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Research article

Global solvability of 3D non-isothermal incompressible nematic liquid crystal flows

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Abstract: We are concerned with the initial value problem of non-isothermal incompressible nematic liquid crystal flows in $\mathbb{R}^3$. Through some time-weighted a priori estimates, we prove the global existence of a strong solution provided that $\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right)\left(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2\right)$ is reasonably small, which extends the corresponding Li's (Methods Appl. Anal. 2015 [4]) and Ding-Huang-Xia's (Filomat 2013 [2]) results to the whole space $\mathbb{R}^3$ and non-isothermal case. Furthermore, we also derive the algebraic decay estimates of the solution.

Keywords: inhomogeneous case; incompressible nematic liquid crystal flows; global strong solution; vacuum
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1. Introduction

The motion of incompressible non-isothermal nematic liquid crystal flows can be governed by the following simplified version of the Ericksen-Leslie equations (see [12, 13])

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u + \nabla p = -\operatorname{div}(\nabla d \odot \nabla d), \\ c_v[(\rho\theta)_t + \operatorname{div}(\rho u\theta)] - \kappa\Delta\theta = 2\mu|\mathfrak{D}(u)|^2 + |\Delta d + |\nabla d|^2 d|^2, \\ d_t + u\cdot\nabla d = \Delta d + |\nabla d|^2 d, \quad |d| = 1, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

which is equipped with the following initial conditions

$$(\rho, u, \theta, d)(x, 0) = (\rho_0, u_0, \theta_0, d_0)(x) \quad \text{for } x \in \mathbb{R}^3. \tag{1.2}$$

Here, ρ, u, θ and d, p are the density, velocity, absolute temperature, macroscopic average of the nematic liquid crystal orientation field and pressure respectively. The $\nabla d \odot \nabla d$ denotes a matrix whose ij-th entry $(1 \leq i, j \leq N)$ is $\partial_{x_i} d \cdot \partial_{x_j} d$. Indeed, $\nabla d \odot \nabla d = (\nabla d)^\top \nabla d$, where $(\nabla d)^\top$ denotes the transpose of the 3×3 matrix ∇d. $\mathfrak{D}(u)$ denotes the deformation tensor given by

$$\mathfrak{D}(u) = \frac{1}{2}(\nabla u + \nabla u^T).$$

The constant $\mu > 0$ is the viscosity coefficient. The positive constants c_v and κ are respectively the heat capacity and ratio of the heat conductivity coefficient to the heat capacity.

The full Ericksen-Leslie model and the simplified version are macroscopic continuum descriptions of the time evolution of the materials, under the influence of both the flow velocity field u and the microscopic orientation configurations d of rod-like liquid crystals. Mathematically, system (1.1) is a strongly coupled system between the nonhomogeneous incompressible Navier-Stokes equations of fluid dynamics and the transported heat flows of harmonic maps. When d is a constant vector and $|d| = 1$, (1.1) reduces to the nonhomogeneous heat-conducting Navier-Stokes equations, which have been discussed in numerous studies on the global well-posedness of strong solutions(please see [1, 3, 14–16, 18–20] and references therein). Due to the strong coupling terms and strong nonlinear terms, the system (1.1) the well-posedness is harder to study than Navier-Stokes equations.

If we don't take into account $(1.1)_3$, the system (1.1) becomes the classical inhomogeneous incompressible nematic liquid crystal flows. In the presence of a vacuum, under the following compatibility condition

$$-\mu\Delta u_0 - \nabla p_0 - \lambda \mathrm{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0} g \tag{1.3}$$

for some $(p_0, g) \in H^1 \times L^2$, Wen-Ding [17] first established the local existence and uniqueness of a strong solution. Later, Li [4] and Ding et al. [2] respectively investigated the global existence of a strong solution to the 3D initial boundary value problem and to the 3D Cauchy problem with reasonably small data; they extended Wen-Ding's result [17] to the global one. At the same time, Li [4] considered the case in 2D-bounded domains (see also [5]). It should be pointed out that the crucial techniques of proofs in [2,4,5] cannot be directly adapted to the situation for the Cauchy problem of the 2D equations. One reason is that when $\Omega = \mathbb{R}^2$ becomes unbounded, the standard Sobolev embedding inequality is critical, and it seems difficult to bound the L^r-norm $(r > 2)$ of the velocity u just in terms of $\| \sqrt{\rho} u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$. Motivated by [10], by introducing a weighted function to the density, as well as the Hardy-type inequality proposed in [7] by Lions, the $\|\rho^\alpha u\|_{L^r}$ $(r > 2, \alpha > 0)$ could be controlled in terms of $\| \sqrt{\rho} u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$.Thus, Liu et al. [6, 8] generalized the results of [4, 5] to the whole space $\mathbb{R}^2$.

Very recently, the author of [9] applied global well-posedness to the 2D Cauchy problem with a vacuum and large initial data. However, it is rather difficult to investigate the global well-posedness and dynamical behaviors of non-isothermal incompressible nematic liquid crystal flows with large initial data in $\mathbb{R}^3$ using the same method. Since the Sobolev embedding inequality is critical, we cannot derive the a priori estimates for the strong solutions with large initial data, which is the key to extending the global strong solution by Lemma 2.1. Fortunately, motivated by [4], the global strong solutions of the systems (1.1) and (1.2) can be established using some small norm of the initial. But the priori estimates of [4] depend on the boundness of the domain due to the Poincaré inequality, which fails to deal with the situation here. On the other hand, there are also other interesting studies on the large time behavior of solutions, which is absent in [2,4]. Thanks to [3], which established the global existence and large time behavior of strong solutions to the 3D Cauchy problem of Navier-Stokes equations using exponential time-weighted energy estimates. However, due to the strong coupling terms and strong nonlinear terms $\nabla d \odot \nabla d$, $u \cdot \nabla d$ and $|\nabla d|^2 d$ in the system (1.1), we cannot directly adapt their method to our case. Thus, the main purpose of the present study was to investigate the global existence and large time behavior of strong solutions with large oscillations and a vacuum to the problems (1.1) and (1.2), provided that the scaling invariant quantity is properly small which generalized the corresponding results [3,4] to the Cauchy problem of non-isothermal incompressible nematic liquid crystal flows.

Before formulating our main results, we will first explain the notations and conventions used throughout this paper. For simplicity, we set

$$\int f dx = \int_{\mathbb{R}^3} f dx, \quad \mu = c_v = \kappa = 1.$$

For $1 \le r \le \infty$, $k \ge 1$, the Sobolev spaces are defined in a standard way.

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \quad D^{k,r} = D^{k,r}(\mathbb{R}^3) = \{v \in L^1_{\text{loc}}(\mathbb{R}^3) | D^k v \in L^r(\mathbb{R}^3)\}, \\ W^{k,p} = W^{k,p}(\mathbb{R}^3), \quad H^k = W^{k,2}, \quad D^k = D^{k,2}. \end{cases}$$

Our main results in this paper are listed in the following Theorem 1.1.

Theorem 1.1. *Assume that the initial data* $(\rho_0, u_0, \theta_0, d_0)$ *satisfies*

$$0 \le \rho_0 \le \bar{\rho}, \quad \rho_0 \in L^{\frac{3}{2}} \cap H^1 \cap W^{1,6}, \quad (u_0, \theta_0) \in D^{1,2} \cap D^{2,2}, \quad \nabla d_0 \in H^2, \quad |d_0| = 1, \tag{1.4}$$

and the compatibility condition

$$\begin{cases} -\Delta u_0 + \nabla p_0 + \text{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0} g_1, \\ -\Delta \theta_0 - 2|\mathfrak{D}(u_0)|^2 - |\Delta d_0 + |\nabla d_0|^2 d_0|^2 = \sqrt{\rho_0} g_2, \end{cases} \tag{1.5}$$

with $p_0 \in H^1$ *and* $g_1, g_2 \in L^2$. *Let* $\mathbb{K}_0$ *be the following constant*

$$\mathbb{K}_0 := \left(\|\sqrt{\rho_0} u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right)\left(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2\right). \tag{1.6}$$

Then, there is a positive constant ϵ_0*, which depends only on* $\bar{\rho}$ *and initial data, such that if*

$$\mathbb{K}_0 \le \epsilon_0, \tag{1.7}$$

then the problems (1.1) *and* (1.2) *have a unique global solution* (ρ, u, θ, d) *on* $\mathbb{R}^3 \times (0, \infty)$ *satisfying*

$$\begin{cases} 0 \le \rho \in C([0,\infty); L^{\frac{3}{2}} \cap H^1 \cap W^{1,6}), \quad \rho_t \in L^\infty([0,\infty); L^2 \cap L^3), \\ (u, \theta) \in C(0,\infty; D^{1,2} \cap D^{2,2}) \cap L^2(0,\infty; D^{2,6}), \\ (u_t, \theta_t) \in L^2(0,\infty; D^{1,2}), \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty(0,\infty; L^2), \\ \nabla p \in C([0,\infty); L^2) \cap L^2(0,\infty; L^2), \\ \nabla d \in C([0,\infty); H^2) \cap L^2([0,\infty); H^3), \\ d_t \in C([0,\infty); H^1) \cap L^2([0,\infty); H^2). \end{cases} \tag{1.8}$$

Moreover, there exists some positive constant C depending only on $\bar{\rho}$ and initial data such that for all $t \geq 1$,

$$\begin{cases} \|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2 \leq Ct^{-\frac{1}{2}}, \\ \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 \leq Ct^{-\frac{3}{2}}, \\ \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2 \leq Ct^{-\frac{5}{2}}, \\ \|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 \leq Ct^{-\frac{5}{2}}, \\ \|\nabla^2 d\|_{L^2}^2 + \|d_t\|_{L^2}^2 \leq Ct^{-1}, \\ \|\nabla^3 d\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \leq Ct^{-2}. \end{cases} \tag{1.9}$$

Remark 1.1. *When temperature field $\theta \equiv 0$, System* (1.1) *reduces to the classical incompressible nematic liquid crystal flows. Compared to [4] for the 3D-bounded domain, some new difficulties occur. First, Poincare's inequality fails for 3D Cauchy problem, which is key to estimating $\|u\|_{L^2}$; one way to overcome this is to estimate $\|\sqrt{\rho}u\|_{L^2}$ and $\|\nabla u\|_{L^2}$. Furthermore, the higher-order estimates of the smooth solution is independent on T, which is different from [4]. Since the corresponding estimates in [4] depend on T, we have improved the priori estimates of [4]. In particular, we also extend the priori estimates of [4] to the non-isothermal case. To the best of our knowledge, Theorem 1.1 is the first result on global well-posedness for a model of multi-dimensional non-isothermal incompressible nematic liquid crystal flows in a vacuum.*

Remark 1.2. *Using the time-weighted a priori estimates, we extend Zhong's corresponding result [19] to incompressible nematic liquid crystal flows by studying the global well-posedness of the Cauchy problem of inhomogeneous heat-conducting Navier-Stokes equations. Moreover, we derive the algebraic decay estimates of the solution, which is lacking in [19].*

