Existence and controllability of Hilfer fractional neutral differential equations with time delay via sequence method

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Abstract: This paper deals with the existence and approximate controllability outcomes for Hilfer fractional neutral evolution equations. To begin, we explore existence outcomes using fractional computations and Banach contraction fixed point theorem. In addition, we illustrate that a neutral system with a time delay exists. Further, we prove the considered fractional time-delay system is approximately controllable using the sequence approach. Finally, an illustration of our main findings is offered.

Keywords: hilfer fractional derivative; approximate controllability; neutral system; time delay; mild solutions; sequence approach; nonlocal conditions

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1. Introduction

Fraction systems have been demonstrated to be important tools for providing many complex miracles in numerous sectors of science and engineering, and this pairing has received a lot of traction.
recently. Fractional differential equations (FDEs), calculations have become increasingly important in mathematics, see [3, 8, 9, 15, 16, 23, 24, 32, 33]. Hilfer [17] launched a separate sort of derivative, alongside Riemann-Liouville and Caputo fractional derivatives, that is, Hilfer fractional derivative (HFD). For additional information, see [2, 4, 9, 17, 20, 22, 29, 30, 43].

Thermal science, chemical engineering, and mechanics all use the time-fractional advection-reaction-diffusion equation. An analytic solution to this equation is nearly impossible to find. Recently, numeral modalities are provided, including a finite differentiation optimization approach and a homotopy perturbation method. The Taylor’s formula, also known as the Delta function, was employed for three decades to build the replicating kernel space, which has proven to be an excellent technique for three decades, the Taylor’s formula, also known as the Delta function, was used to construct the replicating kernel space and it has proven to be a useful method for resolving different forms. In [1], the authors proposed various new reproductive kernel spaces for numerical approaches to time-fractional advection-reaction-diffusion equations based on Legendre polynomials.

References [2, 9] explored the approximate controllability of semilinear inclusions with respect to HFD. Furati, et al. [7] discussed the existence and uniqueness of a problem involving HFD.

Neutral systems have gotten increasing attention in the present generation because among their widespread applicability in various domains of pragmatic mathematics. Several neutral systems, including heat flow in materials, visco-elasticity, wave propagation, and several natural developments, benefit from neutral systems with or without delay. To know more details on neutral system and its application reader can refer [4, 20, 21, 53].

The advancement of current mathematical control theory has been aided by approximate controllability. The difficulties of approximation controllability of differential systems are extensively used in theory connected to system analysis with control. The system with fractional order generated by the fractional evolution system has attracted attention in recent years, list of these distributions may be found in [21, 52]. Li et al. [26] and He et al. [12] developed a fractal differential model as well as a fractal Duffing-Van der Pol oscillator (DVdP) with two-scale fractal derivatives.

An analytic approximate solution can be obtained using two-scale transforms and the He-Laplace method. He and Ji [1] focused on two-scale mathematics and fractional calculus for thermodynamics, and found it is required to show the information lost owing to the reduced dimensional method. In general, one scale is set by utilization, in which case regular calculus is used, and the other scale is determined by the need to reveal lost information, in which case the continuity assumption is allowed and fractional or fractal calculus must be used. For numerical results of space fractional variable coefficient kdv-modified kdv equation via Fourier spectral approach, see [49, 50]. Many academics are now using the Sequence method to represent the approximate controllability outcomes using Riemann-Liouville fractional derivative, fractional evolution with damping, and an impulsive system. See articles [4–6, 18, 19, 26, 28, 29, 31, 34–42, 44–48, 54, 55, 57] for further information.

Consider

\[
\begin{align*}
\mathcal{D}_{0+}^{\alpha,\beta}[x(\vartheta) - g(\vartheta, x_0)] &= \mathcal{A}x(\vartheta) + \mathcal{A}_1 x(\vartheta - \sigma) + Bu(\vartheta) + G(\vartheta, x(\vartheta - \sigma)), \quad \vartheta > 0, \\
I_{0+}^{(1-\alpha)(1-\beta)} x(\vartheta) &= h(\vartheta), \quad \vartheta \in [-\sigma, 0].
\end{align*}
\]

The Hilfer fractional derivative is symbolized by \(\mathcal{D}_{0+}^{\alpha,\beta}\), whose order and type are \(0 < \beta < 1, 0 \leq \alpha \leq 1\) on Hilbert space \(\mathcal{H}\), \(\mathcal{A}\) refers to a \(C_0\) semigroup \(\{S(\vartheta)\}_{\vartheta \geq 0}\)'s infinitesimal generator.
2. Preliminary results

$C(K, \mathcal{H}) : K \rightarrow \mathcal{H}$ symbolizes the continuous function throughout this paper along with $\|x\|_{C} = \sup_{\theta \in \mathcal{K}} e^{-\theta} \|x(\theta)\|$, where $r$ is a fixed positive constant. Now characterize $C_{1-b}(K, \mathcal{H}) = \{x : x^{1-b}(\theta) \in C(K, \mathcal{H})\}$, $\|\cdot\|_{b}$ represented as $\|x\|_{b} = \sup\{t^{1-b} \|x(t)\|, \theta \in \mathcal{K}\}$, where $(1 - b) = (1 - \alpha)(1 - \beta)$ since $b = \alpha + \beta - \alpha \beta$.

Following are the properties of $\mathcal{A}^{\kappa}$:

(i) $D(\mathcal{A}^{\kappa})$ be a Banach space along $\|u\|_{x} = \|\mathcal{A}^{\kappa} u\|$ for $x \in D(\mathcal{A}^{\kappa})$.

(ii) $S(\theta) : U \rightarrow U_{x}$ for $\theta \geq 0$.

(iii) $\mathcal{A}^{\kappa} S(\theta)x = S(\theta) \mathcal{A}^{\kappa} x$ for all $u \in D(\mathcal{A}^{\kappa})$, $\theta \geq 0$.

(iv) For each $\kappa \in (0, 1)$, $\mathcal{A}^{\kappa} S(\theta)$ is bounded, $\mathbb{N}_{x} > 0$, such that

$$\|\mathcal{A}^{\kappa} S(\theta)\| \leq \frac{\mathbb{N}_{x}}{\theta^{\kappa}}, \quad \theta \in (0, b].$$

Definition 2.1. [33] Suppose $\theta : [d, +\infty) \rightarrow \mathbb{R}$, then RLI is defined as

$$I_{d}^{\beta} G(\theta) = \frac{1}{\Gamma(\beta)} \int_{d}^{\theta} \frac{G(r)}{(\theta - r)^{1-\beta}} dr, \quad \theta > d; \ \beta > 0.$$

Definition 2.2. [33] Type $\beta \in [j - 1, j)$, $j \in \mathbb{Z}$ for $G : [d, +\infty) \rightarrow \mathbb{R}$, the RLD is defined as

$$L D_{d}^{\beta} G(\theta) = \frac{1}{\Gamma(j - \beta)} \frac{d^{j}}{dt^{j}} \int_{d}^{\theta} \frac{G(r)}{(\theta - r)^{\beta+1-j}} dr, \quad \theta > d, \ j - 1 < \beta < j.$$

Definition 2.3. [33] Type $\beta \in [j - 1, j)$, $j \in \mathbb{Z}$ for $G : [d, +\infty) \rightarrow \mathbb{R}$, we have the RLD in the form of

$$D_{d}^{\beta} G(\theta) = \frac{1}{\Gamma(j - \beta)} \int_{d}^{\theta} \frac{G^{(j-1)}(r)}{(\theta - r)^{\beta+1-j}} dr, \quad \theta > d, \ j - 1 < \beta < j.$$

