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Research article

Bootstrapping m-generalized order statistics with variable rank

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Abstract: In this paper, several bootstrap properties of m-generalized order statistics (m-GOSs) with variable rank (central and intermediate) are revealed. We study the inconsistency, weak consistency and strong consistency of bootstrapping central and intermediate m-GOSs when the normalizing constants are assumed to be known or estimated from the re-sampled data using a proper re-sample size. Furthermore, sufficient conditions for the weak and strong consistencies of the bootstrapping distributions of central and intermediate m-GOSs based on the normalizing constant estimators are given. Finally, a simulation study is conducted to determine the optimal bootstrap re-sample size corresponding to the best fitting of the bootstrapping distribution.

Keywords: bootstrap method; consistency; central order statistics; intermediate order statistics; generalized order statistics
Mathematics Subject Classification: 62F20, 62F40, 62G30

1. Introduction

Kamps [22] introduced the model of GOSs as a unified approach to a variety of ordered random variables (RVs), including ordinary order statistics (OOSs), sequential order statistics (SOSs), progressive type II censored order statistics (POSs), order statistics with a non-integer sample size, record values, and Pfeifer's record model. Since the GOSs model unifies the models of ordered RVs, the practical importance of GOSs is evident. For example, in reliability theory, the rth extreme OOS indicates the life-length of an $(n-r+1)$-out-of-n system, whereas the model of SOSs is an extension of the OOSs model that describes specific dependencies or interactions among system components induced by component failures. Furthermore, the POSs model is a valuable tool for collecting information in lifetime tests.

The uniform GOSs are defined via their joint probability density function (PDF) on a unit cone of $\mathbb{R}^n$. More specifically, let $n \in \mathbb{N}$, $k \geq 1$ and $m_1, m_2, ..., m_{n-1} \in \mathbb{R}$ be parameters such that $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j > 0$, $r \in \{1, 2, ..., n-1\}$. If the RVs $U(r, n, \widetilde{m}, k) = U^{\star}_{r:n}$, $r = 1, 2, ..., n$, possess a

PDF of the form

$$f^{U^\star_{1:n},U^\star_{2:n},...,U^\star_{n:n}}(u_1,u_2,...,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}(1-u_j)^{m_j}\right)(1-u_n)^{\gamma_n-1} \tag{1.1}$$

on the cone $\{(u_1,u_2,...,u_n) : 0 \le u_1 \le u_2 \le ... \le u_n < 1\} \subset \mathbb{R}^n$, then they are called uniform GOSs. Furthermore, GOSs based on some distribution function (DF) F can be defined via the quantile transformation $X(r,n,\widetilde{m},k) = X^\star_{r:n} = F^{-1}(U^\star_{r:n}),\ r = 1,2,...,n$, where F^{-1} denotes the quantile function of F. On the other hand, by choosing the parameters appropriately, we can obtain different models of ordered RVs, such as m-GOSs ($m_1 = m_2 = ... = m_{n-1} = m, \gamma_r = k+(n-r)(m+1),\ r = 1,2,...,n$); OOSs, a sub-model of m-GOSs ($m = 0$ and $k = 1$); order statistics with non-integral sample size, a sub-model of m-GOSs ($m = 0, k = \alpha - n + 1$ and $n - 1 < \alpha \in \mathbb{R}$); SOSs ($m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1, i = 1,2,...,n-1, 0 < \alpha_i \in \mathbb{R}, k = \alpha_n$); kth record values ($m_1 = m_2 = ... = m_{n-1} = -1, k \in \mathbb{N}$); POSs with censoring scheme $(R_1,R_2,...,R_M)(m_i = R_i, i = 1,2,...,M-1$, and $m_i = 0, i = M, M+1,...,n-1$ and $k = R_M + 1$); and Pfeifer's record model ($m_i = \beta_i - \beta_{i+1} - 1, i = 1,2,...,n-1, 0 < \beta_i \in \mathbb{R}$ and $k = \beta_n$).

The marginal DF, $\Psi^{(m,k)}_{r:n}(x) = P(X^\star_{r:n} \le x)$, of the rth m-GOS is given in Kamps [22] by

$$\Psi^{(m,k)}_{r:n}(x) = 1 - C_{r-1}\overline{F}^{\gamma_r}(x)\sum_{i=0}^{r-1}\frac{1}{i!C_{r-i-1}}g^i_m(x),$$

where $C_{r-1} = \prod_{i=1}^{r}\gamma_i,\ r = 1,2,...,n, \gamma_n = k$, and $\overline{F}(x) = 1-F(x)$. Moreover, if $m \ne -1$, $(m+1)g_m(x) = G_m(x) = 1 - \overline{F}^{m+1}(x)$ is a DF, whereas $g_{-1}(x) = -\log\overline{F}(x)$. Under the condition $m \ne -1$, the possible limit DFs of the maximum m-GOS and their domains of attraction under linear normalization were derived by Nasri-Roudsari [27]. Moreover, the limit DFs of $\Psi^{(m,k)}_{n:n}(x)$ under power normalization were derived by Nasri-Roudsari [28]. The possible non-degenerate limit DFs and the rate of convergence of the upper extreme m-GOSs were discussed by Nasri-Roudsari and Cramer [29]. The necessary and sufficient conditions for weak convergence as well as the form of the possible limit DFs of extreme, central and intermediate m-GOSs, were derived by Barakat [4].

The bootstrap method, which was first introduced by Efron [20] for independent RVs, is an efficient procedure for solving many statistical problems based on re-sampling from the available data. It enables statisticians to perform statistical inference on a wide range of problems without imposing many structural assumptions on the data-generating random process (cf. [14, 21, 26]). For example, the bootstrap method is used to find standard errors for estimators, confidence intervals for unknown parameters, and p-values for test statistics under a null hypothesis.

There are several forms of the bootstrap method and additionally several other re-sampling methods that are related to it, such as jackknifing, cross-validation, randomization tests, and permutation tests. Let $\mathbf{X}_n = (X_1,X_2,...,X_n)$ be a random sample of size n from an unknown DF F. The idea of the bootstrap technique is to re-sample with replacement from the original sample $\mathbf{X}_n$ and form a bootstrapped version of the original statistic. For $B = B(n) \to \infty$, as $n \to \infty$, assume that $Y_1,Y_2,...,Y_B$ are conditionally independent and identically distributed (i.i.d) RVs with distribution

$$P(Y_1 = X_j|\mathbf{X}_n) = \frac{1}{n},\ j = 1,2,...,n.$$

Hence, $(Y_1,Y_2,...,Y_B)$ is a re-sample of size B from the empirical distribution F_n of F based on $\mathbf{X}_n$. Let $P_j,\ j = 1,2,...,n$, be an independent RV with respective beta distribution $I_x(\gamma_j,1), j = 1,2,...,n$. Thus, P_j follows a power function distribution with exponent $\gamma_j = k + (B-j)(m+1),\ j = 1,2,...,B$. Now, in view of the results of Cramer [17], we can write the rth m-GOS based on the empirical DF F_n in the form

$$Y^{\star}_{r:B} = Y(r, B, m, k) = F_n^{-1}\left(1 - \prod_{j=1}^{r} P_j\right), \ r = 1, 2, ..., B.$$

Moreover, let

$$H^{(m,k)}_{r,n,B}(x) = P\left(\frac{Y^{\star}_{B-r+1:B} - b_B}{a_B} \leq x|\mathbf{X}_n\right), \tag{1.2}$$

be the bootstrap distribution of $a_n^{-1}(X^{\star}_{n-r+1:n} - b_n)$, for suitable normalizing constants $a_n > 0$ and b_n, where n and B are the sample size and re-sample size, respectively.

It has been shown, for many statistics, that the bootstrap method is asymptotically consistent (cf. Efron [20]). That is, the asymptotic distribution of the bootstrap for a given statistic is the same as the asymptotic distribution of the original statistic. Many results for the bootstrap method and its applications can be found in the literature. For instance, the inconsistency, weak consistency and strong consistency for bootstrapping the maximum OOSs under linear normalization were investigated by Athreya and Fukuchi [2] and Fukuchi [21]. They showed that, in a full-sample bootstrap situation, the maximum OOSs fails to be consistent. Later, Barakat et al. [10] extended the results of Fukuchi and Athreya to the GOSs. Barakat et al. [12] obtained similar results for the OOSs with variable ranks as well. Furthermore, bootstrapping OOSs with variable rank under power normalization was investigated by Barakat et al. [13].

The main goal of this paper is to build on the findings of [10] by discussing the consistency of bootstrap central and intermediate GOSs for determining an appropriate re-sample size for known and unknown normalizing constants. Moreover, a simulation study is carried out to explain how the bootstrap sample size can be chosen numerically. This paper is structured as follows. In Section 2, we briefly review the main results concerning the asymptotic behaviour of the m-GOSs with variable rank. Sections 3 and 4 are devoted, respectively, to bootstrapping the intermediate and central m-GOSs. Finally, a simulation study is conducted in Section 5.

We end this section with some motivations that highlight the importance of our work.

Work motivation

The purpose of the bootstrap method is to construct an approximate sampling distribution for the statistic of interest. So, if the statistic of interest S_n follows a certain distribution, we would like our bootstrap distribution S_B to converge to the same distribution. If we do not have this, then we can not trust the inferences made. For i.i.d. samples of size n, the ordinary bootstrap method is known to be consistent in many situations, but it may fail in important examples (cf. [10,12,13,21]). Using bootstrap samples of size B, where $B \to \infty$ and $\frac{B}{n} \to 0$, typically resolves the problem (cf. [10, 12, 13, 21]). However, the choice of B is a key issue for the quality of the convergence (e.g., weak consistency and strong consistency). In this paper, we investigate the strong consistency of bootstrapping central and intermediate m-GOSs for an appropriate choice of re-sample size B for known and unknown normalizing constants. The critical choice problem of B is theoretically addressed in this paper. Furthermore, a simulation study is used to discuss it realistically.

The model of m-GOSs contains two practically important sub-models, OOSs and SOSs, on which this study focuses. For central OOSs and SOSs, one can use the bootstrap method to obtain a confidence interval for the pth population quantile. On the other hand, in many important applications such as flood hazard assessment [18], seismic hazard assessment [1] and analysis of bank operational risk [16], we need an estimator (confidence interval estimate) of an intermediate OOSs (SOSs) quantile. Moreover, it is well known that the asymptotic behavior of intermediate quantiles is one of the main factors in choosing a suitable value of threshold in the peak over threshold (POT) approach and constructing related estimators (the Hill estimators) of the tail index (cf. [9, 19]). Therefore, the study of bootstrapping intermediate OOSs will pave the way to use and improve the modeling of extreme values via the POT approach. This potential application of bootstrapping intermediate OOSs will be the subject of future studies.

2. Auxiliary and preliminary results

In this section, we briefly review the main results concerning the asymptotic behaviour of the intermediate and central m-GOSs, which are related to the present work.

2.1. Intermediate OOSs

The intermediate OOSs have a wide range of important applications. For instance, they can be used to estimate the probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extremes relative to the available sample size (cf. [30]). Furthermore, Pickands [30] has revealed that intermediate OOSs can be applied to construct consistent estimators for the shape parameter of the limiting extremal distribution in the parametric form. Teugels [32] and Mason [25] have also found estimators that are in part based on intermediate OOSs. A sequence $\{X_{r_n:n}\}$ is called a sequence of intermediate OOSs if $r_n \underset{n}{\longrightarrow} \infty$ and $\frac{r_n}{n} \underset{n}{\longrightarrow} 0$ (lower intermediate) or $\frac{r_n}{n} \underset{n}{\longrightarrow} 1$ (upper intermediate), where the symbol ($\underset{n}{\longrightarrow}$) stands for convergence as $n \longrightarrow \infty$. Wu [33] (see also, Leadbetter et al. [23]) revealed that, if $\{r_n\}$ is any nondecreasing intermediate rank sequence, and there exist normalizing constants $a_n > 0$ and b_n such that

$$\Psi^{(0,1)}_{n-r_n+1:n}(a_n x + b_n) = I_{F(a_n x+b_n)}(n - r_n + 1, r_n) \underset{n}{\overset{w}{\longrightarrow}} \Psi^{(0,1)}(x), \tag{2.1}$$

where $\underset{n}{\overset{w}{\longrightarrow}}$ stands for weak convergence, as $n \to \infty$, $\Psi^{(0,1)}_{n-r_n+1:n}(x)$ is the DF of the upper r_nth OOS (upper intermediate), and $\Psi^{(0,1)}(x)$ is a nondegenerate DF, then $\Psi^{(0,1)}(x)$ must be one and only one of the types $\mathcal{N}(U_{i;\alpha}(x)),\ i = 1, 2, 3$, where $\mathcal{N}(.)$ denotes the standard normal DF, and α is a positive constant. Moreover,

$$U_{1;\alpha}(x) = \begin{cases} -\alpha \log \mid x \mid, & x \le 0, \\ \infty, & x > 0, \end{cases} \qquad U_{2;\alpha}(x) = \begin{cases} -\infty, & x \le 0, \\ \alpha \log x, & x > 0, \end{cases}$$

and $U_{3;\alpha}(x) = U_3(x) = x,\ \forall x$. Furthermore, (2.1) is satisfied with $\Psi^{(0,1)}(x) = \mathcal{N}(U_{i;\alpha}(x)),\ i = 1, 2, 3$, if and only if

$$\frac{r_n - n\overline{F}(a_n x + b_n)}{\sqrt{r_n}} \underset{n}{\longrightarrow} U_{i;\alpha}(x). \tag{2.2}$$

In this work we confine ourselves to a very wide intermediate rank sequence which is known as Chibisov's rank, where $\frac{r_n}{n^\omega} \underset{n}{\longrightarrow} l^2$, $0 < \omega < 1$; for more details about Chibisov's rank, see ([6, 7, 15]). When (2.1) is satisfied for this rank, we say that F belongs to the intermediate domain of attraction of $\Psi^{(0,1)}(x) = \mathcal{N}(U_{i;\alpha}(x))$ and write $F \in D^{(l,\omega)}\big(\mathcal{N}(U_{i;\alpha}(x))\big)$. The following lemma is needed for studying the asymptotic distributions of the suitably normalized intermediate m-GOSs.