Remark 1.3. *As mentioned above, He et al. [3] obtained the exponential decay-in-time property for the 3D Cauchy problem of inhomogeneous Navier-Stokes equations with a vacuum [that is,* (1.1) *with $d = 1$ and $\theta = 0$]; however, it seems impossible to derive such a decay property due to the sructural characteristics of* (1.1), *especially given that $\rho_0 \in L^{\frac{3}{2}}(\mathbb{R}^3)$ is required. Thus, the the macroscopic average of the nematic liquid crystal orientation field acts as some significant roles on the large time behaviors of velocity and the temperature.*

Remark 1.4. *It should be noted that* (1.6) *is independent of the temperature. The results indicate that the global existence for inhomogeneous incompressible heat conducting models of viscous media is similar to the nonhomogeneous incompressible nematic liquid crystal flows,and therefore does not depend on further sophistication of the heat conducting model.*

The framework of the proof of Theorem 1.1 is the continuation argument (see [4]). We should point out that the crucial techniques used in [4] cannot be adapted to the situation treated here, since their arguments depend heavily on the boundedness of the domains. Moreover, compared to [3,19], the proof of Theorem 1.1 is much more involved due to the strong coupling terms and strong nonlinear terms $\nabla d \odot \nabla d$, $u \cdot \nabla d$ and $|\nabla d|^2 d$. To overcome the difficulties stated above, we need some new ideas. First, we attempted to get time-in-uniform (weighted) estimates on the $L^\infty(0, T; L^2)$-norm of the gradient of velocity, absolute temperature and the gradient of the macroscopic average of the nematic liquid crystal orientation field. We found that the key way to control the terms $\int |u|^2|\nabla^2 d|^2 dx$, $\int |\nabla d|^2|\nabla u|^2 dx$ and $\int |\nabla d|^2|\nabla^2 d|^2$. Thus, we need to achieve higher integrability of the macroscopic average of the nematic liquid crystal orientation field. Motivated by [3, 19], we assume the priori hypothesis (3.1). Hence, the key step is to complete the proof of the a priori hypothesis, that is, to show (3.2). Based on the regularity properties of the Stokes system and elliptic equations, we can obtain the desired bounds, provided that (1.7) is suitably small. Next, the crucial dissipation estimate of the form $\int_0^T \|\nabla\theta\|_{L^2}^2 dt$ (see Lemma 3.5) is important to derive the time-weighted estimate of $\|\sqrt{\rho}\theta_t\|_{L^2}^2$. However, this argument fails for the situation here, since their estimates dependent on T. To overcome this difficulty, we multiply the momentum equation$(1.1)_2$ by $u\theta$ to recover good bounds for the temperature θ [see (3.81)]. Once these key estimates are obtained, we can obtain the desired global high-order a priori estimates of the solution.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the a priori estimates. Finally, we will give the proof of Theorem 1.1 in Section 4.

2. Preliminaries

We begin with the local existence and uniqueness of strong solutions whose proof can be performed by using standard energy methods (see [1, 17]).

Lemma 2.1. *Assume that* $(\rho_0, u_0, \theta_0, d_0)$ *satisfies* (1.4) *and* (1.5). *Then there exists a small time* $T_0 > 0$ *and a unique strong solution* (ρ, u, θ, d, p) *to the problems* (1.1) *and* (1.2) *in* $\mathbb{R}^3 \times (0, T)$ *satisfying* (1.8).

Next, we have some regularity results for the Stokes equations, which have been proven in [3].

Lemma 2.2. *For* $r \in (1, \infty)$, *if* $F \in L^{\frac{6}{5}} \cap L^r$, *there exists some positive constant C depending only on r such that the unique weak solution* $(u, p) \in D^1 \times L^2$ *to the following Stokes system*

$$\begin{cases} -\mathrm{div}(2\mu\mathfrak{D}(u)) + \nabla p = F, & \text{in } \mathbb{R}^3, \\ \mathrm{div} u = 0, & \text{in } \mathbb{R}^3, \\ u(x) \to 0. & \text{as } |x| \to \infty, \end{cases} \tag{2.1}$$

satisfies

$$\|\nabla u\|_{L^2} + \|p\|_{L^2} \le C\|F\|_{L^{\frac{6}{5}}}, \tag{2.2}$$

and

$$\|\nabla^2 u\|_{L^r} + \|\nabla p\|_{L^r} \le C\|F\|_{L^r}. \tag{2.3}$$

Next, the following Gagliardo-Nirenberg inequalities will be stated(see [11] for the detailed proof).

Lemma 2.3. *Assume that* $f \in H^1$ *and* $g \in L^q \cap D^{1,r}$ *with* $q > 1$ *and* $r > 3$. *Then for any* $p \in [2, 6]$, *there exists a positive constant C, depending only on p, q and r, such that*

$$\|f\|_{L^p} \le C\|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}}, \tag{2.4}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^q}^{\frac{q(r-3)}{3r+q(r-3)}} \|\nabla g\|_{L^r}^{\frac{3r}{3r+q(r-3)}}. \tag{2.5}$$

3. A priori estimates

In this section, we will establish some necessary a priori estimates, which together with the local existence (cf. Lemma 2.1) will complete the proof of Theorem 1.1. To this end, we let (ρ, u, d, θ, p) be a strong solutions of (1.1) and (1.2) in $\mathbb{R}^3 \times [0, T]$. For simplicity, we use the letters C, c_i and $C_i (i = 1, 2, \ldots)$ to denote some positive constant that is dependent on $\bar{\rho}$ and initial data, but are independent of T. We also sometimes write $C(\alpha)$ to emphasize the dependence on α.

We first aim to get the following key a priori estimates on (ρ, u, d, θ, p).

Proposition 3.1. *Assume that*

$$\mathbb{K}_0 \le \epsilon_0,$$

there exists some small positive constant ϵ_0 depending only on $\bar{\rho}$ and initial data such that if (ρ, u, d, θ, p) is a smooth solution of (1.1) *and* (1.2) *on $\mathbb{R}^3 \times (0, T]$ satisfying*

$$\sup_{0 \le t \le T} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \le 2\mathbb{K}_0^{\frac{1}{2}}, \tag{3.1}$$

then the following estimate holds

$$\sup_{0 \le t \le T} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \le \mathbb{K}_0^{\frac{1}{2}}. \tag{3.2}$$

Moreover, we have

$$\sup_{0 \le t \le T} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \le C. \tag{3.3}$$

Proof of Proposition 3.1. (1) Multiplying the transport equation $(1.1)_1$ by $q\rho^{q-1}$ for $\forall q > 1$, integrating by parts and using $(1.1)_5$, we have

$$\frac{d}{dt}\|\rho(t)\|_{L^q}^q = 0.$$

It derives

$$\|\rho(t)\|_{L^q} = \|\rho_0\|_{L^q}. \tag{3.4}$$

(2) Multiplying $(1.1)_2$ by u, and integrating by parts over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho |u|^2 dx + \int |\nabla u|^2 dx \\
&= \int \Big[\Delta d \cdot \nabla d + \nabla\Big(\frac{|\nabla d|^2}{2}\Big)\Big] \cdot u = -\int (u \cdot \nabla) d \cdot \Delta d \\
&\le C\|u\|_{L^6}\|\nabla d\|_{L^3}\|\nabla^2 d\|_{L^2} \\
&\le \frac{1}{4}\|\nabla u\|_{L^2}^2 + C\|\nabla d\|_{L^3}^2\|\nabla^2 d\|_{L^2}^2.
\end{aligned} \tag{3.5}$$

Using the fact that $|d|=1$ and integrating by parts, we infer from $(1.1)_4$ and Hölder's and Gagliardo-Nirenberg inequalities that

$$\begin{aligned}
&\frac{d}{dt}\int|\nabla d|^2dx+\int(|d_t|^2+|\nabla^2 d|^2)dx\\
&=\int|d_t-\Delta d|^2dx=\int|u\cdot\nabla d-|\nabla d|^2d|^2dx\\
&\le C\|u\|_{L^6}^2\|\nabla d\|_{L^3}^2+C\|\nabla d\|_{L^4}^4\\
&\le C\|\nabla d\|_{L^3}^2(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2),
\end{aligned}\tag{3.6}$$

which together with (3.1) and (3.5) yield

$$\begin{aligned}
&\frac{d}{dt}\int\Big(\frac{1}{2}\rho|u|^2+|\nabla d|^2\Big)dx+\int\Big(\frac{3}{4}|\nabla u|^2+|d_t|^2+|\nabla^2 d|^2\Big)dx\\
&\le C\|\nabla d\|_{L^3}^2(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2)\\
&\le C\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2}(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2)\\
&\le C_1\mathbb{K}_0^{\frac{1}{4}}(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2).
\end{aligned}\tag{3.7}$$

This gives rise to

$$\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2+\|\nabla d\|_{L^2}^2)+\frac{1}{4}\|\nabla u\|_{L^2}^2+\|d_t\|_{L^2}^2+\frac{3}{4}\|\nabla^2 d\|_{L^2}^2\le 0,\tag{3.8}$$

provided

$$\mathbb{K}_0\le\epsilon_1:=\min\Big\{1,\frac{1}{(4C_1)^4}\Big\}.$$

Then, integrating (3.8) with respect to t, we arrive at

$$\int(\rho|u|^2+|\nabla d|^2)dx+\int_0^t\int(|\nabla u|^2+|\nabla^2 d|^2)dxdt\le C(\|\sqrt{\rho_0}u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2)\tag{3.9}$$

(3) Multiplying $(1.1)_2$ by u_t, and integrating by parts over $\mathbb{R}^3$, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2dx+\int\rho|u_t|^2dx\\
&=-\int\rho u\cdot\nabla u\cdot u_tdx+\int\nabla d\odot\nabla d:\nabla u_tdx\\
&=\frac{d}{dt}\int\nabla d\odot\nabla d:\nabla udx-\int(\nabla d\odot\nabla d)_t:\nabla u-\int\rho(u\cdot\nabla)u\cdot u_tdx\\
&\le\frac{d}{dt}\int\nabla d\odot\nabla d:\nabla udx+C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2}+C\|\nabla d\|_{L^6}\|\nabla d_t\|_{L^2}\|\nabla u\|_{L^3}\\
&\le\frac{d}{dt}\int\nabla d\odot\nabla d:\nabla udx+C\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2}+\varepsilon(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2)\\
&\quad+C\|\nabla^2 d\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}
\end{aligned}\tag{3.10}$$

(4) Multiplying $(1.1)_4$ by $-\nabla\Delta d$ and then integrating by parts over $\mathbb{R}^3$, it follows from Hölder's and Gagliardo-Nirenberg inequalities that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 d\|_{L^2}+\|\nabla^3 d\|_{L^2}^2=\int\nabla(u\cdot\nabla d)\cdot\nabla\Delta d dx-\int\nabla(|\nabla d|^2 d)\nabla\Delta d dx\\
&=\int(\nabla u\cdot\nabla d)\cdot\nabla\Delta d dx+\sum_{i,j,k=1}^{3}\int u_i\partial_i\partial_j d\cdot\partial_i\partial_k^2 d dx-\int\nabla(|\nabla d|^2 d)\cdot\nabla\Delta d dx\\
&=\int(\nabla u\cdot\nabla d)\cdot\nabla\Delta d dx+\sum_{i,j,k=1}^{3}\int\partial_k u_i\partial_i\partial_j d\cdot\partial_i\partial_k d dx-\int\nabla(|\nabla d|^2 d)\cdot\nabla\Delta d dx\\
&\le C\int|\nabla u||\nabla d||\nabla\Delta d|dx+C\int|\nabla u||\nabla^2 d|^2 dx+C\int|\nabla d||\nabla^2 d||\nabla\Delta d|dx\\
&\quad+C\int|\nabla d|^3|\nabla\Delta d|dx\triangleq\sum_{i=1}^{4}I_i
\end{aligned}\tag{3.11}$$

