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Research article

New iterative scheme for fixed point results of weakly compatible maps in multiplicative G_M−metric space via various contractions with application

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Abstract: In this manuscript, we induced several vivid common fixed point theorems for four maps in the setting of complete multiplicative G_M−metric space via various contractive conditions such as Δ-implicit contractions. Some new definitions and results are introduced in multiplicative G_M− metric space.Moreover, illustrative examples are given to validate our obtained results and an application to a system of nonlinear integral equations are provided to show the novelty of our new results.

Keywords: fixed point; multiplicative G_M−metric space; point of coincidence; weakly compatible maps; Δ-implicit contractions
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1. Introduction

Due to its importance, fixed point theory is an exciting branch of mathematics and has vital, major and basic part in both of applied sciences and pure mathematics such as mathematical modeling, modern optimization, control theory, mathematical economics, and other domains for more details see [1, 3, 5, 11, 20, 31–33]. It has numerous applications in many areas of mathematical science.

In 2008, Bashirov et al. [8] induced the definition of multiplicative metric space beside studying some major properties. After that, Bashirov et al. [9] and Florack et al. [14] also studied some other properties in this space. In 2012, Özavsar and Çevikel [23] presented the notion of multiplicative contraction maps on multiplicative metric space in such a way that multiplicative triangle inequality is used instead of the usual triangular inequality and obtained various existence results of fixed point beside many topological characteristics of multiplicative metric space. In 2013, He et al. [17] showed the existence result of common fixed point of four maps using the weakly commuting condition. Inspired by the work of He et al. [17], Gu and Cho [16] used a contraction condition constructed by virtue of a comparison function to obtain existence results of the common fixed point for four maps. Also, many researchers studied common fixed point theorems using the locally contractive, compatible and weakly compatible conditions respectively (see [1,4,6,7,12,13,15,16,19,29,30]). Recently, Jiang and Gu [18] displayed the notion of ϕ-weakly commutative maps and obtained for four maps several common fixed point theorems.

In 2011, Bhatt et al. [10] introduced the notion of weakly compatible maps and concluded some common fixed point theorem in complex-valued metric space for these maps. In 2020, Alfaqih et al. [2] presented the notion of common coincidence point of two pairs of maps beside using implicit relation with applications to show unified common fixed point theorems in complex-valued metric space.

According to this direction, the purpose of this manuscript is to use the notion of implicit contractions beside other new contractions in multiplicative G_M-metric space to show unique common fixed point results of four weakly compatible maps holding those implicit contractions and the other new contractions. Eventually, we introduce several examples to support new results.

2. Preliminaries

Now, we recall many well-known definitions, concepts and usual terminology that will be used in the sequel of discussion.

Definition 2.1. [8] Assume a nonempty set U and a function $\theta_M : U^2 \longrightarrow [0, +\infty)$ hold the following properties:
(θ_{M1}) $\theta_M(q,t) \geq 1, \quad \forall\ q,t \in U;$
(θ_{M2}) $\theta_M(q,t) = 1 \quad \text{iff} \quad q = t;$
(θ_{M3}) $\theta_M(q,t) = \theta_M(t,q) \quad$ (symmetry);
(θ_{M4}) $\theta_M(q,t) \leq \theta_M(q,h) \,.\, \theta_M(h,t) \quad \forall\ q,t,h \in U$ (multiplicative triangle inequality).
The function θ_M is a multiplicative metric on U and the couple (U, θ_M) is a multiplicative metric space.

Definition 2.2. [22] Suppose V denote to a nonempty set and $G_M : V^3 \longrightarrow \mathbb{R}^+$ to a function verify the following assertions:
(G_{M_1}) $G_M(q,t,h) = 1$ if $q = t = h;$
(G_{M_2}) $G_M(q,q,t) > 1 \quad \forall\ q,t \in V$ with $q \neq t;$
(G_{M_3}) $G_M(q,q,t) \leq G_M(q,t,h) \quad \forall\ q,t,h \in V$ with $t \neq h;$
(G_{M_4}) $G_M(q,t,h) = G_M(q,h,t) = G_M(t,h,q) = \ldots$ (symmetry in all variables);
(G_{M_5}) $G_M(q,t,h) \leq G_M(q,f,f) \,.\, G_M(f,t,h) \quad \forall\ q,t,h,f \in V,$ (rectangular inequality).
Thus, the function G_M is a multiplicative generalized metric or, specifically, multiplicative G_M-metric on V and (V, G_M) is a multiplicative G_M-metric space.

Example 2.3. Suppose a G-metric space $(V, G^\bullet)$ and $G_M : V^3 \longrightarrow \mathbb{R}^+$ is defined by $G_M(q,t,h) = e^{\,G^\bullet(q,t,h)}$ $\forall\ q,t,h \in V$. Then, clearly every G-metric space $(V, G^\bullet)$ generates multiplicative G_M-metric space.

Proposition 2.4. [22] *Assume a multiplicative G_M-metric space (V, G_M), then for all $q, t, h, j \in V$, the following properties is satisfying:*

(1) $G_M(q,t,h) = 1$ *if* $q = t = h$;
(2) $G_M(q,t,h) \leq G_M(q,j,j) \,.\, G_M(t,j,j) \,.\, G_M(h,j,j)$;
(3) $G_M(q,t,h) \leq G_M(q,q,t) \,.\, G_M(q,q,h)$;
(4) $G_M(q,t,t) \leq G_M^2(t,q,q)$.

Remark 2.5. Assume a nonempty set V and $G : V^3 \longrightarrow [0,+\infty)$ has the construction $G(q,t,h) = e^{(|q-t| + |t-h| + |h-q|)}$ $\forall$ $q,t,h \in V$. Furthermore, G is a multiplicative $G-$metric on V and (V, G) is a multiplicative $G-$metric space but G is not $G-$metric on V since the condition (G_1) is not verified. (G_1) $G(q,t,h) = 0$ $\quad if \quad q = t = h$.

Lemma 2.6. [22] *Assume a sequence $\{q_n\}$ in a multiplicative G_M-metric space (V, G_M). If $\{q_n\}$ is a multiplicative G_M-convergent then it is a multiplicative G_M-Cauchy sequence.*

Lemma 2.7. [22] *Suppose a sequence $\{q_n\}$ in a multiplicative G_M-metric space (V, G_M). A sequence $\{q_n\}$ in V is a multiplicative G_M-convergent to $r \in V$ iff $G_M(q_n, r, r) \longrightarrow 1$ as $n \longrightarrow \infty$.*

Lemma 2.8. *Assume two sequences $\{q_n\}$ and $\{t_n\}$ in a multiplicative G_M-metric space (V, G_M) such that $\lim\limits_{n\to\infty} q_n = q$ and $\lim\limits_{n\to\infty} t_n = t$, then $\lim\limits_{n\to\infty} G_M(q_n, t_n, t_n) = G_M(q,t,t)$.*

Proof. For $n \in \mathbb{N}$, we have

$$\begin{aligned} G_M(q_n, t_n, t_n) &\leq G_M(q_n, q, q) \,.\, G_M(q, t_n, t_n) \\ &\leq G_M(q_n, q, q) \,.\, G_M(q, t, t) \,.\, G_M(t, t_n, t_n). \end{aligned}$$

By taking $n \to \infty$, we get

$$\lim_{n\to\infty} G_M(q_n, t_n, t_n) \leq G_M(q,t,t). \tag{2.1}$$

Since

$$\begin{aligned} G_M(q,t,t) &\leq G_M(q, q_n, q_n) \,.\, G_M(q_n, t, t) \\ &\leq G_M(q, q_n, q_n) \,.\, G_M(q_n, t_n, t_n) \,.\, G_M(t_n, t, t). \end{aligned}$$

As $n \to \infty$, we obtain

$$G_M(q,t,t) \leq \lim_{n\to\infty} G_M(q_n, t_n, t_n). \tag{2.2}$$

From (2.1) and (2.2), we find

$$\lim_{n\to\infty} G_M(q_n, t_n, t_n) = G_M(q,t,t).$$

Definition 2.9. [10] Suppose a nonempty set V and a pair of self-maps (J, I) on V. Thus, (J, I) is weakly compatible if

$$Ju = Iu \;\Rightarrow\; JIu = IJu \quad \forall\; u \in V.$$

Definition 2.10. [2] Consider a nonempty set V with P, Q, S and T be four self-maps on V. Further, a point $j \in V$ is
(**1**) a fixed point of P if $Pj = j$;
(**2**) a common fixed point of P and Q if $Pj = Qj = j$;

(**3**) a coincidence point of (T, P) if $Pj = Tj$ and $k \in V$ such that $k = Pj = Tj$ is also a coincidence point of (T, P);

(**4**) a common coincidence point of (T, P) and (S, Q) if there exist $k, r \in V$ such that $Pk = Tk = j$ and $Qr = Sr = j$.

