIIM) and local new iterative method (LNIM) in [5] for solving local fractional differential equations. A Cauchy problem in mathematics asks for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain [6].

In [7] Ivan Yegorov and Peter M. Dower presented the method of characteristics for solving some problem classes under convexity/concavity conditions on Hamilton–Jacobi–Bellman equations in optimal control problems. In [8] Chow, Y. T. , et al. developed a parallel method for solving possibly non-convex time- dependent Hamilton–Jacobi equations arising from optimal control and differential game problems. They proposed an algorithm to overcome the curse of dimensionality when solving HJ PDE. Chow, Y. T. , et al. developed a method for solving a large class of non-convex Hamilton-Jacobi partial differential equations (HJ PDE) see [9]. Mitchell, I. M., et al. described and implemented an algorithm for computing the set of reachable states of a continuous dynamic game in [10]. The differential game is a direct application of a differential equation and game theory. Game theory is an important field in mathematics see [11,12]. In [13] Megahed, A. A., et al. discussed the min-max zero-sum two persons continuous differential games with fuzzy controls and fuzzy state trajectories, they derived the necessary and sufficient conditions for getting the optimal strategies. In [14] Megahed, A. A., et al. studied a differential game problem between governments and terrorist organizations for counterterrorism.

In this paper, we assume that two players (player 1 and player 2) play a min-max game in the time interval $t \in [0, T]$. Suppose that player 1's plan against player 2 is $u(x, t)$ which satisfies the Cauchy initial value problem (CIVP):

$$u_{tt} + p(x)u_t = q(x, t), t \in [0, T], x \in R \tag{1.1}$$

with respect to the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \tag{1.2}$$

In order to increase the profit, the player 1's plan $u(x, t)$ at (x_i, t_i) within $R \times [0, T]$, and the resulting profit can be written as

$$J = \sum_{i=1}^{n} u(x_i, t_i).$$

The problem arises to determine the best (x_i, t_i) in order to maximize J for player 1, the most convenient defense $q(x, t)$ for player 2 in order to minimize J for player 1. Here, we have a non-homogenuous linear partial differential equation that classified as a parabolic partial differential equation. In this work, we obtain the integral equation

$$u(x, t) = f(x) + tg(x) + tP(x)f(x) + I^2[q(x, t)] - I[P(x)u(x, t)], \quad x \in R, \ t \in [0, T] \tag{1.3}$$

that equivalent to the CIVP (1.1) and (1.2), then we will prove the existence and the uniqueness of continuous solution for (1.3) by using the principle of contraction mapping. Also, we will apply on the integral equation (1.3) an analytical method; the classical method of successive approximations (PM)[15] which consists of the construction of a sequence of functions such that the limit of this sequence of functions in the sense of uniform convergence is the solution of the integral equation. In this paper, the rest of it is organized as follows the existence and the uniqueness for the solution are discussed in section 2. The purpose of section 3 is to solve the CIVP using PM. In section 4, we suggest a method for solving CIVP. Section 5 is reserved for application.Finally, the conclusion contains comparison between the two methods.

2. Existence and uniqueness

In [16] El-Sayed, A. M. A., et al. concerned with the Cauchy problem of a delay stochastic differential equation of arbitrary (fractional) orders. They proved the existence (local) of a unique mean square continuous solution and studied the continuous dependence of the solution on the initial random variable.
Here, we study the existence of the solution for the integral equation (1.3), let now the Cauchy initial value problem (1.1) and (1.2). By integrating the equation (1.1) relative to t, we have

$$u_t(x,t) - u_t(x,0) + p(x)u(x,t) - p(x)u(x,0) = I[q(x,t)]$$

$$u_t(x,t) - g(x) + p(x)u(x,t) - p(x)f(x) = I[q(x,t)].$$

By integrating more time relative to t.