It follows from Hölder's, Young's and Gagliardo-Nirenberg inequalities that

$$\begin{aligned}
I_1&\le C\|\nabla u\|_{L^3}\|\nabla d\|_{L^6}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\\
&\le\delta\|\nabla^3 d\|_{L^2}^2+C(\delta)\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2 d\|_{L^2}^2, &(3.12)\\
I_2&\le C\|\nabla u\|_{L^3}\|\nabla^2 d\|_{L^3}^2\\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\\
&\le\delta\|\nabla^3 d\|_{L^2}^2+C(\delta)\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2 d\|_{L^2}^2, &(3.13)\\
I_3&\le C\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla^2 d\|_{L^2}^{\frac{3}{2}}\|\nabla^3 d\|_{L^2}^{\frac{3}{2}}\\
&\le\delta\|\nabla^3 d\|_{L^2}^2+C(\delta)\|\nabla^2 d\|_{L^2}^6, &(3.14)\\
I_4&\le C\|\nabla d\|_{L^6}^2\|\nabla^3 d\|_{L^2}\le C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}\\
&\le\delta\|\nabla^3 d\|_{L^2}^2+C(\delta)\|\nabla^2 d\|_{L^2}^6. &(3.15)
\end{aligned}$$

(5) Notice that (ρ,u,d) satisfies the following Stokes system

$$\begin{cases}
-\Delta u+\nabla p=-\rho u_t-\rho(u\cdot\nabla)u-\operatorname{div}(\nabla d\odot\nabla d), & x\in\mathbb{R}^3,\\
\operatorname{div}u=0, & x\in\mathbb{R}^3,\\
u(x)\to 0, & |x|\to\infty.
\end{cases}\tag{3.16}$$

Applying the standard L^p-estimate to the above system ensures that for any $p\in(1,\infty)$,

$$\|\nabla^2 u\|_{L^p}+\|\nabla p\|_{L^p}\le C\|\rho u_t\|_{L^p}+C\|\rho u\cdot\nabla u\|_{L^p}+C\|\operatorname{div}(\nabla d\odot\nabla d)\|_{L^p},\tag{3.17}$$

from which, after using (3.1), (3.17), and Gagliardo-Nirenberg inequality, we have

$$\|\nabla^2 u\|_{L^2}+\|\nabla p\|_{L^2}\le C\|\rho u_t\|_{L^2}+C\|\rho u\cdot\nabla u\|_{L^2}+C\|\operatorname{div}(\nabla d\odot\nabla d)\|_{L^2}$$

$$
\begin{aligned}
&\leq C\|\sqrt{\rho}u_t\|_{L^2}+C\|u\|_{L^6}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^6}^{\frac{1}{2}}+C\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d\|_{L^6}^{\frac{1}{2}}\\
&\leq C\|\sqrt{\rho}u_t\|_{L^2}+C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}+C\|\nabla^2 d\|_{L^2}^{\frac{3}{2}}\|\nabla^3 d\|_{L^2}^{\frac{1}{2}}\\
&\leq C\|\nabla^2 d\|_{L^2}^{\frac{3}{2}}\|\nabla^3 d\|_{L^2}^{\frac{1}{2}}+\frac{1}{2}\|\nabla^2 u\|_{L^2}+C\|\sqrt{\rho}u_t\|_{L^2}+C\|\nabla u\|_{L^2}^3. \qquad (3.18)
\end{aligned}
$$

that is

$$
\|\nabla^2 u\|_{L^2}+\|\nabla p\|_{L^2}\leq C\|\nabla^2 d\|_{L^2}^{\frac{3}{2}}\|\nabla^3 d\|_{L^2}^{\frac{1}{2}}+C\|\sqrt{\rho}u_t\|_{L^2}+C\|\nabla u\|_{L^2}^3. \qquad (3.19)
$$

Applying the gradient operator to $(1.1)_4$, we get

$$
\nabla d_t-\nabla\Delta d=-\nabla(u\cdot\nabla d)+\nabla(|\nabla d|^2 d). \qquad (3.20)
$$

It follows from (3.20) that

$$
\begin{aligned}
\|\nabla d_t\|_{L^2}^2&=\int|\nabla(\Delta d+|\nabla d|^2 d-u\cdot\nabla d)|^2dx\\
&\leq 2\int(|\nabla\Delta d|^2+|\nabla(|\nabla d|^2 d+u\cdot\nabla d)|^2)dx\\
&\leq 2\int|\nabla\Delta d|^2dx+C\int(|\nabla d|^6+|\nabla d|^2|\nabla^2 d|^2)dx\\
&\quad+C\int(|u|^2|\nabla^2 d|^2+|\nabla u|^2|\nabla d|^2)dx\\
&\leq 2\|\nabla^3 d\|_{L^2}^2+C\|\nabla^2 d\|_{L^2}^6+C\|\nabla d\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2\\
&\quad+C\|u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2+C\|\nabla u\|_{L^3}^2\|\nabla d\|_{L^6}^2\\
&\leq 2\|\nabla^3 d\|_{L^2}^2+C\|\nabla^2 d\|_{L^2}^6+C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}\\
&\quad+C\|\nabla u\|_{L^2}^2\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}+C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2 d\|_{L^2}^2\\
&\leq(2+\delta)(\|\nabla^3 d\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2)+C(\delta)\|\nabla^2 d\|_{L^2}^6+C(\delta)\|\nabla u\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2. \qquad (3.21)
\end{aligned}
$$

By assumption, it follows from Hölder's and Young's inequalities that

$$
\left|\int\nabla d\odot\nabla d:\nabla u dx\right|\leq\frac{1}{4}\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^4}^4. \qquad (3.22)
$$

(6) Multiplying (3.20) by $4|\nabla d|^2\nabla d$ and integrating by parts yields

$$
\begin{aligned}
&\frac{d}{dt}\int|\nabla d|^4dx+4\int(|\nabla d|^2|\nabla^2 d|^2+2|\nabla d|^2|\nabla(\nabla d)|^2)dx\\
&=4\int|\nabla d|^2\nabla d(-\nabla(u\cdot\nabla d)+\nabla(|\nabla d|^2 d))dx\\
&\leq C\int(|\nabla d|^4|\nabla u|+|\nabla d|^3|\nabla^2 d||u|+|\nabla d|^4|\nabla^2 d|+|\nabla d|^6)dx\\
&\leq\int|\nabla d|^2|\nabla^2 d|^2dx+C\int(|\nabla u|^2|\nabla d|^2+|u|^2|\nabla^2 d|^2+|\nabla d|^6)dx,
\end{aligned}
$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d|^4 dx + 3\int |\nabla d|^2|\nabla^2 d|^2 dx\\
&\le C\int |\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6 dx\\
&\le \delta(\|\nabla^3 d\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) + C(\delta)\|\nabla^2 d\|_{L^2}^6 + C(\delta)\|\nabla u\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2. \qquad (3.23)
\end{aligned}$$

Thus, by choosing some constant c_1 suitably large such that

$$\frac{1}{2}|\nabla u|^2 - \nabla d \odot \nabla d : \nabla u + c_1|\nabla d|^4 \ge \frac{1}{4}(|\nabla u|^2 + |\nabla d|^4), \qquad (3.24)$$

then applying (3.10) + (3.11) + (3.21) + (3.23) $\times\, c_1$ and using (3.22), (3.19) and (3.24), and Young's inequality, we then obtain

$$\begin{aligned}
&\frac{d}{dt}\int (|\nabla u|^2 + |\nabla^2 d|^2 + |\nabla d|^4)dx + \int (\rho|u_t|^2 + |\nabla^3 d|^2 + |\nabla d_t|^2)dx\\
&\le C\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2} + \varepsilon\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\\
&\quad + \delta\|\nabla^3 d\|_{L^2}^2 + C(\delta)\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2 d\|_{L^2}^2 + C(\delta)\|\nabla^2 d\|_{L^2}^6\\
&\quad + \delta\|\nabla^2 u\|_{L^2}^2 + C(\delta)\|\nabla u\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2\\
&\le \varepsilon\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C\|\nabla u\|_{L^2}^3\|\nabla^2 d\|_{L^2}^3 + C\|\nabla^2 d\|_{L^2}^6\\
&\quad + \delta\|\nabla^3 d\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^4\|\nabla u\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^2\|\nabla u\|_{L^2}^4\\
&\le \varepsilon\|\sqrt{\rho}u_t\|_{L^2}^2 + \delta\|\nabla^3 d\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)^2. \qquad (3.25)
\end{aligned}$$

By using the Gronwall inequality, (3.9), and choosing δ, ε suitably small, and noticing that $\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \le \sup_{0\le s\le t}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2)$, we have

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + c_1\|\nabla d\|_{L^4}^4) + \int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)dt\\
&\le \exp\Big\{\int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)^2 dt\Big\}(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2)\\
&\le \exp\Big\{C(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2)\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\Big\}(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2)\\
&\le \exp\{C\mathbb{K}_0^{\frac{1}{2}}\}(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2) \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2), \qquad (3.26)
\end{aligned}$$

provided $\mathbb{K}_0 \le \epsilon_1$. Thus, we have

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)dt\\
&\le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2). \qquad (3.27)
\end{aligned}$$

(7) Combining (3.8) and (3.27), we have

$$\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_{L^2}^2+\|\nabla d\|_{L^2}^2)(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2)\le C_2\mathbb{K}_0\le \mathbb{K}_0^{\frac{1}{2}}, \tag{3.28}$$

provided

$$\mathbb{K}_0\le \epsilon_2:=\min\Big\{\epsilon_1,\frac{1}{C_2^2}\Big\}.$$

As a consequence, we directly obtain (3.2). The proof of Proposition 3.1 is finished. □

Lemma 3.1. *Under the conditions of Proposition 3.1, it holds that for $i\in\{0,1\}$*

$$\sup_{0\le t\le T} t\|\nabla^2 d\|_{L^2}^2+\int_0^T t\left(\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)dt\le C, \tag{3.29}$$

$$\sup_{0\le t\le T}\left(t^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^2}^2\right)+\int_0^T t^{\frac{1}{2}}\|\nabla u\|_{L^2}^2dt\le C, \tag{3.30}$$

$$\sup_{0\le t\le T} t\|\nabla u\|_{L^2}^2+\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2dt\le C. \tag{3.31}$$

Proof. 1) Using Hölder's and Gagliardo-Nirenberg inequalities, we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla^2 d\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\\
&\le C\|\nabla(u\cdot\nabla d)\|_{L^2}^2+C\|\nabla(|\nabla d|^2 d)\|_{L^2}^2\\
&\le C\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)\|\nabla d\|_{L^\infty}^2+C\|\nabla d\|_{L^6}^4\|\nabla d\|_{L^6}^2\\
&\quad +C\left(\|u\|_{L^6}^2+\|\nabla d\|_{L^6}^2\right)\|\nabla^2 d\|_{L^3}^2\\
&\le C\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}+C\|\nabla^2 d\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2\\
&\le \frac{1}{2}\|\nabla^3 d\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\|\nabla^2 d\|_{L^2}^2,
\end{aligned}$$

which yields

$$\frac{d}{dt}\left(t\|\nabla^2 d\|_{L^2}^2\right)+t\|\nabla d_t\|_{L^2}^2+\frac{t}{2}\|\nabla^3 d\|_{L^2}^2\le\|\nabla^2 d\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)(t\|\nabla^2 d\|_{L^2}^2).$$

This along with Gronwall's inequality, (3.9) and (3.27) yields the desired (3.29).

