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Research article

Decision-making strategy based on Heronian mean operators for managing complex interval-valued intuitionistic uncertain linguistic settings and their applications

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Abstract: This analysis diagnoses a well-known and dominant theory of complex interval-valued intuitionistic uncertain linguistic (CI-VIUL) settings, which is considered to be a very powerful and capable tool to handle ambiguous sorts of theories. Furthermore, to enhance the features of the newly developed CI-VIUL information, we diagnose the algebraic laws, score value and accuracy value. Moreover, keeping in mind that the Heronian mean (HM) operator is a massive dominant operator that can suggest information on interrelationships, in this manuscript, we develop the CI-VIUL arithmetic HM (CI-VIULAHM) operator, CI-VIUL weighted arithmetic HM (CI-VIULWAHM) operator, CI-VIUL geometric HM (CI-VIULGHM) operator, CI-VIUL weighted geometric HM (CI-VIULWGHM) operator and their well-known achievements in the form of some results, important properties and a discussion of some specific cases. At the end, we check the practicality and usefulness of the initiated approaches, and a multi-attribute decision-making (MADM) technique is implemented for CI-VIUL settings. The reliability of the proposed MADM tool is demonstrated by a computational example that evaluates the impact of the diagnosed approaches on various well-known prevailing theories.

Keywords: complex interval-valued intuitionistic uncertain linguistic sets; arithmetic/geometric Heronian mean operators; decision-making support system
Mathematics Subject Classification: 03E52, 03E72, 28E10, 68T27, 94D05

1. Introduction

Decision-making approaches are the techniques we use to get a decision in many situations, like deciding to cross a canal, choosing a later semester's classes or establishing an extended-term business scheme. Furthermore, human decision-making is frequently learned as a consequence of the sensitive performance of alternative terms on possible options and the values of consequences connected with these decisions. Continuously, a large number of intellectuals have used this notion to take a lot of benefits from it. In 1965, Zadeh [1] employed a decision-making tool, named a fuzzy set (FS), by modifying the range of a crisp set to form the unit interval [0, 1]. The value of a truth grade (TG) (part of a FS) is not greater than one. Therefore, it indicates the data in the real world in a more massive and varied way than the application of crisp sets. Nowadays, there are various research tools and techniques for FS theory in distinct regions of research and practical life. For example, L-FSs were diagnosed in [2], hesitant FSs were employed in [3], rough sets were exposed in [4], bipolar FSs were investigated in [5] and bipolar soft sets were initiated in [6]. The analysis discussed above received much attention by researchers, but their approaches are neglected in various places. For illustration, if an agency allows data toward matters such that the TG is 0.6 and the falsity grade (FG) is 0.3, then 0.6+0.3=0.9≤1; hence, FSs cannot respond to such a dilemma. To address and manage such circumstances, Atanassov [7] initiated the notion of intuitionistic FSs (IFSs), with the condition that the sum of both grades must be less than or equal to 1. Yet what happened, if someone gives the opinion in an interval-valued (I-V) form, in such sort of circumstances, the principle of an I-V IFS (I-VIFS) [8] is massively valuable as compared to the existing notion of IFSs. Nowadays, there are various research tools and techniques for IFS theory in distinct regions of research and in practical life. For instance, linguistic intuitionistic fuzzy information was diagnosed by Liu et al. [9]. Gupta et al. [10] illustrated the theory of the VIKOR approach for the intuitionistic fuzzy linguistic environment. Jana and Pal [11] discovered the theory of bipolar IFSs and their applications. Faizi et al. [12] diagnosed the theory of aggregation operators for hesitant intuitionistic fuzzy linguistic information. Fu et al. [13] utilized the theory of decision-making techniques for I-VIFSs, and the theory of geometric interaction aggregation operators for IFSs was diagnosed by Meg and He [14].

Ramot et al. [15] diagnosed the fundamental theory of complex FSs (CFSs) by giving a new look to the TG by designating it in the form of a complex number lying in the unit disc $|z| \leq 1$. CFSs proved to be massively valuable and well-constructed for managing invaluable and less efficient data used in genuine life dilemmas. The range (unit disc in the complex plane) of CFSs is more modified than the range of FSs (unit interval). A valuable number of intellectuals have exploited a lot of approaches in valuable regions, e.g., neuro-fuzzy sets [16], complex fuzzy logic [17], several properties of CFSs [18] and CFS theory [19]. Further, Dick [20] and Tamir et al. [21,22] also investigated CFSs, and more related investigations are given in [23–25]. Additionally, upon the occurrence of various issues, the complex IFS (CIFS) was exposed by Alkouri and Salleh [26], who proved it to be very valuable for managing awkward and invaluable data. CIFSs cope with such sort of problems, which include TGs and FGs, in the form of complex-valued numbers with real and unreal parts that belong to the unit interval. The exact and meaningful conditions of CIFSs are described in the shapes of $0 \leq m_{\Gamma_{RP}}(\check{k}) + n_{\Gamma_{RP}}(\check{k}) \leq 1$ and $0 \leq m_{\Gamma_{IP}}(\check{k}) + n_{\Gamma_{IP}}(\check{k}) \leq 1$. Given $m_{\Gamma_{IP}}(\check{k}) = n_{\Gamma_{IP}}(\check{k}) = 0$ in CIFSs, we get IFSs. By considering the CIFSs, various people have used the CIFSs and tried to employ them in the fields of different regions, e.g., complex interval-valued IFSs (CI-VIFS) [27], complex intuitionistic fuzzy soft sets [28], knowledge measures [29], quaternion numbers [30], the TODIM method [31], complex intuitionistic fuzzy groups [32], aggregation operators [33], hypersoft sets based on CFS [34], complex intuitionistic fuzzy and neutrosophic sets [35] and decision-making strategies [35–37].

In general, a fuzzy system is any system in which the variables range over states that are fuzzy numbers (FNs) rather than real numbers. These FNs may express linguistic terms such as "very small" and "small". If they do, the variable is stated as the "linguistic variable" (LV), initiated by Zadeh [38]. Every term is described on behalf of a variable with values that are real numbers belonging to a particular range. Furthermore, the mathematical form of 2-tuple LVs was diagnosed in [39], and uncertain LVs were exposed in [40]. The combination of uncertain LVs and IFSs was stated in [41]. Bonferroni mean (BM) operators based on the combination of the work in [41,42] and weighted BM operators based on the combination of uncertain LVs and IFSs are described in [43]. The notion of Hamy operators based on intuitionistic uncertain variables is described in [44], and the fundamental and valuable Bonferroni operators based on complex intuitionistic uncertain variables were developed in [45]. Xu and Wang [46] defined the power aggregation operators based on 2-tuple linguistic sets that are not able to cope with fuzzy types of information. Furthermore, Xu et al. [47] proposed power aggregation operators based on linguistic sets that still contain many issues because they are not able to manage the fuzzy type of information. Similarly, Xu and Wang [48] diagnosed the power geometric operators for multiplicative linguistic preference relations. Assume an enterprise wants to utilize a biometric system in the main offices of some organization. For this, the enterprise decides to call upon some experts for giving their opinions concerning each system. Based on this analysis, they try to select beneficial biometric systems. It is clear / obvious that the existing theories based on FSs, IFSs, I-V FSs, etc., are not able to cope with it, because these theories can cope with one dimension of information at a time. Thus, for the above-cited types of dilemmas, we need to improve the quality and worth of the prevailing theories; hence, the theory of complex I-V intuitionistic uncertain linguistic (CI-VIUL) information is more valuable and efficient for managing two-dimensional information in a singleton set. The real part (amplitude term) and imaginary part (phase term) represent the model and production data of the biometric system. Keeping the value and supremacy of the above-cited theories, we can see that the theory of Heronian mean (HM) operators for CI-VIUL information has not been described yet. Thus, the main challenging task for the experts is to

1) express the information in the shape of CI-VIUL numbers (CI-VIULNs),
2) express various new aggregation operators for evaluating some preferences of experts,
3) diagnose a procedure for evaluating the decision-making problem and
4) find the beneficial optima.

To achieve Objective 1, in this manuscript, we diagnose a well-known theory of CI-VIUL settings, as it is a powerful and capable tool to handle an ambiguous sort of theories. Furthermore, we enhance the features of the CI-VIUL information and diagnose the algebraic laws, score value (SV) and accuracy value (AV) for CI-VIUL settings. To achieve Objective 2, we develop the CI-VIUL arithmetic HM (CI-VIULAHM), CI-VIUL weighted arithmetic HM (CI-VIULWAHM), CI-VIUL geometric HM (CI-VIULGHM), CI-VIUL weighted geometric HM (CI-VIULWGHM) and their well-known achievements in the form of some results, important properties and specific cases. To achieve Objective 3, we check the practicality and usefulness of the invented approaches, and a multi-attribute decision-making (MADM) technique is implemented for CI-VIUL settings. To achieve Objective 4, the reliability of the proposed MADM tool is demonstrated by a computational example that assesses the impact of the diagnosed approaches on various well-known prevailing theories.

The major contribution of this analysis is exposed in the following forms: The diagnosis of a well-known theory of CI-VIUL settings and their algebraic laws, and the revision of various basic existing methodologies in Section 2. The well-known theory of CI-VIUL settings and their algebraic laws, SV and AV for CI-VIUL settings are diagnosed in Section 3. In Section 4, we develop the CI-VIULAHM, CI-VIULWAHM, CI-VIULGHM, CI-VIULWGHM and their well-known achievements in the form of some results, important properties and specific cases. In Section 5, we check the practicality and usefulness of the invented approaches and a MADM technique is implemented for CI-VIUL settings. In Section 6, the reliability of the proposed MADM tool is demonstrated via a computational example that assesses the impact of the diagnosed approaches on various well-known prevailing theories. Section 7 concludes the manuscript.

2. Preliminaries

To diagnose a well-known theory of CI-VIUL settings and their algebraic laws, SV and AV for CI-VIUL settings, we have revised various basic existing methodologies like CI-VIFSs and their operational laws. Further, the mathematical terms $\widetilde{\mathcal{X}_{UNI}}$, $m_{\Gamma_{CI}}$ and $n_{\Gamma_{CI}}$, as described by the universal set, TG and FG respectively.

Definition 1. [27] The mathematical expression

$$\Gamma_{CI}=\left\{\left(m_{\Gamma_{CI}}(\check{\mathfrak{t}}),n_{\Gamma_{CI}}(\check{\mathfrak{t}})\right):\check{\mathfrak{t}}\in\widetilde{\mathcal{X}_{UNI}}\right\},\tag{1}$$

is called a CI-VIFS, where

$$m_{\Gamma_{CI}}(\check{\mathfrak{t}})=\left[m^{-}_{\Gamma_{RP}}(\check{\mathfrak{t}}),m^{+}_{\Gamma_{RP}}(\check{\mathfrak{t}})\right]e^{i2\pi\left(\left[m^{-}_{\Gamma_{IP}}(\check{\mathfrak{t}}),m^{+}_{\Gamma_{IP}}(\check{\mathfrak{t}})\right]\right)}$$

and

$$n_{\Gamma_{CI}}(\check{\mathfrak{t}})=\left[n^{-}_{\Gamma_{RP}}(\check{\mathfrak{t}}),n^{+}_{\Gamma_{RP}}(\check{\mathfrak{t}})\right]e^{i2\pi\left(\left[n^{-}_{\Gamma_{IP}}(\check{\mathfrak{t}}),n^{+}_{\Gamma_{IP}}(\check{\mathfrak{t}})\right]\right)}.$$

The major tools of CI-VIFS are

$$0\le m^{+}_{\Gamma_{RP}}(\check{\mathfrak{t}})+n^{+}_{\Gamma_{RP}}(\check{\mathfrak{t}})\le 1$$

and

$$0\le m^{+}_{\Gamma_{IP}}(\check{\mathfrak{t}})+n^{+}_{\Gamma_{IP}}(\check{\mathfrak{t}})\le 1.$$

The mathematical expression

$$\mathcal{L}_{\Gamma_{CI}}(\check{\mathfrak{t}})=\left[\mathcal{L}^{-}_{\Gamma_{RP}}(\check{\mathfrak{t}}),\mathcal{L}^{+}_{\Gamma_{RP}}(\check{\mathfrak{t}})\right]e^{i2\pi\left(\left[\mathcal{L}^{-}_{\Gamma_{IP}}(\check{\mathfrak{t}}),\mathcal{L}^{+}_{\Gamma_{IP}}(\check{\mathfrak{t}})\right]\right)}=\left[\left(1-m^{-}_{\Gamma_{RP}}(\check{\mathfrak{t}})-n^{-}_{\Gamma_{RP}}(\check{\mathfrak{t}})\right),\left(1-m^{+}_{\Gamma_{RP}}(\check{\mathfrak{t}})-n^{+}_{\Gamma_{RP}}(\check{\mathfrak{t}})\right)\right]e^{i2\pi\left[\left(1-m^{-}_{\Gamma_{IP}}(\check{\mathfrak{t}})-n^{-}_{\Gamma_{IP}}(\check{\mathfrak{t}})\right),\left(1-m^{+}_{\Gamma_{IP}}(\check{\mathfrak{t}})-n^{+}_{\Gamma_{IP}}(\check{\mathfrak{t}})\right)\right]},$$

called the refusal grade in the CI-VIF numbers (CI-VIFNs), is stated by

$$\Gamma_{CI-i}=\left(\left[m^{-}_{\Gamma_{RP-i}},m^{+}_{\Gamma_{RP-i}}\right]e^{i2\pi\left(\left[m^{-}_{\Gamma_{IP-i}},m^{+}_{\Gamma_{IP-i}}\right]\right)},\left[n^{-}_{\Gamma_{RP-i}},n^{+}_{\Gamma_{RP-i}}\right]e^{i2\pi\left(\left[n^{-}_{\Gamma_{IP-i}},n^{+}_{\Gamma_{IP-i}}\right]\right)}\right),i=1,2,\dots,\widetilde{\Sigma}.$$

For the given mathematical form of any two CI-VIFNs:

$$\mathit{\Gamma}_{CI-i}=\left(\left[m^{-}_{\Gamma_{RP-i}},m^{+}_{\Gamma_{RP-i}}\right]e^{i2\pi\left(\left[m^{-}_{\Gamma_{IP-i}},m^{+}_{\Gamma_{IP-i}}\right]\right)},\left[n^{-}_{\Gamma_{RP-i}},n^{+}_{\Gamma_{RP-i}}\right]e^{i2\pi\left(\left[n^{-}_{\Gamma_{IP-i}},n^{+}_{\Gamma_{IP-i}}\right]\right)}\right), i=1,2.$$