3. Results

The next class of real functions is defined in [24] but we added some simple modifications in (Δ_3) condition:

Δ-implicit contractions. Let $\mathcal{M}$ denote to the class of real-valued functions and $\Delta \in \mathcal{M}$ iff $\Delta : [1, +\infty)^5 \to [0, +\infty)$ holding the following conditions:

(Δ_1) Δ is continuous and non-decreasing in every coordinate variable;

(Δ_2) for all $x, y \in [1, +\infty)$, there exists $q_1, q_2 \in (0, +\infty)$ satisfying $q_1 q_2 < 1$ such that

$$x \le \Delta(y, x, y, 1, yx) \Longrightarrow x \le y^{q_1}, \quad x \le \Delta(y, y, x, yx, 1) \Longrightarrow x \le y^{q_2};$$

(Δ_3) for all $\delta \in (0, \frac{1}{2})$, $z > 1$,

$$\max\left\{\Delta(1, 1, z, z, 1), \Delta(1, z, 1, 1, z), \Delta(1, z, 1, z, 1)\right\} = z^{\delta} < z.$$

Now, we introduce our first main theorems in complete multiplicative G_M-metric space.

Theorem 3.1. *Consider a complete multiplicative G_M-metric space (V, G_M) with the mappings $P, Q, S, T : V \longrightarrow V$ such that $P(V) \subseteq S(V)$ and $Q(V) \subseteq T(V)$ verify the following: for $u, h \in V$, $u \neq h$ and $\beta \in (0, 1/2)$,*

$$\begin{cases} G_M(Qh, Pu, Pu) \le \mathcal{N}_1, \\ G_M(Pu, Qh, Qh) \le \mathcal{N}_2, \end{cases} \tag{3.1}$$

where

$$(\star) \begin{cases} \mathcal{N}_1 = \Upsilon \left\{ \begin{array}{c} G_M(Pu, Tu, Tu),\ G_M(Sh, Tu, Tu), \\ G_M(Qh, Sh, Sh),\ G_M(Qh, Tu, Tu), \\ \min\left\{G_M(Pu, Sh, Sh),\ G_M(Pu, Tu, Tu)\right\} \end{array} \right\}, \\ \\ \mathcal{N}_2 = \Theta \left\{ \begin{array}{c} G_M(Qh, Sh, Sh),\ G_M(Pu, Tu, Tu), \\ G_M(Tu, Sh, Sh), \\ \min\left\{G_M(Qh, Tu, Tu),\ G_M(Qh, Sh, Sh)\right\}, \\ G_M(Pu, Sh, Sh) \end{array} \right\}. \end{cases}$$

Since $\Upsilon, \Theta \in \mathcal{M}$. If one of $\{P(V), S(V), Q(V), T(V)\}$ is complete, then the couples (T, P) and (S, Q) have unique common point of coincidence. Furthermore, the four self-maps have unique common fixed point such that both the couples (T, P) and (S, Q) are weakly compatible.

Proof. Let an arbitrary point u_0 in V. Since $P(V) \subseteq S(V)$ and $Q(V) \subseteq T(V)$, then we can construct the sequence $\{u_k\}$ in V such that,

$$\begin{cases} h_{2k} = Pu_{2k} = Su_{2k+1}, \\ h_{2k+1} = Qu_{2k+1} = Tu_{2k+2}. \end{cases} \tag{3.2}$$

for all $k \in \mathbb{N} \cup \{0\}$. Assume that either $P(V)$ or $S(V)$ is complete. We then prove that $\{h_k\}$ is a multiplicative G_M-Cauchy sequence. Using $u = u_{2k}$ and $h = u_{2k+1}$ in inequality related to (Υ), we have

$$\begin{aligned} G_M(h_{2k+1}, h_{2k}, h_{2k}) &= G_M(Qu_{2k+1}, Pu_{2k}, Pu_{2k}) \\ &\leq \Upsilon \left\{ \begin{array}{c} G_M(Pu_{2k}, Tu_{2k}, Tu_{2k}),\ G_M(Su_{2k+1}, Tu_{2k}, Tu_{2k}), \\ G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1}),\ G_M(Qu_{2k+1}, Tu_{2k}, Tu_{2k}), \\ \min\Big\{G_M(Pu_{2k}, Su_{2k+1}, Su_{2k+1}),\ G_M(Pu_{2k}, Tu_{2k}, Tu_{2k})\Big\} \end{array} \right\} \\ &\leq \Upsilon \left\{ \begin{array}{c} G_M(h_{2k}, h_{2k-1}, h_{2k-1}),\ G_M(h_{2k}, h_{2k-1}, h_{2k-1}), \\ G_M(h_{2k+1}, h_{2k}, h_{2k}),\ G_M(h_{2k+1}, h_{2k-1}, h_{2k-1}), \\ \min\Big\{G_M(h_{2k}, h_{2k}, h_{2k}),\ G_M(h_{2k}, h_{2k-1}, h_{2k-1})\Big\} \end{array} \right\}. \end{aligned}$$

From (G_{M_1}) and (G_{M_5}), we obtain

$$G_M(h_{2k+1}, h_{2k}, h_{2k}) \leq \Upsilon \left\{ \begin{array}{c} G_M(h_{2k}, h_{2k-1}, h_{2k-1}),\ G_M(h_{2k}, h_{2k-1}, h_{2k-1}), \\ G_M(h_{2k+1}, h_{2k}, h_{2k}), \\ G_M(h_{2k+1}, h_{2k}, h_{2k}) \,.\, G_M(h_{2k}, h_{2k-1}, h_{2k-1}),\ 1 \end{array} \right\}.$$

By (Δ_2), we obtain

$$G_M(h_{2k+1}, h_{2k}, h_{2k}) \leq \Big[G_M(h_{2k}, h_{2k-1}, h_{2k-1})\Big]^{q_2}. \tag{3.3}$$

Similarly, by taking $u = u_{2k+2}$ and $h = u_{2k+1}$ in inequality related to (Θ), we get successively

$$\begin{aligned} G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}) &= G_M(Pu_{2k+2}, Qu_{2k+1}, Qu_{2k+1}) \\ &\leq \Theta \left\{ \begin{array}{c} G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1}),\ G_M(Pu_{2k+2}, Tu_{2k+2}, Tu_{2k+2}), \\ G_M(Tu_{2k+2}, Su_{2k+1}, Su_{2k+1}), \\ \min\Big\{G_M(Qu_{2k+1}, Tu_{2k+2}, Tu_{2k+2}),\ G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1})\Big\}, \\ G_M(Pu_{2k+2}, Su_{2k+1}, Su_{2k+1}) \end{array} \right\}. \end{aligned}$$

Using (G_{M_1}) and (G_{M_5}) again, we have

$$G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}) \leq \Theta \left\{ \begin{array}{c} G_M(h_{2k+1}, h_{2k}, h_{2k}),\ G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}), \\ G_M(h_{2k+1}, h_{2k}, h_{2k}),\ 1, \\ G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}) \,.\, G_M(h_{2k+1}, h_{2k}, h_{2k}) \end{array} \right\}.$$

Hence, using (Δ_2) again

$$G_{M}(h_{2k+2}, h_{2k+1}, h_{2k+1}) \leq \Big[G_{M}(h_{2k+1}, h_{2k}, h_{2k})\Big]^{q_1}. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned} G_{M}(h_{2k+1}, h_{2k}, h_{2k}) &\leq \Big[G_{M}(h_{2k}, h_{2k-1}, h_{2k-1})\Big]^{q_2} \\ &\leq \Big[G_{M}(h_{2k-1}, h_{2k-2}, h_{2k-2})\Big]^{q_1 q_2} \leq \,..... \leq \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k}}. \end{aligned} \tag{3.5}$$