2) It follows from (3.5) that

$$\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\le C\|\nabla d\|_{L^3}^2\|\nabla^2 d\|_{L^2}^2. \tag{3.32}$$

Multiplying the above inequality by $t^{\frac{1}{2}}$ and integrating it over $[0,T]$, we have

$$\sup_{0\le t\le T}\left(t^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^2}^2\right)+\int_0^T t^{\frac{1}{2}}\|\nabla u\|_{L^2}^2dt$$

$$
\begin{aligned}
&\leq C\int_0^T t^{\frac{1}{2}}\|\nabla d\|_{L^3}^2\|\nabla^2 d\|_{L^2}^2 dt + C\sup_{0\leq t\leq T}\|\sqrt{\rho}u\|_{L^2}^2\int_0^1 t^{-\frac{1}{2}}dt\\
&\quad + C\int_1^T \|\nabla u\|_{L^2}^2 dt\\
&\leq C\sup_{0\leq t\leq T}\|\nabla d\|_{L^2}\sup_{0\leq t\leq T}\left(t\|\nabla^2 d\|_{L^2}^2\right)^{\frac{1}{2}}\int_0^T\|\nabla^2 d\|_{L^2}^2 dt + C\\
&\leq C. \qquad (3.33)
\end{aligned}
$$

Multiplying (3.25) by t, we obtain the desired (3.31) after using Gronwall's inequality. Then, we completed the proof of Lemma 3.1. □

Lemma 3.2. *Under the assumption of Theorem 1.1, it holds that for $i \in \{1,2\}$*

$$
\sup_{0\leq t\leq T} t^i(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T t^i(\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2)dt \leq C, \qquad (3.34)
$$

$$
\sup_{0\leq t\leq T} t\|d_t\|_{L^2}^2 + \int_0^T t\|\nabla d_t\|_{L^2}^2 dt \leq C, \qquad (3.35)
$$

$$
\sup_{0\leq t\leq T} t^i(\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) + \int_0^T t^{\frac{3}{2}}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2)dt \leq C. \qquad (3.36)
$$

Proof. (1) Differentiating $(1.1)_2$ with respect to the time variable t gives

$$
\rho u_{tt} + \rho u\cdot\nabla u_t - \Delta u_t + \nabla p_t = -\rho_t(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u - \operatorname{div}(\nabla d\odot\nabla d)_t. \qquad (3.37)
$$

Multiplying the above equality by u_t and integrating the resulting equality by parts over $\mathbb{R}^3$, we deduce after using $(1.1)_1$ that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2dx + \int|\nabla u_t|^2dx\\
&= \int[(\nabla d\odot\nabla d)_t : \nabla u_t - \rho(u_t\cdot\nabla)u\cdot u_t + \operatorname{div}(\rho u)(u_t + u\cdot\nabla u)\cdot u_t]dx\\
&= \int\{(\nabla d\odot\nabla d)_t : \nabla u_t - \rho(u_t\cdot\nabla)u\cdot u_t - \rho u\cdot\nabla(|u_t|^2) + \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\}dx\\
&\leq C\int\rho|u|(|u||\nabla u||\nabla u_t| + |\nabla u|^2|u_t| + |u||\nabla^2 u||u_t|)dx + C\int\rho|u||\nabla u_t||u_t|dx\\
&\quad + C\int\rho|u_t|^2|\nabla u|dx + C\int|\nabla d||\nabla d_t||\nabla u_t|dx \triangleq \sum_{i=1}^4 I_{2i}. \qquad (3.38)
\end{aligned}
$$

The terms on the right-hand side of (3.38) can be bounded as follows. It follows from Hölder's, Young's and Gagliardo-Nirenberg inequalities that

$$
\begin{aligned}
I_{21} &\leq C\|u\|_{L^6}(\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\|u_t\|_{L^6} + \|u\|_{L^6}\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6})\\
&\quad + C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\leq C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{6}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2,\\
I_{22}+I_{23} &\leq C\|\sqrt{\rho}u_t\|_{L^2}(\|\nabla u_t\|_{L^2}\|u\|_{L^\infty} + \|u_t\|_{L^6}\|\nabla u\|_{L^3})\\
&\leq C\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\\
&\leq \frac{1}{6}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2,\\
I_{24} &\leq C\|\nabla d_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^\infty}\\
&\leq C\|\nabla d_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla^2 d\|_{L^2}^{\frac{1}{2}}\|\nabla^3 d\|_{L^2}^{\frac{1}{2}}\\
&\leq \frac{1}{6}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}.
\end{aligned}
$$

Substituting $I_{21} - I_{24}$ into (3.38), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{2}\|\nabla u_t\|_{L^2}^2\\
&\leq C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}. \qquad (3.39)
\end{aligned}
$$

(2) Differentiating (3.20) with respect to the time variable t gives

$$
\nabla d_{tt} - \Delta\nabla d_t = -\nabla(u\cdot\nabla d)_t + \nabla(|\nabla d|^2 d)_t. \qquad (3.40)
$$

Multiplying (3.40) by ∇d_t, and integrating the resulting equality over $\mathbb{R}^3$, we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla d_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2\\
&\leq C\int|\nabla u_t||\nabla d||\nabla d_t|dx + C\int|\nabla u||\nabla d_t|^2dx + C\int|u_t||\nabla^2 d||\nabla d_t|dx\\
&\quad + C\int|\nabla d|^2|d_t||\nabla^2 d_t|dx + C\int|\nabla d||\nabla d_t||\nabla^2 d_t|dx\\
&\triangleq \sum_{i=1}^{5} I_{4i}. \qquad (3.41)
\end{aligned}
$$

Applying Hölder's and Gagliardo-Nirenberg inequalities, and (3.27), we have

$$
\begin{aligned}
I_{31} &\leq C\|\nabla u_t\|_{L^2}\|\nabla d_t\|_{L^3}\|\nabla d\|_{L^6}\\
&\leq C\|\nabla u_t\|_{L^2}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d_t\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}\\
&\leq \frac{1}{4}\|\nabla u_t\|_{L^2}^2 + \frac{1}{4}\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}^4,\\
I_{42} &\leq C\|\nabla u\|_{L^3}\|\nabla d_t\|_{L^2}\|\nabla d_t\|_{L^6}\\
&\leq C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla d_t\|_{L^2}\|\nabla^2 d_t\|_{L^2}\\
&\leq \frac{1}{16}\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla d_t\|_{L^2}^2,\\
I_{33} &\leq C\|u_t\|_{L^6}\|\nabla^2 d\|_{L^2}\|\nabla d_t\|_{L^3}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|\nabla u_t\|_{L^2}\|\nabla^2 d\|_{L^2}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d_t\|_{L^2}^{\frac{1}{2}}\\
&\leq \frac{1}{4}\|\nabla u_t\|_{L^2}^2+\frac{1}{16}\|\nabla^2 d_t\|_{L^2}^2+C\|\nabla^2 d\|_{L^2}^4\|\nabla d_t\|_{L^2}^2,\\
I_{34}&\leq C\|\nabla d\|_{L^6}^2\|d_t\|_{L^6}\|\nabla^2 d_t\|_{L^2}\\
&\leq C\|\nabla^2 d\|_{L^2}^2\|\nabla d_t\|_{L^2}\|\nabla^2 d_t\|_{L^2}\\
&\leq \frac{1}{16}\|\nabla^2 d_t\|_{L^2}^2+C\|\nabla^2 d\|_{L^2}^4\|\nabla d_t\|_{L^2}^2,\\
I_{35}&\leq C\|\nabla^2 d_t\|_{L^2}\|\nabla d_t\|_{L^3}\|\nabla d\|_{L^6}\\
&\leq C\|\nabla^2 d_t\|_{L^2}^{\frac{3}{2}}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}\\
&\leq \frac{1}{16}\|\nabla^2 d_t\|_{L^2}^2+C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}^4.
\end{aligned}
$$

Inserting the estimates of $I_{4i}(i=1,\ldots,5)$ into (3.41), it follows that

$$
\frac{d}{dt}\|\nabla d_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\leq \frac{1}{4}\|\nabla u_t\|_{L^2}^2+C(\|\nabla^2 d\|_{L^2}^4+\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2})\|\nabla d_t\|_{L^2}^2. \tag{3.42}
$$

Now, the inequality of (3.42) added to (3.39) , we infer that

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2)+\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\\
&\leq C(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)(\|\nabla d_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2)\\
&\quad+C\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2.
\end{aligned} \tag{3.43}
$$

(3) It follows from (3.20) and Hölder's and Gagliardo-Nirenberg inequalities that

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2}^2+\|\nabla p\|_{L^2}^2&\leq C\|\rho u_t\|_{L^2}^2+C\|\rho u\cdot\nabla u\|_{L^2}^2+C\|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^2}^2\\
&\leq C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u\|_{L^6}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}+C\|\nabla d\|_{L^\infty}^2\|\nabla^2 d\|_{L^2}^2\\
&\leq C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2}+C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}\\
&\leq \frac{1}{2}\|\nabla^2 u\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^6+C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2},
\end{aligned}
$$

that is

$$
\|\nabla^2 u\|_{L^2}^2+\|\nabla p\|_{L^2}^2\leq C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^6+C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}, \tag{3.44}
$$

which given (3.9), (3.3), (3.29) and (3.55) yields

$$
\begin{aligned}
\int_0^T(\|\nabla^2 u\|_{L^2}^2+\|\nabla p\|_{L^2}^2)dt&\leq C\int_0^T(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2})dt\\
&\leq C\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2)dt+C\\
&\leq C,
\end{aligned} \tag{3.45}
$$

and

$$
\begin{aligned}
&\int_0^T t^{\frac{3}{2}}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2)dt \\
&\le C\int_0^T t^{\frac{3}{2}}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 + \|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2})dt \\
&\le C\sup_{0\le t\le T} t\|\nabla^2 d\|_{L^2}^2\Big(\int_0^T \|\nabla^2 d\|_{L^2}^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^T t\|\nabla^3 d\|_{L^2}^2 dt\Big)^{\frac{1}{2}} \\
&\quad + C\sup_{0\le t\le T} t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2\int_0^T \|\nabla u\|_{L^2}^2 dt + C \\
&\le C.
\end{aligned} \tag{3.46}
$$

