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Research article

On differential analysis of spacelike flows on normal congruence of surfaces

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Abstract: The present paper examines the differential analysis of flows on normal congruence of spacelike curves with spacelike normal vector in terms of anholonomic coordinates in three dimensional Lorentzian space. Eight parameters, which are related by three partial differential equations, are discussed. Then, it is seen that the curl of tangent vector field does not include any component with principal normal direction. Thus there exists a surface which contains both $s-lines$ and $b-lines$. Also, we examine a normal congruence of surfaces containing the $s-lines$ and $b-lines$. By compatibility conditions, Gauss-Mainardi-Codazzi equations are obtained for this normal congruence of surface. Intrinsic geometric properties of this normal congruence of surfaces are given.

Keywords: anholonomic coordinates; normal congruence; flows of spacelike curve; timelike surface; Gauss-Mainardi-Codazzi equations
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1. Introduction

In general, differential geometry of surfaces examines the geometric structure of smooth surfaces. In this context, there are many studies on differential geometry of surfaces in different spaces such as Euclidean and non-Euclidean spaces. In many of these studies, the arc length of the curves on the surface is used to obtain the distance over the surfaces. For this reason, curves on the surface have an important role in the field of differential geometry. Investigations of curves on the surface, not only their geometric properties but also their physical structures are considered. From this point of view, it is the most preferred way of examining the local differential geometric structure of the curve. In many studies dealing with differential geometric properties of curves, some methods and tools of differential calculus are used. This review makes use of the well-known Frenet-Serret frame

$\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$. Considering that $\sigma = \sigma(s, n, b)$ is a given curve in Euclidean space, where s, n and b are the distance along $s-lines$, $n-lines$ and $b-lines$ respectively. The main object is to obtain the system by the directional derivatives of Frenet-Serret frame [1]. The quantities, the normal deformations of the vector-tube in the directions $\overrightarrow{n}$ and $\overrightarrow{b}$,

$$\xi_{ns} = g(\overrightarrow{n}, \frac{\partial}{\partial n}\overrightarrow{t}) \text{ and } \xi_{bs} = g(\overrightarrow{b}, \frac{\partial}{\partial b}\overrightarrow{t})$$

are firstly introduced in [2], respectively.

Since it has many physical applications, Lorentzian geometry is the most studied geometry among non-Euclidean geometries [3–8]. Also, this geometry is a very common research area of physical problems on integrable systems, soliton theory, fluid dynamics, field theories, etc. [1, 9–11]. Since Lorentz-Minkowski spacetime was extended to a curved spacetime by A. Einstein in order to model nonzero gravitational fields, this geometry has been the mathematical theory which is used by general relativity.

In this study, it is aimed to examine spacelike curve flow on Lorentzian space from a different perspective. The three dimensional real vector space equipped with Lorentzian metric

$$\left\langle \overrightarrow{x}, \overrightarrow{y} \right\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3$$

is named after Lorentzian space and denoted by $\mathbb{E}_1^3$. In second section, the three dimensional vector field and the differential geometric aspects of curvature and torsion of vector lines are investigated by means of anholomonic coordinates. We describe Frenet-Serret frame $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ of a given spacelike space curve in $\mathbb{E}_1^3$ in terms of anholonomic coordinates which includes eight parameters, related by three partial differential equations. It is proved that the curl of tangent vector field has no component in the direction of principal normal vector field. Thus, there exists a surface which contains both $s-lines$ and $b-lines$. For this reason, the expression of this normal congruence is also discussed in the last section. Then, intrinsic geometric properties of this normal congruence of surfaces are also given.

2. Differential analysis of spacelike curve with spacelike normal vector

Suppose that $\sigma = \sigma(s, n, b)$ is a given spacelike curve with spacelike normal vector in three dimensional Lorentzian space. And the distance along $s-lines$, $n-lines$ and $b-lines$ of the curve σ are denoted by s, n and b, respectively. The unit spacelike tangent vector of $s-lines$ and $n-lines$ of the curve σ are given by

$$\overrightarrow{t} = \frac{\partial\sigma}{\partial s}, \qquad \overrightarrow{n} = \frac{\partial\sigma}{\partial n},$$

respectively. Then the unit timelike tangent vector of $b-lines$ is given by [12, 13]

$$\overrightarrow{b} = \frac{\partial\sigma}{\partial b}.$$

A three-dimensional vector field can be considered in terms of anholonomic coordinates which includes eight parameters, related by three partial differential equations [14].

Theorem 1. *Suppose that $\sigma = \sigma(s, n, b)$ is a given spacelike curve Lorentzian space. Directional derivatives of the unit vectors $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ are given as follows :*
i)

$$\frac{\partial}{\partial s}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & \tau & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix}, \tag{1}$$

ii)

$$\frac{\partial}{\partial n}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix} = \begin{bmatrix}0 & \xi_{ns} & \tau + \mu_b\\ -\xi_{ns} & 0 & \operatorname{div} b\\ \tau + \mu_b & \operatorname{div} b & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix}, \tag{2}$$

iii)

$$\frac{\partial}{\partial b}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix} = \begin{bmatrix}0 & \mu_n - \tau & -\xi_{bs}\\ \tau - \mu_n & 0 & -\left(\operatorname{div}\overrightarrow{n} + \kappa\right)\\ -\xi_{bs} & -(\operatorname{div}\overrightarrow{n} + \kappa) & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix}. \tag{3}$$

The normal deformations of the vector-tube in the directions $\overrightarrow{n}$ and $\overrightarrow{b}$ are given as

$$\xi_{ns} = \left\langle \overrightarrow{n}, \frac{\partial \overrightarrow{t}}{\partial n}\right\rangle_L, \qquad \xi_{bs} = \left\langle \overrightarrow{b}, \frac{\partial \overrightarrow{t}}{\partial b}\right\rangle_L$$

and abnormality of $\overrightarrow{n}$ and $\overrightarrow{b}$ are stated as

$$\mu_n = \left\langle \operatorname{curl}\overrightarrow{n}, \overrightarrow{n}\right\rangle_L, \qquad \mu_b = \left\langle \operatorname{curl}\overrightarrow{b}, \overrightarrow{b}\right\rangle_L$$

respectively. Curvature and torsion function of the unit speed spacelike curve $\sigma = \sigma(s, n, b)$ are denoted by $\kappa = \kappa(s, n, b)$ and $\tau = \tau(s, n, b)$ respectively.