Consequently,

$$\begin{aligned} G_{M}(h_{2k+2}, h_{2k+1}, h_{2k+1}) &\leq \Big[G_{M}(h_{2k+1}, h_{2k}, h_{2k})\Big]^{q_1} \\ &\leq \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k} q_1}. \end{aligned} \tag{3.6}$$

For all $k, m \in \mathbb{N}$ with $(k < m)$, we get

$$\begin{aligned} & G_{M}(h_{2m+1}, h_{2k+1}, h_{2k+1}) \\ \leq\;& G_{M}(h_{2m+1}, h_{2m}, h_{2m}) \,.\, G_{M}(h_{2m}, h_{2m-1}, h_{2m-1}) \,..... \\ & \qquad G_{M}(h_{2k+4}, h_{2k+3}, h_{2k+3}) \,.\, G_{M}(h_{2k+3}, h_{2k+2}, h_{2k+2}) \,.\, G_{M}(h_{2k+2}, h_{2k+1}, h_{2k+1}) \\ \leq\;& \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2m}} . \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2m-1} q_1} ..... \\ & \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k+1} q_1} . \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k+1}} . \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k} q_1} \\ \leq\;& \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k} q_1 + (q_1 q_2)^{2k+1} + (q_1 q_2)^{2k+1} q_1 + \,.....\, + (q_1 q_2)^{2m}} \\ \leq\;& \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k} q_1 [1 + (q_1 q_2) + (q_1 q_2)^2 + \,...\,] + (q_1 q_2)^{2k+1} [1 + (q_1 q_2) + (q_1 q_2)^2 + \,...\,]} \\ \leq\;& \Big[G_{M}(h_1, h_0, h_0)\Big]^{(q_1 q_2)^{2k} q_1 (1 + q_2) \, [1 + (q_1 q_2) + (q_1 q_2)^2 + \,.....\,]} \\ \leq\;& \Big[G_{M}(h_1, h_0, h_0)\Big]^{\frac{(q_1 q_2)^{2k} q_1 (1 + q_2)}{1 - (q_1 q_2)}} \longrightarrow 1, \quad as \;\; k, m \longrightarrow \infty, \end{aligned}$$

where $(q_1 q_2 < 1)$ implied that $(q_1 q_2)^{2k} \longrightarrow 0, \quad as \;\; k \longrightarrow \infty.$

This proves that $\{h_k\}$ is multiplicative G_M-Cauchy sequence. From the completeness of (V, G_M), there exists $l \in V$ such that $h_k \longrightarrow l$ as $k \longrightarrow \infty$. Then from Eq (3.2), we find

$$\lim_{k \longrightarrow \infty} P u_{2k} = \lim_{k \longrightarrow \infty} S u_{2k+1} = \lim_{k \longrightarrow \infty} T u_{2k+2} = \lim_{k \longrightarrow \infty} Q u_{2k+1} = l. \tag{3.7}$$

Since $P(V) \subseteq S(V)$, if $l \in S(V)$, then there exists $j \in V$ such that

$$S j = l. \tag{3.8}$$

We will prove that $Qj = Sj$. By putting $u = u_{2k}$ and $h = j$ in inequality related to (Υ), we obtain

$$G_M(Qj, Pu_{2k}, Pu_{2k}) \leq \Upsilon \left\{ \begin{array}{c} G_M(Pu_{2k}, Tu_{2k}, Tu_{2k}),\ G_M(Sj, Tu_{2k}, Tu_{2k}), \\ G_M(Qj, Sj, Sj),\ G_M(Qj, Tu_{2k}, Tu_{2k}), \\ \min\left\{G_M(Pu_{2k}, Sj, Sj),\ G_M(Pu_{2k}, Tu_{2k}, Tu_{2k})\right\} \end{array} \right\}.$$

Putting $k \longrightarrow \infty$ and using Eqs (3.7) and (3.8), we get

$$\begin{aligned} G_M(Qj, l, l) &\leq \Upsilon\left(1,\ 1,\ G_M(Qj, l, l),\ G_M(Qj, l, l),\ 1\right) \\ &= \left[G_M(Qj, l, l)\right]^{\beta}, \end{aligned}$$

which is contradiction from (Δ_3), since $\beta \in (0, 1/2)$, then we obtain

$$\left[G_M(Qj, l, l)\right]^{1-\beta} = 1 \implies G_M(Qj, l, l) = 1,$$

implying thereby $Qj = l$. Hence

$$Qj = l = Sj, \tag{3.9}$$

i.e., l is coincidence point of the couple (S, Q).
As $Q(V) \subseteq T(V)$, there exists $r \in V$ such that

$$Tr = l. \tag{3.10}$$

We will show that $Pr = Tr$. Taking $u = r$ and $h = u_{2k+1}$ in inequality related to (Θ), we obtain

$$G_M(Pr, Qu_{2k+1}, Qu_{2k+1}) \leq \Theta \left\{ \begin{array}{c} G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1}),\ G_M(Pr, Tr, Tr), \\ G_M(Tr, Su_{2k+1}, Su_{2k+1}), \\ \min\left\{G_M(Qu_{2k+1}, Tr, Tr),\ G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1})\right\}, \\ G_M(Pr, Su_{2k+1}, Su_{2k+1}) \end{array} \right\}.$$

Taking $k \longrightarrow \infty$ and using Eqs (3.7) and (3.10), we have

$$G_M(Pr, l, l) \leq \Theta\left(1,\ G_M(Pr, l, l),\ 1,\ 1,\ G_M(Pr, l, l)\right) = \left[G_M(Pr, l, l)\right]^{\beta},$$

using (Δ_3), we get $G_M(Pr, l, l) = 1$ which implies that $Pr = l$. Then

$$Pr = l = Tr, \tag{3.11}$$

i.e., l is also coincidence point of the pair (T, P).
Furthermore, $l \in V$ is common coincidence point for the four maps.

In regard to uniqueness: To prove the uniqueness with respect to the coincidence point, consider $l^* \neq l$ another coincidence point of the four maps. Further, there exists j^*, r^* such that $Qj^* = Sj^* = l^*$ and $Pr^* = Tr^* = l^*$. Putting $u = r^*$ and $h = j$ in (Υ), we get

$$G_M(Qj, Pr^*, Pr^*) \leq \Upsilon \left\{ \begin{array}{c} G_M(Pr^*, Tr^*, Tr^*),\ G_M(Sj, Tr^*, Tr^*), \\ G_M(Qj, Sj, Sj),\ G_M(Qj, Tr^*, Tr^*), \\ \min\left\{G_M(Pr^*, Sj, Sj),\ G_M(Pr^*, Tr^*, Tr^*)\right\} \end{array} \right\}.$$

This implies that

$$\begin{aligned} G_M(l, l^*, l^*) &\leq \Upsilon\Big(1,\ G_M(l, l^*, l^*),\ 1,\ G_M(l, l^*, l^*),\ 1\Big) \\ &= \Big[G_M(l, l^*, l^*)\Big]^{\beta}, \end{aligned}$$

that is contradiction due to (Δ_3), then we have $G_M(l, l^*, l^*) = 1$, i.e., $l = l^*$. Therefore, the couples (T, P) and (S, Q) have unique common point of coincidence.

Consider weak compatibility of the couples (T, P) and (S, Q) and Eqs (3.9), (3.11), we get

$$PTr = TPr, \quad QSj = SQj. \tag{3.12}$$

Thus,

$$Pl = Tl, \quad Ql = Sl, \tag{3.13}$$

i.e., l is coincidence point of the pairs (T, P) and (S, Q).