Lemma 2.1. *(cf. Barakat [4]) Let $m > -1$. Then, for any nondecreasing intermediate variable rank r_n, there exist normalizing constants $a_n > 0$ and b_n such that*

$$\Psi^{(m,k)}_{n-r_n+1:n}(a_n x + b_n) \xrightarrow[n]{w} \Psi^{(m,k)}(x), \tag{2.3}$$

if and only if

$$\frac{r_{n^\star} - n\overline{F}^{m+1}(a_n x + b_n)}{\sqrt{r_{n^\star}}} \xrightarrow[n]{} U(x), \tag{2.4}$$

where $\Psi^{(m,k)}(x)$ is a nondegenerate DF with $\Psi^{(m,k)}(x) = \mathcal{N}(U(x))$, and $n^\star = n + \frac{k}{m+1} - 1$.

Theorem 2.1. *(cf. Barakat [4]) Suppose that $m > -1$, and r_n is a nondecreasing intermediate variable rank. Moreover, let $r_n^\star$ be a variable rank defined by*

$$r_n^\star = r_{S^{-1}(n^\star)},$$

where $S(n) = r_{n^\star}/(r_{n^\star}/n^\star)^{\frac{1}{m+1}}$. Then, there exist normalizing constants $a_n > 0$ and b_n for which (2.3) is satisfied for some nondegenerate DF $\Psi^{(m,k)}(x)$ if and only if there are normalizing constants $\alpha_n > 0$ and β_n for which

$$\Psi^{(0,1)}_{n-r_n^\star+1:n}(\alpha_n x + \beta_n) \xrightarrow[n]{w} \Psi^{(0,1)}(x), \tag{2.5}$$

where $\Psi^{(0,1)}(x)$ is some nondegenerate DF. Equivalently,

$$\frac{r_n^\star - n\overline{F}(\alpha_n x + \beta_n)}{\sqrt{r_n^\star}} \xrightarrow[n]{} U_{i;\alpha}(x), \tag{2.6}$$

with $\Psi^{(0,1)}(x) = \mathcal{N}(U_{i;\alpha}(x)),\ i = 1, 2, 3$. In this case, the normalizing constants a_n and b_n can be chosen as $a_n = \alpha_{S(n)}$ and $b_n = \beta_{S(n)}$. Furthermore, $U(x)$ in (2.4) takes the form $(m + 1)U_{i;\alpha}(x)$.

2.2. Central OOSs

When the rank sequence $r_n \xrightarrow[n]{} \infty$ satisfies the regular condition $r_n = \lambda n + o(\sqrt{n})$, where $0 < \lambda < 1$, r_n is referred to as a central rank sequence, and $X_{r_n:n}$ is called central OOSs. There are numerous distinct results for central OOSs and their applications in the literature. Smirnov [31] showed that, if there exist normalizing constants $c_n > 0$ and d_n such that

$$\Psi^{(0,1)}_{n-r_n+1:n}(c_n x + d_n) = I_{F(c_n x + d_n)}(n - r_n + 1, r_n) \xrightarrow[n]{w} \Psi^{(0,1)}_\lambda(x), \tag{2.7}$$

where $\Psi^{(0,1)}_\lambda(x)$ is some nondegenerate DF, then $\Psi^{(0,1)}_\lambda(x)$ must be one and only one of the types $\mathcal{N}(V_{i;\alpha}(x)),\ i = 1, 2, 3, 4$. Moreover,

$$V_{1;\alpha}(x) = \begin{cases} -\infty, & x \le 0, \\ cx^\alpha, & x > 0,\ c,\ \alpha > 0, \end{cases} \qquad V_{2;\alpha}(x) = \begin{cases} -c|x|^\alpha, & x \le 0,\ c,\ \alpha > 0, \\ \infty, & x > 0, \end{cases}$$

$$V_{3;\alpha}(x) = \begin{cases} -c|x|^\alpha, & x \le 0,\ c,\ \alpha > 0, \\ c_1 x^\alpha, & x > 0,\ c_1 > 0, \end{cases} \qquad V_{4;0}(x) = \begin{cases} -\infty, & x \le -1, \\ 0, & -1 < x \le 1, \\ \infty, & x > 1, \end{cases}$$

where $c = \frac{1}{\sqrt{\lambda(1-\lambda)}}$ and $c_1 = c/A, A > 0$. In that case, we say that the DF F belongs to the domain of normal λ-attraction of the limit type $V_{i,\alpha}(x), i = 1,2,3,4$, written $F \in D^{(\lambda)}(V_{i;\alpha}(x))$. Moreover, Smirnov [31] showed that (2.7) is satisfied if and only if

$$\sqrt{n}\left(\frac{\lambda - \overline{F}(c_n x + d_n)}{C_\lambda}\right) \underset{n}{\longrightarrow} V_{i;\alpha}(x), \tag{2.8}$$

where $C_\lambda = 1/c = \sqrt{\lambda(1-\lambda)}$. The following lemma, which is due to Barakat [4], is a cornerstone of the asymptotic theory of central m-GOSs.

Lemma 2.2. *(cf. Barakat [4]) Let $m > -1$. Moreover, let r_n be a nondecreasing variable rank such that $r_n = \lambda n + o(\sqrt{n})$. Then, there exist normalizing constants $c_n > 0$ and d_n such that*

$$\Psi^{(m,k)}_{n-r_n+1:n}(c_n x + d_n) \overset{w}{\underset{n}{\longrightarrow}} \Psi^{(m,k)}_\lambda(x), \tag{2.9}$$

if and only if

$$\sqrt{n}\left(\frac{\lambda - \overline{F}^{m+1}(c_n x + d_n)}{C_\lambda}\right) \underset{n}{\longrightarrow} V(x), \tag{2.10}$$

where $\Psi^{(m,k)}_\lambda(x)$ is a nondegenerate DF with $\Psi^{(m,k)}_\lambda(x) = \mathcal{N}(V(x))$.

Theorem 2.2. *(cf. Barakat [4]) Let $m > -1$, and let r_n be a nondecreasing variable rank such that $r_n = \lambda n + o(\sqrt{n})$. Then, there exist normalizing constants $c_n > 0$ and d_n for which (2.9) holds, for some nondegenerate DF $\Psi^{(m,k)}_\lambda(x)$, if and only if, for the same normalizing constants c_n and d_n, we have $F \in D^{\lambda(m)}(V_{i;\alpha}(x)),\ i = 1,2,3,4$, where $\lambda(m) = \lambda^{\frac{1}{m+1}}$. That is, in view of (2.10), we have*

$$\sqrt{n}\left(\frac{\lambda(m) - \overline{F}(c_n x + d_n)}{C_{\lambda(m)}}\right) \underset{n}{\longrightarrow} V_{i;\alpha}(x),\ i \in \{1,2,3,4\}. \tag{2.11}$$

Moreover, $\Psi^{(m,k)}_\lambda(x) = \mathcal{N}\left((C^\star_{\lambda_{(m)}}/C^\star_\lambda)(m+1)V_i(x)\right)$, where $C^\star_t = C_t/t$.

3. Bootstrapping intermediate m-GOSs

In this section, we study the asymptotic behaviour of bootstrapping intermediate m-GOSs with rank sequence $r_{_B}$. In other words, we are interested in the limiting distribution of $H^{(m,k)}_{r_{_B},n,B}(x) = P\left(\frac{Y^\star_{B-r_B+1:B} - b_B}{a_B} \le x|\mathbf{X}_n\right)$, for different choices of the re-sample size $B = B(n)$, when the normalizing constants a_n and b_n are either known or unknown.

3.1. Consistency of bootstrapping intermediate m-GOSs when the normalizing constants are known

This subsection investigates the inconsistency, weak consistency and strong consistency of the bootstrap distribution $H^{(m,k)}_{r_B,n,B}(x)$ when the normalizing constants are known. More specifically, it is proved in the next theorem that the full-sample bootstrap (i.e., $B = n$) of $H^{(0,k)}_{r_n,n,n}(x)$ fails to be consistent with $\Psi^{(0,k)}(x)$. Moreover, in the same theorem it is shown that if $m > 0$ and $B = n$, then $H^{(m,k)}_{r_n,n,n}(x)$ is a consistent estimator of $\Psi^{(m,k)}(x)$.

Theorem 3.1. *Let the relation (2.3) be satisfied with $\Psi^{(m,k)}(x) = \mathcal{N}((m+1)U_{i;\alpha}(x)),\ i = 1,2,3$. Then,*

$$H^{(0,k)}_{r_n,n,n}(x) \overset{d}{\underset{n}{\longrightarrow}} \mathcal{N}(Z(x)), \tag{3.1}$$

where $\xrightarrow[n]{d}$ *stands for convergence in distribution as* $n \to \infty$, *and* $Z(x)$ *has a normal distribution with mean* $U_{i;\beta}(x)$ *and a variance of one. Moreover, if* $m > 0$, *then*

$$\sup_{x\in\mathbb{R}} \left| H^{(m,k)}_{r_n,n,n}(x) - \Psi^{(m,k)}(x) \right| \xrightarrow[n]{p} 0, \tag{3.2}$$

where $\xrightarrow[n]{p}$ *stands for convergence in probability as* $n \to \infty$.

Proof. Since all of the limit types in (2.2) are continuous, the convergence in (2.3) is uniform in x. Therefore, we can write

$$H^{(m,k)}_{r_B,n,B}(x) = \mathcal{N}\left(\frac{r_{B^\star} - B\overline{F}_n^{m+1}(a_B x + b_B)}{\sqrt{r_{B^\star}}} \right) + \xi_n(x), \tag{3.3}$$

where $\xi_n(x) \xrightarrow[n]{} 0$ uniformly with respect to x, and $B^\star = B + \frac{k}{m+1} - 1 \xrightarrow[n]{} \infty$. Suppose now that the condition $B = n$ holds true; then, (3.3) can be expressed as

$$H^{(m,k)}_{r_n,n,n}(x) = \mathcal{N}\left(\frac{r_{n^\star} - n\overline{F}_n^{m+1}(a_n x + b_n)}{\sqrt{r_{n^\star}}} \right) + \xi_n(x). \tag{3.4}$$

When $m = 0$, after some routine algebraic calculations we can obtain (3.1), which is the same result of Theorem 4.2 in Barakat et al. [12]. Now, we have to prove (3.2) for $m > 0$ when $B = n$. Since $S(n) \xrightarrow[n]{} \infty$, (2.6) implies

$$\frac{r^\star_{S(n)} - S(n)\overline{F}(\alpha_{S(n)}x + \beta_{S(n)})}{\sqrt{r^\star_{S(n)}}} \xrightarrow[n]{} U_{i;\alpha}(x). \tag{3.5}$$

Furthermore, from the central limit theorem we have

$$\frac{n\bar{F}_n(\alpha_{S(n)}x + \beta_{S(n)}) - n\overline{F}(\alpha_{S(n)}x + \beta_{S(n)})}{\sqrt{n\overline{F}(\alpha_{S(n)}x + \beta_{S(n)})(1 - \overline{F}(\alpha_{S(n)}x + \beta_{S(n)}))}} \xrightarrow[n]{d} Z(x). \tag{3.6}$$

On the other hand, from Chibisov [15] and Barakat [4], $S(n)$ can be written in the form $S(n) = l^{\frac{2m}{m+1}} n^{\frac{1+\omega m}{m+1}}(1 + o(1))$, where $0 < \omega < 1$ and $l > 0$. Consequently, we get

$$\frac{S(n)}{n} = l^{\frac{2m}{m+1}} n^{\frac{1+\omega m}{m+1}-1}(1 + o(1)) \xrightarrow[n]{} 0. \tag{3.7}$$

Moreover, from Barakat [4], it is clear that $r^\star_{S(n)} \sim S(n)\overline{F}(\alpha_{S(n)}x + \beta_{S(n)})$. Thus, by (3.6) and (3.7), we obtain

$$\begin{aligned} &\frac{S(n)\bar{F}_n(\alpha_{S(n)}x + \beta_{S(n)}) - S(n)\overline{F}(\alpha_{S(n)}x + \beta_{S(n)})}{\sqrt{r^\star_{S(n)}}} \\ &= \frac{S(n)}{n} \frac{n\bar{F}_n(\alpha_{S(n)}x + \beta_{S(n)}) - n\overline{F}(\alpha_{S(n)}x + \beta_{S(n)})}{\sqrt{r^\star_{S(n)}}} \end{aligned}$$

$$\sim \frac{S(n)}{n} \frac{n\bar{F}_n(\alpha_{S(n)}x+\beta_{S(n)}) - n\overline{F}(\alpha_{S(n)}x+\beta_{S(n)})}{\sqrt{S(n)\overline{F}(\alpha_{S(n)}x+\beta_{S(n)})(1-\overline{F}(\alpha_{S(n)}x+\beta_{S(n)}))}}$$

$$= \sqrt{\frac{S(n)}{n}} \frac{n\bar{F}_n(\alpha_{S(n)}x+\beta_{S(n)}) - n\overline{F}(\alpha_{S(n)}x+\beta_{S(n)})}{\sqrt{n\overline{F}(\alpha_{S(n)}x+\beta_{S(n)})(1-\overline{F}(\alpha_{S(n)}x+\beta_{S(n)}))}} \underset{n}{\longrightarrow} 0. \tag{3.8}$$

