AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 13633–13663.
DOI: 10.3934/math.2022752
Received: 30 March 2022
Revised: 11 May 2022
Accepted: 17 May 2022
Published: 23 May 2022

Research article

Fractional integral estimations pertaining to generalized γ-convex functions involving Raina's function and applications

Bandar Bin-Mohsin[1], **Muhammad Uzair Awan**[2,*], **Muhammad Zakria Javed**[2], **Artion Kashuri**[3] **and Muhammad Aslam Noor**[4]

[1] Department of Mathematics College of Science King Saud University, Riyadh, Saudi Arabia
[2] Department of Mathematics, Government College University, Faisalabad, Pakistan
[3] Department of Mathematics, Faculty of Technical and Natural Sciences, University "Ismail Qemali", 9400 Vlora, Albania
[4] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

* **Correspondence:** Email: awan.uzair@gmail.com; Tel: +923315098787.

Abstract: In this paper, we derive a new fractional integral identity and using this identity as an auxiliary result, some new trapezium like inequalities essentially using the class of generalized γ-convex functions are established. In order to show the efficiency of the obtained results, we discuss and present some special cases and applications.

Keywords: generalized γ-convexity; Jensen's integral inequality; power mean integral inequality; fractional integral; special means; bounded functions; error estimations
Mathematics Subject Classification: 26D07, 26D10, 26D15, 26A33, 26A51, 60E15

1. Introduction and preliminaries

A function $\mathcal{F} : C \to \mathbb{R}$ is said to be convex, if

$$\mathcal{F}(\vartheta\varpi_1 + (1-\vartheta)\varpi_2) \leq \vartheta\mathcal{F}(\varpi_1) + (1-\vartheta)\mathcal{F}(\varpi_2), \quad \forall \varpi_1, \varpi_2 \in C,\ \vartheta \in [0,1].$$

In recent years the classical concept of convex functions have been extended and generalized in different directions and an extensive research has been done in visualizing the properties of these new classes. For details, see [1–5]. The concept of generalized convex sets was defined by Cortez et al. [6] as follows:

Definition 1.1 ([6]). *Let $\rho, \lambda > 0$ and $\varepsilon = (\varepsilon(0), \ldots, \varepsilon(k), \ldots)$ be a bounded sequence of positive real* *numbers. A non-empty set $\mathcal{I} \subseteq \mathbb{R}$ is said to be generalized convex, if*

$$\varpi_1 + \tau\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1) \in \mathcal{I}, \quad \forall\, \varpi_1, \varpi_2 \in \mathcal{I},\ \tau \in [0, 1].$$

Here $\mathcal{R}^{\varepsilon}_{\lambda,\rho}(z)$ is the Raina's function and is defined as follows:

$$\mathcal{R}^{\varepsilon}_{\lambda,\rho}(z) = \mathcal{R}^{\varepsilon(0),\varepsilon(1),\ldots}_{\lambda,\rho}(z) := \sum_{k=0}^{\infty} \frac{\varepsilon(k)}{\Gamma(\rho k + \lambda)} z^k, \qquad z \in \mathbb{C}, \tag{1.1}$$

where $\rho, \lambda > 0$, with bounded modulus $|z| < M$, and $\varepsilon = \{\varepsilon(0), \varepsilon(1), \ldots, \varepsilon(k), \ldots\}$ is a bounded sequence of positive real numbers. For details, see [7].

The class of generalized convex functions is defined as:

Definition 1.2 ([6]). *Let $\rho, \lambda > 0$ and $\varepsilon = (\varepsilon(0), \ldots, \varepsilon(k), \ldots)$ be a bounded sequence of positive real numbers. A function $\mathcal{F} : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be generalized convex, if*

$$\mathcal{F}(\varpi_1 + \tau\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)) \leq (1 - \tau)\mathcal{F}(\varpi_1) + \tau\mathcal{F}(\varpi_2), \quad \forall\, \varpi_1, \varpi_2 \in \mathcal{I},\ \tau \in [0, 1].$$

For some recent studies regarding generalized convexity, see [6, 8].

We now introduce the class of generalized γ-convex functions.

Definition 1.3. *Let $\gamma : (0, 1) \to \mathbb{R}$ be a real function and $\rho, \lambda > 0$ and $\varepsilon = (\varepsilon(0), \ldots, \varepsilon(k), \ldots)$ be a bounded sequence of positive real numbers. A function $\mathcal{F} : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be generalized γ-convex, if*

$$\mathcal{F}(\varpi_1 + \tau\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)) \leq \gamma(1 - \tau)\mathcal{F}(\varpi_1) + \gamma(\tau)\mathcal{F}(\varpi_2), \quad \forall\, \varpi_1, \varpi_2 \in \mathcal{I},\ \tau \in [0, 1].$$

If the above inequality is reversed then we have the class of generalized γ-convexity.

Remark 1.1. *Note that, if we take $\gamma(t) = t^s, t^{-s}$ and $\gamma(t) = 1$, then we recapture the classes of generalized convex functions, Breckner type of generalized s-convex functions [8], Godunova–Levin type of generalized s-convex functions and generalized P-convex functions, respectively from Definition 1.3. This shows that the class of generalized γ-convex functions is quite unifying as it relates several other classes of the convexity.*

Theory of convex functions also played significant role in the development of theory of inequalities. Many inequalities particulary integral inequalities can be obtained easily using the concept of convex functions, see [9]. In recent years researchers have utilized different approaches in developing new analogues of classical inequalities. For example, Sarikaya et al. [10] elegantly used the concepts of fractional calculus in developing fractional analogues of Hermite–Hadamard's inequality. This paper opened a new venue in this direction and consequently extensive research has been done. For example, Du et al. [11] used the concepts of (s, m)-pre-invex functions and obtained variants of Hermite–Hadamard's inequality. Iqbal et al. [12] used the concepts of conformable fractional calculus and obtained new refinements of Hermite–Hadamard's inequality. Khurshid et al. [13] obtained conformable fractional Hermite–Hadamard's inequality using the class of pre-invex functions. Lei et al. [14] established some new bounds related to Fejér–Hermite–Hadamard type inequality and found their corresponding applications. Liao et al. [15] investigated Sugeno integral with respect to α-pre-invex functions. Erhan et al. [16] derived several Fejér–Hermite–Hadamard type inequalities for conformable fractional integrals. Zhang et al. [17] obtained some new k-fractional integral inequalities containing multiple parameters via generalized (s, m)-preinvexity. Mohammed et al. [18] established generalized Hermite–Hadamard inequalities via the tempered fractional integrals. Mohammed et al. [19] derived a new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals. Iqbal et al. [20] obtained Hermite–Hadamard type inequalities pertaining conformable fractional integrals and their applications. Houas et al. [21] found certain weighted integral inequalities involving the fractional hypergeometric operators. Srivastava et al. [22] established new Chebyshev type inequalities via a general family of fractional integral operators with a modified Mittag–Leffler kernel. Mohammed et al. [23] derived new fractional inequalities of Hermite–Hadamard type involving the incomplete gamma functions. Srivastava et al. [24] obtained some families of Mittag–Leffler type functions and associated operators of fractional calculus. Fernandez et al. [25, 26] investigated series representations for fractional-calculus operators involving generalised Mittag–Leffler functions. Srivastava et al. [27] established some new fractional-calculus connections between Mittag–Leffler functions. Srivastava et al. [28] investigated the study of fractional integral operators involving a certain generalized multi-index Mittag–Leffler function. Srivastava et al. [29] used fractional calculus with an integral operator containing a generalized Mittag–Leffler function in the kernel. Tomovski et al. [30] investigated fractional and operational calculus with generalized fractional derivative operators and Mittag–Leffler type functions. Sahoo et al. [31] derived new fractional integral inequalities for convex functions pertaining to Caputo–Fabrizio operator. Butt et al. [32] obtained new fractional Mercer–Ostrowski type inequalities with respect to monotone function. Qaisar et al. [33] established some new fractional integral inequalities of Hermite–Hadamard's type through convexity. Zhao et al. [34] derived Hermite–Jensen–Mercer type inequalities for Caputo fractional derivatives.

We now discuss some preliminaries which will be helpful in studying the main results of this paper.

Definition 1.4 ([35]). *Given a function $\mathcal{F} : [0, \infty) \to \mathbb{R}$, then*

$$D_\sigma(\mathcal{F})(\vartheta) = \lim_{\epsilon \to 0} \frac{\mathcal{F}(\vartheta + \epsilon\vartheta^{1-\sigma}) - \mathcal{F}(\vartheta)}{\epsilon},$$

for all $\vartheta > 0, \sigma \in (0, 1]$ is called fractional derivative.

We denote $\mathcal{F}^\sigma(\vartheta)$, $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\mathcal{F})$ for $D_\sigma(\mathcal{F})(\vartheta)$.

Theorem 1.1 ([35]). *Let $\sigma \in (0, 1]$ and $\mathcal{F}, g$ be σ-differentiable at a point $\vartheta > 0$. Then*

(1) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\vartheta^n) = n\vartheta^{n-\sigma}$, *for all $n \in \mathbb{R}$.*

(2) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(c) = 0$, *where c is a constant.*

(3) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\varpi_1\mathcal{F}(\vartheta) + \varpi_2 g(\vartheta)) = \varpi_1\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\mathcal{F}(\vartheta)) + \varpi_2\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(g(\vartheta))$, *for all $\varpi_1, \varpi_2 \in \mathbb{R}$.*

(4) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\mathcal{F}(\vartheta)g(\vartheta)) = \mathcal{F}(\vartheta)\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(g(\vartheta)) + g(\vartheta)\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\mathcal{F}(\vartheta))$.

(5) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}\left(\frac{\mathcal{F}(\vartheta)}{g(\vartheta)}\right) = \frac{g(\vartheta)\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\mathcal{F}(\vartheta)) - \mathcal{F}(\vartheta)\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(g(\vartheta))}{(g(\vartheta))^2}$.

(6) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}((\mathcal{F} \circ g)(\vartheta)) = \mathcal{F}'(g(\vartheta))\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(g(\vartheta))$, *for $\mathcal{F}$ differentiable at $g(\vartheta)$.*

In addition, if $\mathcal{F}$ is differentiable, then

$$\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\mathcal{F}(\vartheta)) = \vartheta^{1-\sigma}\frac{d}{d\vartheta}(\mathcal{F}(\vartheta)). \tag{1.2}$$

By applying (1.2), one can compute the following:

(1) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(1) = 0.$

(2) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(e^{c\vartheta}) = c\vartheta^{1-\sigma}e^{c\vartheta}, c \in \mathbb{R}.$

(3) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\sin(c\vartheta)) = c\vartheta^{1-\sigma}\cos(c\vartheta), c \in \mathbb{R}.$

(4) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\cos(c\vartheta)) = -c\vartheta^{1-\sigma}\sin(c\vartheta), c \in \mathbb{R}.$

(5) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}\left(\frac{1}{\sigma}\vartheta^\sigma\right) = 1.$

(6) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\sin\frac{\vartheta^\sigma}{\sigma}) = \cos(\frac{\vartheta^\sigma}{\sigma}).$

(7) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\cos\frac{\vartheta^\sigma}{\sigma}) = -\sin(\frac{\vartheta^\sigma}{\sigma}).$

(8) $\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(e^{\frac{\vartheta^\sigma}{\sigma}}) = e^{(\frac{\vartheta^\sigma}{\sigma})}.$

Theorem 1.2 ([35]). *Let $\sigma \in (0,1], \mathcal{F} : [\varpi_1, \varpi_2] \to \mathbb{R}$ be continuous on $[\varpi_1, \varpi_2]$ and σ-differentiable on (ϖ_1, ϖ_2) with $0 < \varpi_1 < \varpi_2$. Then. there exists $c \in (\varpi_1, \varpi_2)$ such that*

$$\frac{\mathrm{d}_\sigma}{\mathrm{d}_\sigma\vartheta}(\mathcal{F})(c) = \frac{\mathcal{F}(\varpi_2) - \mathcal{F}(\varpi_1)}{\frac{\varpi_2{}^\sigma}{\sigma} - \frac{\varpi_1{}^\sigma}{\sigma}}.$$

Definition 1.5 ([36]). *Let $\sigma \in (0,1]$ and $0 \leq \varpi_1 < \varpi_2$. A function $\mathcal{F} : [\varpi_1, \varpi_2] \to \mathbb{R}$ is σ-fractional integrable on $[\varpi_1, \varpi_2]$ if the integral*

$$\int_{\varpi_1}^{\varpi_2} \mathcal{F}(x)\mathrm{d}_\sigma x := \int_{\varpi_1}^{\varpi_2} \mathcal{F}(x)x^{\sigma-1}\mathrm{d}x,$$

exists and is finite.