Now, we prove that l is common fixed point of P, Q, S and T. Putting $u = l$ and $h = j$ in inequality related to (Υ), we obtain

$$G_M(Qj, Pl, Pl) \leq \Upsilon \left\{ \begin{array}{c} G_M(Pl, Tl, Tl),\ G_M(Sj, Tl, Tl), \\ G_M(Qj, Sj, Sj),\ G_M(Qj, Tl, Tl), \\ \min\left\{G_M(Pl, Sj, Sj),\ G_M(Pl, Tl, Tl)\right\} \end{array} \right\}.$$

This tends to

$$\begin{aligned} G_M(l, Pl, Pl) &\leq \Upsilon\Big(1,\ G_M(l, Pl, Pl),\ 1,\ G_M(l, Pl, Pl),\ 1\Big) \\ &= \Big[G_M(l, Pl, Pl)\Big]^{\beta}, \end{aligned}$$

which contradicts (Δ_3), then we obtain $G_M(l, Pl, Pl) = 1$, which tends to $l = Pl$. Then, $l = Pl = Tl$. By a similar way, we can show $l = Ql = Sl$. This means that

$$Pl = Tl = l = Ql = Sl. \tag{3.14}$$

Hence, l is common fixed point of the couples (T, P) and (S, Q).

The conclusion of uniqueness with respect to common fixed point of P, Q, S and T is more easy result of the conclusion of uniqueness with respect to common point of coincidence of the pairs (T, P) and (S, Q). Also, the conclusion is similar in case $l \in T(V)$ and either $Q(V)$ or $T(V)$ is complete . This completes the conclusion.

Example 3.2. Let $V = [1,\infty)$. Define $G_M : V^3 \longrightarrow [1,\infty)$ as

$$G_M(k,r,j) = \left(\left|\frac{k}{r}\right|^{\bullet} . \left|\frac{r}{j}\right|^{\bullet} . \left|\frac{j}{k}\right|^{\bullet} \right),$$

since

$$|k|^{\bullet} = \begin{cases} k & \text{if} \quad k \geq 1; \\ 1/k & \text{if} \quad k < 1. \end{cases} \tag{3.15}$$

Then, (V, G_M) is multiplicative G_M-metric space, see in Example 1.3 of [22].

Let $P, Q, S, T : [1,\infty) \longrightarrow [1,\infty)$ be defined as

$$Pj = \sqrt{\frac{j+1}{2}}, \quad Qj = 1, \quad Sj = j, \quad Tj = \frac{j+1}{2}, \qquad \forall \quad j \in V.$$

It is clearly that $P(V) \subseteq S(V)$ and $Q(V) \subseteq T(V)$.

Also, Let Υ, Θ be defined by

$$\Upsilon(m,p,r,t,w) = \Theta(m,p,r,t,w) = m,$$

for all $m,p,r,t,w \in V$.

Then Υ and Θ are Δ-implicit contractions, since Υ, Θ satisfy the following:

(i) It is clearly that Υ and Θ are continuous and non-decreasing in every coordinate variable;

(ii) for every $x, y \in [1,\infty)$, if $x \leq \Upsilon(y,x,y,1,yx) = y$, then there exist $q_1 = 1 \in (0,\infty)$ and if $x \leq \Theta(y,y,x,yx,1) = y$, then there exist $q_2 = 1 \in (0,\infty)$;

(iii) for every $z > 1$,

$$\max\Big\{\Upsilon(1,1,z,z,1), \Upsilon(1,z,1,1,z), \Upsilon(1,z,1,z,1)\Big\} = z^{\beta} < z,$$

and

$$\max\Big\{\Theta(1,1,z,z,1), \Theta(1,z,1,1,z), \Theta(1,z,1,z,1)\Big\} = z^{\beta} < z.$$

We next prove the condition (3.1) is verified:

Step 1: with respect to inequality related to (Υ),

$$\begin{aligned} L.H.S. \quad &= \quad G_M(Qh,Pu,Pu) = \left|\frac{Qh}{Pu}\right|^{\bullet} . \left|\frac{Pu}{Pu}\right|^{\bullet} . \left|\frac{Pu}{Qh}\right|^{\bullet} \\ &= \quad \left|\frac{1}{\sqrt{\frac{u+1}{2}}}\right|^{\bullet} . |1|^{\bullet} . \left|\sqrt{\frac{u+1}{2}}\right|^{\bullet} = \frac{u+1}{2}. \end{aligned}$$

$$
\begin{aligned}
R.H.S. &= \Upsilon \left\{ \begin{array}{c} G_M(Pu,Tu,Tu),\ G_M(Sh,Tu,Tu), \\ G_M(Qh,Sh,Sh),\ G_M(Qh,Tu,Tu), \\ \min\Big\{G_M(Pu,Sh,Sh),\ G_M(Pu,Tu,Tu)\Big\} \end{array} \right\} \\
&= G_M(Pu,Tu,Tu) = \left|\frac{Pu}{Tu}\right|^{\bullet} . \left|\frac{Tu}{Tu}\right|^{\bullet} . \left|\frac{Tu}{Pu}\right|^{\bullet} \\
&= \left|\frac{1}{\sqrt{\frac{u+1}{2}}}\right|^{\bullet} . |1|^{\bullet} . \left|\sqrt{\frac{u+1}{2}}\right|^{\bullet} = \frac{u+1}{2}.
\end{aligned}
$$

Step 2: with respect to inequality related to (Θ),

$$
\begin{aligned}
L.H.S. &= G_M(Pu,Qh,Qh) = \left|\frac{Pu}{Qh}\right|^{\bullet} . \left|\frac{Qh}{Qh}\right|^{\bullet} . \left|\frac{Qh}{Pu}\right|^{\bullet} \\
&= \left|\sqrt{\frac{u+1}{2}}\right|^{\bullet} . |1|^{\bullet} . \left|\frac{1}{\sqrt{\frac{u+1}{2}}}\right|^{\bullet} = \frac{u+1}{2}.
\end{aligned}
$$

$$
\begin{aligned}
R.H.S. &= \Theta \left\{ \begin{array}{c} G_M(Qh,Sh,Sh),\ G_M(Pu,Tu,Tu), \\ G_M(Tu,Sh,Sh), \\ \min\Big\{G_M(Qh,Tu,Tu),\ G_M(Qh,Sh,Sh)\Big\}, \\ G_M(Pu,Sh,Sh) \end{array} \right\} \\
&= G_M(Qh,Sh,Sh) = \left|\frac{Qh}{Sh}\right|^{\bullet} . \left|\frac{Sh}{Sh}\right|^{\bullet} . \left|\frac{Sh}{Qh}\right|^{\bullet} = \left|\frac{1}{u}\right|^{\bullet} . |1|^{\bullet} . |u|^{\bullet} = u^2.
\end{aligned}
$$

Hence, we observe that $\Upsilon, \Theta \in \mathcal{M}$ satisfy the condition (3.1). Also, it is clearly that 1 is a unique common coincidence point of (P, T) and (Q, S), and also a unique common fixed point of mappings P, Q, S and T. Consequently, Theorem 3.1 is supported by this example.

Now, we introduce a new shape of Theorem 3.1 with different contractive condition.

Theorem 3.3. *Assume (V, G_M) and the maps $P, Q, S, T : V \longrightarrow V$ be the same in Theorem 3.1 satisfy the following: for $u, h \in V$, $u \neq h$,*

$$
\left\{ \begin{array}{lll} G_M(Qh,Pu,Pu) & \leq & \Big[\ \mathcal{L}_1\ \Big]^{\beta}, \qquad (3.16.1) \\ G_M(Pu,Qh,Qh) & \leq & \Big[\ \mathcal{L}_2\ \Big]^{\beta}, \qquad (3.16.2) \end{array} \right. \tag{3.16}
$$

since

$$(\star\star)\begin{cases} \mathcal{L}_1 = \max\left\{\begin{array}{c} G_M(Pu,Tu,Tu),\ G_M(Sh,Tu,Tu), \\ G_M(Qh,Sh,Sh),\ G_M(Qh,Tu,Tu), \\ \min\Big\{G_M(Pu,Sh,Sh),\ G_M(Pu,Tu,Tu)\Big\} \end{array}\right\}, \\ \\ \mathcal{L}_2 = \max\left\{\begin{array}{c} G_M(Qh,Sh,Sh),\ G_M(Pu,Tu,Tu), \\ G_M(Tu,Sh,Sh), \\ \min\Big\{G_M(Qh,Tu,Tu),\ G_M(Qh,Sh,Sh)\Big\}, \\ G_M(Pu,Sh,Sh) \end{array}\right\}. \end{cases}$$

where $\beta \in \left(0, \frac{1}{2}\right)$. *If one of* $\{P(V), S(V), Q(V), T(V)\}$ *is complete, then the couples* (T, P) *and* (S, Q) *have unique common point of coincidence. Further, the four self-maps have unique common fixed point in* V *if the couples* (T, P) *and* (S, Q) *are weakly compatible.*