$$u(x,t) - u(x,0) - tg(x) + I[p(x)u(x,t)] - tp(x)f(x) = I^2[q(x,t)]$$

$$u(x,t) = f(x) + tg(x) + tp(x)f(x) + I^2[q(x,t)] - I[p(x)u(x,t)]$$

Here, we prove the equivalence between the (CIVP) (1.1) and (1.2) and the integral equation (1.3). Now, we get

$$u_t \;=\; g(x) \;+\; p(x)\,f(x) \;+\; I[q(x,t)] \;-\; p(x)\,u(x,t)$$

$$u_{tt} \;=\; q(x,t) \;-\; p(x)\,u_t$$
$$u_{tt} \;+\; p(x)\,u_t \;=\; q(x,t) \;-\; p(x)\,u_t \;+ p(x)\,u_t \;=\; q(x,t)$$

$$u(x,0) \;=\; f(x) + 0.g(x) + 0.p(x)\,f(x) \;+\; \int_0^0 \int_0^0 q(x,t)\,dt \;=\; f(x)$$
$$u_t(x,0) \;=\; g(x) \;+\; p(x)\,f(x) \;+\; \int_0^0 q(x,t)\,dt \;-\; p(x)\,u(x,0) \;=\; g(x)$$

Therefore the integral equation (1.3) is equivalent to the Cauchy initial value problem (1.1) and (1.2), so the solution of the Cauchy initial value problem exists.

Now, consider the integral equation (1.3) with the following assumptions:

1. $p, f, g : R \longrightarrow R$ are continuous functions and there exist M, M_1, M_2 such that $|p(x)| \leq M$, $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$.
2. $q : R \times I \longrightarrow R$ is continuous function and there exists M_3 such that $|q(x,t)| \leq M_3$ for all $x \in R$.

Let C be the space of all real valued functions which are continuous, define the operator T as the following

$$T(u(x,t)) = f(x) + tg(x) + tp(x)f(x) + I^2[q(x,t)] - I[p(x)u(x,t)]$$

For the uniqueness of the solution of the integral equation (1.3) we have the following theorem.

Theorem 1. Let the assumptions 1 and 2 be satisfied. If $MT < 1$, then the integral equation (1.3) has a unique solution $u \in C$.

Proof:
Now, we define $S = \{u \in C : |u(x,t)| \leq r; r = \frac{M_1 + TM_2 + TMM_1 + \frac{T^2}{2}M_3}{1-TM}\}$.
Then the operator T maps S into S, since for $u \in S$

$$|u(x,t)| = |f(x) + tg(x) + tp(x)f(x) + I^2[q(x,t)] - I[p(x)u(x,t)]|$$

$$|u(x,t)| \leq |f(x)| + |tg(x)| + |tp(x)f(x)| + I^2|q(x,t)| + I|p(x)||u(x,t)|$$

$$|u(x,t)| \leq |f(x)| + T|g(x)| + T|p(x)||f(x)| + \frac{T^2}{2}|q(x,t)| + T|p(x)||u(x,t)|$$

$$|u(x,t)| \leq M_1 + TM_2 + TMM_1 + \frac{T^2}{2}M_3 + TM|u(x,t)|$$

$$(1 - MT)|u| \leq M_1 + TM_2 + TMM_1 + \frac{T^2}{2}M_3$$

$$|u| \leq \frac{M_1 + TM_2 + TMM_1 + \frac{T^2}{2}M_3}{(1-MT)} = r.$$

Moreover, it is easy to see that S is a closed subset of C.

To show that T is a contraction mapping, let u, v be two solutions in S, then

$$Tu - Tv = I[p(x)v(x,t)] - I[p(x)u(x,t)]$$

$$|Tu - Tv| \leq I[|p(x)||u(x,t) - v(x,t)|]$$

$$|Tu - Tv| \leq MI[|u(x,t) - v(x,t)|]$$

$$||Tu - Tv|| \leq MT||u - v||.$$

Since $MT < 1$, then T is a contraction mapping and T has a unique fixed point in S.
Thus, there exists a unique solution for (1.3).

3. Picard method

In [17] Hashem, H. H. G. treated with PM for coupled systems of Chandrasekhar quadratic integral equations, studied the existence, the uniqueness and proved the uniform convergence for the solution.