Definition 2.4. [33] $0 \leq \beta \leq 1$, $0 < \alpha < 1$, for $G(\theta)$, then the HFD is

$$D_{d}^{\alpha, \beta} G(\theta) = (I_{d}^{(1-\beta)} D_{d}^{(1-\alpha)(1-\beta)} G)(\theta).$$
Remark 2.5. [17] RLJ and CFD’s Hilfer fractional derivatives are characterized as follows:

\[ D^{\alpha,\beta}_0 \mathcal{G}(\theta) = \begin{cases} \frac{d}{d\theta} (1-\beta)^{1-\beta} \mathcal{G}(\theta) = l^{1-\beta} \mathcal{G}(\theta), \beta = 0, 0 < \beta < 1, d = 0; \\ 1^{1-\beta} \frac{d}{d\theta} \mathcal{G}(\theta) = e^{1-\beta} \mathcal{G}(\theta), \beta = 1, 0 < \beta < 1, d = 0. \end{cases} \]

Definition 2.6. [10, 11] He’s fractional derivative:

In fractal space, fractional evolution equations are established using He’s fractional derivative. The fractional evolution equation is converted into its traditional form via He’s fractional real transform, and the solutions are obtained using the homotopy perturbation method.

Definition 2.7. [14] Two-scale fractal derivative:

The standard differential derivatives and the two-scale fractal derivative are conformable. The two-scale transform is used to convert the nonlinear Zhiber-Shabat oscillator with the fractal derivatives to the traditional model.

\[ \frac{\partial \theta}{\partial x^\sigma} = \Gamma(1 + \alpha) \lim_{x \to x_0} \frac{\theta - T_0}{(x - x_0)^\alpha}, \]

where \( x_0 \) is the smallest scale beyond which there is no physical understanding and it is the porous size. Refer [10, 11, 14], for the variational iteration method refer [13].

Definition 2.8. [33] \( x(\cdot; w) \in C((0, d], \vartheta) \) is a mild solution of (1.1) only if for all \( w \in L^2(K, \mathcal{H}) \), the integral equation

\[ x(\theta) = \begin{cases} R_{\sigma,\beta}(\theta)[h(0) - g(0, h(0))] + g(\theta, x_\theta) + \int_0^\theta Q_\beta(\theta - \tau)[A_0(x, \tau)]d\tau, & \text{for } \theta > 0, \\ h(\theta), & \theta \in [-\sigma, 0), \end{cases} \]

where

\[ Q_\beta(\theta) = \theta^{\beta - 1} V_\beta(\theta); \quad R_{\sigma,\beta}(\theta) = l_{0-}^{1-\beta} Q_\beta(\theta); \quad V_\beta(\theta) = \int_0^\infty \beta \theta N_\beta(\theta) S(\theta^\beta \vartheta)dt. \]

(2.1) implies

\[ x(\theta) = R_{\sigma,\beta}(\theta)[h(0) - g(0, h(0))] + g(\theta, x_\theta) + \int_0^\theta (\theta - \tau)^{\beta - 1} V_\beta(\theta - \tau)[A_0(x, \tau)]d\tau, + A_1 x(-\sigma) + B u(\tau) + G(\tau, x(\tau - \sigma))]d\tau, \quad \text{for } \theta \in K. \]

Wright function: \( N_\beta(\theta) \):

\[ N_\beta(\theta) = 1 + \sum_{k=1}^\infty \frac{(-\theta)^k}{(k-1)! \Gamma(1 - dk)}, \quad 0 < d < 1, \quad \theta \in C, \]

where \( N_\beta(\theta) \) belongs to \( (0, \infty) \) satisfying

\[ \int_0^\infty \theta^\alpha N_\beta(\theta)dt = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \beta \nu)}; \quad \int_0^\infty N_\beta(\theta)dt = 1. \quad \theta \geq 0. \]
Lemma 2.9. \([17, 56]\)

- The \(V_\beta(\vartheta)\) is continuous.
- For \(\vartheta > 0\), \(|R_{a,\beta}(\vartheta)|\), \(|Q_\beta(\vartheta)|\) are strongly continuous.
- For \(\vartheta > 0\) and for all \(x \in \mathcal{H}\), then

\[
\|R_{a,\beta}(\vartheta)x\| \leq \frac{\mathbb{N}\vartheta^{b-1}}{\Gamma(\alpha(1 - \beta) + \beta)}\|x\|, \\
\|Q_\beta(\vartheta)x\| \leq \frac{\mathbb{N}\vartheta^{\beta-1}}{\Gamma(\beta)}\|x\|, \text{ (or) } \|V_\beta(\vartheta)x\| = \frac{\mathbb{N}}{\Gamma(\beta)}\|x\|.
\]

Lemma 2.10. \([29]\) In any case \(x \in B, \ k \in (0, 1)\), then

\[
\mathcal{A}V_\beta(\vartheta)x = \mathcal{A}^{1 - \beta}V_\beta(\vartheta)\mathcal{A}^xu; \\
\|\mathcal{A}^xV_\beta(\vartheta)\| \leq \frac{\beta C_1 \Gamma(2 - k)}{\vartheta^k \Gamma(1 + \beta(1 - k))}, \quad 0 < \vartheta \leq b.
\]

3. Existence results

In order to obtain the existence of mild solution for the system (1.1), the following assumptions are made.

\(\mathcal{F}_1\) : There exists \(\mathbb{N} \geq 0\), such that the semigroup \(S(\vartheta)\) is uniformly bounded on \(\mathcal{H}\),

\[
\sup_{\vartheta \in [0, \infty)} \|S(\vartheta)\| \leq \mathbb{N}.
\]

\(\mathcal{F}_2\) : There exist \(\vartheta_1 \in (0, \beta)\) and \(\eta \in L^\infty(\mathcal{K}, \mathbb{R}^+), \ x \in \mathcal{H}\), the function \(G(\vartheta, x)\) is continuous at \(\vartheta\) then

\[
\|G(\vartheta, x_1) - G(\vartheta, x_2)\| \leq \eta \vartheta^{1-b}\|x_1 - x_2\|_{\mathcal{H}}, \quad x_1, x_2 \in \mathcal{H},
\]

with

\[
\max_{\vartheta \in [0, \vartheta_1]} \|G(\vartheta, 0)\| = \mathbb{N}_0.
\]

\(\mathcal{F}_3\) :

\[
\max \left\{ \|h\|_{C^k}, \frac{\mathbb{N}}{\Gamma(\alpha(1 - \beta) + \beta)}\|h(0)\| + \frac{d^{1-b}\mathbb{N}_b}{\Gamma(\beta)}\Delta_1\|w\|_{C} + \frac{\mathbb{N}_d d^{(1-b)+\beta}}{\Gamma(1 + \beta)} \right\} < q.
\]

\(\mathcal{F}_4\) : \(g : (0, b] \times \mathcal{J} \to B\) is continuous and there is a \(k \in (0, 1)\) such that \(g \in D(\mathcal{A}^k), \ x, \hat{x} \in \mathbb{C}, \ \vartheta \in \mathcal{J}, \ \mathcal{A}^k g(\cdot, x)\) is strongly measurable, there exist \(L_1, L_2 > 0\) and \(\mathcal{A}^k g(\vartheta, \cdot)\) satisfies

\[
\|\mathcal{A}^k g(\vartheta, x) - \mathcal{A}^k g(\vartheta, \hat{x})\| \leq \vartheta^{1-b}L_1\|x(\vartheta) - \hat{x}(\vartheta)\|, \\
\|\mathcal{A}^k g(\vartheta, x)\| \leq L_2(1 + \vartheta^{1-b}\|x(\vartheta)\|).
\]