Thus, multiplying (3.43) by t^i and using Gronwall's inequality, by (3.9), (3.27) and (3.45), we immediately arrive at

$$
\sup_{0\le t\le T} t^i(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T t^i(\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2)dt \le C. \tag{3.47}
$$

(4) We infer from (3.20), Hölder's and Gagliardo-Nirenberg inequalities and $|d| = 1$ that

$$
\begin{aligned}
\|\nabla^3 d\|_{L^2}^2 &\le C(\|\nabla d_t\|_{L^2}^2 + \||\nabla u||\nabla d|\|_{L^2}^2 + \||u||\nabla^2 d|\|_{L^2}^2 + \||\nabla d|^3\|_{L^2}^2 + \||\nabla^2 d||\nabla d|\|_{L^2}^2) \\
&\le C(\|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^\infty}^2 + \|u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2 + \|\nabla d\|_{L^6}^6 + \|\nabla^2 d\|_{L^3}^2\|\nabla d\|_{L^6}^2) \\
&\le C\|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2} + C\|\nabla u\|_{L^2}^2\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2} \\
&\quad + C\|\nabla^2 d\|_{L^2}^6 + C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2} \\
&\le \frac{1}{2}\|\nabla^3 d\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla^2 d\|_{L^2}^4 + C\|\nabla u\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2 \\
&\quad + C\|\nabla^2 d\|_{L^2}^6.
\end{aligned}
$$

that is

$$
\|\nabla^3 d\|_{L^2}^2 \le C\|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla^2 d\|_{L^2}^4 + C\|\nabla u\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^6, \tag{3.48}
$$

which together with (3.9), (3.29), (3.55) and (3.60) yields that

$$
\sup_{0\le t\le T} t^i\|\nabla^3 d\|_{L^2}^2) \le C. \tag{3.49}
$$

(5) Differentiating $(1.1)_4$ with respect to t, multiplying the resulting equality by d_t and then integrating by parts over $\mathbb{R}^3$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |d_t|^2 dx + \int |\nabla d_t|^2 dx \\
&\le C\int |u_t||\nabla d||d_t|dx + C\int |\nabla d_t||\nabla d||d_t|dx + C\int |\nabla d|^2|d_t|^2 dx \\
&\triangleq \sum_{i=1}^3 I_{3i}.
\end{aligned} \tag{3.50}
$$

Applying Hölder's and Gagliardo-Nirenberg inequalities, we derive

$$
\begin{aligned}
I_{41} &\le C\|d_t\|_{L^2}\|u_t\|_{L^6}\|\nabla d\|_{L^3} \\
&\le C\|d_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{1}{4}\|\nabla u_t\|_{L^2}^2 + C\|d_t\|_{L^2}^2, \qquad (3.51)\\
I_{42} + I_{43} &\le C\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\|d_t\|_{L^2} + C\|d_t\|_{L^2}\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 \\
&\le C\|\nabla d_t\|_{L^2}\|\nabla^2 d\|_{L^2}^{\frac{1}{2}}\|\nabla^3 d\|_{L^2}^{\frac{1}{2}}\|d_t\|_{L^2} + C\|d_t\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla^2 d\|_{L^2}^2 \\
&\le \frac{1}{2}\|\nabla d_t\|_{L^2}^2 + C(\|\nabla^2 d\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2})\|d_t\|_{L^2}^2. \qquad (3.52)
\end{aligned}
$$

Hence, one gets

$$
\frac{d}{dt}\|d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \le \frac{1}{4}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla^2 d\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2})\|d_t\|_{L^2}^2. \qquad (3.53)
$$

Multiplying it by t and applying Gronwall's inequality, we have

$$
\sup_{0\le t\le T} t\|d_t\|_{L^2}^2 + \int_0^T t\|\nabla d_t\|_{L^2}^2 dt \le C. \qquad (3.54)
$$

Finally, combining (3.46), (3.49) and (3.47), we have the desired (3.36). Thus, we finished the proof of Lemma 3.2. □

Lemma 3.3. *Under the assumption of Theorem 1.1, it holds that*

$$
\sup_{0\le t\le T} t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2 + \int_0^T t^{\frac{3}{2}}\|\sqrt{\rho}u_t\|_{L^2}^2 dt \le C, \qquad (3.55)
$$

$$
\sup_{0\le t\le T} t^{\frac{5}{2}}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t^{\frac{5}{2}}\|\nabla u_t\|_{L^2}^2 dt \le C, \qquad (3.56)
$$

$$
\sup_{0\le t\le T} t^{\frac{5}{2}}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2) + \int_0^T t^{\frac{3}{2}}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2) dt \le C. \qquad (3.57)
$$

Proof. 1) Given (3.25), one obtains from (3.44) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 \\
&\le \frac{d}{dt}\int \nabla d\odot\nabla d : \nabla u dx + C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2} \\
&\quad + C\|\nabla d\|_{L^3}\|\nabla d_t\|_{L^2}\|\nabla u\|_{L^6} \\
&\le \frac{d}{dt}\int \nabla d\odot\nabla d : \nabla u dx + \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + \delta\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 \\
&\quad + C\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2}\|\nabla d_t\|_{L^2}^2 \\
&\le \frac{d}{dt}\int \nabla d\odot\nabla d : \nabla u dx + \Big(\frac{1}{4}+\delta\Big)\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6
\end{aligned}
$$

$$+ C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2} + C\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2}\|\nabla d_t\|_{L^2}^2 \tag{3.58}$$

Multiplying it by $t^{\frac{3}{2}}$, integrating it over $[0, T]$ and then choosing δ reasonably small, we derive

$$\begin{aligned}
&\sup_{0\le t\le T}\left(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2\right) + \int_0^T t^{\frac{3}{2}}\|\sqrt{\rho}u_t\|_{L^2}^2 dt\\
&\le C\sup_{0\le t\le T} t^{\frac{3}{2}}\|\nabla u\|_{L^2}\|\nabla d\|_{L^3}\|\nabla d\|_{L^6} + C\int_0^T t^{\frac{1}{2}}\|\nabla u\|_{L^2}^2 dt\\
&\quad + C\int_0^T t^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\nabla d\|_{L^3}\|\nabla d\|_{L^6} dt + C\int_0^T t^{\frac{3}{2}}\|\nabla u\|_{L^2}^6 dt\\
&\quad + C\int_0^T t^{\frac{3}{2}}\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2} dt + C\int_0^T t^{\frac{3}{2}}\|\nabla^2 d\|_{L^2}\|\nabla d_t\|_{L^2}^2 dt\\
&\le \frac{1}{2}\sup_{0\le t\le T} t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2 + C\sup_{0\le t\le T} t^{\frac{3}{2}}\|\nabla^2 d\|_{L^2}^3 + C\sup_{0\le t\le T} t^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}\int_0^T \|\nabla^2 d\|_{L^2}^2 dt\\
&\quad + C\sup_{0\le t\le T} t\|\nabla u\|_{L^2}^4\int_0^T t^{\frac{1}{2}}\|\nabla u\|_{L^2}^2 dt + C\sup_{0\le t\le T} t^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}\int_0^T t\|\nabla d_t\|_{L^2}^2 dt +\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla^3 d\|_{L^2} t^{\frac{1}{2}}\|\nabla^2 d\|_{L^2})\int_0^T \|\nabla^2 d\|_{L^2}^2 dt + C\\
&\le C.
\end{aligned} \tag{3.59}$$

2) Multiplying (3.39) by $t^{\frac{5}{2}}$ and using Gronwall's inequality, we infer from (3.39) that

$$\begin{aligned}
&\sup_{0\le t\le T}\left(t^{\frac{5}{2}}\|\sqrt{\rho}u_t\|_{L^2}^2\right) + \int_0^T t^{\frac{5}{2}}\|\nabla u_t\|_{L^2}^2 dt\\
&\le C\int_0^T t^{\frac{3}{2}}\|\sqrt{\rho}u_t\|_{L^2}^2 dt + C\int_0^T t^{\frac{5}{2}}\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2 dt\\
&\quad + C\int_0^T t^{\frac{5}{2}}\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2} dt\\
&\le C\sup_{0\le t\le T}\left(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2 t\|\nabla u\|_{L^2}^2\right)\int_0^T \|\nabla^2 u\|_{L^2}^2 dt + C\\
&\quad + C\sup_{0\le t\le T}\left(t^2\|\nabla^3 d\|_{L^2}^2 t\|\nabla^2 d\|_{L^2}^2\right)^{\frac{1}{2}}\int_0^T t\|\nabla d_t\|_{L^2}^2 dt\\
&\le C.
\end{aligned} \tag{3.60}$$

Combining (3.60), (3.59) and (3.44), we have the desired (3.57). □

Lemma 3.4. *Under the assumption of Theorem 1.1, it holds that*

$$\sup_{0\le t\le T}(\|\nabla\rho\|_{L^2\cap L^6} + \|\rho_t\|_{L^2\cap L^3}) + \int_0^T \left(t^{\frac{5}{2}}\|\nabla^2 u\|_{L^6}^2 + t^{\frac{5}{2}}\|\nabla p\|_{L^6}^2 + t^2\|\nabla^4 d\|_{L^2}^2\right)dt \le C. \tag{3.61}$$

Proof. (1) It follows from Lemma 2.2 and Gagliardo-Nirenberg and Hölder's inequalities that for $r \in (3, 6)$,

$$\|\nabla^2 u\|_{L^r} \le C\|\rho u_t\|_{L^r} + C\|\rho u\cdot\nabla u\|_{L^r} + C\|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^r}$$

$$
\begin{aligned}
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{3r-6}{2r}} + C\|u\|_{L^6}\|\nabla u\|_{L^{\frac{6r}{6-r}}} + C\|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^r}\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + C\|\nabla u\|_{L^2}^{\frac{6(r-1)}{5r-6}}\|\nabla u\|_{W^{1,r}}^{\frac{4r-6}{5r-6}} + C\|\nabla^2 d\|_{L^2}^{\frac{3}{r}}\|\nabla^3 d\|_{L^2}^{\frac{2r-3}{r}}\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + C\|\nabla u\|_{L^2}^{\frac{6r-6}{r}} + \frac{1}{2}\|\nabla^2 u\|_{L^r} + C\|\nabla^2 d\|_{L^2}^{\frac{3}{r}}\|\nabla^3 d\|_{L^2}^{\frac{2r-3}{r}},
\end{aligned}
$$

which directly deduces that

$$
\|\nabla^2 u\|_{L^r} \le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + C\|\nabla u\|_{L^2}^{\frac{6r-6}{r}} + C\|\nabla^2 d\|_{L^2}^{\frac{3}{r}}\|\nabla^3 d\|_{L^2}^{\frac{2r-3}{r}}. \tag{3.62}
$$