We have,

$$\mathit{\Gamma}_{CI-1}\oplus\mathit{\Gamma}_{CI-2}=\begin{pmatrix}\left[m^{-}_{\Gamma_{RP-1}}+m^{-}_{\Gamma_{RP-2}}-m^{-}_{\Gamma_{RP-1}}m^{-}_{\Gamma_{RP-2}},m^{+}_{\Gamma_{RP-1}}+m^{+}_{\Gamma_{RP-2}}-m^{+}_{\Gamma_{RP-1}}m^{+}_{\Gamma_{RP-2}}\right]\\ e^{i2\pi\left[m^{-}_{\Gamma_{IP-1}}+m^{-}_{\Gamma_{IP-2}}-m^{-}_{\Gamma_{IP-1}}m^{-}_{\Gamma_{IP-2}},m^{+}_{\Gamma_{IP-1}}+m^{+}_{\Gamma_{IP-2}}-m^{+}_{\Gamma_{IP-1}}m^{+}_{\Gamma_{IP-2}}\right]},\\ \left[n^{-}_{\Gamma_{RP-1}}n^{-}_{\Gamma_{RP-2}},n^{+}_{\Gamma_{RP-1}}n^{+}_{\Gamma_{RP-2}}\right]e^{i2\pi\left[n^{-}_{\Gamma_{IP-1}}n^{-}_{\Gamma_{IP-2}},n^{+}_{\Gamma_{IP-1}}n^{+}_{\Gamma_{IP-2}}\right]}\end{pmatrix} \quad (2)$$

$$\mathit{\Gamma}_{CI-1}\otimes\mathit{\Gamma}_{CI-2}=\begin{pmatrix}\left[m^{-}_{\Gamma_{RP-1}}m^{-}_{\Gamma_{RP-2}},m^{+}_{\Gamma_{RP-1}}m^{+}_{\Gamma_{RP-2}}\right]e^{i2\pi\left[m^{-}_{\Gamma_{IP-1}}m^{-}_{\Gamma_{IP-2}},m^{+}_{\Gamma_{IP-1}}m^{+}_{\Gamma_{IP-2}}\right]},\\ \left[n^{-}_{\Gamma_{RP-1}}+n^{-}_{\Gamma_{RP-2}}-n^{-}_{\Gamma_{RP-1}}n^{-}_{\Gamma_{RP-2}},n^{+}_{\Gamma_{RP-1}}+n^{+}_{\Gamma_{RP-2}}-n^{+}_{\Gamma_{RP-1}}n^{+}_{\Gamma_{RP-2}}\right]\\ e^{i2\pi\left[n^{-}_{\Gamma_{IP-1}}+n^{-}_{\Gamma_{IP-2}}-n^{-}_{\Gamma_{IP-1}}n^{-}_{\Gamma_{IP-2}},n^{+}_{\Gamma_{IP-1}}+n^{+}_{\Gamma_{IP-2}}-n^{+}_{\Gamma_{IP-1}}n^{+}_{\Gamma_{IP-2}}\right]}\end{pmatrix} \quad (3)$$

$$\Phi_{SC}\mathit{\Gamma}_{CI-1}=\begin{pmatrix}\left[1-\left(1-m^{-}_{\Gamma_{RP-1}}\right)^{\Phi_{SC}},1-\left(1-m^{+}_{\Gamma_{RP-1}}\right)^{\Phi_{SC}}\right]e^{i2\pi\left[1-\left(1-m^{-}_{\Gamma_{IP-1}}\right)^{\Phi_{SC}},1-\left(1-m^{+}_{\Gamma_{IP-1}}\right)^{\Phi_{SC}}\right]},\\ \left[n^{-\Phi_{SC}}_{\Gamma_{RP-1}},n^{+\Phi_{SC}}_{\Gamma_{RP-1}}\right]e^{i2\pi\left[n^{-\Phi_{SC}}_{\Gamma_{IP-1}},n^{+\Phi_{SC}}_{\Gamma_{IP-1}}\right]}\end{pmatrix} \quad (4)$$

$$\mathit{\Gamma}^{\Phi_{SC}}_{CI-1}=\begin{pmatrix}\left[m^{-\Phi_{SC}}_{\Gamma_{RP-1}},m^{+\Phi_{SC}}_{\Gamma_{RP-1}}\right]e^{i2\pi\left[m^{-\Phi_{SC}}_{\Gamma_{IP-1}},m^{+\Phi_{SC}}_{\Gamma_{IP-1}}\right]},\\ \left[1-\left(1-n^{-}_{\Gamma_{RP-1}}\right)^{\Phi_{SC}},1-\left(1-n^{+}_{\Gamma_{RP-1}}\right)^{\Phi_{SC}}\right]e^{i2\pi\left[1-\left(1-n^{-}_{\Gamma_{IP-1}}\right)^{\Phi_{SC}},1-\left(1-n^{+}_{\Gamma_{IP-1}}\right)^{\Phi_{SC}}\right]}\end{pmatrix}. \quad (5)$$

Definition 2. [27] For the given mathematical form of any two CI-VIFNs:

$$\mathit{\Gamma}_{CI-i}=\left(\left[m^{-}_{\Gamma_{RP-i}},m^{+}_{\Gamma_{RP-i}}\right]e^{i2\pi\left(\left[m^{-}_{\Gamma_{IP-i}},m^{+}_{\Gamma_{IP-i}}\right]\right)},\left[n^{-}_{\Gamma_{RP-i}},n^{+}_{\Gamma_{RP-i}}\right]e^{i2\pi\left(\left[n^{-}_{\Gamma_{IP-i}},n^{+}_{\Gamma_{IP-i}}\right]\right)}\right), i=1,2,$$

the SV and AV are diagnosed as

$$\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-1})=\frac{1}{4}\left(m^{-}_{\Gamma_{RP-1}}-n^{-}_{\Gamma_{RP-1}}+m^{-}_{\Gamma_{IP-1}}-n^{-}_{\Gamma_{IP-1}}+m^{+}_{\Gamma_{RP-1}}-n^{+}_{\Gamma_{RP-1}}+m^{+}_{\Gamma_{IP-1}}-n^{+}_{\Gamma_{IP-1}}\right), \quad (6)$$

$$\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-1})=\frac{1}{4}\left(m^{-}_{\Gamma_{RP-1}}+n^{-}_{\Gamma_{RP-1}}+m^{-}_{\Gamma_{IP-1}}+n^{-}_{\Gamma_{IP-1}}+m^{+}_{\Gamma_{RP-1}}+n^{+}_{\Gamma_{RP-1}}+m^{+}_{\Gamma_{IP-1}}+n^{+}_{\Gamma_{IP-1}}\right). \quad (7)$$

It is clear that $\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-1})\in[-1,1]$, and $\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-1})\in[0,1]$. Some relations for Eqs (6) and (7) are diagnosed here:

1) $\mathit{\Gamma}_{CI-1}>\mathit{\Gamma}_{CI-2}$, if $\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-1})>\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-2})$ or $\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-1})>\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-2})$;
2) $\mathit{\Gamma}_{CI-1}<\mathit{\Gamma}_{CI-2}$, if $\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-1})<\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-2})$ or $\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-1})<\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-2})$;
3) $\mathit{\Gamma}_{CI-1}=\mathit{\Gamma}_{CI-2}$, if $\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-1})=\bar{\bar{\zeta}}(\mathit{\Gamma}_{CI-2})$ or $\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-1})=\bar{\bar{\mathfrak{F}}}(\mathit{\Gamma}_{CI-2})$.

Definition 3. [38] The mathematical expression

$$\eta=\left\{\eta_0,\eta_1,\eta_2,\ldots,\eta_{\overline{\overline{k_{SC}}}-1}\right\}, \quad (8)$$

is called linguistic term set (LTS) with an odd $\overline{\overline{k_{SC}}}$ in the availability of the below points:

1) If $\overline{\overline{k_{SC}}} > \overline{\overline{k_{SC}}}'$, then $\eta_{\overline{\overline{k_{SC}}}} > \eta_{\overline{\overline{k_{SC}}}'}$;
2) $neg(\eta_{\overline{\overline{k_{SC}}}}) = \eta_{\overline{\overline{k_{SC}}}'}$ with $\overline{\overline{k_{SC}}} + \overline{\overline{k_{SC}}}' = \overline{\overline{k_{SC}}} + 1$;
3) If $\overline{\overline{k_{SC}}} \geq \overline{\overline{k_{SC}}}'$, $\max\left(\eta_{\overline{\overline{k_{SC}}}}, \eta_{\overline{\overline{k_{SC}}}'}\right) = \eta_{\overline{\overline{k_{SC}}}}$, and if $\overline{\overline{k_{SC}}} \leq \overline{\overline{k_{SC}}}'$, $\max\left(\eta_{\overline{\overline{k_{SC}}}}, \eta_{\overline{\overline{k_{SC}}}'}\right) = \eta_{\overline{\overline{k_{SC}}}}$.

Furthermore, $\eta = \{\eta_i : i \in R\}$, stated linguistic variables (LVs). A mathematical form $\eta = [\eta_{\mu_i}, \eta_{\zeta_s}], \eta_{\mu_i}, \eta_{\zeta_s} \in \eta(i \leq s)$, with $\eta_{\mu_i}, \eta_{\zeta_s}$, is called a uncertain linguistic variable (ULV) [40]. For the given mathematical form of any two ULVs $\eta_1 = [\eta_{\mu_1}, \eta_{\zeta_1}]$ and $\eta_2 = [\eta_{\mu_2}, \eta_{\zeta_2}]$ contained in $\eta_{[0,h]}$, we have

$$\eta_1 \oplus \eta_2 = [\eta_{\mu_1}, \eta_{\zeta_1}] \oplus [\eta_{\mu_2}, \eta_{\zeta_2}] = \left[\eta_{\mu_1+\mu_2-\frac{\mu_1\mu_2}{h}}, \eta_{\zeta_1+\zeta_2-\frac{\zeta_1\zeta_2}{h}}\right], \tag{9}$$

$$\eta_1 \otimes \eta_2 = [\eta_{\mu_1}, \eta_{\zeta_1}] \otimes [\eta_{\mu_2}, \eta_{\zeta_2}] = \left[\eta_{\frac{\mu_1\times\mu_2}{h}}, \eta_{\frac{\zeta_1\times\zeta_2}{h}}\right], \tag{10}$$

$$\Phi_{SC}\eta_1 = \Phi_{SC}[\eta_{\mu_1}, \eta_{\zeta_1}] = \left[\eta_{h\left(1-\left(1-\frac{\mu_1}{h}\right)^{\Phi_{SC}}\right)}, \eta_{h\left(1-\left(1-\frac{\zeta_1}{h}\right)^{\Phi_{SC}}\right)}\right], \tag{11}$$

$$\eta_1^{\Phi_{SC}} = \left[\eta_{h\left(\frac{\mu_1}{h}\right)^{\Phi_{SC}}}, \eta_{h\left(\frac{\zeta_1}{h}\right)^{\Phi_{SC}}}\right]. \tag{12}$$

Definition 4. [41] The mathematical expression

$$HM^{\mathscr{r}_{SC},\mathscr{s}_{SC}}(\Gamma_{CI-1}, \Gamma_{CI-2}, \dots, \Gamma_{CI-\hat{\Sigma}}) = \left(\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)} \sum_{i=1}^{\hat{\Sigma}} \sum_{\mathfrak{s}=1}^{\hat{\Sigma}} \Gamma_{CI-i}^{\mathscr{r}_{SC}} \Gamma_{CI-\mathfrak{s}}^{\mathscr{s}_{SC}}\right)^{\frac{1}{\mathscr{r}_{SC}+\mathscr{s}_{SC}}}, \tag{13}$$

is called an HM operator, and it has the mathematical form: $HM^{\mathscr{r}_{SC},\mathscr{s}_{SC}}: \Theta^{\hat{\Sigma}} \to \Theta$, by

$$HM(\Gamma_{CI-1}, \Gamma_{CI-2}, \dots, \Gamma_{CI-\hat{\Sigma}}) = \left(\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)} \sum_{i=1}^{\hat{\Sigma}} \sum_{\mathfrak{s}=1}^{\hat{\Sigma}} \Gamma_{CI-i} \Gamma_{CI-\mathfrak{s}}\right), \tag{14}$$

is the HM operator.

3. Complex interval-valued intuitionistic uncertain linguistic variables

This analysis diagnoses a well-known theory of CI-VIUL settings as a powerful and capable tool to handle an ambiguous sort of theories. Furthermore, to enhance the features of the CI-VIUL information, we diagnose the algebraic laws, SV and AV for CI-VIUL settings.