Proof. As Theorem 3.1, let u_0 be arbitrary point in V. From Eq (3.2), assume that either $P(V)$ or $S(V)$ is complete, then prove that $\{h_k\}$ is a multiplicative G_M-Cauchy sequence. Using $u = u_{2k}$ and $h = u_{2k+1}$ in inequality (3.16.1), we get

$$\begin{aligned} G_M(h_{2k+1}, h_{2k}, h_{2k}) &= G_M(Qu_{2k+1}, Pu_{2k}, Pu_{2k}) \\ &\le \left[\max\left\{\begin{array}{c} G_M(Pu_{2k}, Tu_{2k}, Tu_{2k}),\ G_M(Su_{2k+1}, Tu_{2k}, Tu_{2k}), \\ G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1}),\ G_M(Qu_{2k+1}, Tu_{2k}, Tu_{2k}), \\ \min\Big\{G_M(Pu_{2k}, Su_{2k+1}, Su_{2k+1}),\ G_M(Pu_{2k}, Tu_{2k}, Tu_{2k})\Big\} \end{array}\right\}\right]^{\beta} \\ &\le \left[\max\left\{\begin{array}{c} G_M(h_{2k}, h_{2k-1}, h_{2k-1}),\ G_M(h_{2k}, h_{2k-1}, h_{2k-1}) \\ G_M(h_{2k+1}, h_{2k}, h_{2k}), \\ G_M(h_{2k+1}, h_{2k}, h_{2k}) \,.\, G_M(h_{2k}, h_{2k-1}, h_{2k-1}),\ 1 \end{array}\right\}\right]^{\beta}, \qquad \Big(\text{using } (G_{M_1}), (G_{M_5})\Big) \end{aligned}$$

implying thereby,

$$G_M(h_{2k+1}, h_{2k}, h_{2k}) \le G_M^{\beta}(h_{2k+1}, h_{2k}, h_{2k}) \,.\, G_M^{\beta}(h_{2k}, h_{2k-1}, h_{2k-1}). \tag{3.17}$$

Thus,

$$G_M(h_{2k+1}, h_{2k}, h_{2k}) \le \Big[G_M(h_{2k}, h_{2k-1}, h_{2k-1})\Big]^{\rho}, \tag{3.18}$$

since $0 < \rho = \dfrac{\beta}{1-\beta} < 1$. Similarly, by putting $u = u_{2k+2}$ and $h = u_{2k+1}$ in inequality (3.16.2), we have

$$\begin{aligned} G(h_{2k+2}, h_{2k+1}, h_{2k+1}) &= G(Pu_{2k+2}, Qu_{2k+1}, Qu_{2k+1}) \\ &\leq \left[\max \left\{ \begin{array}{c} G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1}),\ G_M(Pu_{2k+2}, Tu_{2k+2}, Tu_{2k+2}), \\ G_M(Tu_{2k+2}, Su_{2k+1}, Su_{2k+1}), \\ \min\left\{ G_M(Qu_{2k+1}, Tu_{2k+2}, Tu_{2k+2}),\ G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1}) \right\} \\ G_M(Pu_{2k+2}, Su_{2k+1}, Su_{2k+1}) \end{array} \right\} \right]^{\beta} \\ &\leq \left[\max \left\{ \begin{array}{c} G_M(h_{2k+1}, h_{2k}, h_{2k}),\ G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}), \\ G_M(h_{2k+1}, h_{2k}, h_{2k}),\ 1, \\ G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}) \,.\, G_M(h_{2k+1}, h_{2k}, h_{2k}) \end{array} \right\} \right]^{\beta}, \qquad \left(\text{from } (G_{M_1}), (G_{M_5})\right) \end{aligned}$$

that become,

$$G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}) \leq G_M^{\beta}(h_{2k+2}, h_{2k+1}, h_{2k+1}) \,.\, G_M^{\beta}(h_{2k+1}, h_{2k}, h_{2k}). \tag{3.19}$$

Then,

$$G_M(h_{2k+2}, h_{2k+1}, h_{2k+1}) \leq \left[G_M(h_{2k+1}, h_{2k}, h_{2k}) \right]^{\rho}. \tag{3.20}$$

Now, from (3.20), we get

$$\begin{aligned} G_M(h_{k+1}, h_k, h_k) &\leq \left[G_M(h_k, h_{k-1}, h_{k-1}) \right]^{\rho} \\ &\leq \left[G_M(h_{k-1}, h_{k-2}, h_{k-2}) \right]^{\rho^2} \leq \ ..... \ \leq \left[G_M(h_1, h_0, h_0) \right]^{\rho^k}. \end{aligned} \tag{3.21}$$

For all $k, m \in \mathbb{N}$ with $(k < m)$, we obtain

$$\begin{aligned} G_M(h_m, h_k, h_k) &\leq G_M(h_m, h_{m-1}, h_{m-1}) \,.\, G_M(h_{m-1}, h_{m-2}, h_{m-2}) \ ....\ G_M(h_{k+1}, h_k, h_k) \\ &\leq \left[G_M(h_1, h_0, h_0) \right]^{\rho^{m-1}} \,.\, \left[G_M(h_1, h_0, h_0) \right]^{\rho^{m-2}} \ ....\ \left[G_M(h_1, h_0, h_0) \right]^{\rho^k} \\ &\leq \left[G_M(h_1, h_0, h_0) \right]^{\rho^k (1 + \rho + ... + \rho^{m-k-1})} \\ &\leq \left[G_M(h_1, h_0, h_0) \right]^{\frac{\rho^k}{1-\rho}} \longrightarrow 1, \quad as \ \ k, m \longrightarrow \infty, \end{aligned}$$

since $(0 < \rho < 1)$ tends to $\rho^k \longrightarrow 0, \quad as \ \ k \longrightarrow \infty$.

This illustrates that $\{h_k\}$ is multiplicative G_M−Cauchy sequence. Since (V, G_M) is complete, then there exists $l \in V$ such that $h_k \longrightarrow l$ as $k \longrightarrow \infty$. Then from Eq (3.2), we get Eq (3.7).

Since $P(V) \subseteq S(V)$, if $l \in S(V)$, then there exists $j \in V$ such that Eq (3.8) is verified.

We will show that $Qj = Sj$. Taking $u = u_{2k}$ and $h = j$ in inequality (3.16.1), we have

$$G_M(Qj, Pu_{2k}, Pu_{2k}) \leq \left[\max \left\{ \begin{array}{c} G_M(Pu_{2k}, Tu_{2k}, Tu_{2k}),\ G_M(Sj, Tu_{2k}, Tu_{2k}), \\ G_M(Qj, Sj, Sj),\ G_M(Qj, Tu_{2k}, Tu_{2k}), \\ \min\left\{G_M(Pu_{2k}, Sj, Sj),\ G_M(Pu_{2k}, Tu_{2k}, Tu_{2k})\right\} \end{array} \right\} \right]^{\beta}.$$

Taking $k \longrightarrow \infty$ and using Eqs (3.7) and (3.8), we get

$$\begin{aligned} G_M(Qj, l, l) &\leq \left[\max\left\{1,\ 1,\ G_M(Qj, l, l),\ G_M(Qj, l, l),\ 1\right\}\right]^{\beta} \\ &= \left[G_M(Qj, l, l)\right]^{\beta}, \end{aligned}$$

which is contradiction with respect to (Δ_3). Hence, $G_M(Qj, l, l) = 1$, which tends to $l = Qj$. Thus, Eq (3.9) is satisfied.

Similarly, since $Q(V) \subseteq T(V)$, there exists $r \in V$ such that Eq (3.10) is verified.