Proof. Proof of i) is clear by Frenet-Serret equation for unit speed spacelike curve. So, the proof of ii) and iii) will be given. It is known that for $i = 1, 2, 3$ there exist smooth functions; α_i and β_i where

$$\frac{\partial}{\partial n}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix} = \begin{bmatrix}0 & \alpha_1 & \alpha_2\\ -\alpha_1 & 0 & \alpha_3\\ \alpha_2 & \alpha_3 & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix},$$

$$\frac{\partial}{\partial b}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix}=\begin{bmatrix}0 & \beta_1 & \beta_2\\ -\beta_1 & 0 & \beta_3\\ \beta_2 & \beta_3 & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix}.$$

We need to find these functionals. Firstly, we get

$$\alpha_1=\left\langle\frac{\partial\overrightarrow{t}}{\partial n},\overrightarrow{n}\right\rangle_L=\xi_{ns},\qquad -\beta_2=\left\langle\frac{\partial\overrightarrow{t}}{\partial b},\overrightarrow{b}\right\rangle_L=\xi_{bs}$$

by our assumptions. Then, we also obtain divergence of Serret-Frenet frame fields as follows

$$\begin{aligned}\operatorname{div}\overrightarrow{t}&=\left\langle\overrightarrow{t},\frac{\partial\overrightarrow{t}}{\partial s}\right\rangle_L+\left\langle\overrightarrow{n},\frac{\partial\overrightarrow{t}}{\partial n}\right\rangle_L+\left\langle\overrightarrow{b},\frac{\partial\overrightarrow{t}}{\partial b}\right\rangle_L\\ &=\left\langle\overrightarrow{t},\kappa\overrightarrow{n}\right\rangle_L+\left\langle\overrightarrow{n},\xi_{ns}\overrightarrow{n}+\alpha_2\overrightarrow{b}\right\rangle_L+\left\langle\overrightarrow{b},\beta_1\overrightarrow{n}-\xi_{bs}\overrightarrow{b}\right\rangle_L=\xi_{ns}-\beta_2,\end{aligned}$$

$$\begin{aligned}\operatorname{div}\overrightarrow{n}&=\left\langle\overrightarrow{t},\frac{\partial\overrightarrow{n}}{\partial s}\right\rangle_L+\left\langle\overrightarrow{n},\frac{\partial\overrightarrow{n}}{\partial n}\right\rangle_L+\left\langle\overrightarrow{b},\frac{\partial\overrightarrow{n}}{\partial b}\right\rangle_L\\ &=\left\langle\overrightarrow{t},-\kappa\overrightarrow{t}+\tau\overrightarrow{b}\right\rangle_L+\left\langle\overrightarrow{n},-\xi_{ns}\overrightarrow{t}+\alpha_3\overrightarrow{b}\right\rangle_L+\left\langle\overrightarrow{b},-\beta_1\overrightarrow{t}+\beta_3\overrightarrow{b}\right\rangle_L=-\kappa-\beta_3\end{aligned}$$

and

$$\begin{aligned}\operatorname{div}\overrightarrow{b}&=\left\langle\overrightarrow{t},\frac{\partial\overrightarrow{b}}{\partial s}\right\rangle_L+\left\langle\overrightarrow{n},\frac{\partial\overrightarrow{b}}{\partial n}\right\rangle_L+\left\langle\overrightarrow{b},\frac{\partial\overrightarrow{b}}{\partial b}\right\rangle_L\\ &=\left\langle\overrightarrow{t},\tau\overrightarrow{n}\right\rangle_L+\left\langle\overrightarrow{n},\alpha_2\overrightarrow{t}+\alpha_3\overrightarrow{n}\right\rangle_L+\left\langle\overrightarrow{b},-\xi_{bs}\overrightarrow{t}+\beta_3\overrightarrow{n}\right\rangle_L=\alpha_3.\end{aligned}$$

Therefore, we get

$$\beta_3=-\left(\operatorname{div}\overrightarrow{n}+\kappa\right)\text{ and }\alpha_3=\operatorname{div}b.$$

On the other hand, we also obtain

$$\begin{aligned}\operatorname{curl}\overrightarrow{t}&=\overrightarrow{t}\times_L\frac{\partial\overrightarrow{t}}{\partial s}+\overrightarrow{n}\times_L\frac{\partial\overrightarrow{t}}{\partial n}+\overrightarrow{b}\times_L\frac{\partial\overrightarrow{t}}{\partial b}\\ &=\kappa\overrightarrow{b}-\alpha_2\overrightarrow{t}+\beta_1\overrightarrow{t}=(\beta_1-\alpha_2)\overrightarrow{t}+\kappa\overrightarrow{b},\end{aligned}$$

$$\operatorname{curl}\overrightarrow{n}=\overrightarrow{t}\times_L\frac{\partial\overrightarrow{n}}{\partial s}+\overrightarrow{n}\times_L\frac{\partial\overrightarrow{n}}{\partial n}+\overrightarrow{b}\times_L\frac{\partial\overrightarrow{n}}{\partial b}$$

$$= \tau\overrightarrow{n} + \xi_{ns}\overrightarrow{b} - \operatorname{div} b\overrightarrow{t} + \beta_1\overrightarrow{n}$$
$$= -\operatorname{div}\overrightarrow{b}\overrightarrow{t} + (\tau + \beta_1)\overrightarrow{n} + \xi_{ns}\overrightarrow{b}$$

and

$$\begin{aligned}
\operatorname{curl}\overrightarrow{b} &= \overrightarrow{t} \times_L \frac{\partial\overrightarrow{b}}{\partial s} + \overrightarrow{n} \times_L \frac{\partial\overrightarrow{b}}{\partial n} + \overrightarrow{b} \times_L \frac{\partial\overrightarrow{b}}{\partial b} \\
&= \overrightarrow{t} \times_L (\tau\overrightarrow{n}) + \overrightarrow{n} \times_L (\alpha_2\overrightarrow{t} + \operatorname{div}\overrightarrow{b}\overrightarrow{n}) + \overrightarrow{b} \times_L (-\xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n}) \\
&= \tau\overrightarrow{b} - \alpha_2\overrightarrow{b} + \xi_{bs}\overrightarrow{n} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{t} \\
&= -(\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{t} + \xi_{bs}\overrightarrow{n} + (\tau - \alpha_2)\overrightarrow{b}.
\end{aligned}$$