For convenience

\[
\|\mathcal{A}_1\| = \mathbb{N}_1, \quad \|B\| \leq \mathbb{N}_b; \ \Delta_1 = \frac{d^{(1-b)+\vartheta_1}}{(r_1 + 1)^{(1-\vartheta_1)}}, \quad \Delta_1 = \vartheta = \frac{\beta - 1}{1 - \vartheta_1};
\]

\[
\|\mathcal{A}^x\| = \mathbb{N}_0, \quad \|\mathcal{A}^x\| = \mathbb{N}_3 = \frac{C_1 - 1}{\alpha(1 + \beta \kappa)}; \quad \Delta_2 = \frac{C_1 - 1}{\alpha(1 + \beta \kappa)}.
\]
Theorem 3.1. For every control function \( w(\cdot) \in W \) and the assumptions \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are true then (1.1) has a mild solution on \( C([-\sigma, d]; \mathcal{H}) \).

Proof. \( \Gamma \) has a fixed point in \( \mathcal{H} \):

Define

\[
\Gamma : B_q = \{ x \in C([-\sigma, d], \mathcal{H}) : \| x \| \leq q \},
\]

\[
(\Gamma x)(\theta) = \begin{cases}
\mathcal{R}_{\alpha, \beta}(\theta)[h(0) - g(0, h(0))] + g(\theta, x_0) + \int_0^\theta (\theta - r)^{\alpha-1}V_\beta(\theta - r)\mathcal{A}g(r, x_r)dr \\
+ \int_0^\theta (\theta - r)^{\beta-1}V_\beta(\theta - r)\mathcal{A}_1x(r - \sigma)dr + \int_0^\theta (\theta - r)^{\alpha-1}V_\beta(\theta - r)Bu(r)dr \\
+ \int_0^\theta (\theta - r)^{\beta-1}V_\beta(\theta - r)G(r, x(r - \sigma))dr, \theta \geq 0; \\
h(\theta); \text{ for } -\sigma \leq \theta < 0;
\end{cases}
\]  \hspace{1cm} (3.1)

Step 1: Fix \( q > 0 \) and \( B_q = \{ x \in C([-\sigma, d], \mathcal{H}) : \| x \| \leq q \} \).

\[
q \leq e^{-\theta}[(\Gamma x)(\theta)]_{\mathcal{H}}
\]

\[
\leq \sup \theta^{1-b}e^{-\theta\| \mathcal{R}_{\alpha, \beta}(\theta)[h(0) - g(0, h(0))] + g(\theta, x_0) + \int_0^\theta (\theta - r)^{\alpha-1}V_\beta(\theta - r)\mathcal{A}g(r, x_r)dr \\
+ \int_0^\theta (\theta - r)^{\beta-1}V_\beta(\theta - r)\mathcal{A}_1x(r - \sigma)dr + \int_0^\theta (\theta - r)^{\alpha-1}V_\beta(\theta - r)Bu(r)dr \\
+ \int_0^\theta (\theta - r)^{\beta-1}V_\beta(\theta - r)G(r, x(r - \sigma))dr] \leq \sum_{j=1}^7 J_j,
\]

where

\[
J_1 = \sup \theta^{1-b}e^{-\theta\| \mathcal{R}_{\alpha, \beta}(\theta)[h(0)] \|} \leq \frac{N_0e^{-\theta}}{\Gamma(\alpha(1 - \beta) + \beta)}\| h(0) \|,
\]

\[
J_2 = \sup \theta^{1-b}e^{-\theta\| \mathcal{R}_{\alpha, \beta}(\theta)[g(0) + h(0)] \|} \leq \frac{L_2d^{1-b}N_0N_0e^{-\theta}}{\Gamma(\alpha(1 - \beta) + \beta)},
\]

\[
J_3 = \sup \theta^{1-b}e^{-\theta\| g(\theta, x_0) \|} \leq d^{1-b}N_0e^{-\theta}L_2(1 + \theta^{1-b}\| x(\theta) \|) \leq N_0L_2(1 + q)d^{1-b}e^{-\theta},
\]

\[
J_4 = \sup \theta^{1-b}e^{-\theta\int_0^\theta (\theta - r)^{\alpha-1}\| \mathcal{A}^{1-\kappa}V_\beta(\theta - r)\mathcal{A}^*g(r, x_r) \|dr} \leq d^{1-b}e^{-\theta\| \mathcal{R}_{\alpha, \beta}(\theta)[h(0)] \|} \leq \frac{d^{1-b}e^{-\theta}\beta C_{1-\kappa}\Gamma(1 + \kappa)}{\Gamma(1 + \beta\kappa)} \int_0^\theta (\theta - r)^{\beta\kappa-1}\| \mathcal{A}^*g(r, x_r) \|dr
\]

\[
\leq \frac{d^{1-b}e^{-\theta}\beta C_{1-\kappa}\Gamma(1 + \kappa)}{\Gamma(1 + \beta\kappa)} \int_0^\theta (\theta - r)^{\beta\kappa-1}\| \mathcal{A}^*g(r, x_r) \|dr \leq \frac{d^{1-b}e^{-\theta}\beta C_{1-\kappa}\Gamma(1 + \kappa)}{\Gamma(1 + \beta\kappa)} \int_0^\theta (\theta - r)^{\beta\kappa-1}L_2(1 + \theta^{1-b}\| x(\theta) \|)dr
\]

\[
\leq \frac{d^{1-b}\beta C_{1-\kappa}\Gamma(1 + \kappa)}{\Gamma(1 + \beta\kappa)} \int_0^\theta (\theta - r)^{\beta\kappa-1}L_2(1 + \theta^{1-b}\| x(\theta) \|)dr \leq \frac{d^{1-b}\beta e^{-\theta}\kappa L_2(1 + q)\Delta_2},
\]

\[
J_5 = \sup \theta^{1-b}e^{-\theta\| \mathcal{R}_{\alpha, \beta}(\theta)[h(0)] \|} \leq \frac{d^{1-b}e^{-\theta}\beta C_{1-\kappa}\Gamma(1 + \kappa)}{\Gamma(1 + \beta\kappa)} \int_0^\theta (\theta - r)^{\beta\kappa-1}L_2(1 + \theta^{1-b}\| x(\theta) \|)dr
\]

\[
\]
\[
\mathcal{J}_6 = \sup \theta^{-1-b}e^{-\theta \|d_0d^{-b}\|_{\beta}} \left\| \int_0^\theta (\theta - r)^{-\theta-1} V_p(\theta - r) B_u(r) \, dr \right\|
\]
\[
\leq \frac{N_{d_1-b}e^{-\theta}}{\Gamma(\beta)} \int_0^\theta (\theta - r)^{-\theta-1} ||B_u(r)|| \, dr \leq \frac{d_1^{-b} \|N_{d_0} e^{-\theta}\|_{\beta}}{\Gamma(\beta)} \Delta_1 ||w||_{C},
\]
\[
\mathcal{J}_7 = \sup \theta^{-1-b}e^{-\theta \|d_0d^{-b}\|_{\beta}} \left\| \int_0^\theta (\theta - r)^{-\theta-1} V_p(\theta - r) G(r, x(r - \sigma)) \, dr \right\|
\]
\[
\leq \frac{N_{d_1-b}e^{-\theta}}{\Gamma(\beta)} \int_0^\theta (\theta - r)^{-\theta-1} ||G(r, x(r - \sigma)) - G(r, x(0))|| \, dr
\]
\[
\leq \frac{N_{d_1-b}e^{-\theta}}{\Gamma(\beta)} \left[ \int_0^\theta (\theta - r)^{-\theta-1} \eta \theta^{-1-b} ||x(r - \sigma)|| \, dr + \frac{\|N_0d^\beta\|}{\beta} \right]
\]
\[
\leq \frac{N_{d_1-b}e^{-\theta}}{\Gamma(\beta)} \left[ \eta \Delta_1 q + \frac{\|N_0d^\beta\|}{\beta} \right].
\]