The set of all σ-fractional integrable functions on $[\varpi_1, \varpi_2]$ is denoted by $L^1_\sigma([\varpi_1, \varpi_2])$.

Theorem 1.3 ([37]). *Let $\mathcal{F} : (\varpi_1, \varpi_2) \to \mathbb{R}$ be σ-differentiable and $0 < \sigma \leq 1$. Then for all $\vartheta > \varpi_1$, we have*

$$I^{\varpi_1}_\sigma D^{\varpi_1}_\sigma(\mathcal{F})(\vartheta) = \mathcal{F}(\vartheta) - \mathcal{F}(\varpi_1).$$

Theorem 1.4 ([37]). *(Integration by parts) Let $\mathcal{F}, g : [\varpi_1, \varpi_2] \to \mathbb{R}$ be two functions such that $\mathcal{F}g$ is differentiable. Then*

$$\int_{\varpi_1}^{\varpi_2} \mathcal{F}(x)D^{\varpi_1}_\sigma g(x)\mathrm{d}_\sigma x = (\mathcal{F}g)\mid^{\varpi_2}_{\varpi_1} - \int_{\varpi_1}^{\varpi_2} g(x)D^{\varpi_1}_\sigma \mathcal{F}(x)\mathrm{d}_\sigma x.$$

Theorem 1.5 ([37]). *Let $\mathcal{F} : [\varpi_1, \infty) \to \mathbb{R}$ be such that $\mathcal{F}^{(n)}(\vartheta)$ is continuous and $\sigma \in (n, n+1]$ where $n \in \mathbb{N}$. Then for all $\vartheta \geq \varpi_1$, we have*

$$D^{\varpi_1}_\sigma I^{\varpi_1}_\sigma(\mathcal{F})(\vartheta) = \mathcal{F}(\vartheta).$$

Theorem 1.6 ([38]). *Let $\mathcal{F}:[\varpi_1,\varpi_2]\to\mathbb{R}$ be a continuous function with $\varpi_1<\varpi_2$ and $0<\sigma\leq 1$. Then*

$$|I_{\sigma}^{\varpi_1}(\mathcal{F})(\vartheta)|\leq I_{\sigma}^{\varpi_1}(|\mathcal{F}|)(\vartheta).$$

We also need the following well-known beta functions (complete and incomplete), respectively, for some of our calculations, which are defined as:

$$\mathcal{B}(x,y)=\int_0^1 \vartheta^{x-1}(1-\vartheta)^{y-1}\mathrm{d}\vartheta,\quad \Re(x)>0,\ \Re(y)>0,$$

$$\mathcal{B}_{\rho}(x,y)=\int_0^{\rho} \vartheta^{x-1}(1-\vartheta)^{y-1}\mathrm{d}\vartheta,\quad \Re(x)>0,\ \Re(y)>0,\quad 0<\rho\leq 1.$$

The aim of this paper is to obtain a new integral identity and associated bounds essentially using the concept of generalized γ-convex functions. We also discuss special cases of the main results which shows that the obtained results are quite unifying one. Finally, we also present applications for particular special means with arbitrary positive real numbers, hypergeometric functions, Mittag–Leffler functions, differentiable functions of first order that are in absolute value bounded, and some error estimations of the quadrature formula as well. It is expected that the ideas and techniques of the paper will inspire interested readers.

2. Results and discussion

In this section, we will discuss our main results.

2.1. Auxiliary result

Let us denote, respectively,

$$\mathcal{P}:=[\varpi_1,\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)]$$

and

$$\mathcal{P}^{\circ}:=\left(\varpi_1,\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)\right)$$

which is the interior of $\mathcal{P}$ with $0<\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)$ in the sequel. In order to prove main results of the paper, we need to prove following new auxiliary result.

Lemma 2.1. *Let $\varpi_1,\varpi_2\in\mathbb{R}^+$ with $0<\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)$, and let $\mathcal{F}:\mathcal{P}\to\mathbb{R}$ be a differentiable function on $\mathcal{P}^{\circ}$ for $\sigma\in(0,1]$. If $D_{\sigma}(\mathcal{F})\in L_{\sigma}^{1}(\mathcal{P})$, then*

$$\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}-{\varpi_1}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s$$
$$=\frac{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}-{\varpi_1}^{\sigma}}\left[\int_0^{\frac{1}{2}}\left((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{2\sigma-1}-{\varpi_1}^{\sigma}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma-1}\right)\right.$$
$$\times D_{\sigma}(\mathcal{F})(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\vartheta^{1-\sigma}\mathrm{d}_{\sigma}\vartheta$$

$$+\int_{\frac{1}{2}}^{1}\Big((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{2\sigma-1}-(\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma-1}\Big)$$
$$\times D_{\sigma}(\mathcal{F})(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\vartheta^{1-\sigma}\mathrm{d}_{\sigma}\vartheta\Big].$$

Proof. It suffices to show that

$$\begin{aligned}
I :=& \int_{0}^{\frac{1}{2}}\Big((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{2\sigma-1}-{\varpi_1}^{\sigma}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma-1}\Big)\\
&\times D_{\sigma}(\mathcal{F})(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\vartheta^{1-\sigma}\mathrm{d}_{\sigma}\vartheta\\
&+\int_{\frac{1}{2}}^{1}\Big((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{2\sigma-1}-(\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma-1}\Big)\\
&\times D_{\sigma}(\mathcal{F})(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\vartheta^{1-\sigma}\mathrm{d}_{\sigma}\vartheta\\
=&\int_{0}^{\frac{1}{2}}\Big((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}-{\varpi_1}^{\sigma}\Big)\mathcal{F}'(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\mathrm{d}\vartheta\\
&+\int_{\frac{1}{2}}^{1}\Big((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}\Big)\mathcal{F}'(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\mathrm{d}\vartheta.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
I =& \Big((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}-{\varpi_1}^{\sigma}\Big)\frac{\mathcal{F}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))}{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}\Bigg|_{0}^{\frac{1}{2}}\\
&-\sigma\int_{0}^{\frac{1}{2}}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma-1}\mathcal{F}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\mathrm{d}\vartheta\\
&+\Big((\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}\Big)\frac{\mathcal{F}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))}{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}\Bigg|_{\frac{1}{2}}^{1}\\
&-\sigma\int_{\frac{1}{2}}^{1}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma-1}\mathcal{F}(\varpi_1+\vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))\mathrm{d}\vartheta\\
=&\frac{1}{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}\Bigg[\Bigg(\Bigg(\varpi_1+\frac{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}{2}\Bigg)^{\sigma}-{\varpi_1}^{\sigma}\Bigg)\mathcal{F}\Bigg(\frac{2\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}{2}\Bigg)\\
&-\sigma\int_{\varpi_1}^{\varpi_1+\frac{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}{2}}\mathcal{F}(s)\mathrm{d}_{\sigma}s+\Bigg((\varpi_1+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1))^{\sigma}-\Bigg(\varpi_1+\frac{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2-\varpi_1)}{2}\Bigg)^{\sigma}\Bigg)
\end{aligned}$$

$$\times \mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \sigma \int\limits_{\varpi_1 + \frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}}^{\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_{\sigma} s\Bigg]$$

$$= \frac{1}{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}\Bigg[\left((\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma}\right)\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right)$$

$$-\sigma \int\limits_{\varpi_1}^{\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_{\sigma} s\Bigg].$$

This completes the proof. □

2.2. *New bounds*

In this subsection, using Lemma 2.1, we discuss our main results.

Theorem 2.1. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^{\circ}$ for $\sigma \in (0, 1]$. If $D_{\sigma}(\mathcal{F}) \in L^1_{\sigma}(\mathcal{P})$, and $|\mathcal{F}'|$ is generalized γ-convex function on $\mathcal{P}$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma}} \int\limits_{\varpi_1}^{\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_{\sigma} s\right|$$

$$\leq \frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma}} \left\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3] + |\mathcal{F}'(\varpi_2)|[\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3]\right\},$$

where

$$\mathcal{A}_1 := \int\limits_0^{\frac{1}{2}} {\varpi_1}^{\sigma-1}(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))\gamma^2(1 - \vartheta)\mathrm{d}\vartheta - \int\limits_{\frac{1}{2}}^{1} {\varpi_1}^{\sigma}\gamma^2(1 - \vartheta)\mathrm{d}\vartheta, \tag{2.1}$$

$$\mathcal{A}_2 := \int\limits_0^{\frac{1}{2}} {\varpi_2}^{\sigma-1}(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))\gamma(\vartheta)\gamma(1 - \vartheta)\mathrm{d}\vartheta - \int\limits_{\frac{1}{2}}^{1} {\varpi_2}^{\sigma}\gamma(\vartheta)\gamma(1 - \vartheta)\mathrm{d}\vartheta, \tag{2.2}$$

$$\mathcal{A}_3 := -\int\limits_0^{\frac{1}{2}} {\varpi_1}^{\sigma}\gamma(1 - \vartheta)\mathrm{d}\vartheta + \int\limits_{\frac{1}{2}}^{1} (\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma}\gamma(1 - \vartheta)\mathrm{d}\vartheta, \tag{2.3}$$

$$\mathcal{B}_1 := \int\limits_0^{\frac{1}{2}} {\varpi_2}^{\sigma-1}(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))\gamma^2(\vartheta)\mathrm{d}\vartheta - \int\limits_{\frac{1}{2}}^{1} {\varpi_2}^{\sigma}\gamma^2(\vartheta)\mathrm{d}\vartheta, \tag{2.4}$$

$$\mathcal{B}_2 := \int\limits_0^{\frac{1}{2}} {\varpi_1}^{\sigma-1}(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))\gamma(\vartheta)\gamma(1 - \vartheta)\mathrm{d}\vartheta - \int\limits_{\frac{1}{2}}^{1} {\varpi_1}^{\sigma}\gamma(\vartheta)\gamma(1 - \vartheta)\mathrm{d}\vartheta, \tag{2.5}$$

$$\mathcal{B}_3 := -\int_0^{\frac{1}{2}} \varpi_1{}^{\sigma}\gamma(\vartheta)\mathrm{d}\vartheta + \int_{\frac{1}{2}}^{1} (\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma}\gamma(\vartheta)\mathrm{d}\vartheta. \tag{2.6}$$

Proof. Using Lemma 2.1, generalized γ-convexity of $x^{\sigma-1}$ and $-x^{\sigma}$ $(x > 0)$ for $\sigma \in (0, 1]$, $|\mathcal{F}'|$ is generalized γ-convex, and property of the modulus, we have

$$\begin{aligned}
&\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)} \mathcal{F}(s)\mathrm{d}_{\sigma}s\right| \\
\leq &\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \left[\int_0^{\frac{1}{2}} \left((\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}\right)|\mathcal{F}'(\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))|\mathrm{d}\vartheta\right. \\
&\left.+ \int_{\frac{1}{2}}^{1} \left((\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - (\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma}\right)|\mathcal{F}'(\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))|\mathrm{d}\vartheta\right] \\
\leq &\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \left[\int_0^{\frac{1}{2}} \left((\gamma(1-\vartheta)\varpi_1{}^{\sigma-1} + \gamma(\vartheta)\varpi_2{}^{\sigma-1})(\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)) - \varpi_1{}^{\sigma}\right)\right. \\
&\times \left[\gamma(1-\vartheta)|\mathcal{F}'(\varpi_1)| + \gamma(\vartheta)|\mathcal{F}'(\varpi_2)|\right]\mathrm{d}\vartheta \\
&\left.+ \int_{\frac{1}{2}}^{1} \left((\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \gamma(1-\vartheta)\varpi_1{}^{\sigma} - \gamma(\vartheta)\varpi_2{}^{\sigma}\right)\left[\gamma(1-\vartheta)|\mathcal{F}'(\varpi_1)| + \gamma(\vartheta)|\mathcal{F}'(\varpi_2)|\right]\mathrm{d}\vartheta\right].
\end{aligned}$$

The proof is completed. □

We now discuss some special cases of Theorem 2.1.