Applying PM for the CIVP (1.1) and (1.2), the solution is constructed by the sequence

$$u_n(x,t) = u_0 + I_t^2[q(x,t) - P(x)(u_{n-1}(x,t))_t], n = 1, 2, 3, ... \tag{3.1}$$

$$u_0(x,t) = f(x) + tg(x) \tag{3.2}$$

All functions $u_n(x,t)$ are continuous functions and can be written as a sum of successive differences.

$$u_n(x,t) = u_0(x,t) + \sum_{j=1}^{n}(u_j - u_{j-1}).$$

This means that convergence of the sequence $\{u_n\}$ is equivalent to the convergence of the infinite series $\sum_{j=1}^{\infty}(u_j - u_{j-1})$ and the solution will be $u(x,t) = \lim_{n\to\infty} u_n(x,t)$.

If the infinite series $\sum_{j=1}^{\infty}(u_j - u_{j-1})$ converges, then the sequence $\{u_n\}$ will converge to $u(x,t)$.

To prove the uniform convergence of $u_n(x,t)$, we shall consider the associated series

$$\sum_{n=1}^{\infty}(u_n - u_{n-1})$$

For $n = 1$, we get from (3.1) and (3.2)

$$\begin{aligned} u_1(x,t) - u_0(x,t) &= f(x) + tg(x) + I_t^2[q(x,t) - P(x)(u_0(x,t))_t] - f(x) + tg(x) \\ |u_1(x,t) - u_0(x,t)| &= |I_t^2[q(x,t) - P(x)(u_0(x,t))_t]| \end{aligned}$$

$$|u_1(x,t) - u_0(x,t)| \leq \frac{T^2}{2}M^* \tag{3.3}$$

such that $M^* = M_3 - MM_2$

Now, we shall obtain an estimation for $u_n - u_{n-1}, \quad n \geq 2$

$$|u_n(x,t) - u_{n-1}(x,t)| = |I_t^2[q(x,t) - P(x)(u_{n-1}(x,t))_t] - I_t^2[q(x,t) - P(x)(u_{n-2}(x,t))_t]|$$

$$|u_n(x,t) - u_{n-1}(x,t)| = |I_t^2[P(x)(u_{n-2}(x,t))_t - P(x)(u_{n-1}(x,t))_t]|$$

$$|u_n(x,t) - u_{n-1}(x,t)| \leq MT|u_{n-1}(x,t) - u_{n-2}(x,t)|$$

Putting $n = 2$, then using (3.3)

$$u_2(x,t) - u_1(x,t) = I_t^2[q(x,t) - P(x)(u_1(x,t))_t] - I_t^2[q(x,t) - P(x)(u_0(x,t))_t]$$

$$|u_2(x,t)-u_1(x,t)| \leq M|I_t^2[(u_1(x,t))_t-(u_0(x,t))_t]|$$
$$|u_2(x,t)-u_1(x,t)| \leq MT|u_1(x,t)-u_0(x,t)|$$
$$|u_2(x,t)-u_1(x,t)| \leq MT\frac{M^*T^2}{2}$$
$$u_3(x,t)-u_2(x,t)=I_t^2[q(x,t)-P(x)(u_2(x,t))_t]-I_t^2[q(x,t)-P(x)(u_1(x,t))_t]$$
$$|u_3(x,t)-u_2(x,t)| \leq M|I_t^2[(u_2(x,t))_t-(u_1(x,t))_t]|$$
$$|u_3(x,t)-u_2(x,t)| \leq MT|u_2(x,t)-u_1(x,t)|$$
$$|u_3(x,t)-u_2(x,t)| \leq M^2T^2\frac{M^*T^2}{2}$$

Repeating this technique, we obtain the general estimate for the terms of the series:

$$|u_n(x,t)-u_{n-1}(x,t)| \leq M^{n-1}T^{n-1}\frac{M^*T^2}{2}.$$

Since $MT<1$, then the uniform convergence of $\sum_{n=1}^{\infty}(u_n-u_{n-1})$ is proved and so the sequence $u_n(x,t)$ is uniformly convergent.

$$u(x,t)=\lim_{n\to\infty} f(x)+tg(x)+I_t^2[q(x,t)-p(x)(u_{n-1}(x,t))_t]$$
$$= f(x)+tg(x)+I_t^2[q(x,t)-p(x)(u(x,t))_t].$$