We will prove that $Pr = Tr$. Putting $u = r$ and $h = u_{2k+1}$ in inequality (3.16.2), we find

$$G_M(Pr, Qu_{2k+1}, Qu_{2k+1}) \leq \left[\max \left\{ \begin{array}{c} G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1}),\ G_M(Pr, Tr, Tr), \\ G_M(Tr, Su_{2k+1}, Su_{2k+1}), \\ \min\left\{G_M(Qu_{2k+1}, Tr, Tr),\ G_M(Qu_{2k+1}, Su_{2k+1}, Su_{2k+1})\right\}, \\ G_M(Pz, Su_{2k+1}, Su_{2k+1}) \end{array} \right\} \right]^{\beta}.$$

Hence,

$$\begin{aligned} G_M(Pr, l, l) &\leq \left[\max\left\{1,\ G_M(Pr, l, l),\ 1,\ 1,\ G_M(Pr, l, l)\right\}\right]^{\beta} \\ &= \left[G_M(Pr, l, l)\right]^{\beta}, \end{aligned}$$

from (Δ_3), we have $G_M(Pr, l, l) = 1$ which implies that $Pr = l$. Thus Eq (3.11) is hold.

Hence, $l \in V$ is common point of coincidence for the four maps.

Uniqueness: To illustrate the uniqueness with respect to the coincidence point, assume that $l^* \neq l$ be another coincidence point of the four maps beside the other hypotheses as the same method in Theorem 3.1, then inequality (3.16.1), became

$$G_M(Qj, Pr^*, Pr^*) \leq \left[\max \left\{ \begin{array}{c} G_M(Pr^*, Tr^*, Tr^*),\ G_M(Sj, Tr^*, Tr^*), \\ G_M(Qj, Sj, Sj),\ G_M(Qj, Tr^*, Tr^*), \\ \min\left\{G_M(Pr^*, Sj, Sj),\ G_M(Pr^*, Tr^*, Tr^*)\right\} \end{array} \right\} \right]^{\beta}.$$

Tending thereby

$$\begin{aligned} G_M(l, l^*, l^*) &\leq \left[\max\left\{1,\ G_M(l, l^*, l^*),\ 1,\ G_M(l, l^*, l^*),\ 1\right\}\right]^\beta \\ &= \left[G_M(l, l^*, l^*)\right]^\beta, \end{aligned}$$

which tends to $G_M(l, l^*, l^*) = 1$ (from Δ_3), i.e., $l = l^*$. Consequently, the couples (T, P) and (S, Q) have unique common coincidence point.

By using weak compatibility of (T, P) and (S, Q) and Eqs (3.9), (3.11), we get Eqs (3.12) and (3.13). Hence, l is coincidence point of (T, P) and (S, Q).

Now, we illustrate that f is a common fixed point of P, Q, S and T. Taking $u = l$ and $h = j$ in inequality (3.16.1), we have

$$G_M(Qj, Pl, Pl) \leq \left[\max\left\{\begin{array}{c} G_M(Pl, Tl, Tl),\ G_M(Sj, Tl, Tl), \\ G_M(Qj, Sj, Sj),\ G_M(Qj, Tl, Tl), \\ \min\left\{G_M(Pl, Sj, Sj),\ G_M(Pl, Tl, Tl)\right\} \end{array}\right\}\right]^\beta.$$

This implies to

$$\begin{aligned} G_M(l, Pl, Pl) &\leq \left[\max\left\{1,\ G_M(l, Pl, Pl),\ 1,\ G_M(l, Pl, Pl),\ 1\right\}\right]^\beta \\ &= \left[G_M(l, Pl, Pl)\right]^\beta, \end{aligned}$$

which is contradiction from (Δ_3), then we obtain $G_M(l, Pl, Pl) = 1$, which implies to $l = Pl$. Then, $l = Pl = Tl$. Similarly, we can prove $l = Ql = Sl$. This means that Eq (3.14) is satisfied.

Therefore, l is common fixed point of (T, P) and (S, Q).

As we prove the uniqueness of the common fixed point of the four maps, similarly, we can show the uniqueness of the common point of coincidence of the four maps. Beside, the conclusion is similar in case $l \in T(V)$ and either $Q(V)$ or $T(V)$ is complete. This completes the conclusion.

Example 3.4. Let $V = \mathbb{R}$. Define $G_M : V^3 \longrightarrow [1, \infty)$ as

$$G_M(j, k, r) = \xi^{\left(|j-k| + |k-r| + |r-j|\right)},$$

where $j, k, r \in V$ and $\xi > 1$.

It is clearly that G_M is multiplicative generalized metric [22] on V and the pair (V, G_M) is called multiplicative G_M-metric space.

Let $P, Q, S, T : V \longrightarrow V$ be defined as

$$Pj = 2 = Qj, \quad Sj = \frac{1}{2}j + 1, \quad Tj = \frac{1}{4}j + \frac{3}{2}, \qquad \forall\ j \in \mathbb{R}.$$

Then, $P(V) \subseteq S(V)$ and $Q(V) \subseteq T(V)$.

Next, We show that the condition (3.16) is satisfied:

Step 1: from condition (3.16.1),

$$G_M(Qh, Pu, Pu) = 1 \leq \left[\max \left\{ \begin{array}{c} G_M(Pu, Tu, Tu),\ G_M(Sh, Tu, Tu), \\ G_M(Qh, Sh, Sh),\ G_M(Qh, Tu, Tu), \\ \min\Big\{G_M(Pu, Sh, Sh),\ G_M(Pu, Tu, Tu)\Big\} \end{array} \right\} \right]^{\beta}.$$

Step 2: from condition (3.16.2),

$$G_M(Pu, Qh, Qh) = 1 \leq \left[\max \left\{ \begin{array}{c} G_M(Qh, Sh, Sh),\ G_M(Pu, Tu, Tu), \\ G_M(Tu, Sh, Sh), \\ \min\Big\{G_M(Qh, Tu, Tu),\ G_M(Qh, Sh, Sh)\Big\}, \\ G_M(Pu, Sh, Sh) \end{array} \right\} \right]^{\beta}.$$

Therefore, we note that the condition (3.16) is satisfied. It is clearly that 2 is unique common coincidence point of (T, p) and (S, Q), and also unique common fixed point of maps P, Q, S and T. Then, Theorem 3.3 is verified by this example.

Next, we induce a new thoerem which is an extend to Theorem 3.3.

Theorem 3.5. *Suppose a complete multiplicative G_M-metric space (V, G_M) and the mappings P, Q, S and T be the same in Theorem 3.1 and $\mathcal{L}_1$, $\mathcal{L}_2$ are defined as in Theorem 3.3 hold the following: for $u, h \in V,\ u \neq h$,*

$$\begin{cases} G_M(Qh, Pu, Pu) \leq \Big[\psi_1\big(\mathcal{L}_1\big)\Big]^{\beta}, & (3.22.1) \\ G_M(Pu, Qh, Qh) \leq \Big[\psi_2\big(\mathcal{L}_2\big)\Big]^{\beta}, & (3.22.2) \end{cases} \tag{3.22}$$

where $\beta \in \left(0, \frac{1}{2}\right)$ and $\psi_1, \psi_2 : [0, \infty) \longrightarrow [0, \infty)$ is a continuous and monotone increasing functions such that $\psi_1(0), \psi_2(0) < t$ for all $t > 0$. If one of $\{P(V), S(V), Q(V), T(V)\}$ is complete, then the couples (T, P) and (S, Q) have a unique common coincidence point. Also, the four self-maps have unique common fixed point in V if the pairs (T, P) and (S, Q) are weakly compatible.

Proof. By a similar way of Theorem 3.3 beside using the concept of continuous and monotone increasing functions we can conclude the proof of Theorem 3.5.

Example 3.6. Let V, G_M, P, Q, S, T be defined in Example 3.4 and let

$$\psi_1(j) = j^{1/\beta} = \psi_2(j),$$

such that $\beta \in \left(0, \frac{1}{2}\right)$.