(I) If we take $\gamma(\vartheta) = \vartheta$ in Theorem 2.1, we have

$$\begin{aligned}
&\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)} \mathcal{F}(s)\mathrm{d}_{\sigma}s\right| \\
\leq &\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \left\{\frac{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma}}{8}[|\mathcal{F}'(\varpi_1)| + 3|\mathcal{F}'(\varpi_2)|]\right. \\
&+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)|\mathcal{F}'(\varpi_1)|\left[\frac{11\varpi_1{}^{\sigma-1} + 5\varpi_2{}^{\sigma-1}}{192}\right] \\
&+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)|\mathcal{F}'(\varpi_2)|\left[\frac{5\varpi_1{}^{\sigma-1}}{192} + \frac{\varpi_2{}^{\sigma-1}}{64}\right] - \frac{\varpi_1{}^{\sigma}}{8}[|\mathcal{F}'(\varpi_1)| + |\mathcal{F}'(\varpi_2)|] \\
&\left.+\frac{\varpi_1\varpi_2{}^{\sigma-1}}{24}[2|\mathcal{F}'(\varpi_1)| + |\mathcal{F}'(\varpi_2)|] - \frac{\varpi_2{}^{\sigma}}{24}[2|\mathcal{F}'(\varpi_1)| + 7|\mathcal{F}'(\varpi_2)|]\right\}.
\end{aligned}$$

(II) If we choose $\gamma(\vartheta) = 1$ in Theorem 2.1, we get

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\{[|\mathcal{F}'(\varpi_1)|+|\mathcal{F}'(\varpi_2)|]$$
$$\times\left[\frac{\varpi_1\varpi_2{}^{\sigma-1}-\varpi_2{}^{\sigma}-\varpi_1{}^{\sigma}+(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}}{2}+\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)(\varpi_1{}^{\sigma-1}+\varpi_2{}^{\sigma-1})}{8}\right]\right\}.$$

(III) If we take $\gamma(\vartheta)=\vartheta^{s}$ in Theorem 2.1, we obtain

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}^*_1+\mathcal{A}^*_2+\mathcal{A}^*_3]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}^*_1+\mathcal{B}^*_2+\mathcal{B}^*_3]\},$$

where

$$\mathcal{A}^*_1:=\frac{\varpi_1{}^{\sigma}}{2s+1}\left[1-\frac{1}{2^{2s}}\right]+\frac{\varpi_1{}^{\sigma-1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2s+1}\left[\frac{1}{2s+2}-\frac{1}{2^{2s+2}(2s+2)}-\frac{1}{2^{2s+2}}\right],\tag{2.7}$$

$$\mathcal{A}^*_2:=\mathcal{B}_{\frac{1}{2}}(1+s,1+s)\varpi_2{}^{\sigma-1}(\varpi_1+\varpi_2)+\varpi_2{}^{\sigma-1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\mathcal{B}_{\frac{1}{2}}(1+s,1+s)$$
$$-\varpi_2{}^{\sigma}\mathcal{B}(1+s,1+s),\tag{2.8}$$

$$\mathcal{A}^*_3:=\frac{1}{2^{s+1}(s+1)}\left[\varpi_1{}^{\sigma}+(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}2^{s+1}\right],\tag{2.9}$$

$$\mathcal{B}^*_1:=\frac{\varpi_2{}^{\sigma-1}}{2^{2s+1}(2s+1)}\left[(\varpi_1+\varpi_2+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)-2^{2s+1}\varpi_2\right],\tag{2.10}$$

$$\mathcal{B}^*_2:=\varpi_1{}^{\sigma-1}\mathcal{B}_{\frac{1}{2}}(1+s,1+s)(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))$$
$$-\varpi_1{}^{\sigma-1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\mathcal{B}(1+s,1+s),\tag{2.11}$$

$$\mathcal{B}^*_3:=-\frac{\varpi_1{}^{\sigma}}{2^{s+1}(s+1)}+(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\left[1-\frac{1}{2^{s+1}(s+1)}\right].\tag{2.12}$$

(IV) If we choose $\gamma(\vartheta)=\vartheta^{-s}$ in Theorem 2.1, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}^{**}_1+\mathcal{A}^{**}_2+\mathcal{A}^{**}_3]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}^{**}_1+\mathcal{B}^{**}_2+\mathcal{B}^{**}_3]\},$$

where

$$\mathcal{A}_1^{**} := \frac{\varpi_1{}^{\sigma}}{1-2s}\left[1-\frac{1}{2^{-2s}}\right]+\frac{\varpi_1{}^{\sigma-1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{1-2s}\left[\frac{1}{2-2s}-\frac{1}{2^{2-2s}(2-2s)}-\frac{1}{2^{2-2s}}\right], \tag{2.13}$$

$$\mathcal{A}_2^{**} := \mathcal{B}_{\frac{1}{2}}(1+s,1+s)\varpi_2{}^{\sigma-1}(\varpi_1+\varpi_2)+\varpi_2{}^{\sigma-1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\mathcal{B}_{\frac{1}{2}}(1-s,1-s) - \varpi_2{}^{\sigma}\mathcal{B}(1-s,1-s), \tag{2.14}$$

$$\mathcal{A}_3^{**} := \frac{1}{2^{1-s}(1-s)}\left[\varpi_1{}^{\sigma}+(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}2^{1-s}\right], \tag{2.15}$$

$$\mathcal{B}_1^{**} := \frac{\varpi_2{}^{\sigma-1}}{2^{1-2s}(1-2s)}\left[(\varpi_1+\varpi_2+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)-2^{1-2s}\varpi_2\right], \tag{2.16}$$

$$\mathcal{B}_2^{**} := \varpi_1{}^{\sigma-1}\mathcal{B}_{\frac{1}{2}}(1-s,1-s)(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)) - \varpi_1{}^{\sigma-1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\mathcal{B}(1-s,1-s), \tag{2.17}$$

$$\mathcal{B}_3^{**} := -\frac{\varpi_1{}^{\sigma}}{2^{1-s}(1-s)}+(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\left[1-\frac{1}{2^{1-s}(1-s)}\right]. \tag{2.18}$$

Theorem 2.2. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^{\circ}$ for $\sigma \in (0,1]$. If $D_{\sigma}(\mathcal{F}) \in L^1_{\sigma}(\mathcal{P})$, and $|\mathcal{F}'|^q$ is generalized γ-convex function on $\mathcal{P}$, for $q \geq 1$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\left\{\mathcal{H}^{1-\frac{1}{q}}\left[\mathcal{H}_1|\mathcal{F}'(\varpi_1)|^q +\mathcal{H}_2|\mathcal{F}'(\varpi_2)|^q\right]^{\frac{1}{q}}+\mathcal{K}_{\eta}{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta 1}|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta 2}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{H} := \frac{(2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma+1}-(2\varpi_1)^{\sigma+1}}{2^{\sigma+1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)(\sigma+1)}-\frac{\varpi_1{}^{\sigma}}{2}, \tag{2.19}$$

$$\mathcal{H}_1 := \int\limits_0^{\frac{1}{2}}\left[(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}\right]\gamma(1-\vartheta)\mathrm{d}\vartheta, \tag{2.20}$$

$$\mathcal{H}_2 := \int\limits_0^{\frac{1}{2}}\left[(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}\right]\gamma(\vartheta)\mathrm{d}\vartheta, \tag{2.21}$$

$$\mathcal{K}_{\eta} := \frac{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}}{2}-\frac{(2\varpi_1+2\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma+1}-(2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma+1}}{2^{\sigma+1}(\sigma+1)\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}, \tag{2.22}$$

$$\mathcal{K}_{\eta 1} := \int\limits_{\frac{1}{2}}^{1}\left[(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right]\gamma(1-\vartheta)\mathrm{d}\vartheta, \tag{2.23}$$

$$\mathcal{K}_{\eta_2} := \int_{\frac{1}{2}}^{1} \left[(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - (\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} \right] \gamma(\vartheta) \mathrm{d}\vartheta. \tag{2.24}$$

Proof. Using Lemma 2.1 and property of the modulus, we have

$$\begin{aligned}
&\left| \mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2} \right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma}} \int_{\varpi_1}^{\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_{\sigma} s \right| \\
\leq & \frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma}} \left[\int_0^{\frac{1}{2}} \left((\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma} \right) |\mathcal{F}'(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))| \mathrm{d}\vartheta \right. \\
& \left. + \int_{\frac{1}{2}}^{1} \left((\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - (\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} \right) |\mathcal{F}'(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))| \mathrm{d}\vartheta \right].
\end{aligned}$$

By power mean integral inequality and generalized γ-convexity of $|\mathcal{F}'|^q$, we get

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left((\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma} \right) |\mathcal{F}'(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))| \mathrm{d}\vartheta \\
\leq & \left(\int_0^{\frac{1}{2}} \left((\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma} \right) \mathrm{d}\vartheta \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^{\frac{1}{2}} \left((\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma} \right) |\mathcal{F}'(\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))|^q \mathrm{d}\vartheta \right)^{\frac{1}{q}} \\
\leq & \mathcal{H}^{1-\frac{1}{q}} \left(|\mathcal{F}'(\varpi_1)|^q \int_0^{\frac{1}{2}} \left((\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma} \right) \gamma(1-\vartheta) \mathrm{d}\vartheta \right. \\
& \left. + |\mathcal{F}'(\varpi_2)|^q \int_0^{\frac{1}{2}} \left((\varpi_1 + \vartheta \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - {\varpi_1}^{\sigma} \right) \gamma(\vartheta) \mathrm{d}\vartheta \right)^{\frac{1}{q}}.
\end{aligned}$$

Similarly,

$$\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)|\mathcal{F}'(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))|\mathrm{d}\vartheta$$

$$\leq\left(\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)\mathrm{d}\vartheta\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)|\mathcal{F}'(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))|^{q}\mathrm{d}\vartheta\right)^{\frac{1}{q}}$$

$$\leq\mathcal{K}_{\eta}{}^{1-\frac{1}{q}}\left(|\mathcal{F}'(\varpi_1)|^{q}\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)\gamma(1-\vartheta)\mathrm{d}\vartheta\right.$$

$$\left.+|\mathcal{F}'(\varpi_2)|^{q}\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)\gamma(\vartheta)\mathrm{d}\vartheta\right)^{\frac{1}{q}},$$

which completes the proof. □

We now discuss some special cases of Theorem 2.2.