Therefore, we get

$$\mu_s = \left\langle \operatorname{curl}\overrightarrow{t}, \overrightarrow{t} \right\rangle_L = \left\langle (\beta_1 - \alpha_2)\overrightarrow{t} + \kappa\overrightarrow{b}, \overrightarrow{t} \right\rangle_L = \beta_1 - \alpha_2,$$

$$\mu_n = \left\langle \operatorname{curl}\overrightarrow{n}, \overrightarrow{n} \right\rangle_L = \left\langle -\operatorname{div}\overrightarrow{b}\overrightarrow{t} + (\tau + \beta_1)\overrightarrow{n} + \xi_{ns}\overrightarrow{b}, \overrightarrow{n} \right\rangle_L = \tau + \beta_1,$$

$$\mu_b = \left\langle \operatorname{curl}\overrightarrow{b}, \overrightarrow{b} \right\rangle_L = \left\langle -(\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{t} + \xi_{bs}\overrightarrow{n} + (\tau - \alpha_2)\overrightarrow{b}, \overrightarrow{b} \right\rangle_L = \alpha_2 - \tau.$$

Thus, we obtain

$$\beta_1 = \mu_n - \tau, \qquad \alpha_2 = \tau + \mu_b.$$

Finally, if we substitute obtained values of the smooth functions; α_i and β_i for $i = 1, 2, 3$, then we get

$$\frac{\partial}{\partial n}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 0 & \xi_{ns} & \tau + \mu_b \\ -\xi_{ns} & 0 & \operatorname{div} b \\ \tau + \mu_b & \operatorname{div} b & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix},$$

$$\frac{\partial}{\partial b}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 0 & \mu_n - \tau & -\xi_{bs} \\ \tau - \mu_n & 0 & -\left(\operatorname{div}\overrightarrow{n} + \kappa\right) \\ -\xi_{bs} & -(\operatorname{div}\overrightarrow{n} + \kappa) & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix}.$$

□

Corollary 2. *The following relation between abnormalities of* $\overrightarrow{t}, \overrightarrow{n}$ *and* $\overrightarrow{b}$ *is given by*

$$\mu_s + \tau = \frac{1}{2}(\mu_s + \mu_n - \mu_b).$$

Above equation proves that important results involving Dupin theorem. This means that all coordinate surfaces intersect along common curvature lines in a triply orthogonal coordinate system.

Remark 3. *In vectorial analysis, the vector operator* curl *describes the infinitesimal circulation of a vector field in three-dimensional space. So, the* curl *of a vector field refers to the idea of how a fluid may rotate. It is seen that*

$$\operatorname{curl}\overrightarrow{t} = \mu_s\overrightarrow{t} + \kappa\overrightarrow{b} \tag{4}$$

by proof of above theorem. The results of Eq (4) will be discussed in the next sections. Since $\operatorname{curl}\overrightarrow{t}$ *does not include any component in the direction of principal normal* $\overrightarrow{n}$*, then there exists a surface which contains both* $s-lines$ *and* $b-lines$*.*

Considering the identity curl grad $f = 0$, we obtain

$$\begin{aligned}
\operatorname{curl}\operatorname{grad} f &= \overrightarrow{t} \times_L \left(\frac{\partial}{\partial s}(\overrightarrow{t}\frac{\partial f}{\partial s} + \overrightarrow{n}\frac{\partial f}{\partial n} + \overrightarrow{b}\frac{\partial f}{\partial b})\right) + \overrightarrow{n} \times_L \left(\frac{\partial}{\partial n}(\overrightarrow{t}\frac{\partial f}{\partial s} + \overrightarrow{n}\frac{\partial f}{\partial n} + \overrightarrow{b}\frac{\partial f}{\partial b})\right) \\
&+ \overrightarrow{b} \times_L \left(\frac{\partial}{\partial b}(\overrightarrow{t}\frac{\partial f}{\partial s} + \overrightarrow{n}\frac{\partial f}{\partial n} + \overrightarrow{b}\frac{\partial f}{\partial b})\right) \\
&= \overrightarrow{t} \times_L \left(\frac{\partial \overrightarrow{t}}{\partial s}\frac{\partial f}{\partial s} + \overrightarrow{t}\frac{\partial^2 f}{\partial s^2} + \frac{\partial \overrightarrow{n}}{\partial s}\frac{\partial f}{\partial n} + \overrightarrow{n}\frac{\partial^2 f}{\partial s \partial n} + \frac{\partial \overrightarrow{b}}{\partial s}\frac{\partial f}{\partial b} + \overrightarrow{b}\frac{\partial^2 f}{\partial s \partial b}\right) \\
&+ \overrightarrow{n} \times_L \left(\frac{\partial \overrightarrow{t}}{\partial n}\frac{\partial f}{\partial s} + \overrightarrow{t}\frac{\partial^2 f}{\partial n \partial s} + \frac{\partial \overrightarrow{n}}{\partial n}\frac{\partial f}{\partial n} + \overrightarrow{n}\frac{\partial^2 f}{\partial n^2} + \frac{\partial \overrightarrow{b}}{\partial n}\frac{\partial f}{\partial b} + \overrightarrow{b}\frac{\partial^2 f}{\partial n \partial b}\right) \\
&+ \overrightarrow{b} \times_L \left(\frac{\partial \overrightarrow{t}}{\partial b}\frac{\partial f}{\partial s} + \overrightarrow{t}\frac{\partial^2 f}{\partial b \partial s} + \frac{\partial \overrightarrow{n}}{\partial n}\frac{\partial f}{\partial n} + \overrightarrow{n}\frac{\partial^2 f}{\partial b \partial n} + \frac{\partial \overrightarrow{b}}{\partial b}\frac{\partial f}{\partial b} + \overrightarrow{b}\frac{\partial^2 f}{\partial b^2}\right) \\
&= \frac{\partial f}{\partial s}\operatorname{curl}\overrightarrow{t} + \frac{\partial f}{\partial n}\operatorname{curl}\overrightarrow{n} + \frac{\partial f}{\partial b}\operatorname{curl}\overrightarrow{b} + \overrightarrow{t} \times_L \left(\overrightarrow{t}\frac{\partial^2 f}{\partial s^2} + \overrightarrow{n}\frac{\partial^2 f}{\partial s \partial n} + \overrightarrow{b}\frac{\partial^2 f}{\partial s \partial b}\right) \\
&+ \overrightarrow{n} \times_L \left(\overrightarrow{t}\frac{\partial^2 f}{\partial n \partial s} + \overrightarrow{n}\frac{\partial^2 f}{\partial n^2} + \overrightarrow{b}\frac{\partial^2 f}{\partial n \partial b}\right) + \overrightarrow{b} \times_L \left(\overrightarrow{t}\frac{\partial^2 f}{\partial b \partial s} + \overrightarrow{n}\frac{\partial^2 f}{\partial b \partial n} + \overrightarrow{b}\frac{\partial^2 f}{\partial b^2}\right) \\
&= \frac{\partial f}{\partial s}\operatorname{curl}\overrightarrow{t} + \frac{\partial f}{\partial n}\operatorname{curl}\overrightarrow{n} + \frac{\partial f}{\partial b}\operatorname{curl}\overrightarrow{b} + \left(\frac{\partial^2 f}{\partial b \partial n} - \frac{\partial^2 f}{\partial n \partial b}\right)\overrightarrow{t} \\
&+ \left(\frac{\partial^2 f}{\partial s \partial b} - \frac{\partial^2 f}{\partial b \partial s}\right)\overrightarrow{n} + \left(\frac{\partial^2 f}{\partial s \partial n} - \frac{\partial^2 f}{\partial n \partial s}\right) = \overrightarrow{0}.
\end{aligned}$$