Definition 5. The mathematical expression

$$\Gamma_{CIU} = \left\{\left([\eta_{\mu_i}, \eta_{\zeta_{\mathfrak{s}}}], \left(\mathscr{m}_{\Gamma_{CIU}}(\check{\mathfrak{f}}), \mathscr{n}_{\Gamma_{CIU}}(\check{\mathfrak{f}})\right)\right) : \check{\mathfrak{f}} \in \widetilde{\mathcal{X}_{UNI}}\right\}. \tag{15}$$

is called a CI-VIUL set, where

$$m_{\Gamma_{CIU}}(\breve{\mathfrak{t}}) = [m^{-}_{\Gamma_{RP}}(\breve{\mathfrak{t}}), m^{+}_{\Gamma_{RP}}(\breve{\mathfrak{t}})]e^{i2\pi([m^{-}_{\Gamma_{IP}}(\breve{\mathfrak{t}}), m^{+}_{\Gamma_{IP}}(\breve{\mathfrak{t}})])}$$

and

$$n_{\Gamma_{CIU}}(\breve{\mathfrak{t}}) = [n^{-}_{\Gamma_{RP}}(\breve{\mathfrak{t}}), n^{+}_{\Gamma_{RP}}(\breve{\mathfrak{t}})]e^{i2\pi([n^{-}_{\Gamma_{IP}}(\breve{\mathfrak{t}}), n^{+}_{\Gamma_{IP}}(\breve{\mathfrak{t}})])}.$$

The major tools of CI-VIUL settings are

$$0 \le m^{+}_{\Gamma_{RP}}(\breve{\mathfrak{t}}) + n^{+}_{\Gamma_{RP}}(\breve{\mathfrak{t}}) \le 1$$

and

$$0 \le m^{+}_{\Gamma_{IP}}(\breve{\mathfrak{t}}) + n^{+}_{\Gamma_{IP}}(\breve{\mathfrak{t}}) \le 1$$

with

$$\eta_{\mu_i}, \eta_{\zeta_{\mathfrak{s}}} \in \hat{\eta}(i \le \mathfrak{s}).$$

The mathematical form,

$$\mathcal{L}_{\Gamma_{CIU}}(\breve{\mathfrak{t}}) = [\mathcal{L}^{-}_{\Gamma_{RP}}(\breve{\mathfrak{t}}), \mathcal{L}^{+}_{\Gamma_{RP}}(\breve{\mathfrak{t}})]e^{i2\pi([\mathcal{L}^{-}_{\Gamma_{IP}}(\breve{\mathfrak{t}}), \mathcal{L}^{+}_{\Gamma_{IP}}(\breve{\mathfrak{t}})])} = \left[\left(1 - m^{-}_{\Gamma_{RP}}(\breve{\mathfrak{t}}) - n^{-}_{\Gamma_{RP}}(\breve{\mathfrak{t}})\right), \left(1 - m^{+}_{\Gamma_{RP}}(\breve{\mathfrak{t}}) - n^{+}_{\Gamma_{RP}}(\breve{\mathfrak{t}})\right)\right] e^{i2\pi\left[\left(1 - m^{-}_{\Gamma_{IP}}(\breve{\mathfrak{t}}) - n^{-}_{\Gamma_{IP}}(\breve{\mathfrak{t}})\right), \left(1 - m^{+}_{\Gamma_{IP}}(\breve{\mathfrak{t}}) - n^{+}_{\Gamma_{IP}}(\breve{\mathfrak{t}})\right)\right]},$$

diagnoses the refusal grade and CI-VIULNs stated by

$$\Gamma_{CIU-i} = \left([\eta_{\mu_i}, \eta_{\zeta_{\mathfrak{s}}}], \left([m^{-}_{\Gamma_{RP-i}}, m^{+}_{\Gamma_{RP-i}}]e^{i2\pi([m^{-}_{\Gamma_{IP-i}}, m^{+}_{\Gamma_{IP-i}}])}, [n^{-}_{\Gamma_{RP-i}}, n^{+}_{\Gamma_{RP-i}}]e^{i2\pi([n^{-}_{\Gamma_{IP-i}}, n^{+}_{\Gamma_{IP-i}}])}\right)\right), i, \mathfrak{s} = 1,2, \ldots, \widetilde{\Sigma}.$$

For the given mathematical form of any two CI-VIULNs

$$\Gamma_{CIU-i} = \left([\eta_{\mu_i}, \eta_{\zeta_{\mathfrak{s}}}], \left([m^{-}_{\Gamma_{RP-i}}, m^{+}_{\Gamma_{RP-i}}]e^{i2\pi([m^{-}_{\Gamma_{IP-i}}, m^{+}_{\Gamma_{IP-i}}])}, [n^{-}_{\Gamma_{RP-i}}, n^{+}_{\Gamma_{RP-i}}]e^{i2\pi([n^{-}_{\Gamma_{IP-i}}, n^{+}_{\Gamma_{IP-i}}])}\right)\right), i = 1,2.$$

We have,

$$\Gamma_{CIU-1} \oplus \Gamma_{CIU-2} = \left(\begin{array}{c} \left[\eta_{\mu_1+\mu_2-\frac{\mu_1\mu_2}{h}}, \eta_{\zeta_1+\zeta_2-\frac{\zeta_1\zeta_2}{h}}\right], \\ \left(\begin{array}{c} \left[m^{-}_{\Gamma_{RP-1}} + m^{-}_{\Gamma_{RP-2}} - m^{-}_{\Gamma_{RP-1}}m^{-}_{\Gamma_{RP-2}}, m^{+}_{\Gamma_{RP-1}} + m^{+}_{\Gamma_{RP-2}} - m^{+}_{\Gamma_{RP-1}}m^{+}_{\Gamma_{RP-2}}\right] \\ e^{i2\pi\left[m^{-}_{\Gamma_{IP-1}} + m^{-}_{\Gamma_{IP-2}} - m^{-}_{\Gamma_{IP-1}}m^{-}_{\Gamma_{IP-2}}, m^{+}_{\Gamma_{IP-1}} + m^{+}_{\Gamma_{IP-2}} - m^{+}_{\Gamma_{IP-1}}m^{+}_{\Gamma_{IP-2}}\right]}, \\ \left[n^{-}_{\Gamma_{RP-1}}n^{-}_{\Gamma_{RP-2}}, n^{+}_{\Gamma_{RP-1}}n^{+}_{\Gamma_{RP-2}}\right]e^{i2\pi\left[n^{-}_{\Gamma_{IP-1}}n^{-}_{\Gamma_{IP-2}}, n^{+}_{\Gamma_{IP-1}}n^{+}_{\Gamma_{IP-2}}\right]} \end{array}\right) \end{array}\right), \tag{16}$$

$$\Gamma_{CIU-1} \otimes \Gamma_{CIU-2} =$$

$$\left(\begin{array}{c}\left[\eta_{\frac{\mu_1\times\mu_2}{h}}, \eta_{\frac{\zeta_1\times\zeta_2}{h}}\right],\\ \left(\begin{array}{c}\left[m^-_{\Gamma_{RP-1}} m^-_{\Gamma_{RP-2}}, m^+_{\Gamma_{RP-1}} m^+_{\Gamma_{RP-2}}\right] e^{i2\pi\left[m^-_{\Gamma_{IP-1}} m^-_{\Gamma_{IP-2}}, m^+_{\Gamma_{IP-1}} m^+_{\Gamma_{IP-2}}\right]},\\ \left[n^-_{\Gamma_{RP-1}} + n^-_{\Gamma_{RP-2}} - n^-_{\Gamma_{RP-1}} n^-_{\Gamma_{RP-2}}, n^+_{\Gamma_{RP-1}} + n^+_{\Gamma_{RP-2}} - n^+_{\Gamma_{RP-1}} n^+_{\Gamma_{RP-2}}\right]\\ e^{i2\pi\left[n^-_{\Gamma_{IP-1}} + n^-_{\Gamma_{IP-2}} - n^-_{\Gamma_{IP-1}} n^-_{\Gamma_{IP-2}}, n^+_{\Gamma_{IP-1}} + n^+_{\Gamma_{IP-2}} - n^+_{\Gamma_{IP-1}} n^+_{\Gamma_{IP-2}}\right]}\end{array}\right)\end{array}\right), \quad (17)$$

$$\Phi_{SC}\Gamma_{CIU-1} =$$

$$\left(\begin{array}{c}\left[\eta_{h\left(1-\left(1-\frac{\mu_1}{h}\right)^{\Phi_{SC}}\right)}, \eta_{h\left(1-\left(1-\frac{\zeta_1}{h}\right)^{\Phi_{SC}}\right)}\right],\\ \left(\begin{array}{c}\left[1-\left(1-m^-_{\Gamma_{RP-1}}\right)^{\Phi_{SC}}, 1-\left(1-m^+_{\Gamma_{RP-1}}\right)^{\Phi_{SC}}\right] e^{i2\pi\left[1-\left(1-m^-_{\Gamma_{IP-1}}\right)^{\Phi_{SC}}, 1-\left(1-m^+_{\Gamma_{IP-1}}\right)^{\Phi_{SC}}\right]},\\ \left[n^{-\Phi_{SC}}_{\Gamma_{RP-1}}, n^{+\Phi_{SC}}_{\Gamma_{RP-1}}\right] e^{i2\pi\left[n^{-\Phi_{SC}}_{\Gamma_{IP-1}}, n^{+\Phi_{SC}}_{\Gamma_{IP-1}}\right]}\end{array}\right)\end{array}\right), \quad (18)$$

$$\Gamma_{CIU-1}^{\Phi_{SC}} =$$

$$\left(\begin{array}{c}\left[\eta_{h\left(\frac{\mu_1}{h}\right)^{\Phi_{SC}}}, \eta_{h\left(\frac{\zeta_1}{h}\right)^{\Phi_{SC}}}\right],\\ \left(\begin{array}{c}\left[m^{-\Phi_{SC}}_{\Gamma_{RP-1}}, m^{+\Phi_{SC}}_{\Gamma_{RP-1}}\right] e^{i2\pi\left[m^{-\Phi_{SC}}_{\Gamma_{IP-1}}, m^{+\Phi_{SC}}_{\Gamma_{IP-1}}\right]},\\ \left[1-\left(1-n^-_{\Gamma_{RP-1}}\right)^{\Phi_{SC}}, 1-\left(1-n^+_{\Gamma_{RP-1}}\right)^{\Phi_{SC}}\right] e^{i2\pi\left[1-\left(1-n^-_{\Gamma_{IP-1}}\right)^{\Phi_{SC}}, 1-\left(1-n^+_{\Gamma_{IP-1}}\right)^{\Phi_{SC}}\right]}\end{array}\right)\end{array}\right). \quad (19)$$

Definition 6. For the given mathematical form of any two CI-VIULNs

$$\Gamma_{CIU-1} = \left(\left[\eta_{\mu_1}, \eta_{\zeta_1}\right], \left(\left[m^-_{\Gamma_{RP-i}}, m^+_{\Gamma_{RP-i}}\right] e^{i2\pi\left(\left[m^-_{\Gamma_{IP-i}}, m^+_{\Gamma_{IP-i}}\right]\right)}, \left[n^-_{\Gamma_{RP-i}}, n^+_{\Gamma_{RP-i}}\right] e^{i2\pi\left(\left[n^-_{\Gamma_{IP-i}}, n^+_{\Gamma_{IP-i}}\right]\right)}\right)\right),$$

the SV and AV are diagnosed as

$$\bar{\bar{\zeta}}(\Gamma_{CIU-1}) = \frac{1}{\max(\mu_1,\zeta_1)}(\mu_1+\zeta_1) \times \frac{1}{4}\left(\begin{array}{c} m^-_{\Gamma_{RP-1}} - n^-_{\Gamma_{RP-1}} + m^-_{\Gamma_{IP-1}} - n^-_{\Gamma_{IP-1}}\\ +m^+_{\Gamma_{RP-1}} - n^+_{\Gamma_{RP-1}} + m^+_{\Gamma_{IP-1}} - n^+_{\Gamma_{IP-1}}\end{array}\right), \quad (20)$$

$$\bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-1}) = \frac{1}{\max(\mu_1,\zeta_1)}(\mu_1+\zeta_1) \times \frac{1}{4}\left(\begin{array}{c} m^-_{\Gamma_{RP-1}} + n^-_{\Gamma_{RP-1}} + m^-_{\Gamma_{IP-1}} + n^-_{\Gamma_{IP-1}}\\ +m^+_{\Gamma_{RP-1}} + n^+_{\Gamma_{RP-1}} + m^+_{\Gamma_{IP-1}} + n^+_{\Gamma_{IP-1}}\end{array}\right). \quad (21)$$

It is clear that $\bar{\bar{\zeta}}(\Gamma_{CIU-1}) \in [-1,1]$ and $\bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-1}) \in [0,1]$. Some relations for Eqs (20) and (21) are diagnosed as

1) $\Gamma_{CIU-1} > \Gamma_{CIU-2}$, if $\bar{\bar{\zeta}}(\Gamma_{CIU-1}) > \bar{\bar{\zeta}}(\Gamma_{CIU-2})$ or $\bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-1}) > \bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-2})$;
2) $\Gamma_{CIU-1} < \Gamma_{CIU-2}$, if $\bar{\bar{\zeta}}(\Gamma_{CIU-1}) < \bar{\bar{\zeta}}(\Gamma_{CIU-2})$ or $\bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-1}) < \bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-2})$;
3) $\Gamma_{CIU-1} = \Gamma_{CIU-2}$, if $\bar{\bar{\zeta}}(\Gamma_{CIU-1}) = \bar{\bar{\zeta}}(\Gamma_{CIU-2})$ or $\bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-1}) = \bar{\bar{\mathfrak{F}}}(\Gamma_{CIU-2})$.

4. Arithmetic/geometric Heronian mean operators for CI-VIUL settings

The HM operator is a massive dominant operator that can suggest information on interrelationships. However, in the past, it was applied to the theory and for the purposes of discrimination and resulted in many exploratory inventions. With the availability of a superior HM operator, we develop the CI-VIULAHM operator, CI-VIULWAHM operator, CI-VIULGHM operator, CI-VIULWGHM operator and their well-known achievements in the form of some results, important properties, and specific cases.

Definition 7. The CI-VIULAHM operator is simplified and analyzed by

$$CI-VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}: \Theta^{\widehat{\Sigma}} \rightarrow \Theta, \text{by}$$

$$CI-VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=\left(\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma_{CIU-i}^{\mathcal{r}sc}\Gamma_{CIU-\mathfrak{s}}^{\mathcal{s}sc}\right)^{\frac{1}{\mathcal{r}sc+\mathcal{s}sc}}. \quad (22)$$

Using Eq (22), we diagnosed the result.