Thus, the existence of a solution is proved.
To prove the uniqueness, let $v(x,t)$ be a continuous solution of (1.3). Then

$$v(x,t)=f(x)+tg(x)+I_t^2[q(x,t)-P(x)(v(x,t))_t]$$
$$v(x,t)-u_n(x,t)=f(x)+tg(x)+I_t^2[q(x,t)-p(x)(v(x,t))_t]-f(x)+tg(x)$$
$$+I_t^2[q(x,t)-P(x)(u_{n-1}(x,t))_t]$$
$$|v(x,t)-u_n(x,t)| \leq M|I_t^2[(u_{n-1}(x,t))_t-(v(x,t))_t]|$$
$$|v(x,t)-u_n(x,t)| \leq MT|v(x,t)-u_{n-1}(x,t)|$$

We shall obtain $v(x,t)-u_{n-1}(x,t), \quad n=1,\ 2,\ 3,....$

$$v(x,t)-u_0(x,t)=f(x)+tg(x)+I_t^2[q(x,t)-p(x)(v(x,t))_t]-f(x)+tg(x)$$
$$|v(x,t)-u_0(x,t)|=|I_t^2[q(x,t)-p(x)(v(x,t))_t]|$$
$$|v(x,t)-u_0(x,t)| \leq \frac{t^2}{2}M_3+M|I_t^2(v(x,t))_t|$$
$$|v(x,t)-u_0(x,t)| \leq \frac{T^2}{2}M_3+MT|v(x,t)-f(x)-tg(x)+tg(x)|$$
$$|v(x,t)-u_0(x,t)| \leq \frac{T^2}{2}M_3+MT|v(x,t)-u_0(x,t)|+MT|tg(x)|$$

$$(1 - MT)|v(x,t) - u_0(x,t)| \leq \frac{T^2}{2} M_3 + MT^2 M_2$$

$$|v(x,t) - u_0(x,t)| \leq \frac{\frac{T^2}{2} M_3 + MT^2 M_2}{(1 - MT)}$$

$$|v(x,t) - u_1(x,t)| = |I_t^2[p(x)((u_0(x,t))_t - (v(x,t))_t)]|$$

$$|v(x,t) - u_1(x,t)| \leq MT|v(x,t) - u_0(x,t)|$$

$$|v(x,t) - u_1(x,t)| \leq MT \frac{\frac{T^2}{2} M_3 + MT^2 M_2}{(1 - MT)}$$

$$|v(x,t) - u_2(x,t)| = |I_t^2[p(x)((u_1(x,t))_t - (v(x,t))_t)]|$$

$$|v(x,t) - u_2(x,t)| \leq MT|v(x,t) - u_1(x,t)|$$

$$|v(x,t) - u_2(x,t)| \leq M^2T^2 \frac{\frac{T^2}{2} M_3 + MT^2 M_2}{(1 - MT)}$$

By repeating this technique, we obtain

$$|v(x,t) - u_n(x,t)| \leq M^n T^n \frac{\frac{T^2}{2} M_3 + MT^2 M_2}{(1 - MT)}.$$

Since $MT < 1$. Hence $\lim_{n\to\infty} u_n(x,t) = v(x,t) = u(x,t)$ which completes the proof.

4. Suggested method

Here, we obtain an iterative form of the integral equation (1.3), the solution is constructed by the sequence

$$u_n(x,t) = f(x) + tg(x) + tP(x)f(x) + I^2[q(x,t)] - I[P(x)u_{n-1}(x,t)], n = 1, 2, 3, ... \tag{4.1}$$

$$u_0(x,t) = f(x) + tg(x) \tag{4.2}$$

All functions $u_n(x,t)$ are continuous and can be written as the sum of successive differences.

$$u_n(x,t) = u_0(x,t) + \sum_{j=1}^{n} (u_j - u_{j-1}).$$

This means that the convergence of the sequence $\{u_n\}$ is equivalent to the convergence of the infinite series $\sum_{j=1}^{\infty} (u_j - u_{j-1})$ and the solution will be

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

.