On the one hand, it follows from (3.9), (3.3) and (3.56) that for $0 < t \le 1$,

$$
\begin{aligned}
\int_0^1 \|\nabla u\|_{L^\infty}dt &\le C\int_0^1 (\|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^r})dt\\
&\le C\sup_{0\le t\le 1}(t\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{6-r}{4r}}\Big(\int_0^1 t\|\nabla u_t\|_{L^2}^2 dt\Big)^{\frac{3r-6}{4r}}\Big(\int_0^1 t^{-\frac{2r}{r+6}}dt\Big)^{\frac{r+6}{4r}}\\
&\quad + C\sup_{0\le t\le 1}(\|\nabla u\|_{L^2}^2)^{\frac{2r-3}{r}}\int_0^1 \|\nabla u\|_{L^2}^2 dt + C\int_0^1 \|\nabla^3 d\|_{L^2}^2 dt + C\\
&\le C.
\end{aligned}
\tag{3.63}
$$

On the other hand, using (3.56), (3.36), (3.29) and (3.55), we obtain that

$$
\begin{aligned}
\int_0^T t^{\frac{1}{2}}\|\nabla^2 u\|_{L^r} &\le C\int_1^T (\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + C\|\nabla u\|_{L^2}^{\frac{6r-6}{r}} + \|\nabla^2 d\|_{L^2}^{\frac{3}{r}}\|\nabla^3 d\|_{L^2}^{\frac{2r-3}{r}})dt\\
&\le C\sup_{1\le t\le T}(t^{\frac{5}{2}}\|\sqrt{\rho}u_t\|_{L^2}^2)\Big(\int_1^T t^{\frac{5}{2}}\|\nabla u_t\|_{L^2}^2 dt\Big)^{\frac{3r-6}{4r}}\Big(\int_1^T t^{-\frac{3r-6}{4r}}dt\Big)^{\frac{r+6}{4r}}\\
&\quad + C\sup_{1\le t\le T}\Big\{(t\|\nabla^2 d\|_{L^2}^2)^{\frac{3}{2r}}(t^2\|\nabla^3 d\|_{L^2}^2)^{\frac{r-3}{2r}}\Big\}\Big(\int_1^T t\|\nabla^3 d\|_{L^2}^2 dt\Big)^{\frac{1}{2}}\Big(\int_1^T t^{-\frac{2r-3}{r}}\Big)^{\frac{1}{2}}\\
&\quad + C\sup_{1\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)^{\frac{6r-6}{2r}}\int_1^T t^{-\frac{8r-9}{2r}}dt\\
&\le C,
\end{aligned}
\tag{3.64}
$$

which leads to

$$
\begin{aligned}
\int_1^T \|\nabla u\|_{L^\infty}dt &\le C\int_1^T \|\nabla u\|_{L^2}^{\frac{2r-6}{5r-6}}\|\nabla^2 u\|_{L^r}^{\frac{3r}{5r-6}}dt\\
&\le C\sup_{1\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)^{\frac{2r-6}{10r-12}}\Big(\int_0^T t^{\frac{1}{2}}\|\nabla^2 u\|_{L^r}\Big)^{\frac{3r}{5r-6}}\Big(t^{-\frac{6r-9}{4r-12}}dt\Big)^{\frac{2r-6}{5r-6}}\\
&\le C.
\end{aligned}
\tag{3.65}
$$

Combining (3.63) and (3.65), one obtains

$$
\int_0^T \|\nabla u\|_{L^\infty}dt \le C. \tag{3.66}
$$

(2) Differentiating the continuity equation $(1.1)_1$ with respect to x_i gives rise to

$$(\rho_{x_i})_t + \nabla\rho_{x_i} \cdot u + \nabla\rho \cdot u_{x_i} = 0. \tag{3.67}$$

Multiplying (3.67) by $s|\rho_{x_i}|^{s-2}\rho_{x_i}$ ($s = \{2, 6\}$) and integrating the resulting equation over $\mathbb{R}^3$ gives

$$\frac{d}{dt}\|\nabla\rho\|_{L^2\cap L^6} \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^2\cap L^6}. \tag{3.68}$$

It follows from Gronwall's inequality and (3.66) that

$$\|\nabla\rho\|_{L^2\cap L^6} \le C\|\nabla\rho_0\|_{L^2\cap L^6}. \tag{3.69}$$

Noticing the following facts

$$\begin{aligned}\|\rho_t\|_{L^2\cap L^3} &\le C\|u\|_{L^6}(\|\nabla\rho\|_{L^3} + \|\nabla\rho\|_{L^6})\\ &\le C\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^2\cap L^6} \le C\|\nabla\rho_0\|_{L^2\cap L^6}.\end{aligned} \tag{3.70}$$

(3) Taking ∇ operator to $(1.1)_4$, we get

$$-\nabla^2\Delta d = \nabla^2(|\nabla d|^2 d - u \cdot \nabla d - d_t). \tag{3.71}$$

Using the L^2-estimates of an elliptic system, we derive

$$\begin{aligned}\|\nabla^4 d\|_{L^2} &\le C(\|\nabla^2 d_t\|_{L^2} + \|\nabla^2(u \cdot \nabla d)\|_{L^2} + \|\nabla^2(|\nabla d|^2 d)\|_{L^2})\\ &\le C\|\nabla^2 d_t\|_{L^2} + C\||\nabla^2 u||\nabla d|\|_{L^2} + C\||\nabla u||\nabla^2 d|\|_{L^2} + C\||u||\nabla^3 d|\|_{L^2}\\ &\quad + C\||\nabla d|^2|\nabla^2 d|\|_{L^2} + C\||\nabla^2 d|^2\|_{L^2} + C\||\nabla d||\nabla^3 d|\|_{L^2}\\ &\le C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^\infty}\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^3}\|\nabla^2 d\|_{L^6}\\ &\quad + C\|u\|_{L^6}\|\nabla^3 d\|_{L^3} + C\|\nabla d\|_{L^6}^2\|\nabla^2 d\|_{L^6} + C\|\nabla^2 d\|_{L^\infty}\|\nabla^2 d\|_{L^2}\\ &\quad + C\|\nabla d\|_{L^6}\|\nabla^3 d\|_{L^3}\\ &\le C\|\nabla^2 d\|_{L^2}^{\frac{1}{2}}\|\nabla^3 d\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 d\|_{L^2}\\ &\quad + C(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\|\nabla^3 d\|_{L^2} + C\|\nabla^2 d_t\|_{L^2} + \frac{1}{2}\|\nabla^4 d\|_{L^2},\end{aligned} \tag{3.72}$$

which yields to

$$\begin{aligned}\|\nabla^4 d\|_{L^2}^2 &\le C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^3 d\|_{L^2}^2\\ &\quad + C(\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4)\|\nabla^3 d\|_{L^2}^2 + C\|\nabla^2 d_t\|_{L^2}^2,\end{aligned} \tag{3.73}$$

multiplying t^2 by (3.73), and integrating the resultant in $(0, T)$, and using (3.3), (3.21),(3.31), (3.34)and (3.36), one obtains

$$\int_0^T t^2\|\nabla^4 d\|_{L^2}^2 dt \le C. \tag{3.74}$$

(4) According to (3.17), we have

$$\|\nabla^2 u\|_{L^6}^2 + \|\nabla p\|_{L^6}^2 \le C\|\rho u_t\|_{L^6}^2 + C\|\rho u \cdot \nabla u\|_{L^6}^2 + C\|\mathrm{div}(\nabla d \odot \nabla d)\|_{L^6}^2$$

$$
\begin{aligned}
&\le C\|u_t\|_{L^6}^2 + C\|u\|_{L^\infty}^2\|\nabla u\|_{L^6}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla^2 d\|_{L^6}^2 \\
&\le C\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3. \qquad (3.75)
\end{aligned}
$$

This together with (3.34), (3.36), (3.36) and (3.3) gives

$$
\begin{aligned}
&\int_0^T t^2(\|\nabla^2 u\|_{L^6}^2 + \|\nabla p\|_{L^6}^2)dt \\
&\le C\int_0^T t^2(\|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + \|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3)dt \\
&\le C\sup_{0\le t\le T}(t^2\|\nabla^2 u\|_{L^2}^2)\int_0^T(\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2)dt \\
&\quad + C\sup_{0\le t\le T}(t^2\|\nabla^3 d\|_{L^2}^2)\int_0^T(\|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2)dt + C \\
&\le C, \qquad (3.76)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T t^{\frac{5}{2}}(\|\nabla^2 u\|_{L^6}^2 + \|\nabla p\|_{L^6}^2)dt \\
&\le C\int_0^T t^{\frac{5}{2}}(\|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + \|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3)dt \\
&\le C\sup_{0\le t\le T}(t^{\frac{5}{2}}\|\nabla^2 u\|_{L^2}^2)\int_0^T(\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2)dt \\
&\quad + C\sup_{0\le t\le T}(t^2\|\nabla^3 d\|_{L^2}^2)\Big(\int_0^T\|\nabla^2 d\|_{L^2}^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^T t\|\nabla^3 d\|_{L^2}^2 dt\Big)^{\frac{1}{2}} + C \\
&\le C. \qquad (3.77)
\end{aligned}
$$

This ends the proof of Lemma 3.4. □

Lemma 3.5. *Under the assumption of Theorem 1.1, it holds that for $i \in \{0, 1\}$,*

$$
\sup_{0\le t\le T}(t^{\frac{i}{2}}\|\sqrt{\rho}\theta\|_{L^2}^2) + \int_0^T t^{\frac{i}{2}}\|\nabla\theta\|_{L^2}^2 dt \le C, \qquad (3.78)
$$

$$
\sup_{0\le t\le T}(t^{\frac{3i}{2}}\|\nabla\theta\|_{L^2}^2) + \int_0^T t^{\frac{3i}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^2 dt \le C. \qquad (3.79)
$$

Proof. 1) Multiplying $(1.1)_3$ by θ and integrating by parts, one obtains

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho|\theta|^2dx + \int|\nabla\theta|^2dx \\
&= \int|\mathfrak{D}(u)|^2\theta dx + \int(\Delta d + |\nabla d|^2 d)\cdot(\Delta d + |\nabla d|^2 d)\theta dx \\
&\le C\int|\nabla u|^2\theta dx + C\int|\nabla d||\nabla^3 d|\theta dx + \int|\nabla d|^4\theta dx
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int |\nabla\theta||\nabla d||\nabla^2 d|dx + C\int |\nabla d|^2|\nabla^2 d|\theta dx \\
&=: \sum_{i=1}^{5} R_i.
\end{aligned}
\tag{3.80}
$$