Therefore, from (3.5) and (3.8) we get

$$\frac{r^\star_{S(n)} - S(n)\bar{F}_n(\alpha_{S(n)}x+\beta_{S(n)})}{\sqrt{r^\star_{S(n)}}} = \frac{r^\star_{S(n)} - S(n)\overline{F}(\alpha_{S(n)}x+\beta_{S(n)})}{\sqrt{r^\star_{S(n)}}}$$

$$- \frac{S(n)\bar{F}_n(\alpha_{S(n)}x+\beta_{S(n)}) - S(n)\overline{F}(\alpha_{S(n)}x+\beta_{S(n)})}{\sqrt{r^\star_{S(n)}}} \underset{n}{\longrightarrow} U_{i;\alpha}(x). \tag{3.9}$$

From (3.9), we can write $\bar{F}_n(\alpha_{S(n)}x+\beta_{S(n)}) = \left(\frac{r^\star_{S(n)}}{S(n)}\right)\left(1 - \frac{U_{i;\alpha}(x)}{\sqrt{r^\star_{S(n)}}}(1+o(1))\right)$, which implies

$$\bar{F}_n^{\,m+1}(\alpha_{S(n)}x+\beta_{S(n)}) = \left(\frac{r^\star_{S(n)}}{S(n)}\right)^{m+1}\left(1 - \frac{(m+1)U_{i;\alpha}(x)}{\sqrt{r^\star_{S(n)}}}(1+o(1))\right). \tag{3.10}$$

Furthermore, from Barakat [4], it can be noted that $r^\star_{S(n)} \sim r_{n^\star}$ and $\frac{r^\star_{S(n)}}{S(n)} = \left(\frac{r_{n^\star}}{n}\right)^{\frac{1}{m+1}}$. Thus, by using (3.10), we have

$$\frac{r_{n^\star} - n\bar{F}_n^{\,m+1}(a_nx+b_n)}{\sqrt{r_{n^\star}}} = \omega_n + (m+1)U_{i;\alpha}(x)\tau_n(1+o(1)), \tag{3.11}$$

where

$$\omega_n = \frac{r_{n^\star} - n\left(\frac{r^\star_{S(n)}}{S(n)}\right)^{m+1}}{\sqrt{r_{n^\star}}} = \frac{r_{n^\star} - n\left(\frac{r_{n^\star}}{n}\right)}{\sqrt{r_{n^\star}}} = 0, \tag{3.12}$$

and

$$\tau_n = \frac{n\left(\frac{r^\star_{S(n)}}{S(n)}\right)^{m+1}}{\sqrt{r_{n^\star}}\sqrt{r^\star_{S(n)}}} = \frac{n\left(\frac{r_{n^\star}}{n}\right)}{\sqrt{r^2_{n^\star}}} = 1. \tag{3.13}$$

Substituting from (3.12) and (3.13) into (3.11), we get

$$\frac{r_{n^\star} - n\bar{F}_n^{\,m+1}(a_nx+b_n)}{\sqrt{r_{n^\star}}} \underset{n}{\longrightarrow} (m+1)U_{i;\alpha}(x), \tag{3.14}$$

which proves (3.2), and the proof is completed. □

Theorem 3.2. *Assume that there exist normalizing constants $a_n > 0$ and b_n from which the relation (2.3) holds, with $\Psi^{(m,k)}(x) = \mathcal{N}((m+1)U_{i;\alpha}(x)),\ i = 1,2,3$. Let $S(B) = o(n)$. Then*

$$\sup_{x\in\mathbb{R}} \left| H^{(m,k)}_{r_B,n,B}(x) - \Psi^{(m,k)}(x) \right| \overset{p}{\underset{n}{\longrightarrow}} 0. \tag{3.15}$$

Moreover, if B is chosen such that $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{S(B)}}} < \infty, \forall\, \lambda \in (0,1)$, *then*

$$\sup_{x\in\mathbb{R}} \left| H^{(m,k)}_{r_B,n,B}(x) - \Psi^{(m,k)}(x) \right| \xrightarrow[n]{w.p.1} 0, \tag{3.16}$$

where $\xrightarrow[n]{w.p.1}$ *stands for convergence with probability one as* $n \to \infty$ *(almost sure convergence).*

Proof. By noting that $S(B) \xrightarrow[n]{} \infty$, (2.6) implies

$$\frac{r^{\star}_{S(B)} - S(B)\overline{F}(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}} \xrightarrow[n]{} U_{i;\alpha}(x).$$

Define the statistic $\mathcal{K}_{r^{\star}_{S(n)},n,B}(x)$ by the relation

$$\mathcal{K}_{r^{\star}_{S(B)},n,B}(x) = \frac{r^{\star}_{S(B)} - S(B)\bar{F}_n(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}}.$$

Therefore, we get

$$E\left(\mathcal{K}_{r^{\star}_{S(B)},n,B}(x)\right) = \frac{r^{\star}_{S(B)} - S(B)\overline{F}(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}} \xrightarrow[n]{} U_{i;\alpha}(x), \tag{3.17}$$

and

$$\mathrm{Var}\left(\mathcal{K}_{r^{\star}_{S(B)},n,B}(x)\right) = \frac{S(B)}{n} \frac{\overline{F}(\alpha_{S(B)}x + \beta_{S(B)})(1 - \overline{F}(\alpha_{S(B)}x + \beta_{S(B)}))}{\tilde{r}^{\star}_{S(B)}} \sim \frac{S(B)}{n} \sim o(1) \xrightarrow[n]{} 0, \tag{3.18}$$

where $\tilde{r}^{\star}_{S(B)} = \frac{r^{\star}_{S(B)}}{S(B)} \sim \overline{F}(\alpha_{S(B)}x+\beta_{S(B)})(1-\overline{F}(\alpha_{S(B)}x+\beta_{S(B)}))$. By combining (3.17) and (3.18), we obtain

$$\frac{r^{\star}_{S(B)} - S(B)\bar{F}_n(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}} \xrightarrow[n]{p} U_{i;\alpha}(x), \tag{3.19}$$

which proves (3.15). In order to prove (3.16), it is sufficient to show that the convergence in (3.19) is w.p.1. First note that

$$\frac{r^{\star}_{S(B)} - S(B)\bar{F}_n(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}} = \frac{r^{\star}_{S(B)} - S(B)\overline{F}(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}}$$
$$- \frac{S(B)\bar{F}_n(\alpha_{S(B)}x + \beta_{S(B)}) - S(B)\overline{F}(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}},$$

and

$$\frac{r^{\star}_{S(B)} - S(B)\overline{F}(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}} \xrightarrow[n]{} U_{i;\alpha}(x).$$

Hence, the convergence in (3.19) becomes w.p.1 if we can show that

$$\frac{S(B)\bar{F}_n(\alpha_{S(B)}x+\beta_{S(B)})-S(B)\overline{F}(\alpha_{S(B)}x+\beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}}$$
$$=\sqrt{\frac{S(B)}{n}}\frac{n\bar{F}_n(\alpha_{S(B)}x+\beta_{S(B)})-n\overline{F}(\alpha_{S(B)}x+\beta_{S(B)})}{\sqrt{n\,\tilde{r}^{\star}_{S(B)}}}\xrightarrow[n]{w.p.1}0.$$

By the Borel-Cantelli lemma, it is sufficient to prove that

$$\sum_{n=1}^{\infty}P\left(\sqrt{\frac{S(B)}{n}}\left|\frac{n\bar{F}_n(\alpha_{S(B)}x+\beta_{S(B)})-n\overline{F}(\alpha_{S(B)}x+\beta_{S(B)})}{\sqrt{n\,\tilde{r}^{\star}_{S(B)}}}\right|>\epsilon\right)<\infty$$

for every $\epsilon>0$. For every $\theta>0$, we have

$$\sqrt{\frac{S(B)}{n}}\log P\left(\sqrt{\frac{S(B)}{n}}\left(\frac{n\bar{F}_n(\alpha_{S(B)}x+\beta_{S(B)})-n\overline{F}(\alpha_{S(B)}x+\beta_{S(B)})}{\sqrt{n\tilde{r}^{\star}_{S(B)}}}\right)>\epsilon\right)$$
$$=\sqrt{\frac{S(B)}{n}}\log P\left(\frac{n\bar{F}_n(\alpha_{S(B)}x+\beta_{S(B)})-n\overline{F}(\alpha_{S(B)}x+\beta_{S(B)})}{\sqrt{n\tilde{r}^{\star}_{S(B)}}}>\epsilon\sqrt{\frac{n}{S(B)}}\right)$$
$$=\sqrt{\frac{S(B)}{n}}\log P\left(e^{\theta\mathcal{P}_{n,B}(x)}>e^{\theta\epsilon\sqrt{\frac{n}{S(B)}}}\right),$$

where

$$\mathcal{P}_{n,B}(x)=\frac{n\bar{F}_n(\alpha_{S(B)}x+\beta_{S(B)})-n\overline{F}(\alpha_{S(B)}x+\beta_{S(B)})}{\sqrt{n\,\tilde{r}^{\star}_{S(B)}}}.$$

From Markov's inequality, we get

$$\sqrt{\frac{S(B)}{n}}\log P\left(e^{\theta\mathcal{P}_{n,B}(x)}>e^{\theta\epsilon\sqrt{\frac{n}{S(B)}}}\right)\leq\sqrt{\frac{S(B)}{n}}\log\left(e^{-\theta\epsilon\sqrt{\frac{n}{S(B)}}}E\left(e^{\theta\mathcal{P}_{n,B}(x)}\right)\right)$$
$$\sim\sqrt{\frac{S(B)}{n}}\left(-\theta\epsilon\sqrt{\frac{n}{S(B)}}+\log M_B(\theta)\right)=-\theta\epsilon+\sqrt{\frac{S(B)}{n}}\log M_B(\theta)\xrightarrow[n]{}-\theta\epsilon,$$

where $M_B(\theta)$ denotes the moment generating function of the standard normal distribution. Consequently, for sufficiently large n, we get

$$\sum_{n=1}^{\infty}P\left(\sqrt{\frac{S(B)}{n}}\left(\frac{n\bar{F}_n(\alpha_{S(B)}x+\beta_{S(B)})-n\overline{F}(\alpha_{S(B)}x+\beta_{S(B)})}{\sqrt{n\,\tilde{r}^{\star}_{S(B)}}}\right)>\epsilon\right)$$
$$=\sum_{n=1}^{\infty}e^{\log P\left(\sqrt{\frac{S(B)}{n}}\mathcal{P}_{n,B}(x)>\epsilon\right)}\leq\sum_{n=1}^{\infty}e^{-\theta\epsilon\sqrt{\frac{n}{S(B)}}}<\infty.$$

By a similar argument, for every $\epsilon > 0$, it can be shown that

$$\sum_{n=1}^{\infty} P\left(\sqrt{\frac{S(B)}{n}} \left(\frac{n\bar{F}_n(\alpha_{S(B)}x + \beta_{S(B)}) - n\overline{F}(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{n\,\tilde{r}^{\star}_{S(B)}}} \right) < -\epsilon \right) < \infty.$$

Since the condition $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{S(B)}}} < \infty,\ \forall\ \lambda \in (0,1)$, ensures the convergence of the series $\sum_{n=1}^{\infty} \exp\left\{-\theta\epsilon \sqrt{\frac{n}{S(B)}}\right\}$ for every $\epsilon > 0$, we obtain

$$\frac{r^{\star}_{S(B)} - S(B)\bar{F}_n(\alpha_{S(B)}x + \beta_{S(B)})}{\sqrt{r^{\star}_{S(B)}}} \xrightarrow[n]{w.p.1} U_{i;\alpha}(x). \tag{3.20}$$

From (3.20), we can write

$$\bar{F}_n(\alpha_{S(B)}x + \beta_{S(B)}) = \bar{F}_n(a_B x + b_B) = \left(\frac{r^{\star}_{S(B)}}{S(B)}\right)\left(1 - \frac{U_{i;\alpha}(x)}{\sqrt{r^{\star}_{S(B)}}}(1 + o(1))\right),\ w.p.1,$$

which implies

$$\bar{F}_n^{\,m+1}(a_B x + b_B) = \left(\frac{r^{\star}_{S(B)}}{S(B)}\right)^{m+1}\left(1 - \frac{(m+1)U_{i;\alpha}(x)}{\sqrt{r^{\star}_{S(B)}}}(1 + o(1))\right),\ w.p.1.$$

Therefore, we get

$$\frac{r_{B^\star} - B\overline{F}^{m+1}(a_B x + b_B)}{\sqrt{r_{B^\star}}} = \eta_n + (m+1)U_{i;\alpha}(x)\vartheta_n(1 + o(1)),\ w.p.1,$$

where

$$\eta_n = \frac{r_{B^\star} - B\left(\frac{r^{\star}_{S(B)}}{S(B)}\right)^{m+1}}{\sqrt{r_{B^\star}}} = \frac{r_{B^\star} - B\left(\frac{r_{B^\star}}{B}\right)}{\sqrt{r_{B^\star}}} = 0,$$

and

$$\vartheta_n = \frac{B\left(\frac{r^{\star}_{S(B)}}{S(B)}\right)^{m+1}}{\sqrt{r_{B^\star}}\sqrt{r^{\star}_{S(B)}}} = \frac{B\left(\frac{r_{B^\star}}{B}\right)}{\sqrt{r^2_{B^\star}}} = 1.$$

Consequently,

$$\frac{r_{B^\star} - B\overline{F}^{m+1}(a_B x + b_B)}{\sqrt{r_{B^\star}}} \xrightarrow[n]{w.p.1} (m+1)U_{i;\alpha}(x) = U(x).$$

Thus, (3.16) is proved. This completes the proof. □

3.2. Bootstrap consistency of the intermediate m-GOSs with unknown normalizing constants

One of the most important problems in statistical modeling is to reduce the required knowledge about the DF of the population from which the available data is obtained. In the bootstrap method, this situation leads to the case of unknown normalizing constants. In the rest of this section, we investigate the consistency property of the bootstrap intermediate m-GOSs when the normalizing constants are unknown.