By using of above relations, we get

$$0 = (\frac{\partial^2 f}{\partial b \partial n} - \frac{\partial^2 f}{\partial n \partial b} + \frac{\partial f}{\partial s}\mu_s - \frac{\partial f}{\partial n}\operatorname{div}\overrightarrow{b} - \frac{\partial f}{\partial b}(\operatorname{div}\overrightarrow{n} + \kappa))\overrightarrow{t}$$

$$+ (\frac{\partial^2 f}{\partial s \partial b} - \frac{\partial^2 f}{\partial b \partial s} + \frac{\partial f}{\partial n}\mu_n + \frac{\partial f}{\partial b}\xi_{bs})\overrightarrow{n} + (\frac{\partial^2 f}{\partial s \partial n} - \frac{\partial^2 f}{\partial n \partial s} + \frac{\partial f}{\partial s}\kappa + \frac{\partial f}{\partial n}\xi_{ns} - \frac{\partial f}{\partial b}\mu_b)\overrightarrow{b}.$$

This gives the following relations

$$\frac{\partial^2 f}{\partial b \partial n} - \frac{\partial^2 f}{\partial n \partial b} = -\frac{\partial f}{\partial s}\mu_s + \frac{\partial f}{\partial n}\operatorname{div}\overrightarrow{b} + \frac{\partial f}{\partial b}(\operatorname{div}\overrightarrow{n} + \kappa), \tag{5}$$

$$\frac{\partial^2 f}{\partial s \partial b} - \frac{\partial^2 f}{\partial b \partial s} = -\frac{\partial f}{\partial n}\mu_n - \frac{\partial f}{\partial b}\xi_{bs}, \tag{6}$$

$$\frac{\partial^2 f}{\partial s \partial n} - \frac{\partial^2 f}{\partial n \partial s} = -\frac{\partial f}{\partial s}\kappa - \frac{\partial f}{\partial n}\xi_{ns} + \frac{\partial f}{\partial b}\mu_b. \tag{7}$$

In general, the mixed derivatives of order two don't commute. This means that s, n and b represent anholonomic coordinates.

Theorem 4. *The intrinsic representations of* grad $\overrightarrow{t}$, grad $\overrightarrow{n}$ *and* grad $\overrightarrow{b}$ *give following conditions on geometric parameters* $\kappa, \tau, \mu_s, \mu_b, \operatorname{div}\overrightarrow{n}, \operatorname{div}\overrightarrow{b}, \xi_{ns}, \xi_{bs}$ *by the compatibility of the linear systems*

$$\frac{\partial \xi_{ns}}{\partial b} - \frac{\partial(\mu_n - \tau)}{\partial n} = (\mu_b + \mu_n)(\operatorname{div}\overrightarrow{n} + \kappa) + (-\xi_{bs} + \xi_{ns})\operatorname{div}\overrightarrow{b} - \mu_s\kappa, \tag{8}$$

$$\frac{\partial(\mu_b + \tau)}{\partial b} + \frac{\partial \xi_{bs}}{\partial n} = (\xi_{ns} - \xi_{bs})(\operatorname{div}\overrightarrow{n} + \kappa) + (\mu_b + \mu_n)\operatorname{div}\overrightarrow{b}, \tag{9}$$

$$\frac{\partial \operatorname{div}\overrightarrow{b}}{\partial b} + \frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial n} = -\xi_{bs}\xi_{ns} + (\mu_b + \tau)(\tau - \mu_n) + \operatorname{div}^2\overrightarrow{b} - (\operatorname{div}\overrightarrow{n} + \kappa)^2 - \mu_s\tau, \tag{10}$$

$$\frac{\partial(\mu_n - \tau)}{\partial s} - \frac{\partial \kappa}{\partial b} = \xi_{bs}2\tau - \mu_n(\xi_{ns} + \xi_{bs}), \tag{11}$$

$$\frac{\partial \xi_{bs}}{\partial s} = \kappa(\operatorname{div}\overrightarrow{n} + \kappa) - \tau^2 + (\mu_b + 2\tau)\mu_n - \xi_{bs}^2, \tag{12}$$

$$\frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial s} + \frac{\partial \tau}{\partial b} = -(\operatorname{div}\overrightarrow{n} + 2\kappa)\xi_{bs} + \operatorname{div}\overrightarrow{b}\mu_n, \tag{13}$$

$$\frac{\partial \xi_{ns}}{\partial s} - \frac{\partial \kappa}{\partial n} = -(\mu_b + \tau)\tau - \kappa^2 - \xi_{ns}^2 + (\mu_n - \tau)\mu_b, \tag{14}$$

$$\frac{\partial(\mu_b + \tau)}{\partial s} = -\tau\xi_{ns} + \kappa\operatorname{div}\overrightarrow{b} - (\mu_b + \tau)\xi_{ns} - \xi_{bs}\mu_b, \tag{15}$$

$$\frac{\partial \operatorname{div}\overrightarrow{b}}{\partial s} - \frac{\partial \tau}{\partial n} = -(\mu_b + 2\tau)\kappa - \operatorname{div}\overrightarrow{b}\xi_{ns} - (\operatorname{div}\overrightarrow{n} + \kappa)\mu_b. \tag{16}$$

Proof. By using Eq (5), we may write

$$\frac{\partial^2 \overrightarrow{t}}{\partial b \partial n} - \frac{\partial^2 \overrightarrow{t}}{\partial n \partial b} = -\frac{\partial \overrightarrow{t}}{\partial s}\mu_s + \frac{\partial \overrightarrow{t}}{\partial n}\operatorname{div}\overrightarrow{b} + \frac{\partial \overrightarrow{t}}{\partial b}(\operatorname{div}\overrightarrow{n} + \kappa).$$