Combining $\mathcal{J}_1$ to $\mathcal{J}_7$, we get
\[
e^{-\theta \|\Gamma x(\theta)\|} \leq \frac{N_{e^{-\theta}}}{\Gamma(\alpha(1 - \beta) + \beta)} ||h(0)|| + \frac{L_2 d_1^{-b}N_0N_{e^{-\theta}}}{\Gamma(\alpha(1 - \beta) + \beta)} \Delta_1 + \frac{N_0 L_2 (1 + q)d_1^{-b}e^{-\theta}}{\Gamma(\beta)} \Delta_1 \|w\|_C
\]
\[
+ \frac{\|N_1d_1^{-b}e^{-\theta}\|_\beta}{\Gamma(\beta)} \left[ \eta \Delta_1 q + \frac{\|N_0d^\beta\|}{\beta} \right]
\]
\[
\leq \frac{N_{e^{-\theta}}}{\Gamma(\alpha(1 - \beta) + \beta)} \left[ ||h(0)|| + L_2 d_1^{-b} + L_2 (1 + q)d_1^{-b}e^{-\theta} \left[ N_0 + d^\beta \Delta_2 \right] \right]
\]
\[
+ \frac{\|N_1d_1^{-b}e^{-\theta}\|_\beta}{\Gamma(\beta)} \left[ ||N_1 q + N_b||_C \Delta_1 + d_1^{-b} [\eta \Delta_1 q + \frac{\|N_0d^\beta\|}{\beta}] \right]
\]
\[
\leq \mathbb{P}^* + L_2 (1 + q)d_1^{-b}e^{-\theta} \left[ N_0 + d^\beta \Delta_2 \right] + \frac{\|N_1d_1^{-b}e^{-\theta}\|_\beta}{\Gamma(\beta)} \Delta_1 (N_1 + d_1^{-b} \eta) q.
\]

Where
\[
\mathbb{P}^* = \frac{N}{\Gamma(\alpha(1 - \beta) + \beta)} \left[ ||h(0)|| + L_2 d_1^{-b} + \frac{d_1^{-b} \|N_0 \|_{\beta}}{\Gamma(\beta)} \Delta_1 \|w\|_C + \frac{\|N_0d_{1^{-b}+\beta}\|_{\beta}}{\Gamma(\beta + 1)} \right].
\]

A positive constant $q$ appearing from the norm $\| \cdot \|^C$, we have
\[
q \geq \frac{N_{d_1^{-b}}}{\Gamma(\beta)} \Delta_1 (N_1 + d_1^{-b} \eta) + d_1^{-b} \left[ N_0 + d^\beta \Delta_2 \right] > 0,
\]
and the radius of the sphere

\[ q \geq \max \left\{ \|h\|_C^n, \frac{N}{\Gamma(\alpha(1 - \beta) + \beta)} \left[ \|h(0)\| + L_2 d^{1 - \beta} \right] + \frac{d^{1 - \beta} N N_b}{\Gamma(\beta)} \Delta_1 \|w\|_C + \frac{N N_0 d^{2(1 - \beta) + \beta}}{\Gamma(\beta + 1)} \right\}, \tag{3.3} \]

From (3.2) and (3.3) we are getting a contradiction to \( F \). Therefore \( \| \Gamma x \|^* \leq q \).

**Step 2:** Contraction: For every \( \vartheta \in (0, d] \) using \( (F_2) \) and there exists constants \( x_1, x_2 \in \mathbb{C}(\sigma, d]; \mathcal{H} \), we obtain

\[
\|(\Gamma x_2)(\vartheta) - \Gamma x_1(\vartheta)\| = \left\| \left[ (\vartheta - \vartheta^*) V_{\beta}(\vartheta - \vartheta)[A_0(x_2) \right.ight.
\left. - A_0(x, x_1)] + [A_1(x_2 - \sigma) - A_1(x_1 - \sigma)] + [G(x, x_2 - \sigma) - G(x, x_1 - \sigma)] \right\|
\leq N_0 d^{1 - \beta} L_1 + d^{1 - \beta} \Delta_2
\leq N_0 + d^{1 - \beta} \Delta_2 d^{1 - \beta} L_1\|x_1 - x_2\|_b
\|
\leq \left( N_0 + d^{1 - \beta} \Delta_2 \right) d^{1 - \beta} L_1\|x_1 - x_2\|_b
\|
\leq \frac{N_0 + d^{1 - \beta} \Delta_2}{\Gamma(\beta)}\left( \int_0^\vartheta (\vartheta - \vartheta^*) d\vartheta \right)\|x_2 - x_1\|_C^*
\leq e^{\vartheta^*}\frac{N_0 + d^{1 - \beta} \Delta_2}{\Gamma(\beta)}\left( \int_0^\vartheta (\vartheta - \vartheta^*) d\vartheta \right)\|x_2 - x_1\|_C^*.
\]

From the definition of \( r \) from (3.2), we obtain

\[
\|\Gamma x_2 - \Gamma x_1\|_C^* \leq r^*\|x_2 - x_1\|_C^*, \quad r^* < 1.
\]

Therefore \( \Gamma \) is contraction on \( \mathbb{C}(\sigma, d]; \mathcal{H} \). Hence \( x \) has a fixed point of \( \Gamma \), i.e., it is a mild solution of (1.1). \( \square \)

4. Filter system

By referring the articles [44,58], we have given a filter design for our system (1.1) shown in Figure 1 and it shows a rough diagram format, it contributes to the structure’s practicality by reducing the number of input sources.

(a) Product modulators 1 and 2 accept the \( A \) and \( g(x, x_1), u(\vartheta) \) and \( B \) gives the outputs as \( A_0(x, x_1) \) and \( Bu(x) \).

(b) Product modulator 3 accepts the input \( [h(0) - g(0, h(0))] \) and \( R_{\alpha, \beta}(\vartheta) \) at time \( \vartheta = 0 \), gives the output as \( R_{\alpha, \beta}(\vartheta)[h(0) - g(0, h(0))] \).
(c) $A_1$ and $x(r-\sigma)$, produced $A_1 x(r-\sigma)$.
(d) $Q_{pl}(\vartheta - r)$, $G(r, x(r - \sigma))$ are the inputs. Over $\vartheta$, the inputs are joined and multiplied with an integrator output.
(e) $Q_{pl}(\vartheta - r)$, $A_1 x(r-\sigma)$ are the inputs. Over $\vartheta$, the inputs are joined and multiplied with an integrator output.
(f) The following integrators sum up with the above mentioned modulators over the period $\vartheta$,

$$I_1 = \int_0^\vartheta (\vartheta - r)^{\beta-1} Q_{pl}(\vartheta - r) A_1 x(r-\sigma) \, dt,$$
$$I_2 = \int_0^\vartheta (\vartheta - r)^{\beta-1} Q_{pl}(\vartheta - r) A_1 x(r - \sigma) \, dt,$$
$$I_3 = \int_0^\vartheta (\vartheta - r)^{\beta-1} Q_{pl}(\vartheta - r) B u(r) \, dt,$$
$$I_4 = \int_0^\vartheta (\vartheta - r)^{\beta-1} Q_{pl}(\vartheta - r) G(r, x(r-\sigma)) \, dt,$$

where $Q_{pl}(\vartheta - r) = (\vartheta - r)^{\beta-1} V_{pl}(\vartheta - r)$.