Suppose now that the normalizing constants a_n and b_n are unknown, and they are estimated from the sample data, $\mathbf{X}_n = (X_1, X_2, ..., X_n)$. Let $\hat{a}_B$ and $\hat{b}_B$ be the estimators of a_n and b_n, based on $\mathbf{X}_n$, respectively, and

$$\hat{H}^{(m,k)}_{r_B,n,B}(x) = P\left(\frac{Y^{\star}_{B-r_B+1:B} - \hat{b}_B}{\hat{a}_B} \leq x|\mathbf{X}_n\right) \tag{3.21}$$

be the bootstrap distribution of $\Psi^{(m,k)}_{n-r_n+1:n}(a_n x + b_n)$ with the estimated normalizing constants, $\hat{a}_B$ and $\hat{b}_B$. The sufficient conditions for $\hat{H}^{(m,k)}_{r_B,n,B}(x)$ to be consistent are explored in the next theorem. The idea of this theorem was originally given in Theorem 2.6 for maximum OOSs in [21].

Theorem 3.3. *Assume that $\hat{a}_B$, $\hat{b}_B$ and $B = B(n)$ are such that the following three conditions are satisfied:*

$$(C_1)\ H^{(m,k)}_{r_B,n,B}(x) \xrightarrow[n]{w.p.1} \Psi^{(m,k)}(x), \quad (C_2)\ \frac{\hat{a}_B}{a_B} \xrightarrow[n]{w.p.1} 1, \text{ and } (C_3)\ \frac{\hat{b}_B - b_B}{a_B} \xrightarrow[n]{w.p.1} 0.$$

Then,

$$\sup_{x\in\mathbb{R}} \left|\hat{H}^{(m,k)}_{r_B,n,B}(x) - \Psi^{(m,k)}(x)\right| \xrightarrow[n]{w.p.1} 0. \tag{3.22}$$

Moreover, if we replace "$\xrightarrow[n]{w.p.1}$" with "$\xrightarrow[n]{p}$" in the conditions (C_1)–(C_3), the convergence (3.22) remains true.

Proof. First, we note that the condition (C_1) is equivalent to

$$\frac{r_{B^\star} - B\bar{F}_n^{\,m+1}(a_B x + b_B)}{\sqrt{r_{n^\star}}} \xrightarrow[n]{} U(x).$$

Furthermore, $\forall\ \epsilon > 0$, the conditions (C_2) and (C_3) imply

$$(1-\epsilon)a_B < \hat{a}_B < (1+\epsilon)a_B, \tag{3.23}$$

and

$$b_B - \epsilon a_B < \hat{b}_B < b_B + \epsilon a_B, \tag{3.24}$$

respectively. By fixing $x > 0$, the relations (3.23) and (3.24) yield

$$B\bar{F}_n^{\,m+1}\Big((1+\epsilon)x + \epsilon)a_B + b_B\Big) \leqslant B\bar{F}_n^{\,m+1}(\hat{a}_B x + \hat{b}_B) \leqslant B\bar{F}_n^{\,m+1}\Big((1-\epsilon)x - \epsilon)a_B + b_B\Big).$$

Therefore,

$$\limsup_{n\to\infty} \frac{r_{B^\star} - B\bar{F}_n^{\,m+1}(\hat{a}_B x + \hat{b}_B)}{\sqrt{r_{B^\star}}} \leqslant U\Big((1+\epsilon)x + \epsilon\Big)$$

$$\leqslant U\Big((1-\epsilon)x-\epsilon\Big) \leqslant \liminf_{n\to\infty} \frac{r_{B^\star} - B\bar{F}_n^{\,m+1}(\hat{a}_B x + \hat{b}_B)}{\sqrt{r_{B^\star}}}.$$

Since $U(x)$ is continuous, we get

$$\lim_{n\to\infty} \frac{r_{B^\star} - B\bar{F}_n^{\,m+1}(\hat{a}_B x + \hat{b}_B)}{\sqrt{r_{B^\star}}} = U(x).$$

For $x < 0$, the same limit relation can be accomplished using a similar argument. Consequently, (3.22) is proved. If the conditions (C_1)–(C_3) hold in probability, then for any subsequence $\{n_i\}_{i=1}^{\infty}$, there exists a subsequence $\{n_{i^*}\}_{i^*=1}^{\infty}$, such that the conditions (C_1)–(C_3) hold w.p.1. An application of the first part of the theorem, based on the new subsequence, yields

$$\sup_{x\in\mathbb{R}} \left|\hat{H}_{r_{B(n_{i^*})},n_{i^*},B(n_{i^*})}(x) - \Psi^{(m,k)}(x)\right| \xrightarrow[n]{w.p.1} 0.$$

This completes the proof of the theorem. □

In the next theorem, an appropriate choice of estimators of the normalizing constants which satisfy conditions (C_2) and (C_3) of Theorem 3.3 is accomplished for each domain of attraction. More specifically, we get a specific choice of $\hat{a}_B$ and $\hat{b}_B$ from which (3.22) holds true.

Theorem 3.4. *Assume that* $r_n^{\star\prime} = \frac{n}{S(B)} r^\star_{S(B)}$, $r_n^{\star\prime\prime} = \frac{n}{S(B)}\left(r^\star_{S(B)} + \sqrt{r^\star_{S(B)}}\right)$, *$x^o$ is the right endpoint of F, and $\hat{x}^o = X_{r_n^\star:n}$. The estimators $\hat{a}_B$ and $\hat{b}_B$ can be chosen, respectively, as*

(i) $\hat{a}_B = \hat{x}^o - F_n^{-1}\left(\frac{r^\star_{S(B)}}{S(B)}\right) = X_{r_n^\star:n} - X_{r_n^{\star\prime}:n}$ *and* $\hat{b}_B = X_{r_n^\star:n}$, *if* $F \in D^{(l,\omega)}\Big(\mathcal{N}((m+1)U_{1;\alpha}(x))\Big)$,

(ii) $\hat{a}_B = F_n^{-1}\left(\frac{r^\star_{S(B)}}{S(B)}\right) = X_{r_n^{\star\prime}:n}$ *and* $\hat{b}_B = 0$, *if* $F \in D^{(l,\omega)}\Big(\mathcal{N}((m+1)U_{2;\alpha}(x))\Big)$,

(iii) $\hat{a}_B = F_n^{-1}\left(\frac{r^\star_{S(B)}+\sqrt{r^\star_{S(B)}}}{S(B)} - \frac{r^\star_{S(B)}}{S(B)}\right) = X_{r_n^{\star\prime}:n} - X_{r_n^{\star\prime\prime}:n}$ *and* $\hat{b}_B = F_n^{-1}\left(\frac{r^\star_{S(B)}}{S(B)}\right) = X_{r_n^{\star\prime}:n}$, *if* $F \in D^{(l,\omega)}\Big(\mathcal{N}((m+1)U_3(x))\Big)$.

Moreover, If $S(B) = o(n)$, then

$$\sup_{x\in\mathbb{R}} \left|\hat{H}^{(m,k)}_{r_B,n,B}(x) - \Psi^{(m,k)}(x)\right| \xrightarrow[n]{p} 0. \tag{3.25}$$

Furthermore, if $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{S(B)}}} < \infty$ for each $\lambda \in (0,1)$, then (3.25) holds w.p.1.

Proof. Let $F \in D^{(l,\omega)}\Big(\mathcal{N}((m+1)U_{1;\alpha}(x))\Big)$. In order to prove conditions (C_2) and (C_3), we have to show that

$$\frac{\hat{a}_B}{a_B} = \frac{X_{r_n^\star:n} - X_{r_n^{\star\prime}:n}}{a_B} \xrightarrow[n]{} 1, \tag{3.26}$$

and

$$\frac{\hat{b}_B - b_B}{a_B} = \frac{X_{r_n^\star:n} - x^o}{a_B} \xrightarrow[n]{} 0, \tag{3.27}$$

both in probability or w.p.1. Firstly, let us consider the case of convergence in probability. Clearly,

$$\frac{\hat{a}_B}{a_B} = \frac{X_{r_n^\star:n} - x^o}{\alpha_n} \times \frac{\alpha_n}{a_B} - \frac{X_{r_n^{\star\prime}:n} - x^o}{a_B}.$$

Hence, (3.26) and (3.27) are proved if we can show that

$$\frac{\alpha_n}{a_B} \underset{n}{\longrightarrow} 0, \tag{3.28}$$

and

$$\frac{X_{r_n^{\star\prime}:n} - x^o}{a_B} \overset{p}{\underset{n}{\longrightarrow}} -1. \tag{3.29}$$

On the other hand, for any $\gamma > 0$, Lemma 3.1 in Barakat et al. [11] reveals that

$$\frac{\alpha_n}{a_B} = \frac{\alpha_n}{\alpha_{S(B)}} \sim \frac{\mu^{\frac{-1}{\gamma}}(r_n^\star)}{\mu^{\frac{-1}{\gamma}}(r_{S(B)}^\star)} = \frac{e^{\frac{-1}{\gamma}n^{\frac{\omega}{2}}}}{e^{\frac{-1}{\gamma}(S(B))^{\frac{\omega}{2}}}} = e^{\frac{-1}{\gamma}n^{\frac{\omega}{2}}\left(1-\left(\frac{S(B)}{n}\right)^{\frac{\omega}{2}}\right)} \underset{n}{\longrightarrow} 0,$$

where $\mu(n) = \exp(\sqrt{n})$. Thus, (3.28) is proved. Turning now to prove (3.29), it can be noted that

$$\frac{r_n^{\star\prime} - n\overline{F}(a_B x + b_B)}{\sqrt{r_n^{\star\prime}}} = \frac{r_n^{\star\prime} - n\overline{F}(a_B x + x^o)}{\sqrt{r_n^{\star\prime}}} = \frac{\frac{n}{S(B)} r_{S(B)}^\star - n\overline{F}(a_B x + x^o)}{\sqrt{\frac{n}{S(B)} r_{S(B)}^\star}}$$

$$= \sqrt{\frac{n}{S(B)}}\left(\frac{r_{S(B)}^\star - S(B)\overline{F}(a_B x + x^o)}{\sqrt{r_{S(B)}^\star}}\right) \underset{n}{\longrightarrow} \begin{cases} \infty, & \text{if } x > -1, \\ -\infty, & \text{if } x < -1. \end{cases} \tag{3.30}$$

For every $\epsilon > 0$, (3.30) implies $P\left(\frac{X_{r_n^{\star\prime}:n} - x^o}{a_B} < \epsilon - 1\right) \underset{n}{\longrightarrow} \mathcal{N}(\infty) = 1$, or equivalently,

$$P\left(\frac{X_{r_n^{\star\prime}:n} - x^o}{a_B} > \epsilon - 1\right) \underset{n}{\longrightarrow} 0. \tag{3.31}$$

Similarly,

$$P\left(\frac{X_{r_n^{\star\prime}:n} - x^o}{a_B} < -\epsilon - 1\right) \underset{n}{\longrightarrow} \mathcal{N}(-\infty) = 0. \tag{3.32}$$

By combining (3.31) and (3.32), we get

$$P\left(\left|\frac{X_{r_n^{\star\prime}:n} - x^o}{a_B} + 1\right| > \epsilon\right) \underset{n}{\longrightarrow} 0, \tag{3.33}$$

which proves (3.29). Now, let $F \in D^{(l,\omega)}\left(\mathcal{N}((m+1)U_{2;\alpha}(x))\right)$. Condition (C_3) of Theorem 2.4 is clearly proved (since $\hat{b}_B = 0$). Consequently, it is sufficient to prove only condition (C_2). Thus, we have to show that

$$\frac{\hat{a}_B}{a_B} = \frac{X_{r_n^{\star\prime}:n}}{a_B} \underset{n}{\longrightarrow} 1, \tag{3.34}$$

in probability or w.p.1. We start by proving the convergence in probability. It is clear that,

$$\frac{r_n^{\star\prime} - n\overline{F}(a_B x + b_B)}{\sqrt{r_n^{\star\prime}}} = \frac{\frac{n}{S(B)} r_{S(B)}^\star - n\overline{F}(a_B x)}{\sqrt{\frac{n}{S(B)} r_{S(B)}^\star}}$$

$$= \sqrt{\frac{n}{S(B)}}\left(\frac{r^\star_{S(B)} - S(B)\overline{F}(a_B x)}{\sqrt{r^\star_{S(B)}}}\right) \underset{n}{\longrightarrow} \begin{cases}\infty, & \text{if } x > 1,\\ -\infty, & \text{if } x < 1.\end{cases} \tag{3.35}$$

For every $\epsilon > 0$, (3.35) implies $P\left(\frac{X_{r_n^{\star\prime}:n}}{a_B} < \epsilon + 1\right) \underset{n}{\longrightarrow} \mathcal{N}(\infty) = 1$, which is equivalent to

$$P\left(\frac{X_{r_n^{\star\prime}:n}}{a_B} > \epsilon + 1\right) \underset{n}{\longrightarrow} 0. \tag{3.36}$$

Arguing similarly, we get

$$P\left(\frac{X_{r_n^{\star\prime}:n}}{a_B} < -\epsilon + 1\right) \underset{n}{\longrightarrow} \mathcal{N}(-\infty) = 0. \tag{3.37}$$