Then, $\psi_1(j)$ and $\psi_2(j)$ satisfy the conditions in Theorem 3.5 and we have:

Step 1: from the condition (3.22.1),

$$G_M(Qh,Pu,Pu) = 1 \le \left[\max\left\{\begin{array}{c} G_M(Pu,Tu,Tu),\ G_M(Sh,Tu,Tu), \\ G_M(Qh,Sh,Sh),\ G_M(Qh,Tu,Tu), \\ \min\left\{G_M(Pu,Sh,Sh),\ G_M(Pu,Tu,Tu)\right\} \end{array}\right\}\right]^{\beta}$$

$$= \left[\psi_1\left(\max\left\{\begin{array}{c} G_M(Pu,Tu,Tu),\ G_M(Sh,Tu,Tu), \\ G_M(Qh,Sh,Sh),\ G_M(Qh,Tu,Tu), \\ \min\left\{G_M(Pu,Sh,Sh),\ G_M(Pu,Tu,Tu)\right\} \end{array}\right\}\right)\right]^{\beta}.$$

Step 2: from condition (3.22.2),

$$G_M(Pu,Qh,Qh) = 1 \le \left[\max\left\{\begin{array}{c} G_M(Qh,Sh,Sh),\ G_M(Pu,Tu,Tu), \\ G_M(Tu,Sh,Sh), \\ \min\left\{G_M(Qh,Tu,Tu),\ G_M(Qh,Sh,Sh)\right\}, \\ G_M(Pu,Sh,Sh) \end{array}\right\}\right]^{\beta}$$

$$= \left[\psi_2\left(\max\left\{\begin{array}{c} G_M(Qh,Sh,Sh),\ G_M(Pu,Tu,Tu), \\ G_M(Tu,Sh,Sh), \\ \min\left\{G_M(Qh,Tu,Tu),\ G_M(Qh,Sh,Sh)\right\}, \\ G_M(Pu,Sh,Sh) \end{array}\right\}\right)\right]^{\beta}.$$

Now, we present a new corollary with contraction have two mappings.

Corollary 3.7. *Consider a complete multiplicative G_M−metric space (V, G_M) with the maps $Q, P : V \longrightarrow V$ satisfy the following: for $u, h \in V$, $u \neq h$,*

$$G_M(Ph,Pu,Pu) \le G_M^{\beta}(Qh,Qu,Qu)$$

where $\beta \in (0,1)$. If $Q(V)$ or $P(V)$ is complete, then the couple (Q,P) has unique coincidence point. Further, the couple (Q,P) has a unique fixed point in V if it is weakly compatible.

Example 3.8. Let V and G_M be the same in Example 3.4 and let $Q, P : V \longrightarrow V$ be defined by

$$Qj = \frac{j+1}{2}, \quad Pj = 1, \quad \forall\ j \in \mathbb{R}.$$

Let $u, h \in V$ with $u \neq h$, then we get

$$G_M(Ph, Pu, Pu) = 1 \;\leq\; \varsigma^{\beta\,|u-h|} = G_M^{\beta}(Qh, Qu, Qu).$$

Clearly, 1 is unique coincidence point of (Q, P) and also unique fixed point of maps (Q, P). Consequently, all the assertions in Corollary 3.7 are verified by this example.

4. Application

It is very common that many researchers proved different kind of linear and nonliear Volterra and Fredhlom type integral equations by using various contractions principle. Rasham et al. [26] proved a significant fixed point results for sufficient conditions to solve two system of nonlinear integral equations. For more fixed point results having applications related to integral equations (see [21,25,27,28]).

Theorem 4.1. *Let (V, G_M) be a complete multiplicative G_M−metric space and the map $Q : V \to V$ satisfies the following condition for all u, h, v with $u \neq h \neq v$, then*

$$G_M(Qu, Qh, Qv) \leq G_M^{\beta}(u, h, v)$$

where $\beta \in (0, 1)$. If $Q(V)$ is complete then Q has coincidence point. Also, Q has a unique fixed point in V if it is weakly compatible.

Proof. The proof of Theorem 4.1 is similar to our main Theorem 3.1.

Now we are presenting an application for nonlinear integral equations to find out the unique common solution. Let $E = V([0, 1], \mathbb{R}^+)$ be the set of all continuous functions on $[0, 1]$. Consider the integral equations

$$\acute{u}(\upsilon) = \int_0^{\upsilon} K_1(\upsilon, t, \acute{u}(t))\, dt, \tag{4.1}$$

$$\hbar(\upsilon) = \int_0^{\upsilon} K_2(\upsilon, t, \hbar(t))\, dt, \tag{4.2}$$

$$v(\upsilon) = \int_0^{\upsilon} K_3(\upsilon, t, v(t))\, dt, \tag{4.3}$$

for all $\upsilon \in [0, 1]$, where $K_1, K_2\; K_3$ are functions from $[0, 1] \times [0, 1] \times E$ to $\mathbb{R}$.

$$e^{|K_1(\upsilon,t,\acute{u}(t)) - K_2(\upsilon,t,\hbar(t))| + |K_2(\upsilon,t,\hbar(t)) - K_3(\upsilon,t,v(t))| + |K_3(\upsilon,t,v(t)) - K_1(\upsilon,t,\acute{u}(t))|}$$
$$\leq \frac{1}{1 + G_M(Q\acute{u}, Q\hbar, Qv)} \,.\, G_M(Q\acute{u}, Q\hbar, Qv).$$

For $\acute{u} \in V([0, 1], \mathbb{R}^+)$, define supremum norm as: $\|\acute{u}\|_\tau = \sup_{k\in[0,1]} \{e^{|\acute{u}(k)|}\}$. Define

$$G_M(\acute{u}, \hbar, v) = \left[\sup_{k\in[0,1]} \left\{e^{(|\acute{u}(k)-\hbar(k)| + |\hbar(k)-v(k)| + |v(k)-\acute{u}(k)|)}\right\}\right] = \|e^{(|\acute{u}-\hbar| + |\hbar - v| + |v - \acute{u}|)}\|_\tau,$$

for all $\acute{u}, \hbar, v \in V([0,1], \mathbb{R}^+)$, with these settings, $\left(V([0,1], \mathbb{R}^+), G_M\right)$ becomes a complete multiplicative G_M-metric space.

Now, we prove the following theorem to ensure the uniqueness and existence of a solution of nonlinear integral equations (4.1), (4.2) and (4.3).

Theorem 4.2. *Hypothesize conditions (i) and (ii) are satisfied:*
(i) $K_1, K_2, K_3 : [0,1] \times [0,1] \times V([0,1], \mathbb{R}^+) \to \mathbb{R}$*;*
(ii) Define $Q : V([0,1], \mathbb{R}^+) \to V([0,1], \mathbb{R}^+)$ *by*

$$\begin{aligned} (Q\acute{u})(\upsilon) &= \int_0^\upsilon K_1(\upsilon, t, \acute{u}(t))\, dt, \\ (Q\hbar)(\upsilon) &= \int_0^\upsilon K_2(\upsilon, t, \hbar(t))\, dt, \\ (Qv)(\upsilon) &= \int_0^\upsilon K_3(\upsilon, t, v(t))\, dt. \end{aligned}$$

Assume that $\beta \in (0,1)$ *for each* $\upsilon, t \in [0,1]$ *and* $\acute{u}, \hbar, v \in V([0,1], \mathbb{R}^+)$*, where*

$$G_M(Q\acute{u}, Q\hbar, Qv) = \|e^{|(Q\acute{u})(\upsilon)-(Q\hbar)(\upsilon)|+|(Q\hbar)(\upsilon)-(Qv)(\upsilon)|+|(Qv)(\upsilon)-(Q\acute{u})(\upsilon)|}\|_\tau\,.$$

Then (4.1), (4.2) *and* (4.3) *possess a unique solution.*

Proof. By definition *(ii)*

$$\begin{aligned} & e^{|(Q\acute{u})(\upsilon)-(Q\hbar)(\upsilon)|+|(Q\hbar)(\upsilon)-(Qv)(\upsilon)|+|(Qv)(\upsilon)-(Q\acute{u})(\upsilon)|} \\ = & \int_0^\upsilon e^{|K_1(\upsilon,t,\acute{u}(t))-K_2(\upsilon,t,\hbar(t))|+|K_2(\upsilon,t,\hbar(t))-K_3(\upsilon,t,v(t))|+|K_3(\upsilon,t,v(t))-K_1(\upsilon,t,\acute{u}(t))|}\, dt \\ \leq & \int_0^\upsilon \frac{1}{1+G_M(Q\acute{u}, Q\hbar, Qv)}\,.\, G_M(Q\acute{u}, Q\hbar, Qv)\, dt. \end{aligned}$$