In fact, multiplying $(1.1)_2$ by $u\theta$ and integrating the resulting equation over $\mathbb{R}^3$ gives

$$
\begin{aligned}
R_1 \le\ & C\int \rho|u_t||u||\theta|dx + C\int \rho|u|^2|\nabla u||\theta|dx + C\int |u||\nabla u||\nabla\theta|dx \\
&+ C\int p|u||\nabla\theta|dx + C\int |\nabla d|^2|\nabla u||\theta|dx + C\int |\nabla d|^2|u||\nabla\theta|dx \\
\le\ & \frac{1}{20}\|\nabla\theta\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\int |u|^2|\nabla u|^2dx + C\int \rho|u|^2|\theta|^2dx \\
&+ C\int |p|^2|u|^2dx + C\int |\nabla d|^2|\nabla u||\theta|dx + C\int |\nabla d|^2|u||\nabla\theta|dx \\
\le\ & \frac{1}{20}\|\nabla\theta\|_{L^2}^2 + C\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2 + C\|\sqrt{\rho}u\|_{L^2}\|u\|_{L^6}\|\theta\|_{L^6}^2 \\
&+ C\|p\|_{L^3}^2\|u\|_{L^6}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}\|\nabla d\|_{L^3}\|u\|_{L^6}\|\nabla\theta\|_{L^2} \\
&+ C\|\nabla d\|_{L^\infty}\|\nabla d\|_{L^3}\|\nabla u\|_{L^2}\|\theta\|_{L^6} \\
\le\ & \frac{1}{20}\|\nabla\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2} + C\|\sqrt{\rho}u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2 \\
&+ C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|p\|_{L^2}\|p\|_{L^6}\|\nabla u\|_{L^2}^2 + C\|\nabla^3 d\|_{L^2}\|\nabla^2 d\|_{L^2}^2\|\nabla u\|_{L^2} \\
\le\ & \Big(\frac{1}{20} + C\mathbb{K}_0^{\frac14}\Big)\|\nabla\theta\|_{L^2}^2 + C\|p\|_{L^2}\|\nabla p\|_{L^2}\|\nabla u\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2 \\
&+ C\|\nabla u\|_{L^2}^6 + C\|\nabla^2 d\|_{L^2}^6 + C\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^{\frac23} + C\|\nabla^3 d\|_{L^2}^2\|\nabla^2 d\|_{L^2} \\
\le\ & \Big(\frac{1}{20} + C_3\mathbb{K}_0^{\frac14}\Big)\|\nabla\theta\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^{\frac23} + C\|\nabla u\|_{L^2}^{\frac{16}{3}} \\
&+ C\|\nabla^3 d\|_{L^2}^2\|\nabla^2 d\|_{L^2} + C\|\nabla^2 d\|_{L^2}^9 + C\|\nabla^2 d\|_{L^2}\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{3.81}
$$

where we have used the following fact that

$$
\begin{aligned}
\|p\|_{L^2}\|\nabla p\|_{L^2} \le\ & C(\|\rho u_t\|_{L^{\frac65}} + \|\rho u\cdot\nabla u\|_{L^{\frac65}} + \|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{\frac65}}) \\
&\cdot(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2} + \|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^2}) \\
\le\ & C(\|\sqrt{\rho}u\|_{L^2} + \|\sqrt{\rho}u\cdot\nabla u\|_{L^2} + \|\nabla d\|_{L^3}\|\nabla^2 d\|_{L^2}) \\
&\cdot(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2} + \|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3}) \\
\le\ & C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2 + C\|\nabla d\|_{L^3}^2\|\nabla^2 d\|_{L^2}^2 \\
&+ C\|\nabla d\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2 \\
\le\ & C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2} + C\|\nabla^2 d\|_{L^2}^2 \\
&+ C\|\nabla^3 d\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^6 \\
\le\ & C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^{\frac23} + C\|\nabla u\|_{L^2}^{\frac{16}{3}} + C\|\nabla^2 d\|_{L^2}^2 \\
&+ C\|\nabla^3 d\|_{L^2}^2\|\nabla^2 d\|_{L^2} + C\|\nabla u\|_{L^2}^6 + C\|\nabla^2 d\|_{L^2}^9.
\end{aligned}
\tag{3.82}
$$

It follows from (3.9) , (3.3), Hölder's inequality and the Gagliardo-Nirenberg inequality that

$$
\begin{aligned}
R_2 &\le C\|\nabla^3 d\|_{L^2}\|\nabla d\|_{L^3}\|\theta\|_{L^6}\\
&\le \frac{1}{20}\|\nabla\theta\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^2,\\
R_3 &\le C\|\theta\|_{L^6}\|\nabla d\|_{L^3}\||\nabla d|^3\|_{L^2}\\
&\le C\|\nabla\theta\|_{L^2}\|\nabla d\|_{H^1}\|\nabla d\|_{L^6}^3\\
&\le \frac{1}{20}\|\nabla\theta\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^6,\\
R_4 &\le C\|\nabla\theta\|_{L^2}\|\nabla^2 d\|_{L^6}\|\nabla d\|_{L^3}\\
&\le \frac{1}{20}\|\nabla\theta\|_{L^2}^2 + C\|\nabla^3 d\|_{L^2}^2\|\nabla^2 d\|_{L^2},\\
R_5 &\le C\|\theta\|_{L^6}\||\nabla d|^2\|_{L^3}\|\nabla^2 d\|_{L^6}\\
&\le C\|\nabla\theta\|_{L^2}\|\nabla d\|_{L^6}^2\|\nabla^3 d\|_{L^2}\\
&\le \frac{1}{20}\|\nabla\theta\|_{L^2}^2 + C\|\nabla^3 d\|_{L^2}^2\|\nabla^2 d\|_{L^2}.
\end{aligned}
$$

Collecting the above estimates, choosing δ to be reasonably small and applying $\mathbb{K}_0 \le \epsilon_3 := \min\left\{\epsilon_2, \left(\frac{1}{4C_3}\right)^4\right\}$, we get that

$$
\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^{\frac{2}{3}} + C\|\nabla^3 d\|_{L^2}^2\|\nabla^2 d\|_{L^2}\\
&\quad + C\|\nabla u\|_{L^2}^{\frac{16}{3}} + C\|\nabla^2 d\|_{L^2}^9 + C\|\nabla^2 d\|_{L^2}\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{3.83}
$$

Integrating the above inequality with respect to t, after using (3.9), (3.3) and (3.45) one obtains that

$$
\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_{L^2}^2 + \int_0^T \|\nabla\theta\|_{L^2}^2 dt \le C. \tag{3.84}
$$

(2) Multiplying (3.83) by $t^{\frac{1}{2}}$, we derive from (3.3), (3.55), (3.56), (3.29), (3.45), (3.31) and (3.84) that

$$
\begin{aligned}
&\sup_{0\le t\le T}(t^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_{L^2}^2) + \int_0^T t^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^2 dt\\
&\le C\int_0^T t^{-\frac{1}{2}}\|\sqrt{\rho}\theta\|_{L^2}^2 dt + C\int_0^T t^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}^2 dt + C\int_0^T t^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^{\frac{2}{3}} dt\\
&\quad + C\int_0^T t^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{16}{3}} dt + C\int_0^T t^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}^9 dt + C\int_0^T t^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}\|\nabla u\|_{L^2}^2 dt\\
&\le C\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_{L^2}^2\int_0^1 t^{-\frac{1}{2}}dt + C\int_1^T\|\nabla\theta\|_{L^2}^2 dt + C\int_0^1\|\sqrt{\rho}u_t\|_{L^2}^2 dt\\
&\quad + C\int_1^T t^{-2}(t^{\frac{5}{2}}\|\nabla u_t\|_{L^2}^2)dt + C\sup_{0\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)^{\frac{1}{3}}\int_0^T\|\nabla^2 u\|_{L^2}^2 dt
\end{aligned}
$$

$$
\begin{aligned}
&+C\sup_{0\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)^{\frac{1}{3}}\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac{8}{3}}\int_0^T\|\nabla u\|_{L^2}^2dt\\
&+C\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac{1}{2}}\sup_{0\le t\le T}\|\nabla^2 d\|_{L^2}^6\int_0^T\|\nabla^2 d\|_{L^2}^2dt\\
&+C\sup_{0\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)^{\frac{1}{3}}\int_0^T\|\nabla^2 u\|_{L^2}^2dt\\
\le{}& C. \qquad (3.85)
\end{aligned}
$$

(3) In view of the standard estimate for an elliptic system, one obtains

$$
\begin{aligned}
\|\nabla^2\theta\|_{L^2}^2 &\le C(\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\rho u\cdot\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^4}^4+\||\Delta d+|\nabla d|^2d|\|_{L^4}^4)\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u\|_{L^6}^2\|\nabla\theta\|_{L^3}^2+C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3\\
&\quad+C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3+C\|\nabla d\|_{L^\infty}^4\|\nabla d\|_{L^4}^4\\
&\le\frac{1}{2}\|\nabla^2\theta\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3\\
&\quad+C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3+C\|\nabla^2 d\|_{L^2}^5\|\nabla^3 d\|_{L^2}^2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\|\nabla^2\theta\|_{L^2}^2 \le{}& \|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3\\
&+C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3+C\|\nabla^2 d\|_{L^2}^5\|\nabla^3 d\|_{L^2}^2. \qquad (3.86)
\end{aligned}
$$

(4) Multiplying $(1.1)_3$ by θ_t and integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\theta|^2dx+\int\rho|\theta_t|^2dx\\
&=-\int\rho u\cdot\nabla\theta\theta_t dx+2\int|\mathfrak{D}(u)|^2\theta_t dx+\int|\Delta d+|\nabla d|^2d|^2\theta_t dx\\
&=K_1+K_2+K_3. \qquad (3.87)
\end{aligned}
$$

By using (3.4), (3.86) and Hölder's, Young's and the Gagliardo-Nirenberg inequalities, we have

$$
\begin{aligned}
K_1 &\le C\|u\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta\|_{L^2}\\
&\le\delta\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\\
&\le\delta\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^4+C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2,\\
K_2 &= 2\frac{d}{dt}\int|\mathfrak{D}(u)|^2\theta dx-2\int(|\mathfrak{D}(u)|^2)_t\theta dx\\
&\le 2\frac{d}{dt}\int|\mathfrak{D}(u)|^2\theta dx+C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^3}\|\theta\|_{L^6}\\
&\le 2\frac{d}{dt}\int|\mathfrak{D}(u)|^2\theta dx+C\|\nabla u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla\theta\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\frac{d}{dt}\int |\mathfrak{D}(u)|^2\theta dx + C\|\nabla u_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2,\\
K_3 &= \frac{d}{dt}\int |\Delta d + |\nabla d|^2 d|^2\theta dx - \int (|\Delta d + |\nabla d|^2 d|^2)_t\theta dx\\
&\leq \frac{d}{dt}\int |\Delta d + |\nabla d|^2 d|^2\theta dx + C\|\theta\|_{L^\infty}(\|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^4}^2)\\
&\quad \cdot (\|\nabla^2 d_t\|_{L^2} + \|d_t\|_{L^6}\|\nabla d\|_{L^2}\|\nabla d\|_{L^3} + \|\nabla d\|_{L^6}\|\nabla d_t\|_{L^3})\\
&\leq \frac{d}{dt}\int |\Delta d + |\nabla d|^2 d|^2\theta dx + C\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}}(\|\nabla^2 d\|_{L^2} + \|\nabla^2 d\|_{L^2}^{\frac{3}{2}})\\
&\quad \cdot (\|\nabla^2 d_t\|_{L^2} + \|\nabla d_t\|_{L^2}\|\nabla^2 d\|_{L^2}^{\frac{1}{2}} + \|\nabla^2 d\|_{L^2}^2\|\nabla d_t\|_{L^2})\\
&\leq \frac{d}{dt}\int |\Delta d + |\nabla d|^2 d|^2\theta dx + C\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}\|\nabla^2 d\|_{L^2}^2\\
&\quad + C\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}\|\nabla^2 d\|_{L^2}\\
&\leq \frac{d}{dt}\int |\Delta d + |\nabla d|^2 d|^2\theta dx + \delta\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2\|\nabla^2 d\|_{L^2}^4\\
&\quad + C\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}\\
&\leq \frac{d}{dt}\int |\Delta d + |\nabla d|^2 d|^2\theta dx + \delta\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla\theta\|_{L^2}^2\\
&\quad + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3 + C\|\nabla^2 d\|_{L^2}^5\|\nabla^3 d\|_{L^2}^2\\
&\quad + C\|\nabla\theta\|_{L^2}^2\|\nabla^2 d\|_{L^2}^4 + C\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}.
\end{aligned}
$$