(I) If we take $\gamma(\vartheta)=\vartheta$ in Theorem 2.2, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{*}|\mathcal{F}'(\varpi_1)|^{q}+\mathcal{H}_2^{*}|\mathcal{F}'(\varpi_2)|^{q}]^{\frac{1}{q}}\right.$$
$$\left.+\mathcal{K}_{\eta}{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta 1}^{*}|\mathcal{F}'(\varpi_1)|^{q}+\mathcal{K}_{\eta 2}^{*}|\mathcal{F}'(\varpi_2)|^{q}]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{H}_1^{*}:=\frac{(2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma+1}}{2^{\sigma+2}\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\right)^{2}(\sigma+1)}\left[\frac{(2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)(\sigma+2)}{\sigma+2}\right]$$
$$-\frac{3\varpi_1{}^{\sigma}}{8}-\frac{\varpi_1{}^{\sigma+1}}{\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\right)^{2}(\sigma+1)}\left[\frac{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)(\sigma+2)}{(\sigma+2)}\right],\tag{2.25}$$

$$\mathcal{H}_2^* := \frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+1}}{2^{\sigma+2}\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)\right)^2(\sigma+1)} \times \left[\frac{(2\varpi_1)^{\sigma+2} + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+2) - (2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))}{(\sigma+2)}\right] - \frac{\varpi_1{}^{\sigma}}{8}, \tag{2.26}$$

$$\mathcal{K}_{\eta_1}^* := \frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+1}}{2^{\sigma+2}\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)\right)^2(\sigma+1)} \left[\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+2) + (2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))}{(\sigma+2)}\right], + (\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma}\left[\frac{1}{8} - \frac{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^2}{\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)\right)^2(\sigma+1)(\sigma+2)}\right], \tag{2.27}$$

$$\mathcal{K}_{\eta_2}^* := \frac{3(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma}}{8} + \frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+1}}{2^{\sigma+2}\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)\right)^2(\sigma+1)} \times \left[\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+2) - (2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))}{\sigma+2}\right] + \frac{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+1}}{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+1)}\left[\frac{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)) - \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+2)}{\sigma+2}\right]. \tag{2.28}$$

(II) If we choose $\gamma(\vartheta) = 1$ in Theorem 2.2, we get

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \int_{\varpi_1}^{\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)} \mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq \frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{**}(|\mathcal{F}'(\varpi_1)|^q + |\mathcal{F}'(\varpi_2)|^q)]^{\frac{1}{q}} + \mathcal{K}_{\eta}{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta_1}{}^{**}(|\mathcal{F}'(\varpi_1)|^q + |\mathcal{F}'(\varpi_2)|^q)]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{H}_1^{**} := \frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+1}}{2^{\sigma+1}\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)\right)^2(\sigma+1)} - \frac{\varpi_1{}^{\sigma}}{2} - \frac{\varpi_1{}^{\sigma+1}}{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+1)}, \tag{2.29}$$

$$\mathcal{K}_{\eta_1}^{**} := \frac{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma}}{2} - \frac{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+1}}{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+1)} + \frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+1}}{2^{\sigma+1}\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+1)}. \tag{2.30}$$

(III) If we take $\gamma(\vartheta) = \vartheta^s$ in Theorem 2.2, we obtain

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}$$
$$\times\left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{H}_2^{\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}+\mathcal{K}_{\eta}{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta1}^{\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta2}^{\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{H}_1^{\star}:=\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\right)^{\sigma}\mathcal{B}_{\frac{1}{2}}(1+\sigma,1+s),\tag{2.31}$$

$$\mathcal{H}_2^{\star}:=\frac{\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\right)^{\sigma}}{2^{\sigma+s+1}(\sigma+s+1)},\tag{2.32}$$

$$\mathcal{K}_{\eta1}^{\star}:=\int_{\frac{1}{2}}^{1}\left[(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right](1-\vartheta)^s\mathrm{d}\vartheta,\tag{2.33}$$

$$\mathcal{K}_{\eta2}^{\star}:=\int_{\frac{1}{2}}^{1}\left[(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right]\vartheta^s\mathrm{d}\vartheta.\tag{2.34}$$

(IV) If we choose $\gamma(\vartheta)=\vartheta^{-s}$ in Theorem 2.2, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}$$
$$\times\left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{\star\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{H}_2^{\star\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}+\mathcal{K}_{\eta}{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta1}^{\star\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta2}^{\star\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{H}_1^{\star\star}:=\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\right)^{\sigma}\mathcal{B}_{\frac{1}{2}}(1+\sigma,1-s),\tag{2.35}$$

$$\mathcal{H}_2^{\star\star}:=\frac{\left(\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)\right)^{\sigma}}{2^{\sigma-s+1}(\sigma-s+1)},\tag{2.36}$$

$$\mathcal{K}_{\eta1}^{\star\star}:=\int_{\frac{1}{2}}^{1}\left[(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right](1-\vartheta)^{-s}\mathrm{d}\vartheta,\tag{2.37}$$

$$\mathcal{K}_{\eta2}^{\star\star}:=\int_{\frac{1}{2}}^{1}\left[(\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right]\vartheta^{-s}\mathrm{d}\vartheta.\tag{2.38}$$

Theorem 2.3. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ for $\sigma \in (0, 1]$. If $D_\sigma(\mathcal{F}) \in L^1_\sigma(\mathcal{P})$, and $|\mathcal{F}'|^q$ is γ-convex function on $\mathcal{P}$, for $q \geq 1$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)} \mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \left[\mathcal{H}\left|\mathcal{F}'\left(\frac{C_1}{\mathcal{H}}\right)\right| + \mathcal{K}_\eta\left|\mathcal{F}'\left(\frac{C_2}{\mathcal{K}_\eta}\right)\right|\right],$$

where

$$C_1 := \frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^2}{4(\sigma+2)\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}\left[\frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - 2^{\sigma-1}{\varpi_1}^\sigma(\sigma+2)}{2^\sigma}\right] \tag{2.39}$$
$$- \frac{\sigma{\varpi_1}^{\sigma+2}}{2\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+2)}, \tag{2.40}$$
$$C_2 := \frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^2}{4(\sigma+2)\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}\left[\frac{(2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - 2^{\sigma-1}(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma(\sigma+2)}{2^\sigma}\right]$$
$$- \frac{\sigma(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^{\sigma+2}}{2\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)(\sigma+2)}, \tag{2.41}$$

and $\mathcal{H}$, $\mathcal{K}_\eta$ are defined as in Theorem 2.2 with the assumption that $\gamma(1-\vartheta) + \gamma(\vartheta) = 1$.

Proof. By power mean integral inequality and generalized γ-convexity of $|\mathcal{F}'|^q$, we have

$$\begin{aligned}(\gamma(1-\vartheta)|\mathcal{F}'(\varpi_1)| + \gamma(\vartheta)|\mathcal{F}'(\varpi_2)|)^q &\leq \gamma(1-\vartheta)|\mathcal{F}'(\varpi_1)|^q + \gamma(\vartheta)|\mathcal{F}'(\varpi_2)|^q \\ &\leq |\mathcal{F}'(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))|^q,\end{aligned}$$

which shows that $|\mathcal{F}'|$ is also generalized γ-convex.

By using Lemma 2.1 and property of the modulus, we get

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int_{\varpi_1}^{\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)} \mathcal{F}(s)d_\sigma s\right|$$
$$\leq \frac{\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma}\left[\int_0^{\frac{1}{2}}\left((\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma\right)|\mathcal{F}'(\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))|\mathrm{d}\vartheta\right.$$
$$\left. + \int_{\frac{1}{2}}^1\left((\varpi_1 + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - (\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma\right)|\mathcal{F}'(\varpi_1 + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1))|\mathrm{d}\vartheta\right].$$

Applying Jensen's integral inequality for convex functions, we have

$$\int_0^{\frac{1}{2}}\left((\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}\right)|\mathcal{F}'(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))|\mathrm{d}\vartheta$$
$$\leq\left(\int_0^{\frac{1}{2}}\left((\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}\right)\mathrm{d}\vartheta\right)$$
$$\times\left|\mathcal{F}'\left(\frac{\int_0^{\frac{1}{2}}\left((\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}\right)(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))\mathrm{d}\vartheta}{\int_0^{\frac{1}{2}}\left((\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}\right)\mathrm{d}\vartheta}\right)\right|$$
$$=\mathcal{H}\left|\mathcal{F}'\left(\frac{C_1}{\mathcal{H}}\right)\right|.$$

Similarly,

$$\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)|\mathcal{F}'(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))|\mathrm{d}\vartheta$$
$$\leq\left(\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)\mathrm{d}\vartheta\right)$$
$$\times\left|\mathcal{F}'\left(\frac{\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))\mathrm{d}\vartheta}{\int_{\frac{1}{2}}^{1}\left((\varpi_1+\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}-(\varpi_1+\vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1))^{\sigma}\right)\mathrm{d}\vartheta}\right)\right|$$
$$=\mathcal{K}_{\eta}\left|\mathcal{F}'\left(\frac{C_2}{\mathcal{K}_{\eta}}\right)\right|,$$

which completes the proof. □

Remark 2.1. *If we take $\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2-\varpi_1)=\varpi_2-\varpi_1$ and $\gamma(\vartheta)=\vartheta,1,\vartheta^{s},\vartheta^{-s}$, respectively, then we get the results for classical convex functions (see [39]), P-convex functions, s-convex functions, and Godunova–Levin convex functions.*

Remark 2.2. *If we set $\sigma=1$ in Theorems 2.1–2.3, then we get inequalities for classical integral. Moreover, several new results can be found using Hölder–İşcan, Hölder–Power Mean, Chebyshev, Markov, Young and Minkowski inequalities. We omit here their proofs and the details are left to the interested reader.*

3. Applications

In this section, we discuss several applications for the results obtained in the previous section.

3.1. Applications to special means

We begin, this subsection by considering some particular means for arbitrary positive real numbers ϖ_1, ϖ_2 such that $\varpi_1 < \varpi_2$.

(1) The arithmetic mean:

$$\mathcal{A}(\varpi_1, \varpi_2) := \frac{\varpi_1 + \varpi_2}{2}.$$

(2) The generalized logarithmic (σ, r)-th mean:

$$\mathcal{L}_{(\sigma,r)}(\varpi_1, \varpi_2) := \left[\frac{\sigma(\varpi_2^{r+\sigma} - \varpi_1^{r+\sigma})}{(\varpi_2^{\sigma} - \varpi_1^{\sigma})(r+\sigma)}\right]^{\frac{1}{r}}, \quad r \neq 0, -\sigma; \quad r \in \mathbb{R}, \ \sigma \in (0, 1].$$

Now, by making use of the results obtained in Section 2, we give some applications to special means of different positive real numbers.

Proposition 3.1. *Let* $0 < \varpi_1 < \varpi_2, \ r > 1$ *and* $\sigma \in (0, 1]$, *then*

$$\begin{aligned}
&\left|\mathcal{A}^r(\varpi_1, \varpi_2) - \mathcal{L}^r_{(\sigma,r)}(\varpi_1, \varpi_2)\right| \\
\leq &\frac{r(\varpi_2 - \varpi_1)}{\varpi_2^{\sigma} - \varpi_1^{\sigma}} \left\{ \frac{\varpi_2^{\sigma}}{8}[\varpi_1^{r-1} + 3\varpi_2^{r-1}] + (\varpi_2 - \varpi_1)\varpi_1^{r-1}\left[\frac{11\varpi_1^{\sigma-1} + 5\varpi_2^{\sigma-1}}{192}\right]\right. \\
&+ (\varpi_2 - \varpi_1)\varpi_2^{r-1}\left[\frac{5\varpi_1^{\sigma-1}}{192} + \frac{\varpi_2^{\sigma-1}}{64}\right] \\
&\left. - \frac{\varpi_1^{\sigma}}{8}\left[\varpi_1^{r-1} + \varpi_2^{r-1}\right] + \frac{\varpi_1\varpi_2^{\sigma-1}}{24}[2\varpi_1^{r-1} + \varpi_2^{r-1}] - \frac{\varpi_2^{\sigma}}{24}\left[2\varpi_1^{r-1} + 7\varpi_2^{r-1}\right]\right\}.
\end{aligned}$$

Proof. Under the assumptions of Theorem 2.1, if we take $\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\varpi_2 - \varpi_1) = \varpi_2 - \varpi_1$, $\gamma(\vartheta) = \vartheta$ and $\mathcal{F}(\vartheta) = \vartheta^r$ for $\vartheta > 0$, we have the desired result.