This implies

$$\begin{aligned} e^{|(Q\acute{u})(\upsilon)-(Q\hbar)(\upsilon)|+|(Q\hbar)(\upsilon)-(Qv)(\upsilon)|+|(Qv)(\upsilon)-(Q\acute{u})(\upsilon)|} &\leq \frac{G_M(Q\acute{u}, Q\hbar, Qv)}{1+G_M(Q\acute{u}, Q\hbar, Qv)} \\ \|e^{|(Q\acute{u})(\upsilon)-(Q\hbar)(\upsilon)|+|(Q\hbar)(\upsilon)-(Qv)(\upsilon)|+|(Qv)(\upsilon)-(Q\acute{u})(\upsilon)|}\|_\tau &\leq \frac{G_M(Q\acute{u}, Q\hbar, Qv)}{1+G_M(Q\acute{u}, Q\hbar, Qv)} \\ \frac{1+G_M(Q\acute{u}, Q\hbar, Qv)}{G_M(Q\acute{u}, Q\hbar, Qv)} &\leq \frac{1}{\|e^{|(Q\acute{u})(\upsilon)-(Q\hbar)(\upsilon)|+|(Q\hbar)(\upsilon)-(Qv)(\upsilon)|+|(Qv)(\upsilon)-(Q\acute{u})(\upsilon)|}\|_\tau} \\ \frac{1}{G_M(Q\acute{u}, Q\hbar, Qv)} &\leq \frac{1}{\|e^{|(Q\acute{u})(\upsilon)-(Q\hbar)(\upsilon)|+|(Q\hbar)(\upsilon)-(Qv)(\upsilon)|+|(Qv)(\upsilon)-(Q\acute{u})(\upsilon)|}\|_\tau}. \end{aligned}$$

It yields that

$$-\frac{1}{\|e^{|(Q\acute{u})(\upsilon)-(Q\hbar)(\upsilon)|+|(Q\hbar)(\upsilon)-(Qv)(\upsilon)|+|(Qv)(\upsilon)-(Q\acute{u})(\upsilon)|}\|_{\tau}} \leq \frac{-1}{G_M(Q\acute{u}, Q\hbar, Qv)}.$$

The condition of Theorem (4.2) are satisfied, and $G_M(\acute{u}, \hbar, v) = \|e^{(|\acute{u}-\hbar|+|\hbar-v|+|v-\acute{u}|)}\|_{\tau}$. Hence the integral equations (4.1), (4.2) and (4.3) admit a unique common solution.

Example 4.3. Take $E = [0, 1]$. If we put $v = 1$ in (4.1), (4.2) and (4.3), then we get the following integral equations

$$(Q\acute{u}) = \int_0^1 K_1(v, t, \acute{u}(t))\, dt = \int_0^1 \frac{2}{25\,(v + 1 + \acute{u}(t))}\, dt; \tag{4.4}$$

$$(Q\hbar) = \int_0^1 K_2(v, t, \hbar(t))\, dt = \int_0^1 \frac{2}{25\,(v + 1 + \hbar(t))}\, dt; \tag{4.5}$$

$$(Qv) = \int_0^1 K_3(v, t, v(t))\, dt = \int_0^1 \frac{2}{25\,(v + 1 + v(t))}\, dt, \tag{4.6}$$

where

$$\begin{aligned} K_1(v, t, \acute{u}(t)) &= \frac{2}{25\,(v + 1 + \acute{u}(t))}; \\ K_2(v, t, \hbar(t)) &= \frac{2}{25\,(v + 1 + \hbar(t))}; \\ K_3(v, t, v(t)) &= \frac{2}{25\,(v + 1 + v(t))}. \end{aligned}$$

Equations (4.4)–(4.6) are the special case of Eqs (4.1)–(4.3) respectively, where $v \in [0, 1]$.

Proof.

$$\begin{aligned} & G_M(Q\acute{u}, Q\hbar, Qv) \\ = {} & \exp\left\{\|Q\acute{u} - Q\hbar\| + \|Q\hbar - Qv\| + \|Qv - Q\acute{u}\|\right\} \\ = {} & \int_0^1 \exp\Big\{\|K_1(v, t, \acute{u}(t)) - K_2(v, t, \hbar(t))\| + \|K_2(v, t, \hbar(t)) - K_3(v, t, v(t))\| \\ & \qquad + \|K_3(v, t, v(t)) - K_1(v, t, \acute{u}(t))\|\Big\}\, dt \\ = {} & \int_0^1 \exp\bigg\{\left\|\frac{2}{25\,(v + 1 + \acute{u}(t))} - \frac{2}{25\,(v + 1 + \hbar(t))}\right\| + \bigg\|\frac{2}{25\,(v + 1 + \hbar(t))} \\ & \qquad - \frac{2}{25\,(v + 1 + v(t))}\bigg\| + \left\|\frac{2}{25\,(v + 1 + v(t))} - \frac{2}{25\,(v + 1 + \acute{u}(t))}\right\|\bigg\}\, dt \end{aligned}$$

$$
\begin{aligned}
&= e^{2/25} \int_0^1 \exp\left\{ \left\| \frac{1}{v+1+\acute{u}(t)} - \frac{1}{v+1+\hbar(t)} \right\| + \left\| \frac{1}{v+1+\hbar(t)} - \frac{1}{v+1+v(t)} \right\| \right. \\
&\qquad\qquad\qquad\qquad \left. + \left\| \frac{1}{v+1+v(t)} - \frac{1}{v+1+\acute{u}(t)} \right\| \right\} dt \\
&= e^{2/25} \int_0^1 \exp\left\{ \left\| \frac{\hbar(t)-\acute{u}(t)}{(v+1+\acute{u}(t))(v+1+\hbar(t))} \right\| + \left\| \frac{v(t)-\hbar(t)}{(v+1+\hbar(t))(v+1+v(t))} \right\| \right. \\
&\qquad\qquad\qquad\qquad \left. + \left\| \frac{\acute{u}(t)-v(t)}{(v+1+v(t))(v+1+\acute{u}(t))} \right\| \right\} dt \\
&= e^{2/25} \left[\exp\left\{ \int_0^1 \left\| \frac{\hbar(t)-\acute{u}(t)}{(v+1+\acute{u}(t))(v+1+\hbar(t))} \right\| dt \right.\right. \\
&\qquad \left.\left. + \int_0^1 \left\| \frac{v(t)-\hbar(t)}{(v+1+\hbar(t))(v+1+v(t))} \right\| dt + \int_0^1 \left\| \frac{\acute{u}(t)-v(t)}{(v+1+v(t))(v+1+\acute{u}(t))} \right\| dt \right\} \right] \\
&\leq e^{2/25} \left[\exp\left\{ \|\hbar(t)-\acute{u}(t)\| + \|v(t)-\hbar(t)\| + \|\acute{u}(t)-v(t)\| \right\} \right] \\
&\leq G_M^{\beta}(\acute{u}, \hbar, v); \qquad\qquad \beta = e^{2/25}.
\end{aligned}
$$

It follows that

$$G_M(Q\acute{u}, Q\hbar, Qv) \leq G_M^{\beta}(\acute{u}, \hbar, v).$$

Hence, all conditions of Theorem 4.1 hold. The integral equations (4.4), (4.5) and (4.6) have a unique solution by using Theorem 4.1.

5. Conclusions

In this manuscript, we achieve the uniqueness and existence of common fixed point of the pairs (P, T) and (Q, S) that satisfy Δ-implicit contractions as one of the other different contractive conditions. We provide some nontrivial examples for supporting our main theorems. Our main results represent a generalization and extension to the results in the literature. On the other hand, we prove an application for the system of nonlinear integral equations to show that the common solution of defined nonlinear integral inclusions exists and unique. In future we can extend this work, for multivalued mappings, fuzzy mappings, L-fuzzy mappings, bipolar fuzzy mappings, intuitionistic fuzzy mappings.

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Conflict of interest

The authors declare no conflict of interest.

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