If the infinite series $\sum_{j=1}^{\infty} (u_j - u_{j-1})$ converges, then the sequence $\{u_n\}$ will converge to $u(x,t)$. To prove the uniform convergence of $u_n(x,t)$, we shall consider the associated series

$$\sum_{n=1}^{\infty} (u_n - u_{n-1})$$

For $n = 1$, we get from (4.1) and (4.2)

$$u_1(x,t) - u_0(x,t) = tp(x)f(x) + I^2[q(x,t)] - I[p(x)(f(x) + tg(x))]$$
$$= tp(x)f(x) + I^2[q(x,t)] - tp(x)f(x) - \frac{t^2}{2}g(x)$$
$$|u_1(x,t) - u_0(x,t)| \leq \frac{T^2}{2}M_3 + \frac{T^2}{2}M_2$$
$$|u_1(x,t) - u_0(x,t)| \leq \frac{T^2}{2}M^* \tag{4.3}$$

such that $M^* = M_3 + M_2$

Now, we shall obtain an estimation for $u_n - u_{n-1}, \quad n \geq 2$

$$|u_n(x,t) - u_{n-1}(x,t)| = |I[p(x)u_{n-2}(x,t)] - I[p(x)u_{n-1}(x,t)]|$$
$$|u_n(x,t) - u_{n-1}(x,t)| \leq MT|u_{n-1}(x,t) - u_{n-2}(x,t)|$$

For $n = 2$ and using (4.3)

$$u_2(x,t) - u_1(x,t) = I[p(x)u_0(x,t)] - I[p(x)u_1(x,t)]$$
$$|u_2(x,t) - u_1(x,t)| \leq MI[|u_1(x,t) - u_0(x,t)|]$$
$$|u_2(x,t) - u_1(x,t)| \leq MT\frac{T^2}{2}M^*$$
$$|u_2(x,t) - u_1(x,t)| \leq MT\frac{M^*T^2}{2}$$
$$u_3(x,t) - u_2(x,t) = I[p(x)u_1(x,t)] - I[p(x)u_2(x,t)]$$
$$|u_3(x,t) - u_2(x,t)| \leq MT[|u_2(x,t) - u_1(x,t)|]$$
$$|u_3(x,t) - u_2(x,t)| \leq M^2T^2\frac{M^*T^2}{2}$$

By repeating this technique, we obtain the general estimation for the terms of the series:

$$|u_n(x,t) - u_{n-1}(x,t)| \leq M^{n-1}T^{n-1}\frac{M^*T^2}{2}.$$

Since $MT < 1$, then the uniform convergence of $\sum_{n=1}^{\infty}(u_n - u_{n-1})$ is proved and so the sequence $u_n(x,t)$ is uniformly convergent.

$$u(x,t) = \lim_{n\to\infty} f(x) + tg(x) + tP(x)f(x) + I^2[q(x,t)] - I[P(x)u_{n-1}(x,t)]$$

$$= f(x) + tg(x) + tP(x)f(x) + I^2[q(x,t)] - I[P(x)u(x,t)].$$

Thus, the existence of a solution is proved.

To prove the uniqueness, let $v(x,t)$ be a continuous solution of (4.1) and (4.2). Then

$$v(x,t) = f(x) + tg(x) + tP(x)f(x) + I^2[q(x,t)] - I[P(x)v(x,t)]$$
$$v(x,t) - u_n(x,t) = I[p(x)u_{n-1}(x,t)] - I[p(x)v(x,t)]$$
$$|v(x,t) - u_n(x,t)| \leq MT|v(x,t) - u_{n-1}(x,t)|$$

We shall obtain $v(x,t) - u_{n-1}(x,t), \quad n = 1,\ 2,\ 3, ...$

$$v(x,t) - u_0(x,t) = tp(x)f(x) + I^2[q(x,t)] - I[p(x)v(x,t)] + \frac{t^2}{2}p(x)g(x) - \frac{t^2}{2}p(x)g(x)$$
$$v(x,t) - u_0(x,t) = I[p(x)(f(x) + tg(x))] - I[p(x)v(x,t)] + I^2[q(x,t)] - \frac{t^2}{2}p(x)g(x)$$
$$|v(x,t) - u_0(x,t)| \leq MT|v(x,t) - u_0(x,t)| + \frac{t^2}{2}M_3 + MM_2\frac{t^2}{2}$$
$$|v(x,t) - u_0(x,t)| \leq \frac{\frac{T^2}{2}(M_3 + MM_2)}{1 - TM}$$
$$|v(x,t) - u_1(x,t)| \leq |I[p(x)u_0(x,t)] - I[p(x)v(x,t)]|$$
$$|v(x,t) - u_1(x,t)| \leq TM|v(x,t) - u_0(x,t)|$$
$$|v(x,t) - u_1(x,t)| \leq TM\frac{\frac{T^2}{2}(M_3 + MM_2)}{1 - TM}$$
$$|v(x,t) - u_2(x,t)| \leq |I[p(x)u_1(x,t)] - I[p(x)v(x,t)]|$$
$$|v(x,t) - u_2(x,t)| \leq TM|v(x,t) - u_1(x,t)|$$
$$|v(x,t) - u_2(x,t)| \leq T^2M^2\frac{\frac{T^2}{2}(M_3 + MM_2)}{1 - TM}$$