Based on the relations (3.36) and (3.37), we get

$$P\left(\left|\frac{X_{r_n^{\star\prime}:n}}{a_B} - 1\right| > \epsilon\right) \underset{n}{\longrightarrow} 0, \tag{3.38}$$

which proves (3.34). Finally, assume that $F \in D^{(l,\omega)}(\mathcal{N}((m+1)U_3(x)))$. By Theorem 2.4, in order to prove conditions (C_2) and (C_3), it suffices to show that

$$\frac{\hat{a}_B}{a_B} = \frac{X_{r_n^{\star\prime}:n} - X_{r_n^{\star\prime\prime}:n}}{a_B} \underset{n}{\longrightarrow} 1, \tag{3.39}$$

and

$$\frac{\hat{b}_B - b_B}{a_B} = \frac{X_{r_n^{\star\prime}:n} - b_B}{a_B} \underset{n}{\longrightarrow} 0, \tag{3.40}$$

both in probability or w.p.1. We start with the convergence in probability. Write

$$\frac{\hat{a}_B}{a_B} = \frac{X_{r_n^{\star\prime}:n} - X_{r_n^{\star\prime\prime}:n}}{a_B} = \frac{X_{r_n^{\star\prime}:n} - b_B}{a_B} - \frac{X_{r_n^{\star\prime\prime}:n} - b_B}{a_B}. \tag{3.41}$$

Consequently, to prove (3.39) and (3.40), we need to show that

$$\frac{X_{r_n^{\star\prime\prime}:n} - b_B}{a_B} \underset{n}{\longrightarrow} -1, \tag{3.42}$$

and

$$\frac{X_{r_n^{\star\prime}:n} - b_B}{a_B} \underset{n}{\longrightarrow} 0. \tag{3.43}$$

First, we are going to prove (3.42). Since we have

$$\frac{r_n^{\star\prime\prime} - n\overline{F}(a_B x + b_B)}{\sqrt{r_n^{\star\prime\prime}}} = \frac{\frac{n}{S(B)}\left(r^\star_{S(B)} + \sqrt{r^\star_{S(B)}}\right) - n\overline{F}(a_B x + b_B)}{\sqrt{\frac{n}{S(B)}\left(r^\star_{S(B)} + \sqrt{r^\star_{S(B)}}\right)}}$$

$$= \sqrt{\frac{n}{S(B)}}\left(\frac{r^\star_{S(B)} + \sqrt{r^\star_{S(B)}} - S(B)\overline{F}(a_B x + b_B)}{\sqrt{r^\star_{S(B)}\left(1 + 1/\sqrt{r^\star_{S(B)}}\right)}}\right)$$

$$= \sqrt{\frac{n}{S(B)}} \left(\frac{r^{\star}_{S(B)} + \sqrt{r^{\star}_{S(B)}} - S(B)\overline{F}(a_B x + b_B)}{\sqrt{r^{\star}_{S(B)}(1 + o(1))}} \right)$$

$$= \sqrt{\frac{n}{S(B)}} \left(\frac{r^{\star}_{S(B)} - S(B)\overline{F}(a_B x + b_B)}{\sqrt{r^{\star}_{S(B)}(1 + o(1))}} + \frac{\sqrt{r^{\star}_{S(B)}}}{\sqrt{r^{\star}_{S(B)}(1 + o(1))}} \right),$$

from the assumption of the theorem, we have

$$\frac{r^{\star}_{S(B)} - S(B)\overline{F}(a_B x + b_B)}{\sqrt{r^{\star}_{S(B)}(1 + o(1))}} \underset{n}{\longrightarrow} x.$$

Consequently,

$$\frac{r^{\star''}_n - n\overline{F}(a_B x + b_B)}{\sqrt{r^{\star''}_n}} \underset{n}{\longrightarrow} \begin{cases} \infty, & \text{if } x > -1, \\ -\infty, & \text{if } x < -1. \end{cases} \tag{3.44}$$

Thus, for every $\epsilon > 0$, we get $P\left(\frac{X_{r^{\star''}_n:n} - b_B}{a_B} < \epsilon - 1\right) \underset{n}{\longrightarrow} \mathcal{N}(\infty) = 1$, and this implies

$$P\left(\frac{X_{r^{\star''}_n:n} - b_B}{a_B} > \epsilon - 1\right) \underset{n}{\longrightarrow} 0. \tag{3.45}$$

Further, we have

$$P\left(\frac{X_{r^{\star''}_n:n} - b_B}{a_B} < -\epsilon - 1\right) \underset{n}{\longrightarrow} 0. \tag{3.46}$$

Hence, (3.45) and (3.46) lead to

$$P\left(\left|\frac{X_{r^{\star''}_n:n} - b_B}{a_B} + 1\right| > \epsilon\right) \underset{n}{\longrightarrow} 0,$$

which proves (3.42). Secondly, we are going to prove (3.43). Clearly,

$$\frac{r^{\star'}_n - n\overline{F}(a_B x + b_B)}{\sqrt{r^{\star'}_n}} = \frac{\frac{n}{S(B)} r^{\star}_{S(B)} - n\overline{F}(a_B x + b_B)}{\sqrt{\frac{n}{S(B)} r^{\star}_{S(B)}}} =$$

$$\sqrt{\frac{n}{S(B)}} \left(\frac{r^{\star}_{S(B)} - S(B)\overline{F}(a_B x + b_B)}{\sqrt{r^{\star}_{S(B)}}} \right) \underset{n}{\longrightarrow} \begin{cases} \infty, & \text{if } x > 0, \\ -\infty, & \text{if } x < 0. \end{cases} \tag{3.47}$$

For every $\epsilon > 0$, Eq (3.47) leads to $P\left(\frac{X_{r^{\star'}_n:n} - b_B}{a_B} < \epsilon\right) \underset{n}{\longrightarrow} \mathcal{N}(\infty) = 1$, which yields

$$P\left(\frac{X_{r^{\star'}_n:n} - b_B}{a_B} > \epsilon\right) \underset{n}{\longrightarrow} 0. \tag{3.48}$$

In the same way, we get

$$P\left(\frac{X_{r^{\star'}_n:n} - b_B}{a_B} < -\epsilon\right) \underset{n}{\longrightarrow} 0. \tag{3.49}$$

Thus, relations (3.48) and (3.49) imply

$$P\left(\left|\frac{X_{r_n^{\star\prime}:n}-b_B}{a_B}\right|>\epsilon\right)\underset{n}{\longrightarrow}0,$$

which proves (3.43). Finally, in the proof of Parts (i)–(iii), in order to switch to the convergence w.p.1, we argue in the same way that we did at the end of Theorem 3.3's proof. This completes the proof of the theorem. □

4. Bootstrapping central m-GOSs

In this section, the asymptotic behaviour of the bootstrap distribution for central m-GOSs, $H^{*(m,k)}_{r_B,n,B}(x)=P\left(\frac{Y^{\star}_{B-r_B+1:B}-d_B}{c_B}\le x|\mathbf{X}_n\right)$, is considered for different choices of the re-sample size $B=B(n)$ when the normalizing constants c_n and d_n are assumed to be known or unknown.

4.1. Bootstrap consistency of the central m-GOSs with known normalizing constants

The next theorem discusses the consistency of the bootstrap distribution $H^{*(m,k)}_{r_B,n,B}(x)$ in the case of full-sample bootstrap. It is revealed that the full-sample bootstrap distribution fails to be a consistent estimator of the DF $\Psi^{(m,k)}_{\lambda}(x)$.

Theorem 4.1. *Assume that the relation (2.9) is satisfied, where the weak limits are of the form*

$$\Psi^{(m,k)}_{\lambda}(x)=\mathcal{N}\left(\frac{C^{\star}_{\lambda_{(m)}}}{C^{\star}_{\lambda}}(m+1)V_{i,\alpha}(x)\right),\ \ \textit{for } i=1,2,3,4. \tag{4.1}$$

Then,

$$H^{*(m,k)}_{r_n,n,n}(x)\overset{d}{\underset{n}{\longrightarrow}}\mathcal{N}(\Lambda(x)), \tag{4.2}$$

where $\Lambda(x)$ has a normal distribution with mean $\frac{C^{\star}_{\lambda_{(m)}}}{C^{\star}_{\lambda}}(m+1)V_{i;\alpha}(x)$ and variance $\frac{C^{\star}_{\lambda_{(m)}}}{C^{\star}_{\lambda}}(m+1)$.

Proof. Although the convergence in (2.9) does not yield continuous types in general, under the condition $r_n=\lambda n+o(\sqrt{n})$, Barakat and El-Shandidy [5] showed that the convergence is uniform. Consequently,

$$H^{*(m,k)}_{r_n,n,B}(x)=\mathcal{N}\left(\sqrt{B}\,\frac{\lambda-\bar{F}_n^{\;m+1}(c_Bx+d_B)}{C_{\lambda}}\right)+\xi_n(x),$$

where $\xi_n(x)\underset{n}{\longrightarrow}0$ uniformly with respect to x. Now, consider that the condition $B=n$ holds. Since (2.8) is satisfied, we have

$$\sqrt{n}\left(\frac{\lambda(m)-\overline{F}(c_nx+d_n)}{C_{\lambda(m)}}\right)\underset{n}{\longrightarrow}V_{i;\alpha}(x). \tag{4.3}$$

An application of the central limit theorem yields

$$\frac{n\bar{F}_n(c_nx+d_n)-n\overline{F}(c_nx+d_n)}{\sqrt{n\overline{F}(c_nx+d_n)(1-\overline{F}(c_nx+d_n))}}\overset{d}{\underset{n}{\longrightarrow}}Z(x), \tag{4.4}$$

where $Z(x)$ is the standard normal RV. On the other hand, it is clear from relation (4.3) that $\overline{F}(c_nx + d_n) \xrightarrow[n]{} \lambda(m)$, which implies

$$\frac{\sqrt{n\overline{F}(c_nx+d_n)(1-\overline{F}(c_nx+d_n))}}{\sqrt{n}C_{\lambda(m)}} \xrightarrow[n]{} 1. \tag{4.5}$$

The two limit relations (4.3) and (4.5) enable us to apply Khinchin's type theorem on the relation (4.4) to get

$$\begin{aligned}\sqrt{n}\left(\frac{\lambda(m)-\bar{F}_n(c_nx+d_n)}{C_{\lambda(m)}}\right) = {} & \sqrt{n}\left(\frac{\lambda(m)-\overline{F}(c_nx+d_n)}{C_{\lambda(m)}}\right) \\ & -\frac{n\bar{F}_n(c_nx+d_n)-n\overline{F}(c_nx+d_n)}{\sqrt{n}C_{\lambda(m)}} \xrightarrow[n]{} V_{i;\alpha}(x)-Z(x).\end{aligned} \tag{4.6}$$

As a result of (4.6), we get

$$\begin{aligned}\bar{F}_n{}^{m+1}(c_nx+d_n) &= \lambda\left(1-\frac{C_{\lambda(m)}(V_{i,\alpha}(x)-Z(x))}{\lambda(m)\sqrt{n}}(1+o(1))\right)^{m+1} \\ &= \lambda\left(1-\frac{(m+1)C_{\lambda(m)}(V_{i;\alpha}(x)-Z(x))}{\lambda(m)\sqrt{n}}(1+o(1))\right).\end{aligned} \tag{4.7}$$

Consequently,

$$\begin{aligned}\sqrt{n}\left(\frac{\lambda-\bar{F}_n{}^{m+1}(c_nx+d_n)}{C_\lambda}\right) \xrightarrow[n]{} & \frac{(m+1)\lambda C_{\lambda_{(m)}}}{\lambda_{(m)}C_\lambda}(V_{i;\alpha}(x)-Z(x)) \\ = & \frac{C^\star_{\lambda_{(m)}}}{C^\star_\lambda}(m+1)(V_{i;\alpha}(x)-Z(x)).\end{aligned}$$

Hence, the theorem is proved. □

Theorem 4.2. *Under the same conditions of Theorem 4.1, if $B = o(n)$, then*

$$\sup_{x\in\mathbb{R}}\left|H^{*(m,k)}_{r_B,n,B}(x)-\Psi^{(m,k)}_\lambda(x)\right| \xrightarrow[n]{p} 0. \tag{4.8}$$

Furthermore, if B is such that $\sum_{n=1}^\infty P^{\sqrt{\frac{n}{B}}} < \infty$, $\forall\ P \in (0,1)$, then

$$\sup_{x\in\mathbb{R}}\left|H^{*(m,k)}_{r_B,n,B}(x)-\Psi^{(m,k)}_\lambda(x)\right| \xrightarrow[n]{w.p.1} 0. \tag{4.9}$$

Proof. Write $H^{*(m,k)}_{r_B,n,B}(x) = \mathcal{N}(\mathcal{S}_{n,B}(x)) + \xi_n(x)$, where, $\mathcal{S}_{n,B}(x) = \sqrt{B}\left(\frac{\lambda_{(m)}-\bar{F}_n(c_Bx+d_B)}{C_{\lambda_{(m)}}}\right)$, and $\xi_n(x) \xrightarrow[n]{} 0$ uniformly with respect to x. It can be noted that

$$E(\mathcal{S}_{n,B}(x)) = \sqrt{B}\left(\frac{\lambda_{(m)}-\bar{F}(c_Bx+d_B)}{C_{\lambda_{(m)}}}\right) \xrightarrow[n]{} V_{i;\alpha}(x), \tag{4.10}$$

and

$$\mathrm{Var}(\mathcal{S}_{n,B}(x)) = \frac{B\overline{F}(c_Bx + d_B)(1 - \overline{F}(c_Bx + d_B))}{nC^2_{\lambda_{(m)}}} \underset{n}{\longrightarrow} 0. \tag{4.11}$$

Accordingly,

$$\sqrt{B}\left(\frac{\lambda_{(m)} - \bar{F}_n(c_Bx + d_B)}{C_{\lambda_{(m)}}}\right) \xrightarrow[n]{p} V_{i;\alpha}(x). \tag{4.12}$$