For numerical verification if we take $\varpi_1 = 0$, $\varpi_2 = 1, \sigma = 1$ and $r = 2$, then we have $0.08333 \leq 0.1979$. □

Proposition 3.2. *Let* $0 < \varpi_1 < \varpi_2, \ r > 1, q \geq 1,$ *and* $\sigma \in (0, 1]$, *then*

$$\begin{aligned}
&\left|\mathcal{A}^r(\varpi_1, \varpi_2) - \mathcal{L}^r_{(\sigma,r)}(\varpi_1, \varpi_2)\right| \\
\leq &\frac{r(\varpi_2 - \varpi_1)}{\varpi_2^{\sigma} - \varpi_1^{\sigma}} \left\{ {}^{i}\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{**}\varpi_1^{q(r-1)} + \mathcal{H}_2^{**}\varpi_2^{q(r-1)}]^{\frac{1}{q}} + \mathcal{K}_{\mu}^{*1-\frac{1}{q}}[\mathcal{K}_{\mu_1}^{**}\varpi_1^{q(r-1)} + \mathcal{K}_{\mu_2}^{*}\varpi_2^{q(r-1)}]^{\frac{1}{q}} \right\},
\end{aligned}$$

where

$${}^{i}\mathcal{H} := \frac{(\varpi_1 + \varpi_2)^{\sigma+1} - (2\varpi_1)^{\sigma+1}}{2^{\sigma+1}(\varpi_2 - \varpi_1)(\sigma+1)} - \frac{{\varpi_1}^{\sigma}}{2},$$

$$\mathcal{H}_1^{**} := \int_0^{\frac{1}{2}} [((1-\vartheta)\varpi_1 + \vartheta\varpi_2)^{\sigma} - {\varpi_1}^{\sigma}]\,(1-\vartheta)\mathrm{d}\vartheta,$$

$$\mathcal{H}_2^{**} := \int_0^{\frac{1}{2}} \left[((1-\vartheta)\varpi_1 + \vartheta\varpi_2)^\sigma - {\varpi_1}^\sigma\right] \vartheta \mathrm{d}\vartheta,$$

$$\mathcal{K}_\mu^{*} := \frac{(\varpi_2)^\sigma}{2} - \frac{(2\varpi_1 + 2(\varpi_2 - \varpi_1))^{\sigma+1} - (2\varpi_1 + (\varpi_2 - \varpi_1))^{\sigma+1}}{2^{\sigma+1}(\sigma+1)(\varpi_2 - \varpi_1)},$$

$$\mathcal{K}_{\mu_1}^{**} := \int_{\frac{1}{2}}^{1} \left[(\varpi_2)^\sigma - ((1-\vartheta)\varpi_1 + \vartheta\varpi_2)^\sigma\right] (1-\vartheta) \mathrm{d}\vartheta,$$

$$\mathcal{K}_{\mu_2}^{**} := \int_{\frac{1}{2}}^{1} \left[(\varpi_2)^\sigma - ((1-\vartheta)\varpi_1 + \vartheta\varpi_2)^\sigma\right] \vartheta \mathrm{d}\vartheta.$$

Proof. Under the assumptions of Theorem 2.2, if we choose $\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2 - \varpi_1) = \varpi_2 - \varpi_1$, $\gamma(\vartheta) = \vartheta$ and $\mathcal{F}(\vartheta) = \vartheta^r$ for $\vartheta > 0$, we get the desired result.

For numerical verification if we take $\varpi_1 = 0$, $\varpi_2 = 1, \sigma = 1$ and $q = r = 2$, then we have $0.08333 < 0.3484$. □

3.2. Applications to quadrature formula

Let $\mathcal{U}$ be the partition of the points $\varpi_1 = \mu_0 < \mu_1 < \ldots < \mu_{n-1} < \mu_n = \varpi_1 + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2 - \varpi_1)$ of the interval $\mathcal{P}$ for fixed $\lambda, \rho > 0$, and $\varepsilon = \{\varepsilon(0), \varepsilon(1), \ldots, \varepsilon(k), \ldots\}$ and be a bounded sequence of positive real numbers. Now, we consider the following quadrature formula:

$$\int_{\varpi_1}^{\varpi_1 + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s := M_\sigma(\mathcal{F}, \mathcal{U}) + \mathcal{R}_\sigma(\mathcal{F}, \mathcal{U}),$$

where

$$M_\sigma(\mathcal{F}, \mathcal{U}) := \sum_{i=0}^{n-1} \mathcal{F}\left(\frac{2\mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1} - \mu_i)}{2}\right)\left[\frac{(\mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1} - \mu_i))^\sigma - \mu_i^\sigma}{\sigma}\right]$$

is the midpoint version and $\mathcal{R}_\sigma(\mathcal{F}, \mathcal{U})$ denotes the associated approximation error. Here, we are going to derive some new error estimates for the midpoint formula.

Proposition 3.3. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2 - \varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ for $\sigma \in (0, 1]$. If $D_\sigma(\mathcal{F}) \in L_\sigma^1(\mathcal{P})$, and $|\mathcal{F}'|$ is generalized γ-convex function on $\mathcal{P}$, then*

$$|\mathcal{R}_\sigma(\mathcal{F}, \mathcal{U})| \leq \sum_{i=0}^{n-1} \frac{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1} - \mu_i)}{\sigma} \left\{|\mathcal{F}'(\mu_i)| \left[\mathcal{A}_1^{(i)} + \mathcal{A}_2^{(i)} + \mathcal{A}_3^{(i)}\right] + |\mathcal{F}'(\mu_{i+1})| \left[\mathcal{B}_1^{(i)} + \mathcal{B}_2^{(i)} + \mathcal{B}_3^{(i)}\right]\right\},$$

where

$$\mathcal{A}_1^{(i)} := \int_0^{\frac{1}{2}} \mu_i^{\sigma-1}(\mu_i + \vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1} - \mu_i))\gamma^2(1-\vartheta) \mathrm{d}\vartheta - \int_{\frac{1}{2}}^{1} \mu_i^\sigma \gamma^2(1-\vartheta) \mathrm{d}\vartheta,$$

$$\mathcal{A}_2^{(i)} := \int_0^{\frac{1}{2}} \mu_{i+1}^{\sigma-1}(\mu_i + \vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i))\gamma(\vartheta)\gamma(1-\vartheta)d\vartheta - \int_{\frac{1}{2}}^{1} \mu_{i+1}^{\sigma}\gamma(\vartheta)\gamma(1-\vartheta)d\vartheta,$$

$$\mathcal{A}_3^{(i)} := -\int_0^{\frac{1}{2}} \mu_i^{\sigma}\gamma(1-\vartheta)d\vartheta + \int_{\frac{1}{2}}^{1}(\mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i))^{\sigma}\gamma(1-\vartheta)d\vartheta,$$

$$\mathcal{B}_1^{(i)} := \int_0^{\frac{1}{2}} \mu_{i+1}^{\sigma-1}(\mu_i + \vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i))\gamma^2(\vartheta)d\vartheta - \int_{\frac{1}{2}}^{1} \mu_{i+1}^{\sigma}\gamma^2(\vartheta)d\vartheta,$$

$$\mathcal{B}_2^{(i)} := \int_0^{\frac{1}{2}} \mu_i^{\sigma-1}(\mu_i + \vartheta\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i))\gamma(\vartheta)\gamma(1-\vartheta)d\vartheta - \int_{\frac{1}{2}}^{1} \mu_i^{\sigma}\gamma(\vartheta)\gamma(1-\vartheta)d\vartheta,$$

$$\mathcal{B}_3^{(i)} := -\int_0^{\frac{1}{2}} \mu_i^{\sigma}\gamma(\vartheta)d\vartheta + \int_{\frac{1}{2}}^{1}(\mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i))^{\sigma}\gamma(\vartheta)d\vartheta.$$

Proof. Applying Theorem 2.1 on the subintervals $[\mu_i, \mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i)]$ $(i = 0, 1, 2, \ldots, n-1)$ of the partition $\mathcal{U}$, we have

$$\left|\mathcal{F}\left(\frac{2\mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i)}{2}\right)\left[\frac{(\mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i))^{\sigma} - \mu_i^{\sigma}}{\sigma}\right] - \int_{\mu_i}^{\mu_i+\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i)} \mathcal{F}(s)d_{\sigma}s\right|$$
$$\leq \frac{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i)}{\sigma}\left\{|\mathcal{F}'(\mu_i)|\left[\mathcal{A}_1^{(i)} + \mathcal{A}_2^{(i)} + \mathcal{A}_3^{(i)}\right] + |\mathcal{F}'(\mu_{i+1})|\left[\mathcal{B}_1^{(i)} + \mathcal{B}_2^{(i)} + \mathcal{B}_3^{(i)}\right]\right\}.$$

Summing up with respect to i from 0 to $n-1$ and using the properties of the modulus, we get the desired result. □

Proposition 3.4. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\varpi_2 - \varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ for $\sigma \in (0, 1]$. If $D_\sigma(\mathcal{F}) \in L_\sigma^1(\mathcal{P})$, and $|\mathcal{F}'|^q$ is generalized γ-convex function on $\mathcal{P}$, for $q \geq 1$, then*

$$|\mathcal{R}_\sigma(\mathcal{F}, \mathcal{U})| \leq \sum_{i=0}^{n-1} \frac{\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i)}{\sigma}$$
$$\times\left\{\left(\mathcal{H}^{(i)}\right)^{1-\frac{1}{q}}[\mathcal{H}_1^{(i)}|\mathcal{F}'(\mu_i)|^q + \mathcal{H}_2^{(i)}|\mathcal{F}'(\mu_{i+1})|^q]^{\frac{1}{q}} + \left(\mathcal{K}_\eta^{(i)}\right)^{1-\frac{1}{q}}[\mathcal{K}_{\eta 1}^{(i)}|\mathcal{F}'(\mu_i)|^q + \mathcal{K}_{\eta 2}^{(i)}|\mathcal{F}'(\mu_{i+1})|^q]^{\frac{1}{q}}\right\},$$

where

$$\mathcal{H}^{(i)} := \frac{(2\mu_i + \mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i))^{\sigma+1} - (2\mu_i)^{\sigma+1}}{2^{\sigma+1}\mathcal{R}_{\lambda,\rho}^{\varepsilon}(\mu_{i+1}-\mu_i)(\sigma+1)} - \frac{\mu_i^{\sigma}}{2},$$

$$\mathcal{H}_1^{(i)} := \int_0^{\frac{1}{2}} \left[(\mu_i + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma} - \mu_i^{\sigma}\right] \gamma(1-\vartheta)\mathrm{d}\vartheta,$$

$$\mathcal{H}_2^{(i)} := \int_0^{\frac{1}{2}} \left[(\mu_i + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma} - \mu_i^{\sigma}\right] \gamma(\vartheta)\mathrm{d}\vartheta,$$

$$\mathcal{K}_{\eta}{}^{(i)} := \frac{(\mu_i + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma}}{2} - \frac{(2\mu_i + 2\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma+1} - (2\mu_i + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma+1}}{2^{\sigma+1}(\sigma+1)\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i)},$$

$$\mathcal{K}_{\eta 1}^{(i)} := \int_{\frac{1}{2}}^{1} \left[(\mu_i + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma} - (\mu_i + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma}\right] \gamma(1-\vartheta)\mathrm{d}\vartheta,$$

$$\mathcal{K}_{\eta 2}^{(i)} := \int_{\frac{1}{2}}^{1} \left[(\mu_i + \mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma} - (\mu_i + \vartheta\mathcal{R}^{\varepsilon}_{\lambda,\rho}(\mu_{i+1} - \mu_i))^{\sigma}\right] \gamma(\vartheta)\mathrm{d}\vartheta.$$