**Figure 1.** Filter system.
Finally, we move all integrator outputs to the network. As a result, we have our output result \( x(\theta) \).

5. Approximate controllability results

Nonlinear control systems with approximate controllability are operated by fractional-order with time delay.

**Definition 5.1.** Let \( E(\mathcal{G}) = \{x(d; w) : u(\cdot) \in U\} \) be the reachable set of (1.1) at time \( d \). Suppose \( \mathcal{G} \) is identically zero then (1.1) is said to be corresponding linear system and \( E(0) \) is defined as the reachable set of (1.1).

**Definition 5.2.** Suppose \( \bar{E}(\mathcal{G}) = \mathcal{K} \), then (1.1) is approximately controllable at time \( d \) \( (d > \sigma) \), where \( \bar{E}(\mathcal{G}) \) signifies the closure of \( E(\mathcal{G}) \). If \( \bar{E}(0) = \mathcal{K} \) then (1.1) is also approximately controllable.

Following hypotheses are used to prove the main outcome.

\( \mathcal{F}_5 \): For every \( \mu > 0 \) and \( \ell(\cdot) \) from \( \mathcal{Y} \), then there exists \( u(\cdot) \in U \) such that
\[
\|E\ell - EBu\|_{\mathcal{H}} < \mu.
\]

\( \mathcal{F}_6 \): For \( \nu > 0 \) independent of \( \ell(\cdot) \in \mathcal{Y} \) such that
\[
\|Bu(\cdot)\|_{L^2((0, d]; \mathcal{K})} < \nu \|\ell(\cdot)\|_{L^2((0, d]; \mathcal{K})}.
\]

**Lemma 5.3.** Assumptions \((\mathcal{F}_1), (\mathcal{F}_2)\) are true then the mild solutions of (1.1) satisfies
\[
\|x\|_C \leq P_\nu E_\beta(M(N_1 + d^{1-\beta})d^\beta), \quad \text{for all } u(\cdot) \in W,
\]
\[
\|x_1(\cdot) - x_2(\cdot)\|_C \leq \theta E_\beta(M(N_1 + d^{1-\beta})d^\beta)N_1\|w_1 - w_2\|_Y, \quad \text{for all } w_1, w_2(\cdot) \in \mathcal{X},
\]
where
\[
\begin{align*}
P_\nu &= \frac{N}{\Gamma(\alpha(1 - \beta) + \beta)}\|h(0)\| + \frac{d^{1-\beta}NN_1\Delta_1\|w\|_C + \frac{NN_0d(1-b+\beta)}{\Gamma(\beta + 1)}},
\end{align*}
\]
\[
\theta = \frac{NN_1d^{1-\beta}}{\Gamma(\beta)}\Delta_1.
\]

**Proof.** Define \( \mathcal{E} : \mathcal{Y} \rightarrow C((0, d], \mathcal{K}) \) by
\[
\mathcal{E}\sigma = \int_0^d (d - r)^{\beta-1}V_\beta(d - r)\sigma(r)dr, \quad \text{for } \sigma(\cdot) \in \mathcal{Y},
\]
choosing a desired final state function \( \Psi \) and \( \mu > 0 \) then we have
\[
\|\Psi - \mathcal{R}_\alpha(\theta)[h(0) - g(0, h(0))] - g\lambda - \mathcal{E}\mathcal{A}\lambda - \mathcal{E}\mathcal{A}_1\lambda - \mathcal{E}\mathcal{G} \lambda - \mathcal{E}Bw\| < \mu.
\]
For any \( \Psi \in D(\mathcal{A}) \) and \( x_0 \in \mathcal{H} \), there exists \( \ell > 0 \) such that
\[
\mathcal{E}\ell = \Psi - \mathcal{R}_\alpha(\theta)h(0).
\]
In the above

\[(g, \lambda)(\vartheta) = (g(\vartheta, x_\vartheta), \lambda),\]
\[(G, \lambda)(\vartheta) = G(\vartheta, x(\vartheta - \sigma)),\]
\[A_1, \lambda = A_1 x(\vartheta - \sigma),\]

and

\[x_\vartheta(\vartheta) = x(\vartheta; w_\vartheta),\]

is a mild solution (1.1) according to \(w_\vartheta(\cdot)\) belongs to \(K\). Suppose \(x(\cdot, w) = x(\cdot)\) is a mild solution of (1.1) with respect to \(u(\cdot) \in W\) then

\[
\|x(\vartheta)\|_H = \theta^{1-b}\|R_{x, \beta}(\vartheta)[h(0) - g(0, h(0))] + g(\vartheta, x_\vartheta) + \int_0^\vartheta (\vartheta - \tau)^{\beta-1}V_\beta(\vartheta - \tau)[A_1 x(\vartheta - \sigma) + Bu(\tau) + G(\tau, x(\vartheta - \sigma))]d\tau\%
\]

\[
\leq \frac{N}{\Gamma(\alpha(1 - \beta) + \beta)} \|h(0)\| + \frac{L_2 d^{1-b}N_0 N}{\Gamma(\alpha(1 - \beta) + \beta)} + d^{1-b}N_0 L_2 (1 + \vartheta^{1-b}\|x(\vartheta)\|)
\]

\[
+ \frac{N d^{1-b}}{\Gamma(\beta)} \int_0^\vartheta (\vartheta - \tau)^{\beta-1}\|A_1 x(\vartheta - \sigma)\|d\tau
\]

\[
+ \frac{N d^{1-b}}{\Gamma(\beta)} \int_0^\vartheta (\vartheta - \tau)^{\beta-1}\|u(\tau)\|d\tau + \frac{N d^{1-b}}{\Gamma(\beta)} \int_0^\vartheta (\vartheta - \tau)^{\beta-1}\|G(\tau, x(\vartheta - \sigma))\|d\tau
\]

\[
\leq \frac{N \|h(0)\|}{\Gamma(\alpha(1 - \beta) + \beta)} + \frac{L_2 d^{1-b}N_0 N}{\Gamma(\alpha(1 - \beta) + \beta)} + d^{1-b}N_0 L_2 (1 + \vartheta^{1-b}\|x(\vartheta)\|)
\]

\[
+ d^{1-b}N_0 L_2 (1 + q)\Delta_2 + \frac{N N_1 d^{1-b}}{\Gamma(\beta)} \int_0^\vartheta (\vartheta - \tau)^{\beta-1}\|u(\tau)\|d\tau
\]

\[
+ \frac{N N_1 d^{1-b}}{\Gamma(\beta)} \int_0^\vartheta (\vartheta - \tau)^{\beta-1}\|u(\tau)\|d\tau
\]

\[
\leq \frac{N \|h(0)\|}{\Gamma(\alpha(1 - \beta) + \beta)} + \frac{N \|h(0)\|}{\Gamma(1 + \beta)} \left[ N_0 + N_b \|w\|_Y \right]
\]

\[
+ d^{1-b}N_0 L_2 (1 + q)\Delta_2 + \frac{N (N_1 + 1) d^{1-b}}{\Gamma(\beta)} \int_0^\vartheta (\vartheta - \tau)^{\beta-1}[x(\vartheta - \sigma) + G(\vartheta, 0)]d\tau
\]

\[
+ \frac{N \eta d^{1-b} e^{-\vartheta}}{\Gamma(\beta)} \int_0^\vartheta (\vartheta - \tau)^{\beta-1} e^{r(\tau - \sigma)}\|x(\vartheta)\|d\tau.
\] (5.1)