We'll now show that the convergence in (4.12) is w.p.1. For this purpose, write

$$\begin{aligned}\sqrt{B}\left(\frac{\lambda_{(m)} - \bar{F}_n(c_Bx + d_B)}{C_{\lambda_{(m)}}}\right) = {} & \sqrt{B}\left(\frac{\lambda_{(m)} - \overline{F}(c_Bx + d_B)}{C_{\lambda_{(m)}}}\right) \\ & - \sqrt{\frac{B}{n}}\left(\frac{n\bar{F}_n(c_Bx + d_B) - n\overline{F}(c_Bx + d_B)}{\sqrt{n}C_{\lambda_{(m)}}}\right).\end{aligned}$$

On the other hand, the assumptions of the theorem ensure that $\overline{F}(c_Bx + d_B) \underset{n}{\longrightarrow} \lambda_{(m)}$, and $\sqrt{B}\left(\frac{\lambda_{(m)}-\overline{F}(c_Bx+d_B)}{C_{\lambda_{(m)}}}\right) \underset{n}{\longrightarrow} V_{i;\alpha}(x)$. To prove the limit relation $\sqrt{B}\left(\frac{\lambda_{(m)}-\bar{F}_n(c_Bx+d_B)}{C_{\lambda_{(m)}}}\right) \xrightarrow[n]{w.p.1} V_{i;\alpha}(x)$, it is sufficient to show that

$$\sqrt{\frac{B}{n}}\left(\frac{n\bar{F}_n(c_Bx + d_B) - n\overline{F}(c_Bx + d_B)}{\sqrt{n}C_{\lambda_{(m)}}}\right) = \sqrt{\frac{B}{n}}\left(\frac{n\bar{F}_n(c_Bx + d_B) - n\overline{F}(c_Bx + d_B)}{\sqrt{n\overline{F}(c_Bx + d_B)(1 - \overline{F}(c_Bx + d_B)}}\right) \xrightarrow[n]{w.p.1} 0.$$

According to the Borel-Cantelli lemma, we need to show that, for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\sqrt{\frac{B}{n}}\left|\frac{n\bar{F}_n(c_Bx + d_B) - n\overline{F}(c_Bx + d_B)}{\sqrt{n\overline{F}(c_Bx + d_B)(1 - \overline{F}(c_Bx + d_B)}}\right| > \epsilon\right) < \infty.$$

For every $\theta > 0$, we have

$$\begin{aligned}&\sqrt{\frac{B}{n}}\log P\left(\sqrt{\frac{B}{n}}\left(\frac{n\bar{F}_n(c_Bx + d_B) - n\overline{F}(c_Bx + d_B)}{\sqrt{n\overline{F}(c_Bx + d_B)(1 - \overline{F}(c_Bx + d_B)}}\right) > \epsilon\right) \\ &= \sqrt{\frac{B}{n}}\log P\left(\frac{n\bar{F}_n(c_Bx + d_B) - n\overline{F}(c_Bx + d_B)}{\sqrt{n\overline{F}(c_Bx + d_B)(1 - \overline{F}(c_Bx + d_B)}} > \epsilon\sqrt{\frac{n}{B}}\right) \\ &= \sqrt{\frac{B}{n}}\log P\left(e^{\theta\mathcal{T}_{n,B}} > e^{\theta\epsilon\sqrt{\frac{n}{B}}}\right),\end{aligned}$$

where $\mathcal{T}_{n,B}$ is defined by

$$\mathcal{T}_{n,B} = \frac{n\bar{F}_n(c_Bx + d_B) - n\overline{F}(c_Bx + d_B)}{\sqrt{n\overline{F}(c_Bx + d_B)(1 - \overline{F}(c_Bx + d_B)}}.$$

From Markov's inequality, we get

$$\sqrt{\frac{B}{n}}\log P\left(e^{\theta\mathcal{T}_{n,B}} > e^{\theta\epsilon\sqrt{\frac{n}{B}}}\right) \leq \sqrt{\frac{B}{n}}\log\left(e^{-\theta\epsilon\sqrt{\frac{n}{B}}}E\left(e^{\theta\mathcal{T}_{n,B}}\right)\right)$$

$$\sim \sqrt{\frac{B}{n}}\left(-\theta\epsilon\sqrt{\frac{n}{B}} + \log M_B(\theta)\right) = -\theta\epsilon + \sqrt{\frac{B}{n}}\log M_B(\theta) \underset{n}{\longrightarrow} -\theta\epsilon.$$

Consequently, for sufficiently large n, we get

$$\sum_{n=1}^{\infty} P\left(\sqrt{\frac{B}{n}}\left(\frac{n\bar{F}_n(c_Bx+d_B) - n\overline{F}(c_Bx+d_B)}{\sqrt{n\overline{F}(c_Bx+d_B)(1-\overline{F}(c_Bx+d_B))}}\right) > \epsilon\right)$$
$$\sim \sum_{n=1}^{\infty} e^{\log P\left(\sqrt{\frac{B}{n}}\mathcal{T}_{n,B}>\epsilon\right)} \leq \sum_{n=1}^{\infty} e^{-\theta\epsilon\sqrt{\frac{n}{B}}} < \infty.$$

By using the same method, we can show that, for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\sqrt{\frac{B}{n}}\left(\frac{n\bar{F}_n(c_Bx+d_B) - n\overline{F}(c_Bx+d_B)}{\sqrt{n\overline{F}(c_Bx+d_B)(1-\overline{F}(c_Bx+d_B))}}\right) < -\epsilon\right) < \infty.$$

Since the condition, $\sum_{n=1}^{\infty} P^{\sqrt{\frac{n}{B}}} < \infty$, $\forall\ P \in (0,1)$, assures the convergence of the series $\sum_{n=1}^{\infty}\exp\left\{-\theta\epsilon\sqrt{\frac{n}{B}}\right\}$ for every $\epsilon > 0$, we have

$$\sqrt{B}\left(\frac{\lambda_{(m)} - \bar{F}_n(c_Bx+d_B)}{C_{\lambda_{(m)}}}\right) \xrightarrow[n]{w.p.1} V_{i;\alpha}(x). \tag{4.13}$$

Therefore,

$$\begin{aligned} \bar{F}_n^{\,m+1}(c_Bx+d_B) &= \lambda\left(1 - \frac{C_{\lambda(m)}V_{i;\alpha}(x)}{\lambda(m)\sqrt{n}}(1+o(1))\right)^{m+1} \\ &= \lambda\left(1 - \frac{(m+1)C_{\lambda(m)}V_{i;\alpha}(x)}{\lambda(m)\sqrt{n}}(1+o(1))\right),\ w.p.1. \end{aligned} \tag{4.14}$$

Accordingly,

$$\sqrt{n}\left(\frac{\lambda - \bar{F}_n^{\,m+1}(c_Bx+d_B)}{C_\lambda}\right) \xrightarrow[n]{w.p.1} \frac{(m+1)\lambda C_{\lambda_{(m)}}}{\lambda_{(m)}C_\lambda}V_{i;\alpha}(x) = \frac{C^\star_{\lambda_{(m)}}}{C^\star_\lambda}(m+1)V_{i;\alpha}(x),$$

which was to be proved, and this completes the proof of the theorem. □

4.2. Bootstrap consistency of the central m-GOSs for unknown normalizing constants

Assume that the normalizing constants c_n and d_n are unknown and that they must be estimated using the sample data $\mathbf{X}_n = (X_1, X_2, ..., X_n)$. Let $\hat{c}_B$ and $\hat{d}_B$ be the estimators of c_n and d_n based on $\mathbf{X}_n$ and $\hat{H}^{*(m,k)}_{r_B,n,B}(x) = P\left(\frac{Y^\star_{B-r_B+1:B} - \hat{d}_B}{\hat{c}_B} \leq x|\mathbf{X}_n\right)$ be the bootstrap distribution for appropriately normalized central m-GOSs. The next theorem gives sufficient conditions for $\hat{H}^{*(m,k)}_{r_B,n,B}(x)$ to be consistent, where we restrict ourselves to the first three non-degenerate types, corresponding to $V_{i;\alpha}(x), i = 1,2,3$. Clearly, each of these three limit laws has at most one discontinuity point of the first type.

Theorem 4.3. *Suppose that the conditions of Theorem 4.2 are satisfied. Moreover, suppose that $\hat{c}_B$, $\hat{d}_B$, and $B = B(n)$ satisfy the following three conditions:*

$$(K_1)\ H^{*(m,k)}_{r_B,n,B}(x) \xrightarrow[n]{w.p.1} \Psi^{(m,k)}_\lambda(x), \quad (K_2)\ \frac{\hat{c}_B}{c_B} \xrightarrow[n]{w.p.1} 1,\ \textit{and}\ (K_3)\ \frac{\hat{d}_B - c_B}{d_B} \xrightarrow[n]{w.p.1} 0.$$

Then, $\sup_{x\in\mathbb{R}^c}\left|\hat{H}^{*(m,k)}_{r_B,n,B}(x)-\Psi^{(m,k)}_{\lambda}(x)\right|\xrightarrow[n]{w.p.1}0$, *where* $\mathbb{R}^c\in\mathbb{R}$ *is the set of all continuity points of* $\Psi^{(m,k)}_{\lambda}(x)$. *Moreover, this theorem holds if* "$\xrightarrow[n]{w.p.1}$" *is replaced by* "$\xrightarrow[n]{p}$".

Proof. The proof of the theorem is similar to the proof of Theorem 3.3. □

Now, for the consistency of the bootstrap distribution $\hat{H}^{*(m,k)}_{r_B,n,B}(x)$, the next theorem gives a proper choice for the normalizing constants $\hat{c}_B$ and $\hat{d}_B$ satisfying conditions (K_2) and (K_3) in Theorem 4.3 for each domain of attraction of $\mathcal{N}\left(\frac{C^\star_{\lambda_{(m)}}}{C^\star_{\lambda}}(m+1)V_{i;\alpha}(x)\right)$, $i=1,2,3$.

Theorem 4.4. *Let* $r^{'}_n=[\lambda_{(m)}n]+1$, $r^{''}_n=\left[\frac{n}{\sqrt{B}}+n\right]+1$, *and* $r^{'''}_n=\left[\lambda_{(m)}n-\frac{n}{\sqrt{B}}\right]+1$. *Then,*

i. $\hat{c}_B=F_n^{-1}\left(\lambda_{(m)}+\frac{1}{\sqrt{B}}\right)-F_n^{-1}\left(\lambda_{(m)}\right)=X_{r^{''}_n:n}-X_{r^{'}_n:n}$, *and* $\hat{d}_B=F_n^{-1}\left(\lambda_{(m)}\right)=X_{r^{'}_n:n}$, *if* $F\in D^{\lambda_{(m)}}(\mathcal{N}(V_{1;\alpha}(x)))$;

ii. $\hat{c}_B=F_n^{-1}\left(\lambda_{(m)}\right)-F_n^{-1}\left(\lambda_{(m)}-\frac{1}{\sqrt{B}}\right)=X_{r^{'}_n:n}-X_{r^{'''}_n:n}$, *and* $\hat{d}_B=F_n^{-1}\left(\lambda_{(m)}\right)=X_{r^{'}_n:n}$, *if* $F\in D^{\lambda_{(m)}}(\mathcal{N}(V_{2;\alpha}(x)))$;

iii. $\hat{c}_B=F_n^{-1}\left(\lambda_{(m)}+\frac{1}{\sqrt{B}}\right)-F_n^{-1}\left(\lambda_{(m)}\right)=X_{r^{''}_n:n}-X_{r^{'}_n:n}$, *and* $\hat{d}_B=F_n^{-1}\left(\lambda_{(m)}\right)=X_{r^{'}_n:n}$, *if* $F\in D^{\lambda_{(m)}}(\mathcal{N}(V_{3;\alpha}(x)))$.

Moreover, if $B=o(n)$, *then*

$$\sup_{x\in\mathbb{R}}\left|\hat{H}^{*(m,k)}_{r_B,n,B}(x)-\Psi^{(m,k)}_{\lambda}(x)\right|\xrightarrow[n]{p}0. \tag{4.15}$$

Finally, if $\sum_{n=1}^{\infty}P^{\sqrt{\frac{n}{B}}}<\infty$ *for each* $P\in(0,1)$, *then the convergence in (4.15) holds w.p.1.*

Proof. Let $F\in D^{\lambda_{(m)}}(\mathcal{N}(V_{1;\alpha}(x)))$. In view of Theorem 4.3, we need to show that

$$\frac{\hat{c}_B}{c_B}=\frac{X_{r^{''}_n:n}-X_{r^{'}_n:n}}{c_B}\xrightarrow[n]{}1, \tag{4.16}$$

and

$$\frac{\hat{d}_B-d_B}{c_B}=\frac{X_{r^{'}_n:n}-d_B}{c_B}\xrightarrow[n]{}0, \tag{4.17}$$

both in probability or w.p.1. We start with the convergence in probability. Clearly,

$$\frac{\hat{c}_B}{c_B}=\frac{X_{r^{''}_n:n}-X_{r^{'}_n:n}}{c_B}=\frac{X_{r^{''}_n:n}-d_B}{c_B}-\frac{X_{r^{'}_n:n}-d_B}{c_B}.$$