Theorem 1. Considering Definition 5 and Eq (22), we diagnose

$$CI-VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$

$$\left(\begin{array}{c}\left[\eta_{\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\mu_i^{r_{SC}}\mu_{\mathfrak{s}}^{s_{SC}}}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}}, \eta_{\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\zeta_i^{r_{SC}}\zeta_{\mathfrak{s}}^{s_{SC}}}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}}\right], \\ \left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-r_{SC}}_{\Gamma_{RP-i}}m^{-s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}, \\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+r_{SC}}_{\Gamma_{RP-i}}m^{+s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right] e^{i2\pi\left[\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-r_{SC}}_{\Gamma_{IP-i}}m^{-s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}, \\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+r_{SC}}_{\Gamma_{IP-i}}m^{+s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right]}, \\ \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right), \\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right] e^{i2\pi\left[\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right), \\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right]}\end{array}\right)\end{array}\right). \quad (23)$$

Proof. By using Definition 5, we achieve

$$\Gamma^{r_{SC}}_{CIU-i} = \left(\begin{array}{c}\left[\eta_{h\left(\frac{\mu_i}{h}\right)^{r_{SC}}}, \eta_{h\left(\frac{\zeta_i}{h}\right)^{r_{SC}}}\right], \\ \left(\begin{array}{c}\left[m^{-r_{SC}}_{\Gamma_{RP-i}}, m^{+r_{SC}}_{\Gamma_{RP-i}}\right]e^{i2\pi\left[m^{-r_{SC}}_{\Gamma_{IP-i}}, m^{+r_{SC}}_{\Gamma_{IP-i}}\right]}, \\ \left[1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^{r_{SC}}, 1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^{r_{SC}}\right]e^{i2\pi\left[1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^{r_{SC}}, 1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^{r_{SC}}\right]}\end{array}\right)\end{array}\right),$$

$$\Gamma^{r_{SC}}_{CIU-\mathfrak{s}} = \left(\begin{array}{c}\left[\eta_{h\left(\frac{\mu_{\mathfrak{s}}}{h}\right)^{r_{SC}}}, \eta_{h\left(\frac{\zeta_{\mathfrak{s}}}{h}\right)^{r_{SC}}}\right], \\ \left(\begin{array}{c}\left[m^{-r_{SC}}_{\Gamma_{RP-\mathfrak{s}}}, m^{+r_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right]e^{i2\pi\left[m^{-r_{SC}}_{\Gamma_{IP-\mathfrak{s}}}, m^{+r_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right]}, \\ \left[1-\left(1-n^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)^{r_{SC}}, 1-\left(1-n^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)^{r_{SC}}\right]e^{i2\pi\left[1-\left(1-n^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)^{r_{SC}}, 1-\left(1-n^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)^{r_{SC}}\right]}\end{array}\right)\end{array}\right).$$

Then,

$$\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma_{CIU-i}^{r_{SC}}\Gamma_{CIU-\mathfrak{s}}^{s_{SC}}=$$

$$\left(\begin{array}{c}\left[\eta^{\left(1-\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\mu_i^{r_{SC}}\mu_{\mathfrak{s}}^{s_{SC}}}{h}\right)\right)},\eta^{\left(1-\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\zeta_i^{r_{SC}}\zeta_{\mathfrak{s}}^{s_{SC}}}{h}\right)\right)}\right],\\ \left(\begin{array}{c}\left[1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-r_{SC}}_{\Gamma_{RP-i}}m^{-s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right),1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+r_{SC}}_{\Gamma_{RP-i}}m^{+s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)\right]\\ e^{i2\pi\left[1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-r_{SC}}_{\Gamma_{IP-i}}m^{-s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right),1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+r_{SC}}_{\Gamma_{IP-i}}m^{+s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)\right]},\\ \left[\begin{array}{c}\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^-_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^-_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right),\\ \prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^+_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^+_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\end{array}\right]\\ e^{i2\pi\left[\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^-_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^-_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right),\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^+_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^+_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right]}\end{array}\right)\end{array}\right),$$

Thus,

$$\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma_{CIU-i}^{r_{SC}}\Gamma_{CIU-\mathfrak{s}}^{s_{SC}}=$$

$$\left(\begin{array}{c}\left[\eta^{1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\mu_i^{r_{SC}}\mu_{\mathfrak{s}}^{s_{SC}}}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}},\eta^{1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\zeta_i^{r_{SC}}\zeta_{\mathfrak{s}}^{s_{SC}}}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}}\right],\\ \left(\begin{array}{c}\left[\begin{array}{c}1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-r_{SC}}_{\Gamma_{RP-i}}m^{-s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}},\\ 1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+r_{SC}}_{\Gamma_{RP-i}}m^{+s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\end{array}\right]\\ e^{i2\pi\left[1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-r_{SC}}_{\Gamma_{IP-i}}m^{-s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}},1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+r_{SC}}_{\Gamma_{IP-i}}m^{+s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right]},\\ \left[\begin{array}{c}\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^-_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^-_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}},\\ \left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^+_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^+_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\end{array}\right]\\ e^{i2\pi\left[\begin{array}{c}\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^-_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^-_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}},\\ \left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^+_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^+_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\end{array}\right]}\end{array}\right)\end{array}\right),$$

$$CI-VIULAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right)=\left(\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma_{CIU-i}^{r_{SC}}\Gamma_{CIU-\mathfrak{s}}^{s_{SC}}\right)^{\frac{1}{r_{SC}+s_{SC}}}=$$

$$\left(\begin{array}{c}\left[\eta^{\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-\frac{\mu_i^{r_{SC}}\mu_\varsigma^{s_{SC}}}{h}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}},\eta^{\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-\frac{\zeta_i^{r_{SC}}\zeta_\varsigma^{s_{SC}}}{h}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}}\right],\\ \left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-m^{-\,r_{SC}}_{\Gamma_{RP-i}}m^{-\,s_{SC}}_{\Gamma_{RP-\varsigma}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}},\\ \left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-m^{+\,r_{SC}}_{\Gamma_{RP-i}}m^{+\,s_{SC}}_{\Gamma_{RP-\varsigma}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right]\\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-m^{-\,r_{SC}}_{\Gamma_{IP-i}}m^{-\,s_{SC}}_{\Gamma_{IP-\varsigma}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}},\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-m^{+\,r_{SC}}_{\Gamma_{IP-i}}m^{+\,s_{SC}}_{\Gamma_{IP-\varsigma}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right]},\\ \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^{-}_{\Gamma_{RP-\varsigma}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^{+}_{\Gamma_{RP-\varsigma}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right]\\ e^{i2\pi\left[\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^{-}_{\Gamma_{IP-\varsigma}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\varsigma=1}^{\hat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^{+}_{\Gamma_{IP-\varsigma}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right]}\end{array}\right)\end{array}\right).$$

Well-known and major properties, called idempotency, monotonicity and boundedness, for CI-VIUL settings are investigated.

Property 1. Using Eq (23), we discuss some properties such as those following.

1) If $\Gamma_{CIU-i}=\Gamma_{CIU}$, then

$$CI-VIULAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)=\Gamma_{CIU}. \tag{24}$$

2) If $\eta'_{\mu_i}\le\eta_{\mu_i}$, $\eta'_{\zeta_\varsigma}\le\eta_{\zeta_\varsigma}$, ${m^{-}}'_{\Gamma_{RP-i}}\le m^{-}_{\Gamma_{RP-i}}$, ${m^{-}}'_{\Gamma_{IP-i}}\le m^{-}_{\Gamma_{IP-i}}$, ${n^{-}}'_{\Gamma_{RP-i}}\ge n^{-}_{\Gamma_{RP-i}}$, ${n^{-}}'_{\Gamma_{IP-i}}\ge n^{-}_{\Gamma_{IP-i}}$, and ${m^{+}}'_{\Gamma_{RP-i}}\le m^{+}_{\Gamma_{RP-i}}$, ${m^{+}}'_{\Gamma_{IP-i}}\le m^{+}_{\Gamma_{IP-i}}$, ${n^{+}}'_{\Gamma_{RP-i}}\ge n^{+}_{\Gamma_{RP-i}}$, ${n^{+}}'_{\Gamma_{IP-i}}\ge n^{+}_{\Gamma_{IP-i}}$, then

$$\begin{aligned}&CI-VIULAHM^{r_{SC},s_{SC}}\left(\Gamma'_{CIU-1},\Gamma'_{CIU-2},\dots,\Gamma'_{CIU-\hat{\Sigma}}\right)\\ &\le CI-VIULAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right).\end{aligned} \tag{25}$$

3) If $\Gamma_{CIU-A}=min\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)$, and $\Gamma_{CIU-B}=max\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)$, then

$$\Gamma_{CIU-A}\le CI-VIULAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)\le\Gamma_{CIU-B}. \tag{26}$$

Proof. 1) If $\Gamma_{CIU-i} = \Gamma_{CIU}, i = 1,2,\ldots,\widehat{\Sigma}$, then

$$CI - VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right) = \left(\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma_{CIU-i}^{\mathcal{r}sc}\Gamma_{CIU-\mathfrak{s}}^{\mathcal{s}sc}\right)^{\frac{1}{\mathcal{r}sc+\mathcal{s}sc}}$$

$$= \left(\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma_{CIU}^{\mathcal{r}sc}\Gamma_{CIU}^{\mathcal{s}sc}\right)^{\frac{1}{\mathcal{r}sc+\mathcal{s}sc}} = \left(\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma_{CIU}^{\mathcal{r}sc+\mathcal{s}sc}\right)^{\frac{1}{\mathcal{r}sc+\mathcal{s}sc}}$$

$$= \left(\Gamma_{CIU}^{\mathcal{r}sc+\mathcal{s}sc}\right)^{\frac{1}{\mathcal{r}sc+\mathcal{s}sc}} = \Gamma_{CIU}.$$

2) If $\eta'_{\mu_i} \le \eta_{\mu_i}$, $\eta'_{\zeta_\mathfrak{s}} \le \eta_{\zeta_\mathfrak{s}}$, $m^{-\prime}_{\Gamma_{RP-i}} \le m^{-}_{\Gamma_{RP-i}}$, $m^{-\prime}_{\Gamma_{IP-i}} \le m^{-}_{\Gamma_{IP-i}}$, $n^{-\prime}_{\Gamma_{RP-i}} \ge n^{-}_{\Gamma_{RP-i}}$, $n^{-\prime}_{\Gamma_{IP-i}} \ge n^{-}_{\Gamma_{IP-i}}$, and $m^{+\prime}_{\Gamma_{RP-i}} \le m^{+}_{\Gamma_{RP-i}}$, $m^{+\prime}_{\Gamma_{IP-i}} \le m^{+}_{\Gamma_{IP-i}}$, $n^{+\prime}_{\Gamma_{RP-i}} \ge n^{+}_{\Gamma_{RP-i}}$, $n^{+\prime}_{\Gamma_{IP-i}} \ge n^{+}_{\Gamma_{IP-i}}$, then

$$\Gamma^{+\prime\,\mathcal{r}sc}_{CIU-i}\Gamma^{+\prime\,\mathcal{s}sc}_{CIU-\mathfrak{s}} \le \Gamma^{+\,\mathcal{r}sc}_{CIU-i}\Gamma^{+\,\mathcal{s}sc}_{CIU-\mathfrak{s}}$$

$$\Rightarrow \frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma^{+\prime\,\mathcal{r}sc}_{CIU-i}\Gamma^{+\prime\,\mathcal{s}sc}_{CIU-\mathfrak{s}} \le \frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma^{+\,\mathcal{r}sc}_{CIU-i}\Gamma^{+\,\mathcal{s}sc}_{CIU-\mathfrak{s}}$$

$$\Rightarrow \left(\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma^{+\prime\,\mathcal{r}sc}_{CIU-i}\Gamma^{+\prime\,\mathcal{s}sc}_{CIU-\mathfrak{s}}\right)^{\frac{1}{\mathcal{r}sc+\mathcal{s}sc}} \le \left(\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\Gamma^{+\,\mathcal{r}sc}_{CIU-i}\Gamma^{+\,\mathcal{s}sc}_{CIU-\mathfrak{s}}\right)^{\frac{1}{\mathcal{r}sc+\mathcal{s}sc}}.$$

Thus, we acquire

$$CI - VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma'_{CIU-1},\Gamma'_{CIU-2},\ldots,\Gamma'_{CIU-\widehat{\Sigma}}\right) \le CI - VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right).$$

3) If $\Gamma_{CIU-A} = min\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right)$, and $\Gamma_{CIU-B} = max\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right)$, then, using Property (1), we achieve

$$\Gamma_{CIU-A} \le CI - VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right).$$

Then,

$$CI - VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right) \le \Gamma_{CIU-B};$$

thus,

$$\Gamma_{CIU-A} \le CI - VIULAHM^{\mathcal{r}sc,\mathcal{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right) \le \Gamma_{CIU-B}.$$

Additionally, on the availability of parameters, we diagnose various sorts of specific cases:

1) If $\mathscr{s}_{SC} \to 0$ in Eq (23), then we get the CI-VIUL generalized linear descending weighted mean (CI-VIULGLDWM) operator, and we achieve

$$CI - VIULAHM^{\mathscr{r}_{SC},0}\left(\Gamma_{CIU-1}, \Gamma_{CIU-2}, \dots, \Gamma_{CIU-\hat{\Sigma}}\right) = \lim_{\mathscr{s}_{SC}\to 0}\left(\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}\sum_{i=1}^{\hat{\Sigma}}\sum_{\mathfrak{s}=1}^{\hat{\Sigma}}\Gamma_{CIU-i}^{\mathscr{r}_{SC}}\Gamma_{CIU-\mathfrak{s}}^{\mathscr{s}_{SC}}\right)^{\frac{1}{\mathscr{r}_{SC}+\mathscr{s}_{SC}}} =$$

$$\left(\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}\sum_{i=1}^{\hat{\Sigma}}\Gamma_{CIU-i}^{\mathscr{r}_{SC}}\right)^{\frac{1}{\mathscr{r}_{SC}}} =$$

$$\left(\begin{array}{c} \left[\eta^{\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-\frac{\mu_i^{\mathscr{r}_{SC}}}{h}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}}, \eta^{\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-\frac{\zeta_i^{\mathscr{r}_{SC}}}{h}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}}\right], \\ \left(\begin{array}{c} \left[\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-m^{-\mathscr{r}_{SC}}_{\Gamma_{RP-i}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}, \left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-m^{+\mathscr{r}_{SC}}_{\Gamma_{RP-i}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}\right] \\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-m^{-\mathscr{r}_{SC}}_{\Gamma_{IP-i}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}, \left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-m^{+\mathscr{r}_{SC}}_{\Gamma_{IP-i}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}\right]}, \\ \left[\begin{array}{c} \left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^{\mathscr{r}_{SC}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}\right), \\ \left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^{\mathscr{r}_{SC}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}\right) \end{array}\right] \\ e^{i2\pi\left[\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^{\mathscr{r}_{SC}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}\right), \left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^{\mathscr{r}_{SC}}\right)^{(\hat{\Sigma}+1-i)}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{\mathscr{r}_{SC}}}\right)\right]} \end{array}\right) \end{array}\right).$$