By repeating this technique, we obtain

$$|v(x,t) - u_n(x,t)| \leq T^nM^n\frac{\frac{T^2}{2}(M_3 + MM_2)}{1 - TM}.$$

Since $MT < 1$. Hence $\lim_{n\to\infty} u_n(x,t) = v(x,t) = u(x,t)$ which completes the proof.

5. Example

As an application, assume that a coach into a certain team wishes to keep a football player away his team in the time interval $t \in [0,1]$. we consider (1.1) and (1.2) with $p(x) = x,\ f(x) = sinx,\ g(x) = sinx$ and $q(x,t) = e^t(1 + x)sinx$.
Assume that coach's plan against the football player u(x,t) satisfies the Cauchy initial value problem (CIVP):

$$u_{tt} + xu_t = e^t(1 + x)sinx, t \in [0,1], x \in R \tag{5.1}$$

with respect to the initial conditions

$$u(x,0) = sinx, \quad u_t(x,0) = sinx \tag{5.2}$$

has a unique solution $u(x,t) = e^t sinx$.

By Picard formula (3.1) and (3.2), we have

$$u_n(x,t) = u_0 + I_t^2[e^t(1+x)sinx - x(u_{n-1}(x,t))_t], n = 1,2,3,...... \tag{5.3}$$

$$u_0(x,t) = sinx + tsinx \tag{5.4}$$

Putting $n = 1$, then

$$\begin{aligned} u_1(x,t) &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(u_0)_t] \\ &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(sinx + tsinx)_t] \\ &= e^t(1+x)sinx - xsinx - txsinx - \frac{t^2}{2}xsinx. \\ &= e^t sinx + xe^t sinx - xsinx(1 + t + \frac{t^2}{2}). \end{aligned}$$

$n = 2$, then

$$\begin{aligned} u_2(x,t) &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(u_1)_t] \\ &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(e^t sinx + xe^t sinx - xsinx(1 + t + \frac{t^2}{2}))_t] \\ &= e^t sinx - x^2 e^t sinx + x^2 sinx + x^2 tsinx + x^2\frac{t^2}{2!}sinx + x^2\frac{t^3}{3!}sinx \\ &= e^t sinx - x^2 e^t sinx + x^2 sinx[1 + \frac{t}{1!} + \frac{t^2}{2} + \frac{t^3}{3!}] \end{aligned}$$

$n = 3$, then

$$\begin{aligned} u_3(x,t) &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(u_2)_t] \\ &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(e^t sinx - x^2 e^t sinx + x^2 sinx[1 + \frac{t}{1!} + \frac{t^2}{2} + \frac{t^3}{3!}])_t] \\ &= e^t sinx + x^3 e^t sinx - x^3 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}]. \end{aligned}$$

$n = 4$, then

$$\begin{aligned} u_4(x,t) &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(u_3)_t] \\ &= sinx + tsinx + I_t^2[e^t(1+x)sinx - x(e^t sinx + x^3 e^t sinx - x^3 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}])_t] \\ &= e^t sinx - x^4 e^t sinx + x^4 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}]. \end{aligned}$$

Since the sequence $u_n(x,t)$ is uniformly convergent. then $u_n(x,t) = e^t sinx + (-1)^{n-1}x^n e^t sinx + (-1)^n x^n sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ... + \frac{t^{n+1}}{(n+1)!}]$,

Hence $u(x,t) = \lim_{n\to\infty} u_n(x,t) = e^t sinx$, which is the exact solution.