To prove (4.16) and (4.17), it is sufficient to show that

$$\frac{X_{r^{''}_n:n}-d_B}{c_B}\xrightarrow[n]{p}1, \tag{4.18}$$

and

$$\frac{X_{r^{'}_n:n}-d_B}{c_B}\xrightarrow[n]{p}0. \tag{4.19}$$

We will start by proving (4.18). In view of the relations $\left[\frac{n}{\sqrt{B}}+\lambda_{(m)}n\right]=\frac{n}{\sqrt{B}}+\lambda_{(m)}n-\delta$ and $\frac{1}{\sqrt{B}}+\lambda_{(m)}+\frac{1-\delta}{n}\sim\lambda_{(m)}$ for $0\leq\delta<1$, we get

$$\frac{(n-r_n^{''})-n\overline{F}(c_Bx+d_B)}{\sqrt{r_n^{''}(1-\frac{r_n^{''}}{n})}}=\sqrt{n}\frac{\left(1-\frac{1}{\sqrt{B}}-\lambda_{(m)}-\frac{1-\delta}{n}\right)-\overline{F}(c_Bx+d_B)}{\sqrt{\left(\frac{1}{\sqrt{B}}+\lambda_{(m)}+\frac{1-\delta}{n}\right)\left(1-\left(\frac{1}{\sqrt{B}}+\lambda_{(m)}+\frac{1-\delta}{n}\right)\right)}}$$

$$\sim\sqrt{\frac{n}{B}}\left(\sqrt{B}\frac{(1-\lambda_{(m)})-\overline{F}(c_Bx+d_B)}{C_{\lambda_{(m)}}}-\frac{1+\frac{\sqrt{B}}{n}(1-\delta)}{C_{\lambda_{(m)}}}\right)\underset{n}{\longrightarrow}\begin{cases}\infty, & \text{if } x>1,\\ -\infty, & \text{if } x<1.\end{cases}\tag{4.20}$$

Relation (4.20) is a direct consequence of the obvious relations,

$$\frac{1+\frac{\sqrt{B}}{n}(1-\delta)}{C_{\lambda_{(m)}}}\underset{n}{\longrightarrow}\frac{1}{C_{\lambda_{(m)}}}=c \text{ and } \sqrt{B}\frac{(1-\lambda_{(m)})-\overline{F}(c_Bx+d_B)}{C_{\lambda_{(m)}}}\underset{n}{\longrightarrow}cx^{\alpha}.$$

The relation (4.20) yields $P\left(\frac{X_{r_n^{''}:n}-d_B}{c_B}<\epsilon+1\right)\underset{n}{\longrightarrow}\mathcal{N}(\infty)=1$, which is equivalent to

$$P\left(\frac{X_{r_n^{''}:n}-d_B}{c_B}>\epsilon+1\right)\underset{n}{\longrightarrow}0.\tag{4.21}$$

Similarly, we get

$$P\left(\frac{X_{r_n^{''}:n}-d_B}{c_B}<-\epsilon+1\right)\underset{n}{\longrightarrow}\mathcal{N}(-\infty)=0.\tag{4.22}$$

From (4.21) and (4.22), we get $P\left(\left|\frac{X_{r_n^{''}:n}-d_B}{c_B}-1\right|>\epsilon\right)\underset{n}{\longrightarrow}0$, which proves (4.18). Now, we are going to prove (4.19). It is simple to derive that

$$\frac{(n-r_n^{'})-n\overline{F}(c_Bx+d_B)}{\sqrt{r_n^{'}(1-\frac{r_n^{'}}{n})}}=\sqrt{n}\frac{\left(1-\lambda_{(m)}-\frac{1-\delta}{n}\right)-\overline{F}(c_Bx+d_B)}{\sqrt{\left(\lambda_{(m)}+\frac{1-\delta}{n}\right)\left(1-\left(\lambda_{(m)}+\frac{1-\delta}{n}\right)\right)}}$$

$$=\sqrt{\frac{n}{B}}\left(\sqrt{B}\frac{\left(1-\lambda_{(m)}-\frac{1-\delta}{n}\right)-\overline{F}(c_Bx+d_B)}{\sqrt{\left(\lambda_{(m)}+\frac{1-\delta}{n}\right)\left(1-\left(\lambda_{(m)}+\frac{1-\delta}{n}\right)\right)}}\right)\underset{n}{\longrightarrow}\begin{cases}\infty, & \text{if } x>0,\\ -\infty, & \text{if } x<0.\end{cases}\tag{4.23}$$

Thus, from (4.23), we have $P\left(\frac{X_{r_n^{'}:n}-d_B}{c_B}<\epsilon\right)\underset{n}{\longrightarrow}\mathcal{N}(\infty)=1$, which is equivalent to

$$P\left(\frac{X_{r_n^{'}:n}-d_B}{c_B}>\epsilon\right)\underset{n}{\longrightarrow}0.\tag{4.24}$$

Similarly, we get

$$P\left(\frac{X_{r_n^{'}:n}-d_B}{c_B}<-\epsilon\right)\underset{n}{\longrightarrow}0.\tag{4.25}$$

Therefore, by combining the relations (4.24) and (4.25), we get $P\left(\left|\frac{X_{r_n^{'}:n}-d_B}{c_B}\right|>\epsilon\right)\underset{n}{\longrightarrow}0$, which proves (4.19). This completes the proof of Part i. Now, let $F(c_nx+d_n)\in D^{\lambda_{(m)}}(\mathcal{N}(V_{2;\alpha}(x)))$. It is sufficient to establish from Theorem 4.3 that

$$\frac{\hat{c}_B}{c_B}=\frac{X_{r_n^{'}:n}-X_{r_n^{'''}:n}}{c_B}\underset{n}{\longrightarrow}1,\tag{4.26}$$

and

$$\frac{\hat{d}_B - d_B}{c_B} = \frac{X_{r'_n:n} - d_B}{c_B} \xrightarrow[n]{} 0, \tag{4.27}$$

both in probability or w.p.1. We begin with the situation of probability convergence, and we begin with

$$\frac{\hat{c}_B}{c_B} = \frac{X_{r'_n:n} - X_{r'''_n:n}}{c_B} = \frac{X_{r'_n:n} - d_B}{c_B} - \frac{X_{r'''_n:n} - d_B}{c_B}.$$

Hence, to prove (4.26) and (4.27), it is sufficient to show that

$$\frac{X_{r'''_n:n} - d_B}{c_B} \xrightarrow[n]{p} -1, \tag{4.28}$$

and

$$\frac{X_{r'_n:n} - d_B}{c_B} \xrightarrow[n]{p} 0. \tag{4.29}$$

We are going to prove (4.28). By applying the relations $[\lambda_{(m)}n - \frac{n}{\sqrt{B}}] = \lambda_{(m)}n - \frac{n}{\sqrt{B}} - \delta,\ 0 \le \delta < 1$, and $\lambda_{(m)} - \frac{1}{\sqrt{B}} + \frac{1-\delta}{n} \sim \lambda_{(m)}$, as $n \to \infty$, we can deduce that

$$\frac{(n - r'''_n) - n\overline{F}(c_B x + d_B)}{\sqrt{r'''_n(1 - \frac{r'''_n}{n})}} = \sqrt{n}\frac{\left(n - \lambda_{(m)} + \frac{1}{\sqrt{B}} - \frac{1-\delta}{n}\right) - \overline{F}(c_B x + d_B)}{\sqrt{\left(\lambda_{(m)} - \frac{1}{\sqrt{B}} + \frac{1-\delta}{n}\right)\left(1 - \left(\lambda_{(m)} - \frac{1}{\sqrt{B}} + \frac{1-\delta}{n}\right)\right)}}$$

$$\sim \sqrt{\frac{n}{B}}\left(\sqrt{B}\frac{(1 - \lambda_{(m)}) - \overline{F}(c_B x + d_B)}{C_{\lambda_{(m)}}} - \frac{-1 + \frac{\sqrt{B}}{n}(1-\delta)}{C_{\lambda_{(m)}}}\right) \xrightarrow[n]{} \begin{cases} -\infty, & \text{if } |x| > 1, \\ \infty, & \text{if } |x| < 1. \end{cases} \tag{4.30}$$

Thus, on account of (4.30), we get $P\left(\frac{X_{r'''_n:n} - d_B}{c_B} < \epsilon - 1\right) \xrightarrow[n]{} \mathcal{N}(\infty) = 1$, which implies

$$P\left(\frac{X_{r'''_n:n} - d_B}{c_B} > \epsilon - 1\right) \xrightarrow[n]{} 0. \tag{4.31}$$

In a similar vein, we have

$$P\left(\frac{X_{r'''_n:n} - d_B}{c_B} < -\epsilon - 1\right) \xrightarrow[n]{} \mathcal{N}(-\infty) = 0. \tag{4.32}$$

From (4.31) and (4.32), we get $P\left(\left|\frac{X_{r'''_n:n} - d_B}{c_B} + 1\right| > \epsilon\right) \xrightarrow[n]{} 0$. Hence, (4.28) is proved. We turn now to prove (4.29). We start with the obvious limit relation

$$\frac{(n - r'_n) - n\overline{F}(c_B x + d_B)}{\sqrt{r'_n(1 - \frac{r'_n}{n})}} = \sqrt{n}\frac{\left(1 - \lambda_{(m)} - \frac{1-\delta}{n}\right) - \overline{F}(c_B x + d_B)}{\sqrt{\left(\lambda_{(m)} + \frac{1-\delta}{n}\right)\left(1 - \left(\lambda_{(m)} + \frac{1-\delta}{n}\right)\right)}} =$$

$$\sqrt{\frac{n}{B}}\left(\sqrt{B}\frac{\left(1 - \lambda_{(m)} - \frac{1-\delta}{n}\right) - \overline{F}(c_B x + d_B)}{\sqrt{\left(\lambda_{(m)} + \frac{1-\delta}{n}\right)\left(1 - \left(\lambda_{(m)} + \frac{1-\delta}{n}\right)\right)}}\right) \xrightarrow[n]{} \begin{cases} \infty, & \text{if } x > 0, \\ -\infty, & \text{if } x < 0, \end{cases} \tag{4.33}$$

which in turn implies that $P\left(\frac{X_{r_n':n}-d_B}{c_B}<\epsilon\right)\underset{n}{\longrightarrow}\mathcal{N}(\infty)=1$, and hence

$$P\left(\frac{X_{r_n':n}-d_B}{c_B}>\epsilon\right)\underset{n}{\longrightarrow}0. \tag{4.34}$$

Moreover, the limit relation (4.33) yields

$$P\left(\frac{X_{r_n':n}-d_B}{c_B}<-\epsilon\right)\underset{n}{\longrightarrow}0. \tag{4.35}$$

By combining (4.34) and (4.35), we get $P\left(\left|\frac{X_{r_n':n}-d_B}{c_B}\right|>\epsilon\right)\underset{n}{\longrightarrow}0$, which proves (4.29).
Finally, consider the case $F\in D^{\lambda_{(m)}}(\mathcal{N}(V_{3;\alpha}(x)))$. From Theorem 4.3, it suffices to show that

$$\frac{\hat{c}_B}{c_B}=\frac{X_{r_n'':n}-X_{r_n':n}}{c_B}\underset{n}{\longrightarrow}1, \tag{4.36}$$

and

$$\frac{\hat{d}_B-d_B}{c_B}=\frac{X_{r_n':n}-d_B}{c_B}\underset{n}{\longrightarrow}0, \tag{4.37}$$

both in probability or w.p.1. We first focus on the case of the convergence in probability, and we start with

$$\frac{\hat{c}_B}{c_B}=\frac{X_{r_n'':n}-X_{r_n':n}}{c_B}=\frac{X_{r_n'':n}-d_B}{c_B}-\frac{X_{r_n':n}-d_B}{c_B}.$$

Therefore, to prove (4.36) and (4.37), it is sufficient to show that

$$\frac{X_{r_n'':n}-d_B}{c_B}\overset{p}{\underset{n}{\longrightarrow}}1, \tag{4.38}$$

and

$$\frac{X_{r_n':n}-d_B}{c_B}\overset{p}{\underset{n}{\longrightarrow}}0. \tag{4.39}$$

By proceeding in the same manner that we did in Parts i and ii, we can easily show that

$$\frac{(n-r_n'')-n\overline{F}(c_Bx+d_B)}{\sqrt{r_n''(1-\frac{r_n''}{n})}}\underset{n}{\longrightarrow}\begin{cases}\infty, & \text{if } x>1,\\ -\infty, & \text{if } x<1.\end{cases} \tag{4.40}$$

Relation (4.40) yields $P\left(\frac{X_{r_n'':n}-d_B}{c_B}<\epsilon+1\right)\underset{n}{\longrightarrow}\mathcal{N}(\infty)=1$, which is equivalent to

$$P\left(\frac{X_{r_n'':n}-d_B}{c_B}>\epsilon+1\right)\underset{n}{\longrightarrow}0. \tag{4.41}$$

Similarly, we get

$$P\left(\frac{X_{r_n'':n}-d_B}{c_B}<-\epsilon+1\right)\underset{n}{\longrightarrow}\mathcal{N}(-\infty)=0. \tag{4.42}$$

The two limit relations (4.41) and (4.42) yield

$$P\left(\left|\frac{X_{k_n'':n}-d_B}{c_B}-1\right|>\epsilon\right)\underset{n}{\longrightarrow}0,$$

which in turn proves (4.38). On the other hand, the proof of the relation (4.37) follows also by proceeding as we did in Parts i and ii. Finally, in order to transfer to the convergence w.p.1 in Parts i–ii, we argue in the same way we did at the end of the proof of Theorem 3.3. The proof of the theorem is now completed. □

5. Simulation study

In this section, a simulation study is conducted to explain how we can choose the bootstrap re-sample size B numerically, relying on the best fit of the bootstrapping DF of central and intermediate quantiles for different ranks. More specifically, we apply the Kolmogorov-Smirnov (K-S) goodness of fit test to test the null hypothesis H_0 : The central (intermediate) quantiles follow a normal distribution at a 5% significance level. Moreover, we repeat the K-S test many times for different values of B and then select the value of B that corresponds to the largest average p-value for the fit. See Tables 1–3.

In this simulation, three m-GOSs sub-models based on the standard normal distribution are considered. Namely, these are OOSs ($\gamma_i = n - i + 1$), SOSs with $m = 1$ (i.e. $\gamma_i = 2(n - i) + 1$) and SOSs with $m = 2$ (i.e., $\gamma_i = 3(n - i) + 1$). Moreover, the full sample size is $n = 20,000$ (if we enlarge the sample size, the running time becomes very long, especially for the SOSs model), the number of replicates is $M = 1000$, the central ranks are $\lambda = 0.25$ and $\lambda = 0.5$, for the re-sample bootstrap, $B = 100 - 400(50)$, and we choose the intermediate rank to be $r_n = \sqrt{n}$. The results are presented in Tables 1 and 2.