2) If $\mathscr{r}_{SC} \to 0$ in Eq (23), then we get the CI-VIUL generalized linear ascending weighted mean (CI-VIULGLAWM) operator, and we achieve

$$CI - VIULAHM^{0,\mathscr{s}_{SC}}\left(\Gamma_{CIU-1}, \Gamma_{CIU-2}, \dots, \Gamma_{CIU-\hat{\Sigma}}\right) = \lim_{\mathscr{r}_{SC}\to 0}\left(\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}\sum_{i=1}^{\hat{\Sigma}}\sum_{\mathfrak{s}=1}^{\hat{\Sigma}}\Gamma_{CIU-i}^{\mathscr{r}_{SC}}\Gamma_{CIU-\mathfrak{s}}^{\mathscr{s}_{SC}}\right)^{\frac{1}{\mathscr{r}_{SC}+\mathscr{s}_{SC}}} =$$

$$\left(\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}\sum_{i=1}^{\hat{\Sigma}}\Gamma_{CIU-i}^{\mathscr{s}_{SC}}\right)^{\frac{1}{\mathscr{s}_{SC}}} =$$

$$\left(\begin{array}{c}\left[\eta^{\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\frac{\mu_i^{s_{SC}}}{h}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}}, \eta^{\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\frac{\zeta_i^{s_{SC}}}{h}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}}\right], \\ \left(\begin{array}{c}\left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-m^{-s_{SC}}_{\Gamma_{RP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}, \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-m^{+s_{SC}}_{\Gamma_{RP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right] \\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-m^{-s_{SC}}_{\Gamma_{IP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}, \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-m^{+s_{SC}}_{\Gamma_{IP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right]}, \\ \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right), \\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right)\end{array}\right] \\ e^{i2\pi\left[\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right), \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right)\right]}\end{array}\right)\end{array}\right).$$

3) If $r_{SC} = s_{SC} = \frac{1}{2}$ in Eq (23), then we get the CI-VIUL basic HM (CI-VIULBHM) operator, and we achieve

$$CI - VIULAHM^{\frac{1}{2},\frac{1}{2}}\left(\Gamma_{CIU-1}, \Gamma_{CIU-2}, \ldots, \Gamma_{CIU-\widehat{\Sigma}}\right) =$$

$$\left(\begin{array}{c}\left[\eta\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\mu_i^{\frac{1}{2}}\mu_{\mathfrak{s}}^{\frac{1}{2}}}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right),\eta\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\zeta_i^{\frac{1}{2}}\zeta_{\mathfrak{s}}^{\frac{1}{2}}}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)\right],\\ \left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-\frac{1}{2}}_{\Gamma_{RP-i}}m^{-\frac{1}{2}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right),\\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+\frac{1}{2}}_{\Gamma_{RP-i}}m^{+\frac{1}{2}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)\end{array}\right]\\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-\frac{1}{2}}_{\Gamma_{IP-i}}m^{-\frac{1}{2}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right),\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+\frac{1}{2}}_{\Gamma_{IP-i}}m^{+\frac{1}{2}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)\right]},\\ \left[\begin{array}{c}\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^{\frac{1}{2}}\left(1-n^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}},\\ \left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^{\frac{1}{2}}\left(1-n^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\end{array}\right]\\ e^{i2\pi\left[\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^{\frac{1}{2}}\left(1-n^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}},\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^{\frac{1}{2}}\left(1-n^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right]}\end{array}\right)\end{array}\right).$$

4) If $r_{SC}=s_{SC}=1$ in Eq (23), then we get the CI-VIULBHM operator, and we achieve

$$CI-VIULAHM^{1,1}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$

$$\left(\begin{array}{c}\left[\eta^{\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\mu_i^1\mu_{\mathfrak{s}}{}^1}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}},\eta^{\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\frac{\zeta_i^1\zeta_{\mathfrak{s}}{}^1}{h}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}}\right],\\ \left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-1}{}_{\Gamma_{RP-i}}m^{-1}{}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}},\\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+1}{}_{\Gamma_{RP-i}}m^{+1}{}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}\end{array}\right]\\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{-1}{}_{\Gamma_{IP-i}}m^{-1}{}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-m^{+1}{}_{\Gamma_{IP-i}}m^{+1}{}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}\right]},\\ \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^1\left(1-n^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)^1\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^1\left(1-n^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)^1\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}\right)\end{array}\right]\\ e^{i2\pi\left[\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^1\left(1-n^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)^1\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^1\left(1-n^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)^1\right)\right)^{\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}}\right)^{\frac{1}{2}}\right)\end{array}\right]}\end{array}\right)\end{array}\right).$$

Definition 8. The CI-VIULWAHM operator is simplified and diagnosed by

$$CI-VIULWAHM^{r_{SC},s_{SC}}:\Theta^{\widehat{\Sigma}}\to\Theta,\text{by}$$

$$CI-VIULWAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$

$$\left(\frac{2}{\widehat{\Sigma}\left(\widehat{\Sigma}+1\right)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(\widehat{\Sigma}\,\widehat{\Omega}_{W-i}\Gamma_{CIU-i}\right)^{r_{SC}}\left(\widehat{\Sigma}\,\widehat{\Omega}_{W-\mathfrak{s}}\Gamma_{CIU-\mathfrak{s}}\right)^{s_{SC}}\right)^{\frac{1}{r_{SC}+s_{SC}}}. \tag{27}$$

The terms $\widehat{\Omega}_W=\left\{\widehat{\Omega}_{W-1},\widehat{\Omega}_{W-2},\dots,\widehat{\Omega}_{W-\widehat{\Sigma}}\right\}$, shows the weight vector with $\sum_{i=1}^{\widehat{\Sigma}}\widehat{\Omega}_{W-i}=1$, $\widehat{\Omega}_{W-i}\in[0,1]$.

Theorem 2. Using Definition 5 and Eq (27), we achieve

$$CI-VIULWAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$

$$\left(\begin{array}{c}\eta\left[\begin{array}{l}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\left(1-\frac{\mu_i}{h}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-\frac{\mu_{\mathfrak{s}}}{h}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}},\\ \eta\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\left(1-\frac{\zeta_i}{h}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-\frac{\zeta_{\mathfrak{s}}}{h}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right],\\ \left(\begin{array}{c}\left[\begin{array}{l}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\left(1-m^{-}_{\Gamma_{RP-i}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}},\\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\left(1-m^{+}_{\Gamma_{RP-i}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right]\\ e^{i2\pi\left[\begin{array}{l}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\left(1-m^{-}_{\Gamma_{IP-i}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}},\\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\left(1-m^{+}_{\Gamma_{IP-i}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right]},\\ \left[\begin{array}{l}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-\widehat{\Sigma}\widehat{\Omega}_{W-i}}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^{-\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+\widehat{\Sigma}\widehat{\Omega}_{W-i}}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-n^{+\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right]\\ e^{i2\pi\left[\begin{array}{l}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{-\widehat{\Sigma}\widehat{\Omega}_{W-i}}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^{-\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-n^{+\widehat{\Sigma}\widehat{\Omega}_{W-i}}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-n^{+\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right]}\end{array}\right)\end{array}\right). \tag{28}$$

Proof. Omitted.

Theorem 3. Prove that the CI-VIULAHM operator is a certain case of the CI-VIULWAHM operator.
Proof. Assume

$$CI-VIULWAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$
$$\left(\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(\widehat{\Sigma}\,\widehat{\Omega}_{W-i}\Gamma_{CIU-i}\right)^{r_{SC}}\left(\widehat{\Sigma}\,\widehat{\Omega}_{W-\mathfrak{s}}\Gamma_{CIU-\mathfrak{s}}\right)^{s_{SC}}\right)^{\frac{1}{r_{SC}+s_{SC}}}.$$

If $\widehat{\Omega}_W=\left\{\frac{1}{\widehat{\Sigma}},\frac{1}{\widehat{\Sigma}},\dots,\frac{1}{\widehat{\Sigma}}\right\}$, then

$$CI-VIULWAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$

$$\left(\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(\widehat{\Sigma}\,\widehat{\Omega}_{W-i}\Gamma_{CIU-i}\right)^{r_{SC}}\left(\widehat{\Sigma}\,\widehat{\Omega}_{W-\mathfrak{s}}\Gamma_{CIU-\mathfrak{s}}\right)^{s_{SC}}\right)^{\frac{1}{r_{SC}+s_{SC}}} =$$

$$\left(\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(\widehat{\Sigma}\frac{1}{\widehat{\Sigma}}\Gamma_{CIU-i}\right)^{r_{SC}}\left(\widehat{\Sigma}\frac{1}{\widehat{\Sigma}}\Gamma_{CIU-\mathfrak{s}}\right)^{s_{SC}}\right)^{\frac{1}{r_{SC}+s_{SC}}} =$$

$$\left(\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}\sum_{i=1}^{\widehat{\Sigma}}\sum_{\mathfrak{s}=1}^{\widehat{\Sigma}}(\Gamma_{CIU-i})^{r_{SC}}(\Gamma_{CIU-\mathfrak{s}})^{s_{SC}}\right)^{\frac{1}{r_{SC}+s_{SC}}} = CI-$$

$$VIULAHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right).$$

Definition 9. The CI-VIULGHM operator is simplified and diagnosed by

$$CI-VIULGHM^{r_{SC},s_{SC}}:\Theta^{\widehat{\Sigma}}\to\Theta,\text{ by}$$

$$CI-VIULGHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right) = \frac{1}{r_{SC}+s_{SC}}\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}(r_{SC}\Gamma_{CIU-i}+s_{SC}\Gamma_{CIU-\mathfrak{s}})\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}. \quad (29)$$

Theorem 4. Using Definition 5 and Eq (29), we achieve

$$CI-VIULGHM^{r_{SC},s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right) =$$

$$\left(\begin{matrix}\left[\begin{matrix}\eta\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\frac{\mu_i}{h}\right)^{r_{SC}}\left(1-\frac{\mu_{\mathfrak{s}}}{h}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right),\\ \eta\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\frac{\zeta_i}{h}\right)^{r_{SC}}\left(1-\frac{\zeta_{\mathfrak{s}}}{h}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{matrix}\right],\\ \left(\begin{matrix}\left[\begin{matrix}\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-m^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{RP-i}}\right)^{r_{SC}}\left(1-m^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{matrix}\right]\\ e^{i2\pi\left[\begin{matrix}\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-m^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{IP-i}}\right)^{r_{SC}}\left(1-m^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{matrix}\right]},\\ \left[\begin{matrix}\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{-\,r_{SC}}_{\Gamma_{RP-i}}n^{-\,s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}},\\ \left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{+\,r_{SC}}_{\Gamma_{RP-i}}n^{+\,s_{SC}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{matrix}\right]\\ e^{i2\pi\left[\begin{matrix}\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{-\,r_{SC}}_{\Gamma_{IP-i}}n^{-\,s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}},\\ \left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{+\,r_{SC}}_{\Gamma_{IP-i}}n^{+\,s_{SC}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{matrix}\right]}\end{matrix}\right)\end{matrix}\right). \quad (30)$$

Proof. Omitted.

Additionally, in the availability of parameters, we diagnose various sorts of specific cases:

1) If $s_{SC} \to 0$ in Eq (30), then we get the CI-VIUL generalized geometric linear descending weighted mean (CI-VIULGGLDWM) operator

$$CI-VIULGHM^{r_{SC},0}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)=\lim_{s_{SC}\to 0}\left(\frac{1}{r_{SC}+s_{SC}}\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(r_{SC}\Gamma_{CIU-i}+s_{SC}\Gamma_{CIU-\mathfrak{s}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)=\left(\frac{1}{r_{SC}}\left(\prod_{i=1}^{\hat{\Sigma}}r_{SC}\Gamma_{CIU-i}\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)=$$

$$\left(\begin{matrix} \left[\begin{matrix}\eta\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-\frac{\mu_i}{h}\right)^{r_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right),\\ \eta\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-\frac{\zeta_i}{h}\right)^{r_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right)\end{matrix}\right], \\ \left(\begin{matrix} \left[\begin{matrix}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{RP-i}}\right)^{r_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{RP-i}}\right)^{r_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right)\end{matrix}\right] \\ e^{i2\pi\left[\begin{matrix}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{IP-i}}\right)^{r_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{IP-i}}\right)^{r_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right)\end{matrix}\right]}, \\ \left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{-r_{SC}}_{\Gamma_{RP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}},\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{+r_{SC}}_{\Gamma_{RP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right] \\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{-r_{SC}}_{\Gamma_{IP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}},\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{+r_{SC}}_{\Gamma_{IP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}}}\right]} \end{matrix}\right)\end{matrix}\right).$$

2) If $r_{SC} \to 0$ in Eq (30), then we get the CI-VIUL generalized geometric linear ascending weighted mean (CI-VIULGGLAWM) operator

$$CI-VIULGHM^{0,s_{SC}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=\lim_{r_{SC}\to 0}\left(\frac{1}{r_{SC}+s_{SC}}\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}(r_{SC}\Gamma_{CIU-i}+s_{SC}\Gamma_{CIU-\mathfrak{s}})\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)=\left(\frac{1}{s_{SC}}\left(\prod_{i=1}^{\widehat{\Sigma}}s_{SC}\Gamma_{CIU-i}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)=$$

$$\left(\begin{array}{c}\left[\begin{array}{l}\eta^{\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-\frac{\mu_i}{h}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right)},\\ \eta^{\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-\frac{\zeta_i}{h}\right)^{s_{SC}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right)}\end{array}\right],\\ \left(\begin{array}{c}\left[\begin{array}{l}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{RP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{RP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right)\end{array}\right]\\ e^{i2\pi\left[\begin{array}{l}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{IP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right),\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{IP-i}}\right)^{s_{SC}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right)\end{array}\right]},\\ \left[\begin{array}{l}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{-s_{SC}}_{\Gamma_{RP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}},\\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{+s_{SC}}_{\Gamma_{RP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\end{array}\right]\\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{-s_{SC}}_{\Gamma_{IP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}},\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\left(1-n^{+s_{SC}}_{\Gamma_{IP-i}}\right)^{(\widehat{\Sigma}+1-i)}\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)^{\frac{1}{s_{SC}}}\right]}\end{array}\right)\end{array}\right).$$