Now, we Apply our iterative method (4.1) and (4.2), we have

$$u_n(x,t) = u_0 + txsinx + I_t^2[e^t(1+x)sinx] - I_t[xu_{n-1}(x,t)], n = 1,2,3,...... \tag{5.5}$$

$$u_0(x,t) = sinx + tsinx \tag{5.6}$$

Putting $n = 1$, then

$$u_1(x,t) = sinx + tsinx + txsinx + I_t^2[e^t(1+x)sinx] - I_t[xu_0(x,t)]$$
$$= sinx + tsinx + I_t^2[e^t(1+x)sinx] - I_t[x(sinx + tsinx)]$$
$$= e^t sinx + xe^t sinx - xsinx(1 + t + \frac{t^2}{2}).$$

$n = 2$, then

$$u_2(x,t) = sinx + tsinx + txsinx + I_t^2[e^t(1+x)sinx] - I_t[xu_1(x,t)]$$
$$= sinx + tsinx + I_t^2[e^t(1+x)sinx] - I_t^2[e^t sinx + xe^t sinx - xsinx(1 + t + \frac{t^2}{2})]$$
$$= e^t sinx - x^2 e^t sinx + x^2 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!}]$$

$n = 3$, then

$$u_3(x,t) = sinx + tsinx + txsinx + I_t^2[e^t(1+x)sinx] - I_t[xu_2(x,t)]$$
$$= sinx + tsinx + I_t^2[e^t(1+x)sinx] - I_t^2[e^t sinx - x^2 e^t sinx + x^2 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!}]]$$
$$= e^t sinx + x^3 e^t sinx - x^3 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}].$$

Put $n = 4$, then

$$u_4(x,t) = sinx + tsinx + txsinx + I_t^2[e^t(1+x)sinx] - I_t[xu_3(x,t)]$$
$$= sinx + tsinx + I_t^2[e^t(1+x)sinx] - I_t^2[e^t sinx + x^3 e^t sinx - x^3 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}]]$$
$$= e^t sinx - x^4 e^t sinx + x^4 sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}].$$

Since the sequence $u_n(x,t)$ is uniformly convergent, then

$$u_n(x,t) = e^t sinx + (-1)^{n-1} x^n e^t sinx + (-1)^n x^n sinx[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ... + \frac{t^{n+1}}{(n+1)!}].$$

Hence $u(x,t) = \lim_{n\to\infty} u_n(x,t) = e^t sinx$, which is the exact solution.

In Figure 1, we presented a comparison between the approximated solutions at different times to the problem. We noticed that, when the time is increased, the football player's chances to participate in the match until reaching highest participation and it will be the maximum (the top of the waves). After increasing the football player's mistakes, the participation started to decrease until vanishing ($u \leq 0$).

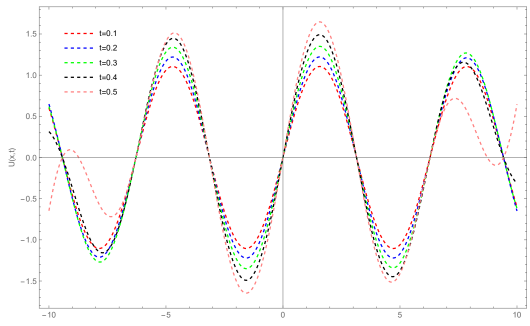

Figure 1. Comparison between the solutions at different times.

At $t = 0.1$, we found from the graph (red wave) $u = 1.10512$ is maximum at $x = 1.58$, $u = 1.10483$ at $x = 7.86$, at $t = 0.2$, we found from the graph (blue wave) $u = 1.22133$ is maximum at $x = 1.58$, $u = 1.21085$ at $x = 7.86$.

Now, we showed the participant chances for player 2 as the following

	$x = 1.58$	$x = 7.86$
$t = 0.1$	$u = 1.10512$	$u = 1.10483$
$t = 0.2$	$u = 1.22133$	$u = 1.21085$
$t = 0.3$	$u = 1.34947$	$u = 1.26727$

6. Conclusions

The main concern of this work is to construct a method for solving the problem (1.1) with initial conditions (1.2), relied on the analysis described the proposed numerical method is very efficient which save time and efforts, we obtained the solution quickly and it helped us in the theoretical and application studies for this problem and its like it.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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