Proof. Using the same technique as in Proposition 3.3 but applying Theorem 2.2, we obtain the desired result. □

3.3. Applications to hypergeometric functions

From relation (1.1), if we set $\rho = 1, \lambda = 0$ and $\sigma(k) = \frac{(\phi)_k(\psi)_k}{(\eta)_k} \neq 0$, where ϕ, ψ and η are parameters may be real or complex values and $(m)_k$ is defined as $(m)_k = \frac{\Gamma(m+k)}{\Gamma(m)}$ and its domain is restricted as $|x| \leq 1$, then we have the following hypergeometric function

$$\mathcal{R}(\phi;\psi;\eta;x) = \sum_{k=0}^{\infty} \frac{(\phi)_k(\psi)_k}{k!(\eta)_k} x^k.$$

Lemma 3.1. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^{\circ}$ for $\sigma \in (0,1]$. If $D_{\sigma}(\mathcal{F}) \in L^1_{\sigma}(\mathcal{P})$, then*

$$\begin{aligned}
&\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \int_{\varpi_1}^{\varpi_1 + \mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1)} \mathcal{F}(s)\mathrm{d}_{\sigma}s \\
=& \frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))^{\sigma} - \varpi_1{}^{\sigma}} \Bigg[\int_0^{\frac{1}{2}} \Big((\varpi_1 + \vartheta\mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))^{2\sigma-1} - \varpi_1{}^{\sigma}(\varpi_1 \\
&+\vartheta\mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))^{\sigma-1}\Big) D_{\sigma}(\mathcal{F})(\varpi_1 + \vartheta\mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))\vartheta^{1-\sigma}\mathrm{d}_{\sigma}\vartheta \\
&+ \int_{\frac{1}{2}}^{1} \Big((\varpi_1 + \vartheta\mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))^{2\sigma-1} - (\varpi_1 + \mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))^{\sigma}(\varpi_1 + \vartheta\mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))^{\sigma-1}\Big) \\
&\times D_{\sigma}(\mathcal{F})(\varpi_1 + \vartheta\mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1))\vartheta^{1-\sigma}\mathrm{d}_{\sigma}\vartheta\Bigg].
\end{aligned}$$

Theorem 3.1. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}(\phi;\psi;\eta;\varpi_2 - \varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^{\circ}$ for $\sigma \in (0,1]$. If $D_{\sigma}(\mathcal{F}) \in L^1_{\sigma}(\mathcal{P})$, and $|\mathcal{F}'|$ is generalized γ-convex function on $\mathcal{P}$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\left\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3]\right\},$$

where $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{B}_1,\mathcal{B}_2$ and $\mathcal{B}_3$ are given by (2.1)–(2.6), respectively.

We now discuss some special cases of Theorem 3.1.

(I) If we take $\gamma(\vartheta)=\vartheta$ in Theorem 3.1, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\left\{\frac{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}}{8}[|\mathcal{F}'(\varpi_1)|+3|\mathcal{F}'(\varpi_2)|]\right.$$
$$+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)|\mathcal{F}'(\varpi_1)|\left[\frac{11\varpi_1{}^{\sigma-1}+5\varpi_2{}^{\sigma-1}}{192}\right]$$
$$+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)|\mathcal{F}'(\varpi_2)|\left[\frac{5\varpi_1{}^{\sigma-1}}{192}+\frac{\varpi_2{}^{\sigma-1}}{64}\right]-\frac{\varpi_1{}^{\sigma}}{8}[|\mathcal{F}'(\varpi_1)|+|\mathcal{F}'(\varpi_2)|]$$
$$\left.+\frac{\varpi_1\varpi_2{}^{\sigma-1}}{24}[2|\mathcal{F}'(\varpi_1)|+|\mathcal{F}'(\varpi_2)|]-\frac{\varpi_2{}^{\sigma}}{24}[2|\mathcal{F}'(\varpi_1)|+7|\mathcal{F}'(\varpi_2)|]\right\}.$$

(II) If we choose $\gamma(\vartheta)=1$ in Theorem 3.1, we get

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\left\{[|\mathcal{F}'(\varpi_1)|+|\mathcal{F}'(\varpi_2)|]\right.$$
$$\left.\times\left[\frac{\varpi_1\varpi_2{}^{\sigma-1}-\varpi_2{}^{\sigma}-\varpi_1{}^{\sigma}+(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}}{2}+\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)(\varpi_1{}^{\sigma-1}+\varpi_2{}^{\sigma-1})}{8}\right]\right\}.$$

(III) If we take $\gamma(\vartheta)=\vartheta^{s}$ in Theorem 3.1, we obtain

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\left\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}_1^*+\mathcal{A}_2^*+\mathcal{A}_3^*]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}_1^*+\mathcal{B}_2^*+\mathcal{B}_3^*]\right\},$$

where $\mathcal{A}_1^*,\mathcal{A}_2^*,\mathcal{A}_3^*,\mathcal{B}_1^*,\mathcal{B}_2^*$ and $\mathcal{B}_3^*$ are given by (2.7)–(2.12), respectively.

(IV) If we choose $\gamma(\vartheta)=\vartheta^{-s}$ in Theorem 3.1, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\left\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}_1^{**}+\mathcal{A}_2^{**}+\mathcal{A}_3^{**}]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}_1^{**}+\mathcal{B}_2^{**}+\mathcal{B}_3^{**}]\right\},$$

where $\mathcal{A}_1^{**},\mathcal{A}_2^{**},\mathcal{A}_3^{**},\mathcal{B}_1^{**},\mathcal{B}_2^{**}$ and $\mathcal{B}_3^{**}$ are given by (2.13)–(2.18), respectively.

Theorem 3.2. *Let $\varpi_1,\varpi_2\in\mathbb{R}^+$ with $0<\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)$, and let $\mathcal{F}:\mathcal{P}\to\mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ for $\sigma\in(0,1]$. If $D_\sigma(\mathcal{F})\in L_\sigma^1(\mathcal{P})$, and $|\mathcal{F}'|^q$ is generalized γ-convex function on $\mathcal{P}$, for $q\geq 1$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\Big\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1|\mathcal{F}'(\varpi_1)|^q+\mathcal{H}_2|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}$$
$$+\mathcal{K}_\eta{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta_1}|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta_2}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\Big\},$$

where $\mathcal{H},\mathcal{H}_1,\mathcal{H}_2,\mathcal{K}_\eta,\mathcal{K}_{\eta_1}$ and $\mathcal{K}_{\eta_2}$ are given by (2.19)–(2.24), respectively.

We now discuss some special cases of Theorem 3.2.

(I) If we take $\gamma(\vartheta)=\vartheta$ in Theorem 3.2, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\Big\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^*|\mathcal{F}'(\varpi_1)|^q+\mathcal{H}_2^*|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}+\mathcal{K}_\eta{}^{1-\frac{1}{q}}$$
$$\times[\mathcal{K}_{\eta_1}^*|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta_2}^*|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\Big\},$$

where $\mathcal{H}_1^*,\mathcal{H}_2^*,\mathcal{K}_{\eta_1}^*$ and $\mathcal{K}_{\eta_2}^*$ are given by (2.25)–(2.28), respectively.

(II) If we choose $\gamma(\vartheta)=1$ in Theorem 3.2, we get

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\Big\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{**}(|\mathcal{F}'(\varpi_1)|^q+|\mathcal{F}'(\varpi_2)|^q)]^{\frac{1}{q}}+\mathcal{K}_\eta{}^{1-\frac{1}{q}}$$
$$\times[\mathcal{K}_{\eta_1}{}^{**}(|\mathcal{F}'(\varpi_1)|^q+|\mathcal{F}'(\varpi_2)|^q)]^{\frac{1}{q}}\Big\},$$

where $\mathcal{H}_1^{**}$ and $\mathcal{K}_{\eta_1}{}^{**}$ are given by (2.29) and (2.30).

(III) If we take $\gamma(\vartheta)=\vartheta^s$ in Theorem 2.2, we obtain

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}$$
$$\times\left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{H}_2^{\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}+\mathcal{K}_{\eta}{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta1}^{\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta2}^{\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where $\mathcal{H}_1^{\star},\mathcal{H}_2^{\star},\mathcal{K}_{\eta1}^{\star}$ and $\mathcal{K}_{\eta2}^{\star}$ are given by (2.31)–(2.34), respectively.

(IV) If we choose $\gamma(\vartheta)=\vartheta^{-s}$ in Theorem 2.2, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}$$
$$\times\left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{\star\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{H}_2^{\star\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}+\mathcal{K}_{\eta}{}^{1-\frac{1}{q}}[\mathcal{K}_{\eta1}^{\star\star}|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta2}^{\star\star}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where $\mathcal{H}_1^{\star\star},\mathcal{H}_2^{\star\star},\mathcal{K}_{\eta1}^{\star\star}$ and $\mathcal{K}_{\eta2}^{\star\star}$ are given by (2.35)–(2.38), respectively.

Theorem 3.3. *Let $\varpi_1,\varpi_2\in\mathbb{R}^+$ with $0<\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)$, and let $\mathcal{F}:\mathcal{P}\to\mathbb{R}$ be a differentiable function on $\mathcal{P}^{\circ}$ for $\sigma\in(0,1]$. If $D_{\sigma}(\mathcal{F})\in L^1_{\sigma}(\mathcal{P})$, and $|\mathcal{F}'|^q$ is γ-convex function on $\mathcal{P}$, for $q\geq 1$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_{\sigma}s\right|$$
$$\leq\frac{\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}(\phi;\psi;\eta;\varpi_2-\varpi_1))^{\sigma}-\varpi_1{}^{\sigma}}\left[\mathcal{H}\left|\mathcal{F}'\left(\frac{C_1}{\mathcal{H}}\right)\right|+\mathcal{K}_{\eta}\left|\mathcal{F}'\left(\frac{C_2}{\mathcal{K}_{\eta}}\right)\right|\right],$$

where C_1 and C_2 are given by (2.40) and (2.41) and $\mathcal{H}$, $\mathcal{K}_{\eta}$ are defined as in Theorem 2.2 with the assumption that $\gamma(1-\vartheta)+\gamma(\vartheta)=1$.