3) If $r_{SC} = s_{SC} = \frac{1}{2}$ in Eq (30), then we get the CI-VIUL basic geometric HM (CI-VIULBGHM) operator

$$CI-VIULGHM^{\frac{1}{2},\frac{1}{2}}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\ldots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$

$$\left(\begin{array}{c}\left[\begin{array}{l}\eta\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\frac{\mu_i}{h}\right)^{\frac{1}{2}}\left(1-\frac{\mu_{\mathfrak{s}}}{h}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)\right)',\\ \eta\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-\frac{\zeta_i}{h}\right)^{\frac{1}{2}}\left(1-\frac{\zeta_{\mathfrak{s}}}{h}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)\right)\end{array}\right],\\ \left(\begin{array}{c}\left[\begin{array}{l}\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-m^-_{\Gamma_{RP-i}}\right)^{\frac{1}{2}}\left(1-m^-_{\Gamma_{RP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}},\\ \left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-m^+_{\Gamma_{RP-i}}\right)^{\frac{1}{2}}\left(1-m^+_{\Gamma_{RP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\end{array}\right]\\ e^{i2\pi\left[\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-m^-_{\Gamma_{IP-i}}\right)^{\frac{1}{2}}\left(1-m^-_{\Gamma_{IP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}},\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-\left(1-m^+_{\Gamma_{IP-i}}\right)^{\frac{1}{2}}\left(1-m^+_{\Gamma_{IP-\mathfrak{s}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right]},\\ \left[\begin{array}{l}\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-n^{-\frac{1}{2}}_{\Gamma_{RP-i}}n^{-\frac{1}{2}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right),\\ \left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-n^{+\frac{1}{2}}_{\Gamma_{RP-i}}n^{+\frac{1}{2}}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)\end{array}\right]\\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-n^{-\frac{1}{2}}_{\Gamma_{IP-i}}n^{-\frac{1}{2}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right),\left(1-\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left(1-n^{+\frac{1}{2}}_{\Gamma_{IP-i}}n^{+\frac{1}{2}}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}\right)\right]}\end{array}\right)\end{array}\right).$$

4) If $r_{SC} = s_{SC} = 1$ in Eq (30), then we get the CI-VIUL geometric line HM (CI-VIULGLHM) operator

$$CI-VIULGHM^{1,1}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\widehat{\Sigma}}\right)=$$

$$\left(\begin{array}{c}\left[\begin{array}{c}\eta\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\frac{\mu_i}{h}\right)^1\left(1-\frac{\mu_{\mathfrak{s}}}{h}\right)^1\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}}\right),\\ \eta\left(1-\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\frac{\zeta_i}{h}\right)^1\left(1-\frac{\zeta_{\mathfrak{s}}}{h}\right)^1\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}}\right)\end{array}\right],\\ \left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(1-\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{RP-i}}\right)\left(1-m^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}},\\ \left(1-\left(1-\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{RP-i}}\right)\left(1-m^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}}\end{array}\right]\\ e^{i2\pi\left[\begin{array}{c}\left(1-\left(1-\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{-}_{\Gamma_{IP-i}}\right)\left(1-m^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}},\\ \left(1-\left(1-\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-m^{+}_{\Gamma_{IP-i}}\right)\left(1-m^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}}\end{array}\right]},\\ \left[\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{-}_{\Gamma_{RP-i}}n^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{+}_{\Gamma_{RP-i}}n^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}}\right]\\ e^{i2\pi\left[\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{-}_{\Gamma_{IP-i}}n^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-n^{+}_{\Gamma_{IP-i}}n^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{2}}\right]}\end{array}\right)\end{array}\right).$$

Property 2. Using Eq (30), we discuss some properties, such as those following.

1) If $\Gamma_{CIU-i}=\Gamma_{CIU}$, then

$$CI-VIULGHM^{\mathscr{r}sc,\mathscr{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)=\Gamma_{CIU}. \tag{31}$$

2) If $\Gamma'_{CIU-i}\leq\Gamma_{CIU-i}, i=1,2,\dots,\hat{\Sigma}$, where

$$\Gamma'_{CIU-i}=\left(\left[\eta'_{\mu_i},\eta'_{\zeta_{\mathfrak{s}}}\right],\left(m'_{\Gamma_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi\left(m'_{\Gamma_{IP-i}}(\breve{\mathfrak{k}})\right)},n'_{\Gamma_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi\left(n'_{\Gamma_{IP-i}}(\breve{\mathfrak{k}})\right)}\right)\right),i,\mathfrak{s}=1,2,\dots,\hat{\Sigma},$$

then

$$\begin{aligned}&CI-VIULGHM^{\mathscr{r}sc,\mathscr{s}sc}\left(\Gamma'_{CIU-1},\Gamma'_{CIU-2},\dots,\Gamma'_{CIU-\hat{\Sigma}}\right)\\&\leq CI-VIULGHM^{\mathscr{r}sc,\mathscr{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right).\end{aligned} \tag{32}$$

3) If $\Gamma_{CIU-A}=min\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)$, and $\Gamma_{CIU-B}=max\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)$, then

$$\Gamma_{CIU-A}\leq CI-VIULGHM^{\mathscr{r}sc,\mathscr{s}sc}\left(\Gamma_{CIU-1},\Gamma_{CIU-2},\dots,\Gamma_{CIU-\hat{\Sigma}}\right)\leq\Gamma_{CIU-B}. \tag{33}$$

Proof. Omitted.

Definition 10. The CI-VIULWGHM operator is simplified by

$$CI-VIULWGHM^{\mathscr{r}_{SC},\mathscr{s}_{SC}}: \Theta^{\widehat{\Sigma}} \rightarrow \Theta, \text{by}$$

$$CI-VIULWGHM^{\mathscr{r}_{SC},\mathscr{s}_{SC}}\left(\Gamma_{CIU-1}, \Gamma_{CIU-2}, \dots, \Gamma_{CIU-\widehat{\Sigma}}\right) = \frac{1}{\mathscr{r}_{SC}+\mathscr{s}_{SC}}\left(\prod_{i=1}^{\widehat{\Sigma}}\prod_{\mathfrak{s}=1}^{\widehat{\Sigma}}\left((\mathscr{r}_{SC}\Gamma_{CIU-i})^{\widehat{\Sigma}\widehat{\Omega}_{W-i}} + (\mathscr{s}_{SC}\Gamma_{CIU-\mathfrak{s}})^{\widehat{\Sigma}\widehat{\Omega}_{W-\mathfrak{s}}}\right)\right)^{\frac{2}{\widehat{\Sigma}(\widehat{\Sigma}+1)}}. \quad (34)$$

The term $\widehat{\Omega}_W = \{\widehat{\Omega}_{W-1}, \widehat{\Omega}_{W-2}, \dots, \widehat{\Omega}_{W-\widehat{\Sigma}}\}$ shows the weight vector with $\sum_{i=1}^{\widehat{\Sigma}}\widehat{\Omega}_{W-i} = 1$, $\widehat{\Omega}_{W-i} \in [0,1]$.

Theorem 5. Using Definition 5 and Eq (34), we obtain

$$CI-VIULWGHM^{\mathscr{r}_{SC},\mathscr{s}_{SC}}\left(\Gamma_{CIU-1}, \Gamma_{CIU-2}, \dots, \Gamma_{CIU-\widehat{\Sigma}}\right) =$$

$$\left(\begin{array}{l} \left[\begin{array}{l}\eta\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-\left(\frac{\mu_i}{h}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-\left(\frac{\mu_{\mathfrak{s}}}{h}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}, \\ \eta\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-\left(\frac{\zeta_i}{h}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-\left(\frac{\zeta_{\mathfrak{s}}}{h}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right], \\ \left(\left[\begin{array}{l}\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-m^{-\hat{\Sigma}\hat{\Omega}_{W-i}}_{\Gamma_{RP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{-\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}, \\ \left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-m^{+\hat{\Sigma}\hat{\Omega}_{W-i}}_{\Gamma_{RP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{+\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right] \\ e^{i2\pi\left[\begin{array}{l}\left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-m^{-\hat{\Sigma}\hat{\Omega}_{W-i}}_{\Gamma_{IP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{-\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}, \\ \left(1-\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-m^{+\hat{\Sigma}\hat{\Omega}_{W-i}}_{\Gamma_{IP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-m^{+\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\end{array}\right]}, \\ \left[\begin{array}{l}\left(1-\left(\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-n^{-}_{\Gamma_{RP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-n^{-}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right), \\ \left(1-\left(\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-n^{+}_{\Gamma_{RP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-n^{+}_{\Gamma_{RP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right] \\ e^{i2\pi\left[\begin{array}{l}\left(1-\left(\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-n^{-}_{\Gamma_{IP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-n^{-}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right), \\ \left(1-\left(\left(\prod_{i=1}^{\hat{\Sigma}}\prod_{\mathfrak{s}=1}^{\hat{\Sigma}}\left(1-\left(1-\left(1-n^{+}_{\Gamma_{IP-i}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-i}}\right)^{r_{SC}}\left(1-\left(1-n^{+}_{\Gamma_{IP-\mathfrak{s}}}\right)^{\hat{\Sigma}\hat{\Omega}_{W-\mathfrak{s}}}\right)^{s_{SC}}\right)\right)^{\frac{2}{\hat{\Sigma}(\hat{\Sigma}+1)}}\right)^{\frac{1}{r_{SC}+s_{SC}}}\right)\end{array}\right]}\end{array}\right)\right).$$

Proof. Omitted.

5. Decision-making procedure for CI-VIUL settings

Decision-making approaches are the techniques we use with data to get a decision in situations like deciding to cross a canal, choosing a later semester's classes or establishing an extended-term business scheme. Furthermore, human decision-making is frequently learned as a consequence of the sensitive performance of alternative terms on the possible options and the values of the consequences connected to these decisions. This analysis describes the MADM technique using initiated approaches.

5.1. Decision-making process

This study states how we employ the MADM procedure in the field of the CI-VIULWAHM operator or CI-VIULWGHM operator. Therefore, to check the practicality and usefulness of the initiated approaches, a MADM technique is implemented for CI-VIUL settings. The reliability of the proposed MADM tool is demonstrated via a computational example that assesses the impact of the diagnosed approaches on various well-known prevailing theories. For this, various mathematical forms of alternatives are given in the form of $\bar{\Phi}_{Al} = \{\bar{\Phi}_{Al-1}, \bar{\Phi}_{Al-2}, \dots, \bar{\Phi}_{Al-\hat{\Sigma}}\}$, similarly, the mathematical shapes of attributes are given in the shape of $\bar{\mathcal{L}}_{At} = \{\bar{\mathcal{L}}_{At-1}, \bar{\mathcal{L}}_{At-2}, \dots, \bar{\mathcal{L}}_{At-\hat{m}}\}$ with weight vectors $\widehat{\Omega}_W = \{\widehat{\Omega}_{W-1}, \widehat{\Omega}_{W-2}, \dots, \widehat{\Omega}_{W-\hat{n}}\}$, expressing the value of experts with $\sum_{i=1}^{\hat{n}} \widehat{\Omega}_{W-i} = 1$. Furthermore, expressions for different experts are given by $\bar{\mathcal{D}}_{Dm} = \{\bar{\mathcal{D}}_{Dm-1}, \bar{\mathcal{D}}_{Dm-2}, \dots, \bar{\mathcal{D}}_{Dm-\hat{\Sigma}}\}$, and their weight vectors are $\widehat{\Omega'}_W = \{\widehat{\Omega'}_{W-1}, \widehat{\Omega'}_{W-2}, \dots, \widehat{\Omega'}_{W-\hat{n}}\}$, showing the opinions of the experts with $\sum_{i=1}^{\hat{n}} \widehat{\Omega'}_{W-i} = 1$. Under the availability of the above data, we diagnose various matrices that are of the shape $\breve{\mathcal{R}}^i, i = 1,2, \dots, \hat{n}$, where the terms included in the matrix are expressed by $\Gamma_{CIU} = \left(\left[\eta_{\mu_i}, \eta_{\zeta_s}\right], \left(m_{\Gamma_{CIU}}(\breve{f}), n_{\Gamma_{CIU}}(\breve{f})\right)\right)$, where $m_{\Gamma_{CIU}}(\breve{f}) = \left[m^-_{\Gamma_{RP}}(\breve{f}), m^+_{\Gamma_{RP}}(\breve{f})\right] e^{i2\pi\left[m^-_{\Gamma_{IP}}(\breve{f}), m^+_{\Gamma_{IP}}(\breve{f})\right]}$, and $n_{\Gamma_{CIU}}(\breve{f}) = \left[n^-_{\Gamma_{RP}}(\breve{f}), n^+_{\Gamma_{RP}}(\breve{f})\right] e^{i2\pi\left[n^-_{\Gamma_{IP}}(\breve{f}), n^+_{\Gamma_{IP}}(\breve{f})\right]}$, with well-known and valuable rules: $0 \leq m^+_{\Gamma_{RP}}(\breve{f}) + n^+_{\Gamma_{RP}}(\breve{f}) \leq 1$ and $0 \leq m^+_{\Gamma_{IP}}(\breve{f}) + n^+_{\Gamma_{IP}}(\breve{f}) \leq 1$ where $\eta_{\mu_i}, \eta_{\zeta_s} \in \hat{\eta}(i \leq s)$. To evaluate the above problem, the decision-making process is diagnosed here.

5.2. Proposed algorithm

Now, the steps of a new technique to describe the problem are shown below.
Step 1. Formulate the matrix by putting the items in the shape of CI-VIULNs.
Step 2. Formulate the CI-VIULN by aggregating the given data with the help of the CI-VIULWAHM operator or CI-VIULWGHM operator.
Step 3. Again, formulate the CI-VIULN by aggregating the given data with the help of the CI-VIULWAHM operator or CI-VIULWGHM operator.
Step 4. Formulate the SV with the availability of aggregated CI-VIULNs.
Step 5. Formulate the ranking values in which the availability of SVs demonstrates the best options.

6. Illustrating example

Data given in [41] are very valuable and informative for determining the beneficial option for taking the best decision from the family of decisions. In [41], it is stated that there are some well-known organizations that try to pick the most beneficial and ideal option among all options. For this, experts give four potential terms, expressed as the family of alternatives, as stated below:

1) $\bar{\Phi}_{Al-1}$: vehicle company,

2) $\bar{\Phi}_{Al-2}$: food company,
3) $\bar{\Phi}_{Al-3}$: computer company,
4) $\bar{\Phi}_{Al-4}$: mobiles company.

For this, experts also give data that expressed the four attributes:

1) $\bar{\mathcal{L}}_{At-1}$: hazard factor,
2) $\bar{\mathcal{L}}_{At-2}$: improvement factor,
3) $\bar{\mathcal{L}}_{At-3}$: social factor,
4) $\bar{\mathcal{L}}_{At-4}$: other factors.