By using Gronwall’s inequality, Mittag-Leffler function

\[E_\beta(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(k\beta + 1)}, \text{ and } \|x\|_C^* = \sup_{\vartheta \in (0, b)} e^{-\vartheta} \|x(\vartheta)\|,
\]
(5.1) implies

\[ \|x\|_C^* \leq \mathbb{P}^* E_\beta (N_1 + d^{1-b} \eta) d^\beta, \]

where

\[ \mathbb{P}^* = \frac{N_1}{\Gamma(\alpha(1-\beta) + \beta)} \left[ \|h(0)\| + L_2 d^{1-b} \right] + \frac{d^{1-b} N_1}{\Gamma(\beta)} \Delta_1 \|w\|_C + \frac{N_1 d^{2(1-b) + \beta}}{\Gamma(\beta + 1)}. \]

Now

\[ \|x_1(\theta) - x_2(\theta)\| \leq e^{-\theta N_1 d^{1-b}} \left( \int_0^\theta (\theta - r)^{\beta-1} dr \right) \|w_2 - w_1\|_C, \]

\[ \|x_1 - x_2\|_C^* \leq \theta E_\beta (N_1 + d^{1-b} \eta) d^\beta \|w_2 - w_1\|_C. \]

This completes the proof. \( \square \)

**Theorem 5.4.** If \((\mathcal{F}_1) - (\mathcal{F}_3)\) are true then (1.1) is approximately controllable.

**Proof.** To verify \(D(\mathcal{A}) \subset \mathcal{H}\) for all \(\Psi \in D(\mathcal{A})\), there is a control \(w_\mu(\cdot) \in W\), such that

\[ \|\Psi - \mathcal{R}_{\alpha,\beta}(d)[h(0) - g(0, h(0))] - g_\alpha - E\mathcal{A}_1 x_\mu - \mathcal{E}\mathcal{G} x_\mu - \mathcal{E}\mathcal{B} w_\mu\| < \mu, \quad \mu > 0. \]  

(5.2)

For any \(x_0\) belongs to \(\mathcal{H}\) and there exists a function \(\ell(\cdot)\) belongs to \(\mathcal{Y}\), then

\[ \mathcal{E}\ell = \Psi - \mathcal{R}_{\alpha,\beta}(d)[h(0) - g(0, h(0))] - g_\alpha. \]

Suppose \(w_1(\cdot)\) belongs to \(W\) and \(\mu > 0\) then from \((\mathcal{F}_3)\) we choose an arbitrary value \(w_2(\cdot)\) belongs to \(W\) such that

\[ \|\Psi - \mathcal{R}_{\alpha,\beta}(d)[h(0) - g(0, h(0))] - g_\alpha - E\mathcal{A}_1 x_1 - \mathcal{E}\mathcal{G} x_1 - \mathcal{E}\mathcal{B} w_2\| < \frac{\mu}{2^\gamma}. \]

(5.3)

From the above we note \(x_1(\theta)\) takes \(x(\theta; w_1)\) and \(x_2(\theta)\) takes \(x(\theta; w_2)\) for \(0 \leq \theta \leq d\).

Again from \((\mathcal{F}_3)\) there exists \(\sigma_2(\cdot) \in W\), such that

\[ \left\| \mathcal{E}[\mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G} x_2 - \mathcal{G} x_1] - \mathcal{E}\mathcal{B} \sigma_2 \right\| < \frac{\mu}{2^\gamma}, \]

we now consider

\[ \left\| B \sigma_2 \right\|_\mathcal{Y} \leq v \left\| \mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G} x_2 - \mathcal{G} x_1 \right\| \]

\[ \leq v (N_1 + d^{1-b} \eta) \|x_2(\cdot) - x_1(\cdot)\| \]

\[ \leq \theta \nu (N_1 + d^{1-b} \eta) E_\beta (N_1 + d^{1-b} \eta) d^\beta \|w_2 - w_1\|_C. \]

Define \(w_3(\theta) = w_2(\theta) - v(\theta), w_3(\cdot) \in W\), then

\[ \|\xi - \mathcal{R}_{\alpha,\beta}(\theta)[h(0) - g(0, h(0))] - g_\alpha - E\mathcal{A}_1 x_2 - \mathcal{E}\mathcal{G} x_2 - \mathcal{E}\mathcal{B} w_3\| \]

\[ \leq \|\xi - \mathcal{R}_{\alpha,\beta}(\theta)[h(0) - g(0, h(0))] - g_\alpha - E\mathcal{A}_1 x_1 - \mathcal{E}\mathcal{G} x_1 - \mathcal{E}\mathcal{B} w_2\| \]
+ \|Efv2 - E[A_1x_2 - A_1x_1 + Gx_2 - Gx_1]\| \leq \left(\frac{1}{2^2} + \frac{1}{2^3}\right)\mu.

Suppose there is a sequence \(\{x_k(\cdot)\} \subset X\), then

\[
\|\xi - \mathcal{R}_{\alpha,\beta}(\vartheta)[h(0) - g(0, h(0))] - \vartheta A_1x_k - \vartheta A_1x_k - \vartheta Gx_k - \vartheta Bw_k\| \leq \left(\frac{1}{2^2} + \cdots + \frac{1}{2^3}\right)\mu.
\]

In the above \(x_k(\cdot) = x(\cdot; w_k)\) for \(0 \leq \vartheta \leq \mu\), and

\[
\|Bw_{k+1} - Bw_k\|_H \leq \theta N(\mathbb{N}_1 + d^{\beta-\eta})\varphi E\varphi(\mathbb{N}(\mathbb{N}_1 + d^{\beta-\eta})\varphi|\varphi(w_k - w_{k-1})|_H.
\]

By referring (5.3) and there exists a \(\chi(\cdot) \in W\), such that

\[
\lim_{k \to \infty} Bw_k(\cdot) = \chi(\cdot) \in Y.
\]

As a result, for every \(\mu > 0\), there exists a positive integer number \(N\), such that

\[
\|Efw_{N+1} - Efw_N\| < \frac{\mu}{2}.
\]

Hence, we get

\[
\|\Psi - \mathcal{R}_{\alpha,\beta}(\vartheta)[h(0) - g(0, h(0))] - \vartheta A_1x_N - \vartheta A_1x_N - \vartheta Gx_N - \vartheta Bw_N\|
\leq \|\Psi - \mathcal{R}_{\alpha,\beta}(\vartheta)[h(0) - g(0, h(0))] - \vartheta A_1x_N - \vartheta A_1x_N - \vartheta Gx_N - \vartheta Bw_{N+1}\|
\leq \|Efw_{N+1} - Efw_N\|
\leq \left(\frac{1}{2^2} + \cdots + \frac{1}{2^3}\right)\mu + \frac{\mu}{2} < \mu.
\]

Therefore (1.1) is approximate controllability. Thus this ends the proof.

\[\square\]

6. Nonlocal condition

Byszewski [15, 16] investigated the idea of “nonlocal conditions”, proving the existence and uniqueness of mild, strong, and classical nonlocal Cauchy problem solutions for semilinear evolution equations. In [51] the authors considered the controllability with nonlocal conditions by utilizing fixed point methods and fractional calculus. A valuable conversation about the nonlocal conditions are given in [25, 27, 51].