By compatibility of the linear systems in Eqs (2) and (3), we obtain

$$\begin{aligned}
\frac{\partial^2 \overrightarrow{t}}{\partial b \partial n} - \frac{\partial^2 \overrightarrow{t}}{\partial n \partial b} &= \frac{\partial}{\partial b}(\xi_{ns}\overrightarrow{n} + (\mu_b + \tau)\overrightarrow{b}) - \frac{\partial}{\partial n}((\mu_n - \tau)\overrightarrow{n} - \xi_{bs}\overrightarrow{b})\\
&= \frac{\partial \xi_{ns}}{\partial b}\overrightarrow{n} + \xi_{ns}\frac{\partial \overrightarrow{n}}{\partial b} + \frac{\partial(\mu_b + \tau)}{\partial b}\overrightarrow{b} + (\mu_b + \tau)\frac{\partial \overrightarrow{b}}{\partial b}\\
&- \frac{\partial}{\partial n}(\mu_n - \tau)\overrightarrow{n} - (\mu_n - \tau)\frac{\partial \overrightarrow{n}}{\partial n} + \frac{\partial \xi_{bs}}{\partial n}\overrightarrow{b} + \xi_{bs}\frac{\partial \overrightarrow{b}}{\partial n}\\
&= \frac{\partial \xi_{ns}}{\partial b}\overrightarrow{n} + \xi_{ns}((\tau - \mu_n)\overrightarrow{t} + (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{b}) + \frac{\partial(-\mu_b - \tau)}{\partial b}\overrightarrow{b}\\
&+ (-\mu_b - \tau)(\xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n}) - \frac{\partial}{\partial n}(\tau - \mu_n)\overrightarrow{n}\\
&- (\tau - \mu_n)(\xi_{ns}\overrightarrow{t} - \operatorname{div}\overrightarrow{b}\,\overrightarrow{b}) - \frac{\partial \xi_{bs}}{\partial n}\overrightarrow{b} - \xi_{bs}((-\mu_b - \tau)\overrightarrow{t} + \operatorname{div}\overrightarrow{b}\,\overrightarrow{n})\\
&= (\xi_{ns}(\tau - \mu_n) + \xi_{bs}(-\mu_b - \tau) - \xi_{ns}(\tau - \mu_n) - \xi_{bs}(-\mu_b - \tau))\overrightarrow{t}\\
&+ (\frac{\partial \xi_{ns}}{\partial b} - (-\mu_b - \tau)(\operatorname{div}\overrightarrow{n} + \kappa) - \frac{\partial}{\partial n}(\tau - \mu_n) - \xi_{bs}\operatorname{div}\overrightarrow{b})\overrightarrow{n}\\
&+ (\xi_{ns}(\operatorname{div}\overrightarrow{n} + \kappa) + \frac{\partial(-\mu_b - \tau)}{\partial b} + (\tau - \mu_n)\operatorname{div}\overrightarrow{b} - \frac{\partial \xi_{bs}}{\partial n})\overrightarrow{b}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{\partial^2 \overrightarrow{t}}{\partial b \partial n} - \frac{\partial^2 \overrightarrow{t}}{\partial n \partial b} &= (\frac{\partial \xi_{ns}}{\partial b} - (\mu_b + \tau)(\operatorname{div}\overrightarrow{n} + \kappa) - \frac{\partial}{\partial n}(\mu_n - \tau) + \xi_{bs}\operatorname{div}\overrightarrow{b})\overrightarrow{n}\\
&+ (-\xi_{ns}(\operatorname{div}\overrightarrow{n} + \kappa) + \frac{\partial(\mu_b + \tau)}{\partial b} - (\mu_n - \tau)\operatorname{div}\overrightarrow{b} + \frac{\partial \xi_{bs}}{\partial n})\overrightarrow{b}.
\end{aligned}$$

On the other hand, we have

$$-\frac{\partial \overrightarrow{t}}{\partial s}\mu_s + \frac{\partial \overrightarrow{t}}{\partial n}\operatorname{div}\overrightarrow{b} + \frac{\partial \overrightarrow{t}}{\partial b}(\operatorname{div}\overrightarrow{n} + \kappa) = -(\kappa\overrightarrow{n})\mu_s + (\xi_{ns}\overrightarrow{n} + (\mu_b + \tau)\overrightarrow{b})\operatorname{div}\overrightarrow{b}$$

$$+ ((\mu_n - \tau)\overrightarrow{n} - \xi_{bs}\overrightarrow{b})(\operatorname{div}\overrightarrow{n} + \kappa)$$

$$= (\kappa\mu_s - \xi_{ns}\operatorname{div}\overrightarrow{b} - (\tau - \mu_n)(\operatorname{div}\overrightarrow{n} + \kappa))\overrightarrow{n}$$

$$+ ((\mu_b + \tau)\operatorname{div}\overrightarrow{b} - \xi_{bs}(\operatorname{div}\overrightarrow{n} + \kappa))\overrightarrow{b}.$$

Therefore, we get

$$\frac{\partial \xi_{ns}}{\partial b} + \frac{\partial(\mu_n - \tau)}{\partial n} = (\mu_b + \mu_n)(\operatorname{div}\overrightarrow{n} + \kappa) + (-\xi_{bs} + \xi_{ns})\operatorname{div} b\overrightarrow{b} - \mu_s\kappa.$$

By equality of the coefficient of binormal vector fields, we obtain

$$\frac{\partial(\mu_b + \tau)}{\partial b} + \frac{\partial \xi_{bs}}{\partial n} = (\xi_{ns} - \xi_{bs})(\operatorname{div}\overrightarrow{n} + \kappa) + (\mu_b + \mu_n)\operatorname{div} b\overrightarrow{b}.$$

By using Eq (5), we have

$$\frac{\partial^2\overrightarrow{n}}{\partial b\partial n} - \frac{\partial^2\overrightarrow{n}}{\partial n\partial b} = \frac{\partial}{\partial b}\left(-\xi_{ns}\overrightarrow{t} + \operatorname{div} b\overrightarrow{b}\right) - \frac{\partial}{\partial n}\left((\tau - \mu_n)\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{b}\right)$$

$$= \left(-\frac{\partial}{\partial b}\xi_{ns} - \xi_{bs}\operatorname{div} b - \frac{\partial}{\partial n}(\tau - \mu_n) + (\tau + \mu_n) - (\operatorname{div}\overrightarrow{n} + \kappa)\right)\overrightarrow{t}$$

$$+ \left(\xi_{ns}\xi_{bs} + \frac{\partial}{\partial b}\operatorname{div} b - (\tau - \mu_n)(\mu_b + \tau) + \frac{\partial}{\partial n}(\operatorname{div}\overrightarrow{n} + \kappa)\right)\overrightarrow{b}.$$

and

$$-\mu_s\frac{\partial n}{\partial s} + \operatorname{div} b\frac{\partial n}{\partial n} + (\operatorname{div}\overrightarrow{n} + \kappa)\frac{\partial n}{\partial b} = \left(\mu_s\kappa - \xi_{ns}\operatorname{div} b + (\operatorname{div}\overrightarrow{n} + \kappa)(\tau - \mu_n)\right)\overrightarrow{t}$$
$$+ \left(-\mu_s + (\operatorname{div} b)^2 - (\operatorname{div}\overrightarrow{n} + \kappa)^2\right)\overrightarrow{b}.$$