This study relies on the fact that the sample central and intermediate quantiles based on the standard normal distribution weakly converge to the normal distribution. Moreover, according, to the results of Sections 3 and 4, we expect that the bootstrapping DFs of central and intermediate quantiles converge to the normal distribution provided that $B \ll n$ (i.e., $B = o(n)$) for central ranks and $S(B) \ll n$ (i.e., $S(B) = o(n)$) for intermediate ranks. Moreover, based on the K-S test for normality and the corresponding p-values, the best value of B for central ranks (that corresponds to the largest p-value) should be chosen such that $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{B}}} < \infty, \ \forall\ \lambda \in (0,1)$, while the best value of B for intermediate ranks should chosen such that $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{S(B)}}} < \infty, \ \forall\ \lambda \in (0,1)$ (see Remark 5.3).

The simulation study is implemented by Mathematica 12.3 via the following algorithm:

Step 1. Select the m-GOSs model. That is, choose the values of γ_i.

Step 2. Generate an ordered sample of size $n = 20000$, say $\mathbf{X}_n$, based on the standard normal distribution by the algorithm of Barakat et al. [8].

Step 3. Choose the central or intermediate rank.

Step 4. Select the bootstrap re-sample size B.

Step 5. Select M random samples, each of which is size B, from $\mathbf{X}_n$ with replacement.

Step 6. Compute the quantile of each sample, and store them in Q_B.

Step 7. By using the K-S test, check the normality of the data sets Q_B to the normal distribution and then compute the p-value.

Step 8. Repeat Steps 5–7 many times (100 times) for each chosen value of B and then compute the average p-value.

Step 9. Repeat Steps 4–8 for different values of B, and then pick the largest average p-value and the corresponding value of B.

Remark 5.1. *In the earlier version of this paper, in order to implement Step 7 of the previous algorithm, we fitted the data sets Q_B to the normal DF by using the K-S test after calculating the sample mean and standard deviation. However, we noted an important issue that the K-S test can be used to fit the* *normal distribution only when parameters are not estimated from the data (cf. [24]). Since our focus here is only on checking the normality of the bootstrap samples, we apply the K-S test to check the normality of the given sample bootstrapping statistics without estimating any parameters.*

Remark 5.2. *According to the results presented in Tables 1 and 2, it is noted that for* $n = 20000$, *the largest average p-values for selected central ranks are always achieved at B, which falls in the interval [100–200]. Moreover, the best average p-values for selected intermediate ranks are achieved at B falling in the interval [200–300]. However, it is shown that the accuracy of the goodness of fit depends on both the selected GOSs model and the selected ranks. Moreover, in view of the results given in Table 3, the average p-value for intermediate OSs increases as the sample size increases with the same bootstrap re-sample size.*

Remark 5.3. *Based on the results of Sections 3 and 4, the best performance of the bootstrapping DFs of the central m-GOSs occurs at the values of B for which* $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{B}}} < \infty, \forall\, \lambda \in (0,1)$. *On the other hand, according to [2], the condition* $\sqrt{B} = o(\frac{2\sqrt{n}}{\log n})$ *is a sufficient condition for* $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{B}}} < \infty, \forall$ $\lambda \in (0,1)$, *which implies that the best performance of the bootstrapping DFs of the central OSs occurs when* $B \ll 800$. *Moreover, in the case of intermediate m-GOSs, the condition* $\sqrt{S(B)} = o(\frac{2\sqrt{n}}{\log n})$ *is a sufficient condition for* $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{S(B)}}} < \infty, \forall\, \lambda \in (0,1)$, *which implies that the best performance of the bootstrapping DFs of intermediate m-GOSs occurs when* $S(B) \ll 800$. *Since in our study we choose the intermediate rank* $r_n = \sqrt{n}$, *this implies* $B \ll 7500$. *Therefore, the simulation output endorses this anticipated result.*

Table 1. The average p-values ($\overline{p}$) for fitting different bootstrap central quantiles of the SOSs and OOSs models to a normal distribution with various values of B.

	$k=1,\ m=1$ (SOSs)		$k=1,\ m=2$ (SOSs)		$k=1,\ m=0$ (OOSs)	
	$\lambda=0.5$	$\lambda=0.25$	$\lambda=0.5$	$\lambda=0.25$	$\lambda=0.5$	$\lambda=0.25$
B	$\overline{p}$	$\overline{p}$	$\overline{p}$	$\overline{p}$	$\overline{p}$	$\overline{p}$
100	0.25239	0.31176	0.34101	0.22219	0.35895⋆	0.26554
150	0.29968⋆	0.24726	0.36257⋆	0.22550	0.31619	0.31344⋆
200	0.17472	0.32276⋆	0.34588	0.22552⋆	0.32378	0.21946
250	0.21397	0.24931	0.31296	0.18478	0.27626	0.24845
300	0.19569	0.29015	0.26269	0.18825	0.29849	0.18454
350	0.19635	0.26083	0.24576	0.18935	0.27687	0.16976
400	0.15829	0.26653	0.26114	0.17181	0.24873	0.17627

Table 2. The average p-values for fitting bootstrap intermediate quantiles of the OOSs and SOSs models to a normal distribution with various values of B.

	Intermediate (OOSs)	Intermediate (SOSs)
B	$\overline{p}\ (r_n = \sqrt{n})$	$\overline{p}\ (r_n^{\star} = \sqrt{n})$
100	0.12289	0.02019
150	0.20122	0.07148
200	0.28733⋆	0.14383
250	0.18223	0.18207
300	0.12546	0.21568⋆
350	0.08254	0.17340
400	0.05879	0.12973

Table 3. The average p-values for fitting bootstrap intermediate quantiles of OOSs to a normal distribution with various values of B when $M = 1000$ and $n = 200,000$.

	Intermediate (OOSs)
B	$\overline{p}$ $(r_n = \sqrt{n})$
100	0.30920
150	0.22490
200	0.25541
250	0.36622
300	0.43944*
350	0.39108
400	0.38285

6. Conclusions

The bootstrap method is an efficient procedure for solving many statistical problems based on re-sampling from the available data. For example, the bootstrap method is used to find standard errors for estimators, confidence intervals for unknown parameters, and p-values for test statistics under a null hypothesis. One of the desired properties of the bootstrapping method is consistency, which guarantees that the limit of the bootstrap distribution is the same as the distribution of the given statistic. In this paper, we investigated the strong consistency of bootstrapping central and intermediate m-GOSs for an appropriate choice of re-sample size for known and unknown normalizing constants. Finally, a simulation study was conducted to to explain how we can choose the bootstrap re-sample size B numerically, relying on the best fit of the bootstrapping DF of central and intermediate quantiles for different ranks.

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Conflict of interest

All authors declare no conflict of interest in this paper.

References

1. J. Adams, G. Atkinson, Development of seismic hazard maps for the proposed 2005 edition of the National Building Code of Canada, *Can. J. Civil Eng.,* **30** (2003), 255–271. https://doi.org/10.1139/l02-070

2. K. B. Athreya, J. Fukuchi, *Bootstrapping extremes of i.i.d random variables*, In Proceedings of the Conference on Extreme Value Theory and Applications, NIST, Maryland, **7** (1994), 23–29.

3. K. B. Athreya, J. Fukuchi, Confidence interval for end point of a c.d.f, via bootstrap, *J. Stat. Plan. Infer.,* **58** (1997), 299–320. https://doi.org/10.1016/S0378-3758(96)00087-0

4. H. M. Barakat, Limit theory of generalized order statistics, *J. Stat. Plan. Infer.,* **137** (2007), 1–11. https://doi.org/10.1016/j.jspi.2005.10.003

5. H. M. Barakat, M. A. El-Shandidy, Some limit theorems of intermediate term of a random number of independent random variables, *Comment. Math. Univ. Ca.,* **31** (1990), 323–336. http://hdl.handle.net/10338.dmlcz/106862

6. H. M. Barakat, A. R. Omar, On limit distributions for intermediate order statistics under power normalization, *Math. Methods Stat.,* **20** (2011), 365–377. https://doi.org/10.3103/S1066530711040053

7. H. M. Barakat, A. R. Omar, On convergence of intermediate order statistics under power normalization, *J. Stat. Appl. Pro.,* **4** (2015), 405–409. http://dx.doi.org/10.12785/jsap/040307

8. H. M. Barakat, M. E. El-Adll, A. E. Aly, Exact prediction intervals for future exponential lifetime based on random generalized order statistics, *J. Comput. Math. Appl.,* **61** (2011), 1366–1378. https://doi.org/10.1016/j.camwa.2011.01.002

9. H. M. Barakat, E. M. Nigm, A. M. Alaswed , The Hill estimators under power normalization, *Appl. Math. Model.,* **45** (2017), 813–822. https://doi.org/10.1016/j.apm.2017.01.028

10. H. M. Barakat, E. M. Nigm, M. E. El-Adll, Bootstrap for extreme generalized order statistics, *Arab. J. Sci. Eng.,* **36** (2011), 1083–1090. https://doi.org/10.1007/s13369-011-0105-1

11. H. M. Barakat, E. M. Nigm, M. H. Harpy, Limit theorems for univariate and bivariate order statistics with variable ranks, *Statistics,* **54** (2020), 737–755. https://doi.org/10.1080/02331888.2020.1772787

12. H. M. Barakat, E. M. Nigm, O. M. Khaled, F. A. Momenkhan, Bootstrap method for order statistics and modeling study of the air pollution, *Commun. Stat.-Simul. C.,* **44** (2015), 1477–1491. https://doi.org/10.1080/03610918.2013.805051

13. H. M. Barakat, E. M. Nigm, O. M. Khaled, Bootstrap method for central and intermediate order statistics under power normalization, *Kybernetika,* **51** (2015), 923–932. Available from: `http://hdl.handle.net/10338.dmlcz/144817.`

14. P. J. Bickel, D. A. Freedman, Some asymptotic theory for the bootstrap, *Ann. Stat.,* **9** (1981), 1196–1217. https://doi.org/10.1214/aos/1176345637

15. D. M. Chibisov, On limit distributions for order statistics, *Theor. Probab. Appl.,* **9** (1964), 142–148. https://doi.org/10.1137/1109021

16. E. W. Cope, G. Mignolia, G. Antonini, R. Ugoccioni, Challenges and pitfalls in measuring operational risk from loss data, *J. Oper. Risk,* **4** (2009), 3–27. https://doi.org/10.21314/JOP.2009.069

17. E. Cramer, *Contributions to generalized order statistics*, Habilitationsschrift, Reprint: University of Oldenburg, 2003.

18. L. de Haan, Fighting the arch-enemy with mathematics, *Stat. Neerl.,* **44** (1990), 45–68. https://doi.org/10.1111/j.1467-9574.1990.tb01526.x

19. L. de Haan, A. Ferreira, *Extreme value theory: An introduction*, New York: Springer, 2006. https://doi.org/10.1007/0-387-34471-3

20. B. Efron, *Bootstrap methods: Another look at the jackknife,* Breakthroughs in statistics, New York: Springe, 1992, 569–593. https://doi.org/10.1007/978-1-4612-4380-9_41

21. J. Fukuchi, *Bootstrapping extremes of random variables*, Doctoral dissertation, Ames, IA: Iowa State University, 1994. https://doi.org/10.31274/rtd-180813-10322
22. U. Kamps, A concept of generalized order statistics, *J. Stat. Plan. Infer.,* **48** (1995), 1–23. https://doi.org/10.1016/0378-3758(94)00147-N
23. M. R. Leadbetter, G. Lindgren, H. Rootzén, *Extremes and related properties of random sequences and processes*, New York: Springer, 1983. https://doi.org/10.1007/978-1-4612-5449-2
24. H. W. Lilliefors, On the Kolmogorov-Smirnov test for normality with mean and variance unknown, *J. Am. Stat. Assoc.,* **62** (1967), 399–402. https://doi.org/10.1080/01621459.1967.10482916
25. D. M. Mason, Laws of large numbers for sums of extreme values, *Ann. Probab.,* **10** (1982), 754–764. Available from: `https://www.jstor.org/stable/2243383.`
26. C. H. Nagaraja, H. N. Nagaraja, Distribution-free approximate methods for constructing confidence intervals for quantiles *Int. Stat. Rev.,* **88** (2020), 75–100. https://doi.org/10.1111/insr.12338
27. D. Nasri-Roudsari, Extreme value theory of generalized order statistics, *J. Stat. Plan. Inf.,* **55** (1996), 281–297. https://doi.org/10.1016/S0378-3758(95)00200-6
28. D. Nasri-Roudsari, Limit distributions of generalized order statistics under power normalization, *Commun. Stat.-Theor. M.,* **28** (1999), 1379–1389. https://doi.org/10.1080/03610929908832362
29. D. Nasri-Roudsari, E. Cramer, On the convergence rates of extreme generalized order statistics, *Extremes,* **2** (1999), 421–447. https://doi.org/10.1023/A:1009904316589
30. J. Pickands III, Statistical inference using extreme order statistics, *Ann. Stat.,* **3** (1975), 119–131. https://www.jstor.org/stable/2958083
31. N. V. Smirnov, Limit distribution for terms of a variational series, *Am. Math. Soc. Trans. Ser.,* **11** (1952), 82–143.
32. J. L. Teugels, Limit theorems on order statistics, *Ann. Probab.,* **9** (1981), 868–880. Available from: `https://www.jstor.org/stable/2243744.`
33. C. Y. Wu, Types of limit distributions for some terms of variational series, *Sci. Sin.,* **15**(1966), 749–762.

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