Apparently, the controllability of neutral differential problems in particular of time delay with nonlocal conditions with respect to Hilfer fractional differential equations has not been explored at this point. Motivated by the articles [25, 53, 56], consider

\[
\begin{cases}
\mathcal{D}_{0+}^{\alpha,\beta}[x(\vartheta) - g(\vartheta, x_0)] = A_1x(\vartheta) + A_1x(\vartheta - \sigma) + Bu(\vartheta) + G(\vartheta, x(\vartheta - \sigma)), \text{ for all } \vartheta > 0,
\{I_{0+}^{(1-\alpha)(1-\beta)}x(\vartheta) + p(\vartheta, \vartheta, \vartheta_2, \vartheta_3, \cdots, \vartheta_n) = h(\vartheta), \vartheta \in [-\sigma, 0]\}.
\end{cases}
\]

(6.1)

Where \(K\) is a positive real, \(0 < t_1 < t_2 < t_3 < \cdots < t_n \leq d\), \(p : C([0, K], H) \to H\) and satisfying the following assumption:

\[\square\]

Theorem 6.2. If the assumptions (F₁)–(F₅) are true then (6.2) is approximately controllable.

Proof. To verify $\overline{D(\mathcal{A})} \subset \mathcal{H}$ for every $\Psi \in D(\mathcal{A})$, suppose there is a control $u_\mu(\cdot) \in U$, then

$$
\|\Psi - R_{x,\mu}(d)[\psi(\theta_1, \theta_2, \ldots, \theta_n)(0) + \hat{h}(0) - g(0, h(0))] - g_\lambda \mu - E\mathcal{A}_1 x_\mu - E\mathcal{G} x_\mu - E\mathcal{B} u_\mu\| < \mu, \ \mu > 0.
$$

(6.3)

For any $x_0 \in \mathcal{H}$ and there exists a function $\ell(\cdot) \in \mathcal{X}$, such that

$$
\mathcal{E}\ell = \Psi - R_{x,\mu}(d)[\psi(\theta_1, \theta_2, \ldots, \theta_n)(0) + \hat{h}(0) - g(0, h(0))] - g_\lambda \mu.
$$

Suppose $u_1(\cdot) \in U$ and $\mu > 0$ then from (F₅) we choose an arbitrary value $v_2(\cdot) \in W$ such that

$$
\|\Psi - R_{x,\mu}(d)[\psi(\theta_1, \theta_2, \ldots, \theta_n)(0) + \hat{h}(0) - g(0, h(0))] - g_\lambda \mu - E\mathcal{A}_1 x_\mu - E\mathcal{G} x_\mu - E\mathcal{B} u_2\| < \frac{\mu}{2^2}.
$$

(6.4)

From the above we note $x_1(\theta)$ takes $x(\theta; u_1)$ and $x_2(\theta)$ takes $x(\theta; u_2)$ for $0 \leq \theta \leq d$.

Again from (F₅) there exists $x_1(\cdot) \in U$, such that

$$
\left\|\mathcal{E}[\mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G} x_2 - \mathcal{G} x_1] - E\mathcal{B} x_2\right\| < \frac{\mu}{2^3},
$$

we now consider

$$
\left\|B x_2\right\|_Y \leq v\left\|\mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G} x_2 - \mathcal{G} x_1\right\|
$$

$$
\leq v(N_1 + d^{1-\beta} \eta)|x_2(\cdot) - x_1(\cdot)|
$$

$$
\leq 2v(N_1 + d^{1-\beta} \eta)E(\mathcal{A}[N_1 + d^{1-\beta} \eta]|u_2 - u_1|_C).
$$

Define $u_3(\theta) = u_2(\theta) - v_2(\theta), u_3(\cdot) \in U$, then
As a result, any \( \mu > 0 \), then
\[
\|\xi - R_{\alpha,\beta}(\theta)[p(\theta_{1}, \theta_{2}, \theta_{3}, \cdots, \theta_{n})(0) + h(0) - g(0, h(0))] - g_{\lambda} - \mathcal{E}A_{1}x_{2} - \mathcal{E}Gx_{2} - \mathcal{E}B_{1}\| \\
\leq \|\xi - R_{\alpha,\beta}(\theta)[p(\theta_{1}, \theta_{2}, \theta_{3}, \cdots, \theta_{n})(0) + h(0)] - \mathcal{E}A_{1}x_{1} - \mathcal{E}Gx_{1} - \mathcal{E}B_{2}\| \\
+ \|\mathcal{E}Bv_{2} - \mathcal{E}[A_{1}x_{2} - A_{1}x_{1} + Gx_{2} - Gx_{1}]\| \leq \left(\frac{1}{2^{2}} + \frac{1}{2^{3}}\right)\mu.
\]

Suppose there is a sequence \( \{x_{k}(\cdot)\} \subset X \), then
\[
\|\xi - R_{\alpha,\beta}(\theta)[p(\theta_{1}, \theta_{2}, \theta_{3}, \cdots, \theta_{n})(0) + h(0)] - \mathcal{E}A_{1}x_{k} - \mathcal{E}Gx_{k} - \mathcal{E}B_{k+1}\| < \left(\frac{1}{2^{2}} + \cdots + \frac{1}{2^{k}}\right)\mu.
\]

In the above \( x_{k}(\cdot) = x(\cdot; u_{k}) \) for \( 0 \leq \theta \leq \mu \), and
\[
\|B_{k+1} - Bu_{k}\|_{H} \leq \theta N(\mathcal{N}) + d^{1-\beta}_{\eta}N(\mathcal{N})d^{\beta}||u_{k}(\cdot) - u_{k-1}(\cdot)||_{H}.
\]

By referring (6.4) and there exists a \( \chi(\cdot) \in U \) such that
\[
\lim_{k \to \infty} Bu_{k}(\cdot) = \chi(\cdot) \in \mathcal{Y}.
\]

As a result, any \( \mu > 0 \), there is a positive integer number \( N \), then
\[
\|\mathcal{E}B_{N+1} - \mathcal{E}B_{N}\| < \frac{\mu}{2}.
\]

Hence, we get
\[
\|\Psi - R_{\alpha,\beta}(\theta)[p(\theta_{1}, \theta_{2}, \theta_{3}, \cdots, \theta_{n})(0) + h(0)] - \mathcal{E}A_{1}x_{N} - \mathcal{E}Gx_{N} - \mathcal{E}B_{N}\| \\
\leq \|\Psi - R_{\alpha,\beta}(\theta)[p(\theta_{1}, \theta_{2}, \theta_{3}, \cdots, \theta_{n})(0) + h(0)] - \mathcal{E}A_{1}x_{N} - \mathcal{E}Gx_{N} - \mathcal{E}B_{N+1}\| \\
+ \|\mathcal{E}B_{N+1} - \mathcal{E}B_{N}\| \leq \left(\frac{1}{2^{2}} + \cdots + \frac{1}{2^{N}}\right)\mu + \frac{\mu}{2} < \mu.
\]

As a consequence, the system (6.1) is approximately controllable. This ends the proof. \( \square \)