By using coefficient of binormal vector field, we get

$$\frac{\partial \operatorname{div}\overrightarrow{b}}{\partial b} + \frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial n} = -\xi_{bs}\xi_{ns} + (\mu_b + \tau)(\tau - \mu_n) + \operatorname{div}^2\overrightarrow{b} - (\operatorname{div}\overrightarrow{n} + \kappa)^2 - \mu_s\tau.$$

Similarly, we have

$$\frac{\partial^2\overrightarrow{t}}{\partial s\partial b}-\frac{\partial^2\overrightarrow{t}}{\partial b\partial s}=\frac{\partial}{\partial s}(\frac{\partial\overrightarrow{t}}{\partial b})-\frac{\partial}{\partial b}(\frac{\partial\overrightarrow{t}}{\partial s})$$

by Eq (6). By compatibility of the linear systems in Eqs (1) and (3), we obtain

$$\frac{\partial^2\overrightarrow{t}}{\partial s\partial b}-\frac{\partial^2\overrightarrow{t}}{\partial b\partial s}=\frac{\partial}{\partial s}((\mu_n-\tau)\overrightarrow{n}-\xi_{bs}\overrightarrow{b})-\frac{\partial}{\partial b}(\kappa\overrightarrow{n})$$

$$=\frac{\partial(\mu_n-\tau)}{\partial s}\overrightarrow{n}+(\mu_n-\tau)\frac{\partial\overrightarrow{n}}{\partial s}-\frac{\partial\xi_{bs}}{\partial s}\overrightarrow{b}-\xi_{bs}\frac{\partial\overrightarrow{b}}{\partial s}-\frac{\partial\kappa}{\partial b}\overrightarrow{n}-\kappa\frac{\partial\overrightarrow{n}}{\partial b}.$$

Then, we get

$$\frac{\partial^2\overrightarrow{t}}{\partial s\partial b}-\frac{\partial^2\overrightarrow{t}}{\partial b\partial s}=\frac{\partial(\tau-\mu_n)}{\partial s}\overrightarrow{n}+(\tau-\mu_n)(\kappa\overrightarrow{t}+\tau\overrightarrow{b})+\frac{\partial\xi_{bs}}{\partial s}\overrightarrow{b}+\xi_{bs}(-\tau\overrightarrow{n})$$

$$-\frac{\partial\kappa}{\partial b}\overrightarrow{n}-\kappa((\tau-\mu_n)\overrightarrow{t}+(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{b})$$

$$=(\frac{\partial(\mu_n-\tau)}{\partial s}-\tau\xi_{bs}-\frac{\partial\kappa}{\partial b})\overrightarrow{n}+((\mu_n-\tau)\tau-\frac{\partial\xi_{bs}}{\partial s}+\kappa(\operatorname{div}\overrightarrow{n}+\kappa))\overrightarrow{b}.$$

Moreover, we have

$$-\frac{\partial\overrightarrow{t}}{\partial n}\mu_n-\frac{\partial\overrightarrow{t}}{\partial b}\xi_{bs}=-(\xi_{ns}\overrightarrow{n}+(\mu_b+\tau)\overrightarrow{b})\mu_n-\Big((\mu_n-\tau)\overrightarrow{n}-\xi_{bs}\overrightarrow{b}\Big)\xi_{bs}$$

$$=(-\xi_{ns}\mu_n-\xi_{bs}(\mu_n-\tau))\overrightarrow{n}+(-(\mu_b+\tau)\mu_n+\xi_{bs}^2)\overrightarrow{b}.$$

This gives the following equation

$$\frac{\partial(\mu_n-\tau)}{\partial s}-\frac{\partial\kappa}{\partial b}=2\tau\xi_{bs}-(\xi_{ns}+\xi_{bs})\,\mu_n).$$

And we obtain

$$\frac{\partial\xi_{bs}}{\partial s}=\kappa(\operatorname{div}\overrightarrow{n}+\kappa)-\tau^2+(\mu_b+2\tau)\mu_n-\xi_{bs}^2$$

by coefficients of binormal vector fields. Similarly, we get

$$\frac{\partial^2\overrightarrow{b}}{\partial s\partial b}-\frac{\partial^2\overrightarrow{b}}{\partial b\partial s}=\frac{\partial}{\partial s}(\frac{\partial\overrightarrow{b}}{\partial b})-\frac{\partial}{\partial b}(\frac{\partial\overrightarrow{b}}{\partial s}).$$

Then, we also have

$$\frac{\partial}{\partial s}(\frac{\partial \overrightarrow{b}}{\partial b}) - \frac{\partial}{\partial b}(\frac{\partial \overrightarrow{b}}{\partial s}) = (-\frac{\partial(\xi_{bs})}{\partial s} + \kappa(\operatorname{div}\overrightarrow{n} + \kappa) - \tau(\tau - \mu_n))\overrightarrow{t}$$

$$+ (-\frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial s} - \frac{\partial \tau}{\partial b} - \kappa\xi_{bs})\overrightarrow{n}.$$

We obtain

$$-\frac{\partial \overrightarrow{b}}{\partial b}\xi_{bs} - \frac{\partial \overrightarrow{b}}{\partial n}\mu_n = \left(\xi_{bs}^2 - \mu_n(\tau + \mu_b)\right)\overrightarrow{t} + (\xi_{bs}(\operatorname{div}\overrightarrow{n} + \kappa) - \mu_n \operatorname{div}\overrightarrow{b})\overrightarrow{n}$$

by Eq (6). Then, we get

$$\frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial s} + \frac{\partial \tau}{\partial b} = -(\operatorname{div}\overrightarrow{n} + 2\kappa)\xi_{bs} + \operatorname{div}\overrightarrow{b}\mu_n.$$

The last three equations can be obtained by similar way to the others. □

3. Normal congruence of surfaces containing $s - lines$ and $b - lines$

There exists a normal congruence of surfaces including the $s - lines$ and $b - lines$ if and only if

$$\mu_n = 0. \tag{17}$$

Theorem 5. *Gauss-Mainardi-Codazzi equations are obtained as follows:*

$$\frac{\partial \tau}{\partial s} + \frac{\partial \kappa}{\partial b} = -2\tau\xi_{bs},$$

$$\frac{\partial \xi_{bs}}{\partial s} = -\xi_{bs}^2 - \tau^2 + \kappa(\operatorname{div}\overrightarrow{n} + \kappa),$$

$$\frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial s} + \frac{\partial \tau}{\partial b} = -\xi_{bs}(\operatorname{div}\overrightarrow{n} + 2\kappa).$$