3.4. Applications to Mittag–Leffler functions

Moreover if we take $\sigma=(1,1,1...)$, $\lambda=1$ and $\rho=\phi$ with $\Re(\phi)>0$ in (1.1), then we obtain well-known Mittag–Leffler function:

$$\mathcal{R}_{\phi}(x)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(1+\phi k)}x^k.$$

Lemma 3.2. *Let $\varpi_1,\varpi_2\in\mathbb{R}^+$ with $0<\mathcal{R}_{\phi}(\varpi_2-\varpi_1)$, and let $\mathcal{F}:\mathcal{P}\to\mathbb{R}$ be a differentiable function on $\mathcal{P}^{\circ}$ for $\sigma\in(0,1]$. If $D_{\sigma}(\mathcal{F})\in L^1_{\sigma}(\mathcal{P})$, then*

$$
\begin{aligned}
&\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\\
=&\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\\
&\times\left[\int_0^{\frac{1}{2}}\left((\varpi_1+\vartheta\mathcal{R}_\phi(\varpi_2-\varpi_1))^{2\sigma-1}-\varpi_1{}^\sigma(\varpi_1+\vartheta\mathcal{R}_\phi(\varpi_2-\varpi_1))^{\sigma-1}\right)\right.\\
&\times D_\sigma(\mathcal{F})(\varpi_1+\vartheta\mathcal{R}_\phi(\varpi_2-\varpi_1))\vartheta^{1-\sigma}\mathrm{d}_\sigma\vartheta\\
&+\int_{\frac{1}{2}}^1\left((\varpi_1+\vartheta\mathcal{R}_\phi(\varpi_2-\varpi_1))^{2\sigma-1}-(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma(\varpi_1+\vartheta\mathcal{R}_\phi(\varpi_2-\varpi_1))^{\sigma-1}\right)\\
&\left.\times D_\sigma(\mathcal{F})(\varpi_1+\vartheta\mathcal{R}_\phi(\varpi_2-\varpi_1))\vartheta^{1-\sigma}\mathrm{d}_\sigma\vartheta\right].
\end{aligned}
$$

Theorem 3.4. *Let $\varpi_1,\varpi_2\in\mathbb{R}^+$ with $0<\mathcal{R}_\phi(\varpi_2-\varpi_1)$, and let $\mathcal{F}:\mathcal{P}\to\mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ for $\sigma\in(0,1]$. If $D_\sigma(\mathcal{F})\in L^1_\sigma(\mathcal{P})$, and $|\mathcal{F}'|$ is generalized γ-convex function on $\mathcal{P}$, then*

$$
\begin{aligned}
&\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|\\
\leq&\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\left\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3]\right\},
\end{aligned}
$$

where $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{B}_1,\mathcal{B}_2$ and $\mathcal{B}_3$ are given by (2.1)–(2.6), respectively.

We now discuss some special cases of Theorem 3.4.

(I) If we take $\gamma(\vartheta)=\vartheta$ in Theorem 3.4, we have

$$
\begin{aligned}
&\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int\limits_{\varpi_1}^{\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|\\
\leq&\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\left\{\frac{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma}{8}[|\mathcal{F}'(\varpi_1)|+3|\mathcal{F}'(\varpi_2)|]\right.\\
&+\mathcal{R}_\phi(\varpi_2-\varpi_1)|\mathcal{F}'(\varpi_1)|\left[\frac{11\varpi_1{}^{\sigma-1}+5\varpi_2{}^{\sigma-1}}{192}\right]\\
&+\mathcal{R}_\phi(\varpi_2-\varpi_1)|\mathcal{F}'(\varpi_2)|\left[\frac{5\varpi_1{}^{\sigma-1}}{192}+\frac{\varpi_2{}^{\sigma-1}}{64}\right]-\frac{\varpi_1{}^\sigma}{8}[|\mathcal{F}'(\varpi_1)|+|\mathcal{F}'(\varpi_2)|]\\
&\left.+\frac{\varpi_1\varpi_2{}^{\sigma-1}}{24}[2|\mathcal{F}'(\varpi_1)|+|\mathcal{F}'(\varpi_2)|]-\frac{\varpi_2{}^\sigma}{24}[2|\mathcal{F}'(\varpi_1)|+7|\mathcal{F}'(\varpi_2)|]\right\}.
\end{aligned}
$$

(II) If we choose $\gamma(\vartheta)=1$ in Theorem 3.4, we get

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}$$
$$\times\left\{[|\mathcal{F}'(\varpi_1)|+|\mathcal{F}'(\varpi_2)|]\left[\frac{\varpi_1\varpi_2{}^{\sigma-1}-\varpi_2{}^\sigma-\varpi_1{}^\sigma+(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma}{2}\right.\right.$$
$$\left.\left.+\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)(\varpi_1{}^{\sigma-1}+\varpi_2{}^{\sigma-1})}{8}\right]\right\}.$$

(III) If we take $\gamma(\vartheta)=\vartheta^s$ in Theorem 3.4, we obtain

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\left\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}_1^*+\mathcal{A}_2^*+\mathcal{A}_3^*]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}_1^*+\mathcal{B}_2^*+\mathcal{B}_3^*]\right\},$$

where $\mathcal{A}_1^*,\mathcal{A}_2^*,\mathcal{A}_3^*,\mathcal{B}_1^*,\mathcal{B}_2^*$ and $\mathcal{B}_3^*$ are given by (2.7)–(2.12), respectively.

(IV) If we choose $\gamma(\vartheta)=\vartheta^{-s}$ in Theorem 3.4, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\left\{|\mathcal{F}'(\varpi_1)|[\mathcal{A}_1^{**}+\mathcal{A}_2^{**}+\mathcal{A}_3^{**}]+|\mathcal{F}'(\varpi_2)|[\mathcal{B}_1^{**}+\mathcal{B}_2^{**}+\mathcal{B}_3^{**}]\right\},$$

where $\mathcal{A}_1^{**},\mathcal{A}_2^{**},\mathcal{A}_3^{**},\mathcal{B}_1^{**},\mathcal{B}_2^{**}$ and $\mathcal{B}_3^{**}$ are given by (2.13)–2.18), respectively.

Theorem 3.5. *Let $\varpi_1,\varpi_2\in\mathbb{R}^+$ with $0<\mathcal{R}_\phi(\varpi_2-\varpi_1)$, and let $\mathcal{F}:\mathcal{P}\to\mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ for $\sigma\in(0,1]$. If $D_\sigma(\mathcal{F})\in L^1_\sigma(\mathcal{P})$, and $|\mathcal{F}'|^q$ is generalized γ-convex function on $\mathcal{P}$, for $q\geq 1$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}{2}\right)-\frac{\sigma}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\int_{\varpi_1}^{\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1)}\mathcal{F}(s)\mathrm{d}_\sigma s\right|$$
$$\leq\frac{\mathcal{R}_\phi(\varpi_2-\varpi_1)}{(\varpi_1+\mathcal{R}_\phi(\varpi_2-\varpi_1))^\sigma-\varpi_1{}^\sigma}\left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1|\mathcal{F}'(\varpi_1)|^q+\mathcal{H}_2|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}+\mathcal{K}_\eta{}^{1-\frac{1}{q}}\right.$$
$$\left.\times[\mathcal{K}_{\eta_1}|\mathcal{F}'(\varpi_1)|^q+\mathcal{K}_{\eta_2}|\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where $\mathcal{H},\mathcal{H}_1,\mathcal{H}_2,\mathcal{K}_\eta,\mathcal{K}_{\eta_1}$ and $\mathcal{K}_{\eta_2}$ are given by (2.19)–(2.24), respectively.

We now discuss some special cases of Theorem 3.5.

(I) If we take $\gamma(\vartheta) = \vartheta$ in Theorem 3.5, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int_{\varpi_1}^{\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{R}_\phi(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^* |\mathcal{F}'(\varpi_1)|^q + \mathcal{H}_2^* |\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right.$$
$$\left. + {\mathcal{K}_\eta}^{1-\frac{1}{q}}[{\mathcal{K}_{\eta 1}}^* |\mathcal{F}'(\varpi_1)|^q + {\mathcal{K}_{\eta 2}}^* |\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where $\mathcal{H}_1^*, \mathcal{H}_2^*, {\mathcal{K}_{\eta 1}}^*$ and ${\mathcal{K}_{\eta 2}}^*$ are given by (2.25)–(2.28), respectively.

(II) If we choose $\gamma(\vartheta) = 1$ in Theorem 3.5, we get

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int_{\varpi_1}^{\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{R}_\phi(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{**} (|\mathcal{F}'(\varpi_1)|^q + |\mathcal{F}'(\varpi_2)|^q)]^{\frac{1}{q}}\right.$$
$$\left. + {\mathcal{K}_\eta}^{1-\frac{1}{q}}[{\mathcal{K}_{\eta 1}}^{**} (|\mathcal{F}'(\varpi_1)|^q + |\mathcal{F}'(\varpi_2)|^q)]^{\frac{1}{q}}\right\},$$

where $\mathcal{H}_1^{**}$ and $\mathcal{K}_1^{**}$ are given by (2.29) and (2.30).

(III) If we take $\gamma(\vartheta) = \vartheta^s$ in Theorem 3.5, we obtain

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int_{\varpi_1}^{\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{R}_\phi(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma}$$
$$\times \left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^\star |\mathcal{F}'(\varpi_1)|^q + \mathcal{H}_2^\star |\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}} + {\mathcal{K}_\eta}^{1-\frac{1}{q}}[{\mathcal{K}_{\eta 1}}^\star |\mathcal{F}'(\varpi_1)|^q + {\mathcal{K}_{\eta 2}}^\star |\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where $\mathcal{H}_1^\star, \mathcal{H}_2^\star, {\mathcal{K}_{\eta 1}}^\star$ and ${\mathcal{K}_{\eta 2}}^\star$ are given by (2.31)–(2.34), respectively.

(IV) If we choose $\gamma(\vartheta) = \vartheta^{-s}$ in Theorem 3.5, we have

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int_{\varpi_1}^{\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{R}_\phi(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma}$$
$$\times \left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1^{\star\star} |\mathcal{F}'(\varpi_1)|^q + \mathcal{H}_2^{\star\star} |\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}} + {\mathcal{K}_\eta}^{1-\frac{1}{q}}[{\mathcal{K}_{\eta 1}}^{\star\star} |\mathcal{F}'(\varpi_1)|^q + {\mathcal{K}_{\eta 2}}^{\star\star} |\mathcal{F}'(\varpi_2)|^q]^{\frac{1}{q}}\right\},$$

where $\mathcal{H}_1^{\star\star}, \mathcal{H}_2^{\star\star}, {\mathcal{K}_{\eta 1}}^{\star\star}$ and ${\mathcal{K}_{\eta 2}}^{\star\star}$ are given by (2.35)–(2.38), respectively.

Theorem 3.6. *Let $\varpi_1, \varpi_2 \in \mathbb{R}^+$ with $0 < \mathcal{R}_\phi(\varpi_2 - \varpi_1)$, and let $\mathcal{F} : \mathcal{P} \to \mathbb{R}$ be a differentiable function on $\mathcal{P}^\circ$ for $\sigma \in (0, 1]$. If $D_\sigma(\mathcal{F}) \in L^1_\sigma(\mathcal{P})$, and $|\mathcal{F}'|^q$ is γ-convex function on $\mathcal{P}$, for $q \geq 1$, then*

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int\limits_{\varpi_1}^{\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{R}_\phi(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}_\phi(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \left[\mathcal{H}\left|\mathcal{F}'\left(\frac{C_1}{\mathcal{H}}\right)\right| + \mathcal{K}_\eta\left|\mathcal{F}'\left(\frac{C_2}{\mathcal{K}_\eta}\right)\right|\right],$$

where C_1 and C_2 are given by (2.40) and (2.41) and $\mathcal{H}$, $\mathcal{K}_\eta$ are defined as in Theorem 2.3 with the assumption that $\gamma(1 - \vartheta) + \gamma(\vartheta) = 1$.

3.5. Applications to bounded functions

In this last section, we discuss applications regarding bounded functions in absolute value of the results obtained from our main results. We suppose that the following condition is satisfied:

$$|\mathcal{F}'| \leq \mathcal{M}.$$

Applying the above condition, we have the following results.