7. Application

Consider
\[
\begin{align*}
D_{0+}^{\alpha, \beta} & \left[ x(\theta, \beta) - \int_{0}^{\beta} c(\beta, u)x(\theta, \beta)d\beta \right] = \frac{\partial}{\partial \beta} x(\theta, \beta) + x(\theta - \sigma, \beta) + G(\theta, x(\theta - \sigma, \beta)) + Bu(\theta, \beta), \ \theta \in (0, 1), \\
x(\theta, 0) &= x(\theta, \pi) = 0, \ \theta \geq 0, \ \text{and} \\
x_{0} & \left[ (1 - \alpha) x(0, \beta) = h(\beta), \ \theta \in [-\sigma, 0], \ \beta \in [0, \pi]. \right.
\end{align*}
\]

The Hilfer fractional derivative is symbolized by \( D_{0+}^{\alpha, \beta} \), whose order and type are \( \frac{3}{2} \), \( 0 \leq \alpha \leq 1 \) and \( I_{0+}^{(1 - \alpha)} \) is the Reimann-Liouville integral of order \( \frac{1}{3}(1 - \alpha) \). The function \( G(\cdot, \cdot) \in L^{2}([0, \pi] \times [0, \pi], \mathbb{R}^{+}) \), for \( m > 0 \).
Abstract form: Considering $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$, $\mathcal{H} = L^2([0, \pi], \mathbb{R})$ which is defined as $\mathcal{A} \nu = \nu''$, $\nu \in D(\mathcal{A})$, where

$$D(\mathcal{A}) = \nu \in \mathcal{H} : \nu, \nu' \text{ are absolutely continuous,}$$

and

$$D(\mathcal{A}) = \nu'' \in \mathcal{H}, \nu(0) = \nu(\pi) = 0.$$ 

Also, $\mathcal{A}$ satisfies $C_1, C_2$ Then, we have

$$\mathcal{A} x = -\sum_{h=1}^{\infty} h^2 \langle x, \zeta_h \rangle \zeta_h, \quad \vartheta \in D(\mathcal{A}), \quad (7.2)$$

where

$$\zeta_h(\beta) = \sqrt{\frac{2}{\pi}} \sin(h\beta), \quad h = 1, 2, \cdots.$$ 

For all $x \in \mathcal{H}$,

$$T(\vartheta)x = \sum_{h=1}^{\infty} e^{-h^2\vartheta} \langle x, \zeta_h \rangle \zeta_h, \quad \|T(\cdot)\| \leq 1.$$ 

The function $g : [0, d] \times \rightarrow \mathcal{H}$ is defined by $g(\vartheta, x_\vartheta) = \int_0^\pi c(\beta, u) x(\vartheta, \beta) d\beta$.

Let $\frac{\partial}{\partial \vartheta} c(\beta, u)$ be measurable, $c(0, u) = c(\pi, u) = 0$, and

$$L = \sqrt{\int_0^\pi \int_0^\pi \left(\frac{\partial}{\partial \beta} c(\beta, u)\right)^2 d\beta dz} < \infty, \quad \int_0^\pi \int_0^\pi c^2(\beta, u) d\beta dz < \infty,$$

c is measurable.

Hence, $X_1(\vartheta) \in D(\mathcal{A}^\frac{1}{2})$ and $\|\mathcal{A}^\frac{1}{2}\|^2 \leq L$.

Therefore,

$$\langle X_1(\vartheta), \zeta_h \rangle = \int_0^\pi \zeta_h(x) \int_0^\pi c(\beta, u) u(\beta) d\beta dx = \frac{1}{h} \sqrt{\frac{2}{\pi}} \left(x(\vartheta), \frac{1}{\sec(h\vartheta)}\right).$$

Let’s define the $B$ as

$$Bu(\vartheta) = \sum_{h=1}^{\infty} \hat{u}_h(\vartheta) \zeta_h, \quad u(\vartheta) = \sum_{h=1}^{\infty} u_h(\vartheta) \zeta_h,$$

where $u_h(\vartheta) = \langle u(\vartheta, \zeta_h) \rangle, h = 0, 1, 2, \cdots$.

$$\hat{u}_h(\vartheta) = \begin{cases} 0, & 0 \leq \vartheta < 1 - \frac{1}{h^2}, \\ u_h(\vartheta), & 1 - \frac{1}{h^2} \leq \vartheta \leq 1; \end{cases} \quad (7.3)$$

the reader can understand by $\|Bu(\cdot)\| \leq \|u(\cdot)\|$, $B$ is bounded linear operator. The linear system of (7.1) is

$$\begin{cases} D_{0+}^{\frac{1}{2}}[x_h(\vartheta) - g(\vartheta)] = u_h(\vartheta), \quad \vartheta \in (0, 1], \beta \in \mathcal{K}, \\ I_{0}^{\frac{1}{2}(1-\alpha)} x_h(0) = h(\vartheta), \quad \vartheta \in [-\sigma, 0], \quad \beta \in [0, \pi]. \end{cases} \quad (7.4)$$
Hence, for any given $f(\cdot)$

$$\rho = \int_0^1 (1 - r)^{-\frac{1}{3}} \mathcal{G}_{\frac{2}{3}}(1 - r)f(r)dr = \sum_{h=1}^{\infty} \rho_h \varsigma_h, \quad \rho_h = \langle \rho, \varsigma_h \rangle.$$  

For any $f(\cdot)$ assume $\hat{u}_h$ as

$$\hat{u}_h = \frac{2h^2}{(1 - e^{-2})} \rho_he^{-h^2(1-\theta)}, \quad \frac{1}{h^2} \leq \theta \leq 1,$$

where

$$\rho_h = \int_{1 - \frac{1}{h^2}}^{1} \int_0^{\infty} (1 - r)^{-\frac{1}{3}} \vartheta N_{\frac{2}{3}}(\vartheta)e^{-h^2(1-\theta)\vartheta^2},$$

$$\int_0^1 (1 - r)^{-\frac{1}{3}} \mathcal{G}_{\frac{2}{3}}(1 - r)Bu(r)dr = \int_0^1 (1 - r)^{-\frac{1}{3}} \mathcal{G}_{\frac{2}{3}}(1 - r)f(r)dr,$$

as a consequence, $\mathcal{F}_6$ is fulfilled. Furthermore, we have

$$\|Bu(\cdot)\|^2 = \sum_{h=1}^{\infty} \int_{1 - \frac{1}{h^2}}^{1} |\hat{u}_h(\vartheta)|^2 d\vartheta = (1 - e^{-2})^{-1} \sum_{h=1}^{\infty} 2h^2 \rho_h^2$$

$$= \frac{3}{2} (1 - e^{-2})^{-1} \sum_{h=1}^{\infty} (1 - e^{-2h^2}) \int_0^1 |f_h(\vartheta)|^2 d\vartheta \leq \frac{3}{2} (1 - e^{-2})^{-1} |f(\cdot)|^2.$$  

As a result, it can be observed that if the conditions $\mathcal{F}_6$, is satisfied then (7.1) is approximately controllable on $\mathcal{H}$.

8. Conclusions

Our study investigates the existence and approximate controllability of HF neutral evolution equations with time delay. The Sequence method was used to derive the approximate controllability outcomes for HFD equations with time delay. An illustration is offered to support the analytical findings at the end and also given a filter diagram to represent the mild solution of the system with neutral term. Next, new research may use the sequence method with infinite delay to extend the Hilfer fractional stochastic differential evolution equations to approximate control results.

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Conflict of interest

Conflicts of interest are not present in this study.
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