For the above four criteria, experts give their opinions in the form of their weight vectors $\widehat{\Omega}'_W = (0.4,0.4,0.2)^T$ for the decision matrix and $\widehat{\Omega}_W = (0.4, 0.3, 0.2, 0.1)^T$ for CI-VIULNs. Now, the steps of a new technique to describe the dilemma are shown below:

Step 1. Formulate the matrices by putting the items in the shape of CI-VIULNs (see Tables 1–3).

Table 1. Matrix $\breve{\mathcal{R}}^1$, including CI-VIULNs.

Alternative/Attribute	$\bar{\mathcal{L}}_{At-1}$	$\bar{\mathcal{L}}_{At-2}$	$\bar{\mathcal{L}}_{At-3}$	$\bar{\mathcal{L}}_{At-4}$
$\bar{\Phi}_{Al-1}$	$\left([\eta_1,\eta_2],\left([0.3,0.4]e^{i2\pi[0.1,0.2]},[0.2,0.3]e^{i2\pi[0.3,0.4]}\right)\right)$	$\left([\eta_1,\eta_2],\left([0.31,0.41]e^{i2\pi[0.11,0.21]},[0.21,0.31]e^{i2\pi[0.31,0.41]}\right)\right)$	$\left([\eta_1,\eta_2],\left([0.32,0.42]e^{i2\pi[0.12,0.22]},[0.22,0.32]e^{i2\pi[0.32,0.42]}\right)\right)$	$\left([\eta_1,\eta_2],\left([0.33,0.43]e^{i2\pi[0.13,0.23]},[0.23,0.33]e^{i2\pi[0.33,0.43]}\right)\right)$
$\bar{\Phi}_{Al-2}$	$\left([\eta_1,\eta_3],\left([0.1,0.3]e^{i2\pi[0.2,0.4]},[0.2,0.3]e^{i2\pi[0.2,0.3]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.11,0.31]e^{i2\pi[0.21,0.41]},[0.21,0.31]e^{i2\pi[0.21,0.31]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.12,0.32]e^{i2\pi[0.22,0.42]},[0.22,0.32]e^{i2\pi[0.22,0.32]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.13,0.33]e^{i2\pi[0.23,0.43]},[0.23,0.33]e^{i2\pi[0.23,0.33]}\right)\right)$
$\bar{\Phi}_{Al-3}$	$\left([\eta_2,\eta_3],\left([0.5,0.6]e^{i2\pi[0.3,0.5]},[0.1,0.2]e^{i2\pi[0.2,0.3]}\right)\right)$	$\left([\eta_2,\eta_3],\left([0.51,0.61]e^{i2\pi[0.31,0.51]},[0.11,0.21]e^{i2\pi[0.21,0.31]}\right)\right)$	$\left([\eta_2,\eta_3],\left([0.52,0.62]e^{i2\pi[0.32,0.52]},[0.12,0.22]e^{i2\pi[0.22,0.32]}\right)\right)$	$\left([\eta_2,\eta_3],\left([0.53,0.63]e^{i2\pi[0.33,0.53]},[0.13,0.23]e^{i2\pi[0.23,0.33]}\right)\right)$
$\bar{\Phi}_{Al-4}$	$\left([\eta_1,\eta_3],\left([0.2,0.6]e^{i2\pi[0.2,0.4]},[0.2,0.3]e^{i2\pi[0.1,0.4]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.21,0.61]e^{i2\pi[0.21,0.41]},[0.21,0.31]e^{i2\pi[0.11,0.41]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.22,0.62]e^{i2\pi[0.22,0.42]},[0.22,0.32]e^{i2\pi[0.12,0.42]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.23,0.63]e^{i2\pi[0.23,0.43]},[0.23,0.33]e^{i2\pi[0.13,0.43]}\right)\right)$

Table 2. Matrix $\breve{\mathcal{R}}^2$, including CI-VIULNs.

Alternative/Attribute	$\bar{\mathcal{L}}_{At-1}$	$\bar{\mathcal{L}}_{At-2}$	$\bar{\mathcal{L}}_{At-3}$	$\bar{\mathcal{L}}_{At-4}$
$\bar{\Phi}_{Al-1}$	$\left([\eta_1,\eta_3],\left([0.4,0.5]e^{i2\pi[0.2,0.3]},[0.1,0.2]e^{i2\pi[0.1,0.2]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.41,0.51]e^{i2\pi[0.21,0.31]},[0.11,0.21]e^{i2\pi[0.11,0.21]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.42,0.52]e^{i2\pi[0.22,0.32]},[0.12,0.22]e^{i2\pi[0.12,0.22]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.43,0.53]e^{i2\pi[0.23,0.33]},[0.13,0.23]e^{i2\pi[0.13,0.23]}\right)\right)$
$\bar{\Phi}_{Al-2}$	$\left([\eta_1,\eta_2],\left([0.2,0.3]e^{i2\pi[0.3,0.5]},[0.1,0.3]e^{i2\pi[0.1,0.2]}\right)\right)$	$\left([\eta_1,\eta_2],\left([0.21,0.31]e^{i2\pi[0.31,0.51]},[0.11,0.31]e^{i2\pi[0.11,0.21]}\right)\right)$	$\left([\eta_1,\eta_2],\left([0.22,0.32]e^{i2\pi[0.32,0.52]},[0.12,0.32]e^{i2\pi[0.12,0.22]}\right)\right)$	$\left([\eta_1,\eta_2],\left([0.23,0.33]e^{i2\pi[0.33,0.53]},[0.13,0.33]e^{i2\pi[0.13,0.23]}\right)\right)$
$\bar{\Phi}_{Al-3}$	$\left([\eta_3,\eta_4],\left([0.2,0.3]e^{i2\pi[0.2,0.3]},[0.1,0.2]e^{i2\pi[0.1,0.2]}\right)\right)$	$\left([\eta_3,\eta_4],\left([0.21,0.31]e^{i2\pi[0.21,0.31]},[0.11,0.21]e^{i2\pi[0.11,0.21]}\right)\right)$	$\left([\eta_3,\eta_4],\left([0.22,0.32]e^{i2\pi[0.22,0.32]},[0.12,0.22]e^{i2\pi[0.12,0.22]}\right)\right)$	$\left([\eta_3,\eta_4],\left([0.23,0.33]e^{i2\pi[0.23,0.33]},[0.13,0.23]e^{i2\pi[0.13,0.23]}\right)\right)$
$\bar{\Phi}_{Al-4}$	$\left([\eta_1,\eta_3],\left([0.2,0.3]e^{i2\pi[0.1,0.2]},[0.2,0.3]e^{i2\pi[0.1,0.3]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.21,0.31]e^{i2\pi[0.11,0.21]},[0.21,0.31]e^{i2\pi[0.11,0.31]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.22,0.32]e^{i2\pi[0.12,0.22]},[0.22,0.32]e^{i2\pi[0.12,0.32]}\right)\right)$	$\left([\eta_1,\eta_3],\left([0.23,0.33]e^{i2\pi[0.13,0.23]},[0.23,0.33]e^{i2\pi[0.13,0.33]}\right)\right)$

Table 3. Matrix $\breve{\mathcal{R}}^3$, including CI-VIULNs.

Alternative/Attribute	$\bar{\mathcal{L}}_{At-1}$	$\bar{\mathcal{L}}_{At-2}$	$\bar{\mathcal{L}}_{At-3}$	$\bar{\mathcal{L}}_{At-4}$
$\bar{\Phi}_{Al-1}$	$\left([\eta_2,\eta_3],\left([0.2,0.6]e^{i2\pi[0.3,0.4]},[0.1,0.2]e^{i2\pi[0.3,0.4]}\right)\right)$	$\left([\eta_2,\eta_3],\left([0.21,0.61]e^{i2\pi[0.31,0.41]},[0.11,0.21]e^{i2\pi[0.31,0.41]}\right)\right)$	$\left([\eta_2,\eta_3],\left([0.22,0.62]e^{i2\pi[0.32,0.42]},[0.12,0.22]e^{i2\pi[0.32,0.42]}\right)\right)$	$\left([\eta_2,\eta_3],\left([0.23,0.63]e^{i2\pi[0.33,0.43]},[0.13,0.23]e^{i2\pi[0.33,0.43]}\right)\right)$
$\bar{\Phi}_{Al-2}$	$\left([\eta_2,\eta_4],\left([0.2,0.3]e^{i2\pi[0.3,0.4]},[0.1,0.2]e^{i2\pi[0.2,0.3]}\right)\right)$	$\left([\eta_2,\eta_4],\left([0.21,0.31]e^{i2\pi[0.31,0.41]},[0.11,0.21]e^{i2\pi[0.21,0.31]}\right)\right)$	$\left([\eta_2,\eta_4],\left([0.22,0.32]e^{i2\pi[0.32,0.42]},[0.12,0.22]e^{i2\pi[0.22,0.32]}\right)\right)$	$\left([\eta_2,\eta_4],\left([0.23,0.33]e^{i2\pi[0.33,0.43]},[0.13,0.23]e^{i2\pi[0.23,0.33]}\right)\right)$
$\bar{\Phi}_{Al-3}$	$\left([\eta_3,\eta_4],\left([0.1,0.5]e^{i2\pi[0.2,0.4]},[0.2,0.3]e^{i2\pi[0.1,0.2]}\right)\right)$	$\left([\eta_3,\eta_4],\left([0.11,0.51]e^{i2\pi[0.21,0.41]},[0.21,0.31]e^{i2\pi[0.11,0.21]}\right)\right)$	$\left([\eta_3,\eta_4],\left([0.12,0.52]e^{i2\pi[0.22,0.42]},[0.22,0.32]e^{i2\pi[0.12,0.22]}\right)\right)$	$\left([\eta_3,\eta_4],\left([0.13,0.53]e^{i2\pi[0.23,0.43]},[0.23,0.33]e^{i2\pi[0.13,0.23]}\right)\right)$

Continued on next page

Alternative/Attribute	$\overline{\mathcal{L}}_{At-1}$	$\overline{\mathcal{L}}_{At-2}$	$\overline{\mathcal{L}}_{At-3}$	$\overline{\mathcal{L}}_{At-4}$
$\overline{\boldsymbol{\Phi}}_{Al-4}$	$\left([\eta_2,\eta_4],\left([0.1,0.3]e^{i2\pi[0.2,0.3]},[0.2,0.4]e^{i2\pi[0.3,0.4]}\right)\right)$	$\left([\eta_2,\eta_4],\left([0.11,0.31]e^{i2\pi[0.21,0.31]},[0.21,0.41]e^{i2\pi[0.31,0.41]}\right)\right)$	$\left([\eta_2,\eta_4],\left([0.12,0.32]e^{i2\pi[0.22,0.32]},[0.22,0.42]e^{i2\pi[0.32,0.42]}\right)\right)$	$\left([\eta_2,\eta_4],\left([0.13,0.33]e^{i2\pi[0.23,0.33]},[0.23,0.43]e^{i2\pi[0.33,0.43]}\right)\right)$

Step 2. Formulate the CI-VIULN by aggregating the given data with the help of the CI-VIULWAHM operator or CI-VIULWGHM operator, and using the values of parameters $r_{SC}, s_{SC} = 1$; then, see the data in Table 4.

Table 4. Aggregated values using CI-VIULWAHM operator.

	$\overline{\mathcal{L}}_{At-1}$	$\overline{\mathcal{L}}_{At-2}$	$\overline{\mathcal{L}}_{At-3}$	$\overline{\mathcal{L}}_{At-4}$
$\overline{\boldsymbol{\Phi}}_{Al-1}$	$\left([\eta_{0.33448},\eta_{0.65666}],\left([0.3998,0.6171]e^{i2\pi[0.2521,0.3796]},[0.0004,0.0049]e^{i2\pi[0.0039,0.019]}\right)\right)$	$\left([\eta_{0.33448},\eta_{0.65666}],\left([0.4121,0.6281]e^{i2\pi[0.2650,0.3921]},[0.00057,0.0058]e^{i2\pi[0.0047,0.0215]}\right)\right)$	$\left([\eta_{0.33448},\eta_{0.65666}],\left([0.4243,0.6392]e^{i2\pi[0.278,0.4045]},[0.00077,0.0069]e^{i2\pi[0.0057,0.027]}\right)\right)$	$\left([\eta_{0.33448},\eta_{0.65666}],\left([0.4364,0.6501]e^{i2\pi[0.2908,0.4169]},[0.0010,0.008]e^{i2\pi[0.0068,0.027]}\right)\right)$
$\overline{\boldsymbol{\Phi}}_{Al-2}$	$\left([\eta_{0.3344},\eta_{0.7272}],\left([0.2162,0.3888]e^{i2\pi[0.3441,0.5512]},[0.00041,0.0088]e^{i2\pi[0.0010,0.0081]}\right)\right)$	$\left([\eta_{0.3344},\eta_{0.7272}],\left([0.2292,0.4011]e^{i2\pi[0.3567,0.5626]},[0.00057,0.0103]e^{i2\pi[0.0013,0.0095]}\right)\right)$	$\left([\eta_{0.3344},\eta_{0.7272}],\left([0.2421,0.4134]e^{i2\pi[0.3691,0.5740]},[0.00078,0.01192]e^{i2\pi[0.00173,0.01104]}\right)\right)$	$\left([\eta_{0.3344},\eta_{0.7272}],\left([0.2550,0.4256]e^{i2\pi[0.3816,0.5852]},[0.0010,0.0137]e^{i2\pi[0.00219,0.01271]}\right)\right)$
$\overline{\boldsymbol{\Phi}}_{Al-3}$	$\left([\eta_{0.6566},\eta_{0.8526}],\left([0.3693,0.587]e^{i2\pi[0.3092,0.5104]},[0.00035,0.0044]e^{i2\pi[0.00041,0.0049]}\right)\right)$	$\left([\eta_{0.6566},\eta_{0.8526}],\left([0.3819,0.5983]e^{i2\pi[0.3218,0.5222]},[0.00049,0.0053]e^{i2\pi[0.00057,0.0058]}\right)\right)$	$\left([\eta_{0.6566},\eta_{0.8526}],\left([0.3943,0.6096]e^{i2\pi[0.3344,0.5338]},[0.00068,0.0063]e^{i2\pi[0.00078,0.0069]}\right)\right)$	$\left([\eta_{0.6566},\eta_{0.8526}],\left([0.4067,0.6208]e^{i2\pi[0.3469,0.5454]},[0.0009,0.0075]e^{i2\pi[0.0010,0.0082]}\right)\right)$
$\overline{\boldsymbol{\Phi}}_{Al-4}$	$\left([\eta_{0.3344},\eta_{0.7897}],\left([0.2278,0.5228]e^{i2\pi[0.2162,0.3895]},[0.0025,0.0198]e^{i2\pi[0.0007,0.02967]}\right)\right)$	$\left([\eta_{0.3344},\eta_{0.7897}],\left([0.2407,0.5346]e^{i2\pi[0.2292,0.4019]},[0.0031,0.02237]e^{i2\pi[0.00095,0.0329]}\right)\right)$	$\left([\eta_{0.3344},\eta_{0.7897}],\left([0.2535,0.5463]e^{i2\pi[0.2421,0.4142]},[0.0038,0.0250]e^{i2\pi[0.0012,0.03641]}\right)\right)$	$\left([\eta_{0.3344},\eta_{0.7897}],\left([0.2663,0.5579]e^{i2\pi[0.2550,0.4264]},[0.0046,0.0279]e^{i2\pi[0.00163,0.04012]}\right)\right)$

Step 3. Again, formulate the CI-VIULN by aggregating the given data with the help of the CI-VIULWAHM operator or CI-VIULWGHM operator and using $r_{SC}, s_{SC} = 1$; then, see the data in Table 5.