Proof. The proof can be stated by using the compatibility conditions of Eqs (10), (11) and (16) in Theorem 4. In the case of $\mu_n = 0$, these equations reduces to Gauss-Mainardi-Codazzi equations for this normal congruence of surfaces. □

Corollary 6. *In the case of $\mu_n = 0$, since the $s - lines$ and $b - lines$ lie on the constituent surfaces Ψ, this means that $\overrightarrow{n}$ is perpendicular to surface. Thus $\overrightarrow{n}$ parallel to the normal vector field $\overrightarrow{N}$ of the surfaces Ψ.*

Proof. By definitions of the vector fields $\overrightarrow{t}$ and $\overrightarrow{b}$, we obtain

$$\frac{\partial \Psi}{\partial s} = \frac{\partial \sigma}{\partial s} = \overrightarrow{t} \text{ and } \frac{\partial \Psi}{\partial b} = \frac{\partial \sigma}{\partial b} = \overrightarrow{b}.$$

Then, we have

$$\frac{\partial \Psi}{\partial s} \times_L \frac{\partial \Psi}{\partial b} = \overrightarrow{t} \times_L \overrightarrow{b} = \overrightarrow{n}.$$

Thus we obtain

$$\overrightarrow{N}=\frac{\frac{\partial\Psi}{\partial s}\times_L\frac{\partial\Psi}{\partial b}}{\left\|\frac{\partial\Psi}{\partial s}\times_L\frac{\partial\Psi}{\partial b}\right\|}=\overrightarrow{n}.$$

□

Remark 7. *The one-parameter family of surfaces* Ψ, *which contain the* $s-lines$ *and* $b-lines$, *are timelike surfaces, since* $\overrightarrow{n}$ *is a spacelike vector field.*

Theorem 8. *The geodesic curvature of* $b-lines$ *of the surface* Ψ *is given as follows*

$$k_{g_b}=\xi_{bs}.$$

And $s-lines$ *are the geodesics of the surface* Ψ.

Proof. It is known that

$$\frac{\partial^2\Psi}{\partial b^2}=\frac{\partial\overrightarrow{b}}{\partial b}=-\xi_{bs}\overrightarrow{t}-(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{n}$$

by Eq (3) in Theorem 1. Then we get the geodesic curvatures of $b-lines$ as follows

$$k_{g_b}=\left\langle\frac{\partial^2\Psi}{\partial b^2},\overrightarrow{n}\times_L\overrightarrow{b}\right\rangle_L$$

$$=\left\langle-\xi_{bs}\overrightarrow{t}-(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{n},\overrightarrow{n}\times_L\overrightarrow{b}\right\rangle_L$$

$$=\left\langle-\xi_{bs}\overrightarrow{t}-(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{n},-\overrightarrow{t}\right\rangle_L=\xi_{bs}.$$

Similarly, we get

$$\frac{\partial^2\Psi}{\partial s^2}=\frac{\partial^2\sigma}{\partial s^2}=\frac{\partial\overrightarrow{t}}{\partial s}=\kappa\overrightarrow{n}$$

by Eq (1) in Theorem 1. So, we get the geodesic curvatures of $s-lines$ as follows

$$k_{g_s}=\left\langle\frac{\partial^2\Psi}{\partial s^2},\overrightarrow{n}\times_L\overrightarrow{t}\right\rangle_L$$

$$=\left\langle\kappa\overrightarrow{n},\overrightarrow{n}\times_L\overrightarrow{t}\right\rangle=\left\langle\kappa\overrightarrow{n},-\overrightarrow{b}\right\rangle=0.$$

This implies that $s-lines$ are the geodesics of the surface Ψ. □

Theorem 9. *The normal curvatures of* $b-lines$ *and* $s-lines$ *of the surface* Ψ *are given as follows*

$$k_{n_b}=-(\operatorname{div}\overrightarrow{n}+\kappa),\qquad k_{n_s}=\kappa$$

respectively.

Proof. Again by using the equation

$$\frac{\partial^2\Psi}{\partial b^2} = -\xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n},$$

we obtain the normal curvatures of $b - lines$ as follows

$$k_{n_b} = \left\langle \frac{\partial^2\Psi}{\partial b^2}, \overrightarrow{n} \right\rangle_L$$

$$= \left\langle -\xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n}, \overrightarrow{n} \right\rangle_L$$

$$= -(\operatorname{div}\overrightarrow{n} + \kappa).$$

And so, the normal curvatures of $s - lines$ are obtained as follows

$$k_{n_s} = \left\langle \frac{\partial^2\Psi}{\partial s^2}, \overrightarrow{n} \right\rangle_L = \left\langle \kappa\overrightarrow{n}, \overrightarrow{n} \right\rangle_L = \kappa.$$

□

Theorem 10. *The geodesic torsion of $b - lines$ and $s - lines$ of the surface Ψ are obtained as follows*

$$\tau_{g_b} = \tau, \qquad \tau_{g_s} = -\tau,$$

respectively.

Proof. We obtain the geodesic torsion of $b - lines$ as follows:

$$\tau_{g_b} = \left\langle \frac{\partial\overrightarrow{n}}{\partial b}, \overrightarrow{n}\times_L\overrightarrow{b} \right\rangle_L$$
$$= \left\langle \tau\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{b}, -\overrightarrow{t} \right\rangle_L = \tau$$

by Eq (3) in Theorem 1. Similarly, the geodesic torsion of $s - lines$ is given as follows

$$\tau_{g_s} = -\left\langle \frac{\partial\overrightarrow{n}}{\partial s}, \overrightarrow{n}\times_L\overrightarrow{t} \right\rangle_L$$
$$= -\left\langle -\kappa\overrightarrow{t} + \tau\overrightarrow{b}, -\overrightarrow{b} \right\rangle_L = -\tau$$

by Eq (1) in Theorem 1. □

Theorem 11. *Gaussian and mean curvatures of the surface Ψ are given as follows*

$$K = \kappa(\operatorname{div}\overrightarrow{n} + \kappa),$$

$$H = -\frac{\operatorname{div} n + 2\kappa}{2},$$

respectively.