Corollary 3.1. *Under the assumptions of Theorem 2.1, the following inequality holds:*

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int\limits_{\varpi_1}^{\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{M}\mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \left\{\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3\right\}.$$

Corollary 3.2. *Under the assumptions of Theorem 2.2, the following inequality holds:*

$$\left|\mathcal{F}\left(\frac{2\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1)}{2}\right) - \frac{\sigma}{(\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \int\limits_{\varpi_1}^{\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1)} \mathcal{F}(s) \mathrm{d}_\sigma s\right|$$
$$\leq \frac{\mathcal{M}\mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1)}{(\varpi_1 + \mathcal{R}^\varepsilon_{\lambda,\rho}(\varpi_2 - \varpi_1))^\sigma - {\varpi_1}^\sigma} \left\{\mathcal{H}^{1-\frac{1}{q}}[\mathcal{H}_1 + \mathcal{H}_2]^{\frac{1}{q}} + {\mathcal{K}_\eta}^{1-\frac{1}{q}}[\mathcal{K}_{\eta_1} + \mathcal{K}_{\eta_2}]^{\frac{1}{q}}\right\}.$$

4. Conclusions

In this paper we obtain a new integral identity and associated bounds essentially using the concept of generalized γ-convex functions. We also discussed special cases of the main results which shown that the obtained results are quite unifying one. Moreover, we also presented several applications for particular special means with arbitrary positive real numbers, hypergeometric functions, Mittag–Leffler functions, differentiable functions of first order that are in absolute value bounded, and some error estimations of the quadrature formula as well. Since the class of generalized γ-convex functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, and mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. Studies relating convexity, partial convexity, and preinvex functions (as contractive operators) may have useful applications in complex interdisciplinary studies, such as maximizing the likelihood from multiple linear regressions involving Gauss–Laplace distribution. For more details, see [40–45]. We hope that our ideas and techniques of this paper will inspire interested readers working in this field.

Acknowledgments

The authors are thankful to the editor and the anonymous reviewers for their valuable comments and suggestions. This research is supported by Project number (RSP-2021/158), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors have no conflicts of interest to declare.

References

1. W. W. Breckner, Stetigkeitsaussagen fiir eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen, *Pupl. Inst. Math.*, **23** (1978), 13–20.
2. S. S. Dragomir, J. Pecarič, L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21** (1995), 335–341.
3. E. K. Godunova, V. I. Levin, Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii, *Vycislitel. Mat. Fiz. Mezvuzov. Sb. Nauc. Trudov*, 1985, 138–142.
4. S. Varošanec, On h-convexity, *J. Math. Anal. Appl.*, **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086
5. S. Wu, M. U. Awan, M. A. Noor, K. I. Noor, S. Iftikhar, On a new class of convex functions and integral inequalities, *J. Inequal. Appl.*, **2019** (2019), 131. https://doi.org/10.1186/s13660-019-2074-y
6. M. J. Vivas-Cortez, R. Liko, A. Kashuri, J. E. H. Hernández, New quantum estimates of trapezium-type inequalities for generalized φ-convex functions, *Mathematics*, **7** (2019), 1047. https://doi.org/10.3390/math7111047
7. R. K. Raina, On generalized Wright's hypergeometric functions and fractional calculus operators, *East As. Math. J.*, **21** (2015), 191–203.
8. M. J. Vivas-Cortez, M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, Trapezium-like inequalities involving k-th order differentiable $\mathcal{R}_\gamma$-convex functions and applications, *Symmetry*, **14** (2022), 448. https://doi.org/10.3390/sym14030448
9. S. S. Dragomir, C. Pearce, Selected topics on Hermite–Hadamard inequalities and applications, *Science Direct Working Paper*, 2003. Available from: `https://ssrn.com/abstract=3158351`.

10. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048

11. T. S. Du, J. G. Liao, Y. J. Li, Properties and integral inequalities of Hadamard–Simpson type for the generalized (s, m)-preinvex functions, *J. Nonlinear Sci. Appl.*, **9** (2016), 3112–3126.

12. A. Iqbal, M. A. Khan, S. Ullah, Y. M. Chu, Some New Hermite–Hadamard-type inequalities associated with conformable fractional integrals and their applications, *J. Funct. Spaces*, **2020** (2020), 9845407. https://doi.org/10.1155/2020/9845407

13. Y. Khurshid, M. A. Khan, Y. M. Chu, Z. A. Khan, Hermite–Hadamard–Fejér inequalities for conformable fractional integrals via pre-invex functions, *J. Funct. Spaces*, **2019** (2019), 3146210. https://doi.org/10.1155/2019/3146210

14. H. Lei, T. S. Du, Some new bounds related to Fejér–Hermite–Hadamard type inequality and their applications, *ScienceAsia*, **45** (2019), 361–370. https://doi.org/10.2306/scienceasia1513-1874.2019.45.361

15. J. G. Liao, S. H. Wu, T. S. Du, The Sugeno integral with respect to α-preinvex functions, *Fuzzy Sets Syst.*, **379** (2020), 102–114. https://doi.org/10.1016/j.fss.2018.11.008

16. S. Erhan, I. Mumcu, Hermite–Hadamard–Fejér type inequalities for conformable fractional integrals, *Misk. Math. Notes*, **20** (2019), 475–488. https://doi.org/10.18514/MMN.2019.2421

17. Y. Zhang, T. S. Du, H. Wang, Some new k-fractional integral inequalities containing multiple parameters via generalized (s, m)-preinvexity, *Ital. J. Pure Appl. Math.*, **40** (2018), 510–527.

18. P. O. Mohammed, M. Z. Sarikaya, D. Baleanu, On the generalized Hermite–Hadamard inequalities via the tempered fractional integrals, *Symmetry*, **12** (2020), 595. https://doi.org/10.3390/sym12040595

19. P. O. Mohammed, I. Brevik, A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals, *Symmetry*, **12** (2020), 610. https://doi.org/10.3390/sym12040610

20. A. Iqbal, M. A. Khan, S. Ullah, Y. M. Chu, A. Kashuri, Hermite–Hadamard type inequalities pertaining conformable fractional integrals and their applications, *AIP Adv.*, **8** (2018), 075101. https://doi.org/10.1063/1.5031954

21. M. Houas, Certain weighted integral inequalities involving the fractional hypergeometric operators, *Scientia, Ser. A, Math. Sci.*, **27** (2016), 87–97.

22. H. M. Srivastava, A. Kashuri, P. O. Mohammed, A. M. Alsharif, J. L. G. Guirao, New Chebyshev type inequalities via a general family of fractional integral operators with a modified Mittag–Leffler kernel, *AIMS Math.*, **6** (2021), 11167–11186. https://doi.org/10.3934/math.2021648

23. P. O. Mohammed, T. Abdeljawad, D. Baleanu, A. Kashuri, F. Hamasalh, P. Agarwal, New fractional inequalities of Hermite–Hadamard type involving the incomplete gamma functions, *J. Inequal. Appl.*, **2020** (2020), 263. https://doi.org/10.1186/s13660-020-02538-y

24. H. M. Srivastava, Some families of Mittag–Leffler type functions and associated operators of fractional calculus, *TWMS J. Pure Appl. Math.*, **7** (2016), 123–145.

25. A. Fernandez, D. Baleanu, H. M. Srivastava, Series representations for fractional-calculus operators involving generalised Mittag–Leffler functions, *Commun. Nonlinear Sci. Numer. Simulat.*, **67** (2019), 517–527. https://doi.org/10.1016/j.cnsns.2018.07.035

26. A. Fernandez, D. Baleanu, H. M. Srivastava, Corrigendum to "Series representations for fractional-calculus operators involving generalised Mittag–Leffler functions" [Commun. Nonlinear Sci. Numer. Simulat. 67 (2019) 517–527], *Commun. Nonlinear Sci. Numer. Simulat.* **82** (2020), 104963. https://doi.org/10.1016/j.cnsns.2019.104963

27. H. M. Srivastava, A. Fernandez, D. Baleanu, Some new fractional-calculus connections between Mittag–Leffler functions, *Mathematics*, **7** (2019), 485. https://doi.org/10.3390/math7060485

28. H. M. Srivastava, M. K. Bansal, P. Harjule, A study of fractional integral operators involving a certain generalized multi-index Mittag–Leffler function, *Math. Meth. Appl. Sci.*, **41** (2018), 6108–6121. https://doi.org/10.1002/mma.5122

29. H. M. Srivastava, Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag–Leffler function in the kernel, *Appl. Math. Comput.*, **211** (2009), 198–210. https://doi.org/10.1016/j.amc.2009.01.055

30. Ž. Tomovski, R. Hilfer, H. M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag–Leffler type functions, *Integr. Transf. Spec. F.*, **21** (2010), 797–814. https://doi.org/10.1080/10652461003675737

31. S. K. Sahoo, P. O. Mohammed, B. Kodamasingh, M. Tariq, Y. S. Hamed, New fractional integral inequalities for convex functions pertaining to Caputo–Fabrizio operator, *Fractal Fract.*, **6** (2022), 171. https://doi.org/10.3390/fractalfract6030171

32. S. I. Butt, A. Nosheen, J. Nasir, K. A. Khan, R. M. Mabela, New fractional Mercer–Ostrowski type inequalities with respect to monotone function, *Math. Probl. Eng.*, **2022** (2022), Article ID 7067543. https://doi.org/10.1155/2022/7067543

33. S. Qaisar, J. Nasir, S. I. Butt, S. Hussain, On some fractional integral inequalities of Hermite–Hadamard's type through convexity, *Symmetry*, **11** (2019), 137. https://doi.org/10.3390/sym11020137

34. J. Zhao, S. I. Butt, J. Nasir, Z. Wang, I. Tlili, Hermite–Jensen–Mercer type inequalities for Caputo fractional derivatives, *J. Funct. Spaces*, **2020** (2020), Article ID 7061549. https://doi.org/10.1155/2020/7061549

35. R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002

36. D. R. Anderson, Taylor's formula and integral inequalities for conformable fractional derivatives, In: P. M. Pardalos, T. M. Rassias, *Contributions in mathematics and engineering*, Springer, 2016, 25–43. https://doi.org/10.1007/978-3-319-31317-7_2

37. T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, **279** (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016

38. O. S. Iyiola, E. R. Nwaeze, Some new results on the new conformable fractional calculus with application using D'Alambert approach, *Progr. Fract. Differ. Appl.*, **2** (2016), 115–122. https://doi.org/10.18576/pfda/020204

39. M. A. Khan, T. Ali, S. S. Dragomir, M. Z. Sarikaya, Hermite–Hadamard type inequalities for conformable fractional integrals, *RACSAM*, **112** (2018), 1033–1048. https://doi.org/10.1007/s13398-017-0408-5
40. X. S. Zhou, C. X. Huang, H. J. Hu, L. Liu, Inequality estimates for the boundedness of multilinear singular and fractional integral operators, *J. Inequal. Appl.*, **2013** (2013), 303. https://doi.org/10.1186/1029-242X-2013-303
41. H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian hypergeometric series*, Chichester: Ellis Halsted Press, 1985.
42. N. S. Barnett, P. Cerone, S. S. Dragomir, J. Roumeliotis, Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval, *J. Inequal. Pure Appl. Math.*, **2** (1999), 1–18.
43. N. S. Barnett, S. S. Dragomir, Some elementary inequalities for the expectation and variance of a random variable whose pdf is defined on a finite interval, *RGMIA Res. Rep. Collect.*, **2** (1999), 1–7.
44. P. Cerone, S. S. Dragomir, On some inequalities for the expectation and variance, *Korean J. Comput. Appl. Math.*, **8** (2000), 357–380. https://doi.org/10.1007/BF02941972
45. J. E. Pečarič, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Vol. 187, New York: Academic Press, 1992.