Table 5. Aggregated values using CI-VIULWAHM and CI-VIULWAHM operators.

	CI-VIULWAHM	CI-VIULWGHM
$\overline{\Phi}_{Al-1}$	$\left(\begin{array}{c}[\eta_{0.08145},\eta_{0.1587}],\\ \left(\begin{array}{c}[0.4818,0.7069]e^{i2\pi[0.3163,0.4598]},\\ [0.00000091,0.00134]e^{i2\pi[0.00104,0.00663]}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\eta_{0.08145},\eta_{0.1587}],\\ \left(\begin{array}{c}[0.2960,0.5174]e^{i2\pi[0.1657,0.2772]},\\ [0.00072,0.0072]e^{i2\pi[0.0059,0.0263]}\end{array}\right)\end{array}\right)$
$\overline{\Phi}_{Al-2}$	$\left(\begin{array}{c}[\eta_{0.0814},\eta_{0.1755}],\\ \left(\begin{array}{c}[0.2747,0.4698]e^{i2\pi[0.4204,0.6413]},\\ [0.0000009,0.00266]e^{i2\pi[0.00022,0.00241]}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\eta_{0.0814},\eta_{0.1755}],\\ \left(\begin{array}{c}[0.1371,0.2857]e^{i2\pi[0.2447,0.4471]},\\ [0.00072,0.0126]e^{i2\pi[0.00169,0.01172]}\end{array}\right)\end{array}\right)$
$\overline{\Phi}_{Al-3}$	$\left(\begin{array}{c}[\eta_{0.1587},\eta_{0.2052}],\\ \left(\begin{array}{c}[0.4485,0.6774]e^{i2\pi[0.3812,0.5996]},\\ [0.0.00006,0.00119]e^{i2\pi[0.0.00009,0.00134]}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\eta_{0.1587},\eta_{0.2052}],\\ \left(\begin{array}{c}[0.2677,0.4851]e^{i2\pi[0.2137,0.4049]},\\ [0.00062,0.0066]e^{i2\pi[0.00072,0.00727]}\end{array}\right)\end{array}\right)$
$\overline{\Phi}_{Al-4}$	$\left(\begin{array}{c}[\eta_{0.08145},\eta_{0.19034}],\\ \left(\begin{array}{c}[0.2881,0.6125]e^{i2\pi[0.2747,0.4706]},\\ [0.00062,0.0069]e^{i2\pi[0.00015,0.01134]}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}[\eta_{0.08145},\eta_{0.19034}],\\ \left(\begin{array}{c}[0.1462,0.4178]e^{i2\pi[0.1371,0.2864]},\\ [0.0039,0.02742]e^{i2\pi[0.00119,0.04031]}\end{array}\right)\end{array}\right)$

Step 4. Formulate the SV with the availability of aggregated CI-VIULNs, as given in Table 6.

Table 6. Expressed SVs using data in Table 5.

	CI-VIULWAHM	CI-VIULWGHM
$\overline{\mathfrak{E}}_{Al-1}$	0.73986	0.46009
$\overline{\mathfrak{E}}_{Al-2}$	0.65919	0.39818
$\overline{\mathfrak{E}}_{Al-3}$	0.93306	0.60145
$\overline{\mathfrak{E}}_{Al-4}$	0.58084	0.32658

Step 5. Formulate the ranking values for the availability of SVs to demonstrate the best options, as given in Table 7.

Table 7. Expressed ranking values.

Methods	Ranking values
CI-VIULWAHM operator	$\overline{\Phi}_{Al-3} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-4}$
CI-VIULWGHM operator	$\overline{\Phi}_{Al-3} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-4}$

Table 7 provides the same ranking results with the same beneficial optimal $\overline{\Phi}_{Al-3}$, as obtained using the CI-VIULWAHM operator and CI-VIULWGHM operator. Figure 1 states the practical form of the data in Table 6.

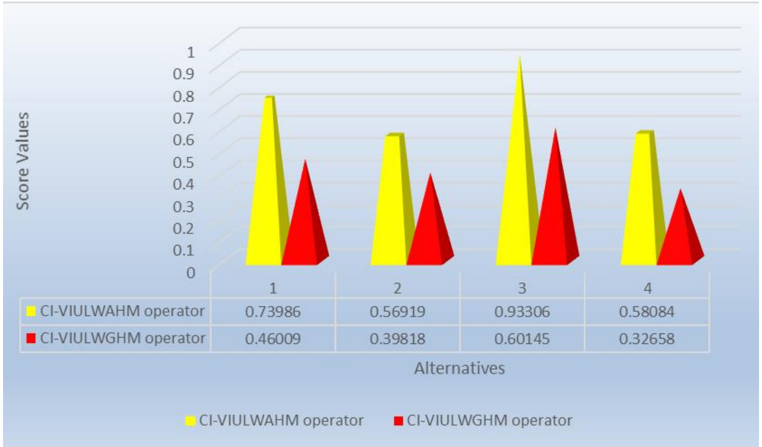

Figure 1. Shown graphical structure of data in Table 6.

6.1. *Influence of parameters*

Here the main theme is to find the stability of the parameters by using their different values and discussing their ranking values. Here, we suggest two main parameters, called r_{SC} and s_{SC}. Using the data in Tables 1–3, we find the influences of parameters on different values. The major analysis of this theory is demonstrated in Table 8 with the help of two well-known theories called CI-VIULWAHM and CI-VIULWGHM operators. First, we try to fix the value of the parameter $s_{SC} = 1$; then, see Table 8.

Table 8. Expressed influences of parameter r_{SC} for $s_{SC} = 1$.

r_{SC}	Operator	Score Values	Ranking Values
1	WAHM	0.7398, 0.6591, 0.9330, 0.5808	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	0.4600, 0.3982, 0.6014, 0.3265	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
2	WAHM	0.618, 0.5574, 0.8554, 0.4652	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	0.2764, 0.2474, 0.4679, 0.1410	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
5	WAHM	0.2582, 0.2251, 0.5497, 0.1721	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	−0.4424, −0.4409, −0.3171, −0.5144	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
7	WAHM	0.2313, 0.1940, 0.5124, 0.1766	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	−0.7208, −0.7302, −0.7182, −0.7668	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
10	WAHM	0.2893, 0.2491, 0.5928, 0.2430	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	−0.9715, −0.9657, −1.0896, −0.9764	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$

Tables 8 and 9 state that for any number of parameters, we can get the same ranking result, then the beneficial optimal value is $\bar{\Phi}_{Al-3}$. Moreover, with the availability of the presented approaches, we further improved the quality of the invented approaches with the help of the comparative analysis diagnosed here.

Table 9. Expressed influences of parameter $\mathcal{s}_{SC}$ for $\mathcal{r}_{SC} = 1$.

$\mathcal{s}_{SC}$	Operator	Score Values	Ranking Values
1	WAHM	0.7398, 0.6591, 0.9330, 0.5808	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	0.4600, 0.3982, 0.6014, 0.3265	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
2	WAHM	0.6681, 0.5905, 0.8739, 0.4815	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	0.3727, 0.3242, 0.5543, 0.1985	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
5	WAHM	0.2579, 0.1919, 0.4958, 0.03177	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	-0.2382, -0.2634, -0.05009, -0.4125	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
7	WAHM	0.03748, -0.03779, 0.2520, -0.1812	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	-0.5289, -0.56151, -0.3913, -0.6661	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
10	WAHM	-0.1822, -0.2731, -0.0081, -0.3821	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	WGHM	-0.7752, -0.8183, -0.6939, -0.8692	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$

6.2. Comparative analysis

Comparative analysis refers to the sensitivity of two or more techniques, data sets or tools. Pattern analysis, filtering and strategic decision-making techniques are different forms of sensitivity analysis. In healthcare, sensitive analysis is performed to compare the large number of medical records, images and other data used to find the supremacy of the decision-making tool. In this strategy, we suggested various existing operators in these forms: the HM operator initiated in [41], partitioned BM (PBM) operator initiated in [42], weighted Bonferroni ordered weighted averaging (WBOWA) operator exposed in [43], Hamy mean (HaM) operators diagnosed in [44] and BM operators proposed in [45]. The information given in [41–45] was diagnosed based on intuitionistic uncertain linguistic sets. The prevailing operators based on intuitionistic uncertain linguistic information were diagnosed in [41–45] and have a lot of limitations because they can deal only with one dimension of information at a time; however, the supremacy of the proposed work is that they can easily deal with two dimensions of information at a time. Table 10 includes a comparative analysis of the initiated and existing operators using the data in Tables 1–3.

Table 10. Results of comparative analysis.

Methods	Operator	Score Values	Ranking Values
Liu et al. [41]	HM	Limited rules, not able to find the solution of the above example	Failed to resolve the above theory
Liu and Liu [42]	PBM	Limited rules, not able to find the solution of the above example	Failed to resolve the above theory
Liu et al. [43]	WBOWA	Limited rules, not able to find the solution of the above example	Failed to resolve the above theory
Liu et al. [44]	HaM	Limited rules, not able to find the solution of the above example	Failed to resolve the above theory
Liu and Zhang [45]	BM	Limited rules, not able to find the solution of the above example	Failed to resolve the above theory
Proposed operators	CI-VIULWAHM	0.7398,0.6591,0.9330,0.5808	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$
	CI-VIULWGHM	0.4600,0.3982,0.6014,0.3265	$\bar{\Phi}_{Al-3} \geq \bar{\Phi}_{Al-1} \geq \bar{\Phi}_{Al-2} \geq \bar{\Phi}_{Al-4}$

The data given in [41–45] have various limitations and, due to these reasons, they cannot give the exact solution of the considered data. The demonstrated approaches have a lot of advantages, and they can easily find the solutions to the awkward and complicated sorts of data. Using the information in Table 10, we get the best optimal value in the form $\overline{\Phi}_{Al-3}$. Furthermore, we try to explain the above result in the form of a graphical structure, like Figure 2. Figure 2 includes four alternatives initiated by six different scholars [41–45]. Therefore, the diagnosed approaches are more dominant as compared to the approaches in [41–45].

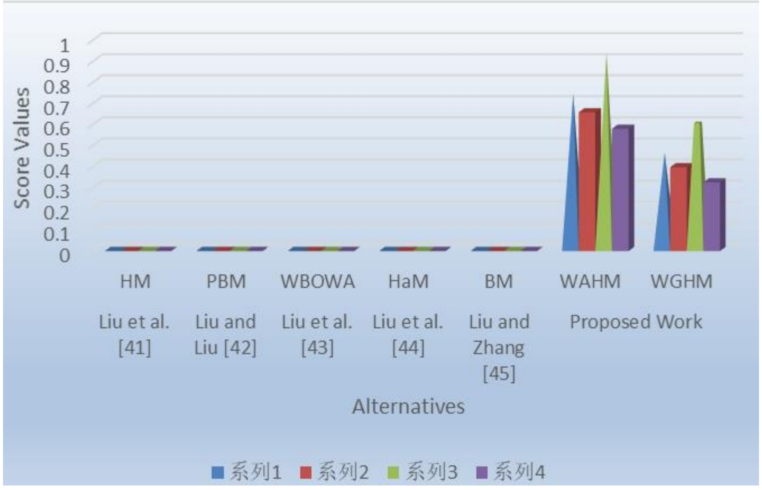

Figure 2. Geometrical shapes of data in Table 10.

7. Conclusions

The theory of CI-VIUL information is more massive and generalized than the existing theories, such as intuitionistic, I-V intuitionistic, intuitionistic fuzzy linguistic, I-V intuitionistic fuzzy linguistic and linguistic sets. The main and most valuable results of this analysis are described below.

1) We diagnosed the well-known theory, called the CI-VIUL setting, as a more powerful and capable tool to handle ambiguous sorts of theories. Furthermore, to enhance the features of the CI-VIUL information, we diagnosed the algebraic laws, SV and AV for CI-VIUL settings.
2) We developed the CI-VIULAHM operator, CI-VIULWAHM operator, CI-VIULGHM operator, CI-VIULWGHM operator and their well-known achievements in the form of some results, important properties and specific cases.
3) We checked the practicality and usefulness of the initiated approaches, and a MADM technique was implemented for CI-VIUL settings.
4) The reliability of the proposed MADM tool was demonstrated by a computational example that assesses the impact of the diagnosed approaches on various well-known prevailing theories.

Decision social networks mostly depend on the individual decisions in the examples. In the upcoming times, we can continue to enhance superior aggregation operators, different types of techniques, new similarity measures, etc., further diagnosing a massively valuable and genuine weight determination technique that can be employed to evaluate awkward and problematic issues in various real-life problems. In addition, we will modify the proposed work for complex spherical FSs [49], T-spherical FSs [50], Pythagorean FSs [51], decision-making [52–55], linear Diophantine FSs [56] and fuzzy N-soft sets [57] to enhance the study of the existing approaches.

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Conflict of interest

The authors declare that they have no conflict of interest. The data used in this manuscript are artificial, and anyone can use it without prior permission of the authors by just citing this article.

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