Proof. The first fundamental form of the surface Ψ is obtained as

$$I = \langle d\Psi, d\Psi\rangle_L = \left\langle \frac{\partial\Psi}{\partial s}ds + \frac{\partial\Psi}{\partial b}db, \frac{\partial\Psi}{\partial s}ds + \frac{\partial\Psi}{\partial b}db \right\rangle_L$$

$$= \left\langle \overrightarrow{t}\,ds + \overrightarrow{b}\,db, \overrightarrow{t}\,ds + \overrightarrow{b}\,db \right\rangle_L$$

$$= ds^2 - db^2.$$

We get $g_{11} = 1$, $g_{12} = 0$ and $g_{22} = -1$. Since normal vector field of the surface Ψ is equal to $\overrightarrow{n}$, we find the second fundamental form as follows

$$II = \left\langle d\Psi, d\overrightarrow{n} \right\rangle_L = \left\langle \frac{\partial\Psi}{\partial s}ds + \frac{\partial\Psi}{\partial b}db, \frac{\partial\overrightarrow{n}}{\partial s}ds + \frac{\partial\overrightarrow{n}}{\partial b}db \right\rangle_L$$

$$= \left\langle \overrightarrow{t}\,ds + \overrightarrow{b}\,db, (\kappa\overrightarrow{t} + \tau\overrightarrow{b})ds + (\tau\overrightarrow{t} + (\mathrm{div}\,\overrightarrow{n} + \kappa\overrightarrow{b})db \right\rangle_L$$

$$= -\kappa ds^2 + (\mathrm{div}\,\overrightarrow{n} + \kappa)db^2.$$

We have $l_{11} = -\kappa$, $l_{12} = 0$ and $l_{22} = \mathrm{div}\,\overrightarrow{n} + \kappa$. Thus, Gaussian curvature K of the surface Ψ is given

$$K = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{-\kappa(\mathrm{div}\,\overrightarrow{n} + \kappa)}{-1} = \kappa(\mathrm{div}\,\overrightarrow{n} + \kappa).$$

And the mean curvature H of the surface Ψ is obtained as

$$H = \frac{g_{11}l_{22} - 2g_{12}l_{12} + g_{22}l_{11}}{2(g_{11}g_{22} - g_{12}^2)} = \frac{(\mathrm{div}\,\overrightarrow{n} + \kappa) + \kappa}{-2} = \frac{\mathrm{div}\,\overrightarrow{n} + 2\kappa}{-2}.$$

□

Corollary 12. *If the following equality is satisfied*

$$\kappa(\mathrm{div}\,\overrightarrow{n} + \kappa) = 0,$$

then the surface Ψ is developable.

Remark 13. *We know that Gaussian curvature of the surface Ψ is found as*

$$K = \kappa(\mathrm{div}\,\overrightarrow{n} + \kappa).$$

By following equation

$$\frac{\partial\xi_{bs}}{\partial s} = -\xi_{bs}^2 + \kappa(\mathrm{div}\,\overrightarrow{n} + \kappa),$$

we obtain that

$$K = \frac{\partial\xi_{bs}}{\partial s} + \xi_{bs}^2.$$

If $b-$ lines are geodesics and $s-$ lines are plane curves, then the surface Ψ is developable.

Corollary 14. *The surface* Ψ *is minimal if and only if*

$$\operatorname{div}\overrightarrow{n} = -2\kappa.$$

Corollary 15. *The surface* Ψ *is a NLS surface if and only if* $\kappa = 1$.

Proof. It is easily seen that the equality

$$\frac{\partial\Psi}{\partial s}\times_L\frac{\partial^2\Psi}{\partial s^2} = \frac{\partial\Psi}{\partial b}$$

is satisfied if and only if $\kappa = 1$. □

Example 16. *Let the surface* $\Psi = \Psi(s, b)$ *containing the* $s-$ *lines and* $b-$ *lines be given as follows*

$$\Psi(s,b) = \left(\frac{1}{2}\sinh\left(s+\sqrt{5}b\right), \frac{1}{2}\cosh\left(s+\sqrt{5}b\right), \frac{\sqrt{5}}{2}s+\frac{1}{2}b\right)$$

where

$$\frac{\partial\Psi}{\partial s}(s,b) = \left(\frac{1}{2}\cosh\left(s+\sqrt{5}b\right), \frac{1}{2}\sinh\left(s+\sqrt{5}b\right), \frac{\sqrt{5}}{2}\right)$$

is a unit spacelike vector field. We obtain that

$$\overrightarrow{t}(s,b) = \left(\frac{1}{2}\cosh\left(s+\sqrt{5}b\right), \frac{1}{2}\sinh\left(s+\sqrt{5}b\right), \frac{\sqrt{5}}{2}\right),$$
$$\overrightarrow{n}(s,b) = \left(\sinh\left(s+\sqrt{5}b\right), \cosh\left(s+\sqrt{5}b\right), 0\right).$$

And we also see that

$$\frac{\partial\Psi}{\partial b}(s,b) = \left(\frac{\sqrt{5}}{2}\cosh\left(s+\sqrt{5}b\right), \frac{\sqrt{5}}{2}\sinh\left(s+\sqrt{5}b\right), \frac{1}{2}\right).$$

Thus, we get

$$\overrightarrow{b}(s,b) = \left(\frac{\sqrt{5}}{2}\cosh\left(s+\sqrt{5}b\right), \frac{\sqrt{5}}{2}\sinh\left(s+\sqrt{5}b\right), \frac{1}{2}\right).$$

Furthermore, we obtain that

$$\kappa(s,b) = \frac{1}{2}, \qquad \tau(s,b) = \frac{\sqrt{5}}{2}.$$

Then, we also have

$$\xi_{bs}(s,b) = \mu_n(s,b) = 0, \qquad \operatorname{div} n(s,b) = -2$$

which implies that

$$k_{n_s}(s,b) = \frac{1}{2}, \qquad \tau_{g_b}(s,b) = \frac{\sqrt{5}}{2}, \qquad \tau_{g_s}(s,b) = -\frac{\sqrt{5}}{2}.$$

Gaussian and mean curvature of the surface $\Psi = \Psi(s, b)$ *are obtained as follows*

$$K(s,b) = -\frac{5}{4}, \qquad H(s,b) = 1,$$

respectively.

4. Conclusions

This study investigates spacelike curves with spacelike normal vector field by means of anholomonic coordinates on Lorentzian space. Frenet-Serret formulas $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ of a given spacelike space curve are described which includes eight parameters related to three partial differential equations. It is proved that the curl of tangent vector field has no component in the direction of principal normal vector field. This means that there exists a surface which contains both $s-lines$ and $b-lines$. For this reason, the expression of this normal congruence is also discussed with intrinsic geometric properties. Finally, an example is stated to explain the obtained results.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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