Inserting $K_i (i = 1, 2, 3)$ into (3.87) and choosing δ suitably small, we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla\theta\|_{L^2}^2 - \psi(t)\Big) + \|\sqrt{\rho}\theta_t\|_{L^2}^2\\
&\leq C\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla\theta\|_{L^2}^2\\
&\quad + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3 + C\|\nabla^2 d\|_{L^2}^5\|\nabla^3 d\|_{L^2}^2\\
&\quad + C\|\nabla\theta\|_{L^2}^2\|\nabla^2 d\|_{L^2}^4 + C\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2},
\end{aligned}
\tag{3.88}
$$

where

$$
\psi(t) := 4\int |\mathfrak{D}(u)|^2\theta dx + 2\int |\Delta d + |\nabla d|^2 d|^2\theta dx \tag{3.89}
$$

satisfies

$$
\begin{aligned}
\psi(t) &\leq C\|\theta\|_{L^6}(\|\nabla u\|_{L^2}\|\nabla u\|_{L^3} + \|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{L^3} + \||\nabla d|^2\|_{L^3}\||\nabla d|^2\|_{L^2})\\
&\leq C\|\nabla\theta\|_{L^2}(\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2} + \|\nabla^2 d\|_{L^2}^{\frac{3}{2}}\|\nabla^3 d\|_{L^2} + C\|\nabla^2 d\|_{L^2}^{\frac{5}{2}})\\
&\leq \frac{1}{2}\|\nabla\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^5.
\end{aligned}
\tag{3.90}
$$

Then, the desired (3.79) follows from Gronwall's inequality, (3.29), (3.36), (3.34), (3.85), (3.86), (3.88) and (3.90). □

Lemma 3.6. *Under the assumption of Theorem 1.1, it holds that*

$$\sup_{0\le t\le T}(t^{\frac{5}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\int_0^T t^{\frac{5}{2}}\|\nabla\theta_t\|_{L^2}^2dt\le C, \tag{3.91}$$

$$\sup_{0\le t\le T}t^{\frac{5}{2}}\|\nabla^2\theta\|_{L^2}^2+\int_0^T(t^{\frac{3}{2}}\|\nabla^2\theta\|_{L^2}^2+t^{\frac{5}{2}}\|\nabla^2\theta\|_{L^6}^2)dt\le C. \tag{3.92}$$

Proof. (1) Applying the operator ∂_t to $(1.1)_3$ and a series of direct computations yields

$$\begin{aligned}&\rho\theta_{tt}+\rho u\cdot\nabla\theta_t-\Delta\theta_t\\&=\operatorname{div}(\rho u)\theta_t+\operatorname{div}(\rho u)u\cdot\nabla\theta-\rho u_t\cdot\nabla\theta+2(|\mathfrak{D}(u)|^2)_t+(|\Delta d+|\nabla d|^2d|^2)_t.\end{aligned} \tag{3.93}$$

Multiplying (3.93) by θ_t in L^2 and integrating by parts over $\mathbb{R}^3$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\theta_t|^2dx+\int|\nabla\theta_t|^2dx\\&=-2\int\rho u\cdot\nabla\theta_t\theta_tdx-\int\rho u\cdot\nabla(u\cdot\nabla\theta\theta_t)dx-\int\rho u_t\cdot\nabla\theta\theta_tdx\\&\quad+2\int(|\mathfrak{D}(u)|^2)_t\theta_tdx+\int(|\Delta d+|\nabla d|^2d|^2)_t\theta_tdx\\&=:\sum_{i=1}^5Z_i.\end{aligned} \tag{3.94}$$

It follows from (3.4), Hölder's inequality and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}Z_1&\le C\|u\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\&\le\delta\|\nabla\theta_t\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}^2\\&\le\delta\|\nabla\theta_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla^2u\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^4,\\Z_2&\le C\int\Big(\rho|u||\nabla u||\nabla\theta||\theta_t|+\rho|u|^2|\nabla^2\theta||\theta_t|+\rho|u|^2|\nabla\theta||\nabla\theta_t|\Big)dx\\&\le C\|u\|_{L^\infty}\|\nabla u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta\|_{L^3}+C\|u\|_{L^6}^2\|\nabla^2\theta\|_{L^2}\|\theta_t\|_{L^6}\\&\quad+C\|u\|_{L^6}^2\|\nabla\theta_t\|_{L^2}\|\nabla\theta\|_{L^6}\\&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{3}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}}\\&\quad+C\|\nabla u\|_{L^2}^2\|\nabla^2\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}\\&\le C\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^2}^3+C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}\\&\quad+\delta\|\nabla\theta_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla^2\theta\|_{L^2}^2\\&\le C\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^2}^3+C\|\sqrt{\rho}\theta_t\|_{L^2}^4+\|\nabla\theta\|_{L^2}^2\|\nabla^2\theta\|_{L^2}^2\\&\quad+\delta\|\nabla\theta_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla^2\theta\|_{L^2}^2,\\Z_3&\le C\|\sqrt{\rho}\theta_t\|_{L^3}\|u_t\|_{L^6}\|\nabla\theta\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\|\nabla\theta\|_{L^2}\\
&\le C\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\|\nabla\theta\|_{L^2}^2\\
&\le \delta\|\nabla\theta_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla\theta\|_{L^2}^4,\\
Z_4 &\le C\|\nabla u\|_{L^3}\|\nabla u_t\|_{L^2}\|\theta_t\|_{L^6}\\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\
&\le \delta\|\nabla\theta_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2}^2,\\
Z_5 &\le C\int|\theta_t|\Big(|\nabla^2 d||\nabla^2 d_t| + |\nabla^2 d||\nabla d||\nabla d_t| + |\nabla d|^2|\nabla^2 d_t|\\
&\quad + |\nabla^2 d||\nabla d|^2|d_t| + |\nabla d|^3|\nabla d_t| + |\nabla d|^4|d_t|\Big)dx\\
&\le C\|\nabla\theta_t\|_{L^2}\Big(\|\nabla^2 d\|_{L^3}\|\nabla^2 d_t\|_{L^2} + \|\nabla^2 d\|_{L^2}\|\nabla d\|_{L^6}\|\nabla d_t\|_{L^6}\\
&\quad + \|\nabla^2 d_t\|_{L^2}\|\nabla d\|_{L^6}^2 + \|\nabla d\|_{L^4}^2\|\nabla^2 d\|_{L^6}\|d_t\|_{L^6} + \|\nabla d\|_{L^6}^2\|\nabla^2 d_t\|_{L^2}\\
&\quad + \|\nabla d\|_{L^6}^3\|\nabla d_t\|_{L^3} + \|d_t\|_{L^6}\||\nabla d|^2\|_{L^3}\||\nabla d|^2\|_{L^3}\Big)\\
&\le \delta\|\nabla\theta_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^4\|\nabla^2 d_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^6\|\nabla d_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}^6\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}^8.
\end{aligned}
$$

Substituting $Z_i (i = 1, 2, \cdots, 5)$ into (3.94) and choosing δ suitably small, we have

$$
\begin{aligned}
&\frac{d}{dt}\int\rho|\theta_t|^2dx + \int|\nabla\theta_t|^2dx\\
&\le C\|\sqrt{\rho}\theta_t\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3\\
&\quad + C\|\nabla\theta\|_{L^2}^2\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla^2\theta\|_{L^2}^2\\
&\quad + C\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^4\|\nabla^2 d_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^6\|\nabla d_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}^6\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}^8\\
&\le C\|\sqrt{\rho}\theta_t\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3\\
&\quad + C(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^4)\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^4\|\nabla u\|_{L^2}^4\\
&\quad + C(\|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^4)(\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}^3 + \|\nabla^2 d\|_{L^2}^5\|\nabla^3 d\|_{L^2}^2)\\
&\quad + C\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^4\|\nabla^2 d_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}^3\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^6\|\nabla d_t\|_{L^2}^2\\
&\quad + C\|\nabla^2 d\|_{L^2}^6\|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2\|\nabla^2 d\|_{L^2}^8.
\end{aligned} \tag{3.95}
$$

(2) Multiplying the above inequality by $t^{\frac{5}{2}}$, integrating the result with respect to t and using Gronwall's inequality, we have

$$
\sup_{0\le t\le T}(t^{\frac{5}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^2) + \int_0^T t^{\frac{5}{2}}\|\nabla\theta_t\|_{L^2}^2dt
$$

$$
\begin{aligned}
&\le C\int_0^T t^{\frac{3}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^2dt + C\sup_{1\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)\int_1^T t^{\frac{3}{2}}\|\nabla^2 u\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}(t^{\frac{5}{2}}\|\nabla^2 u\|_{L^2}^2)\int_0^T(\|\nabla^2 u\|_{L^2}^2+\|\nabla u\|_{L^2}^2)dt\\
&\quad + C\sup_{1\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)^2\sup_{1\le t\le T}\|\nabla\theta\|_{L^2}^2\int_1^T\|\nabla\theta\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}(t^2\|\nabla^3 d\|_{L^2}^2)\int_0^T(t\|\nabla^3 d\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2)dt\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac{5}{2}}\int_0^T\|\nabla^3 d\|_{L^2}^2dt + C\int_0^1\|\nabla u_t\|_{L^2}^2dt\\
&\quad + C\int_1^T t^{\frac{5}{2}}\|\nabla u_t\|_{L^2}^2dt + C\sup_{0\le t\le T}(t\|\nabla\theta\|_{L^2}^2)\int_0^T t^{\frac{3}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla u\|_{L^2}^2)^{\frac{1}{2}}\sup_{0\le t\le T}(t^2\|\nabla^2 u\|_{L^2}^2)^{\frac{1}{2}}\int_0^T t\|\nabla u_t\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}\left\{(t\|\nabla^3 d\|_{L^2}^2)^{\frac{1}{2}}+(t\|\nabla^2 d\|_{L^2}^2)^{\frac{1}{2}}\right\}\int_0^T t^2\|\nabla^2 d_t\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac{1}{2}}\sup_{0\le t\le T}(t^2\|\nabla d_t\|_{L^2}^2)\int_0^T\|\nabla^3 d\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}(t^2\|\nabla d_t\|_{L^2}^2)\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac{1}{2}}\int_0^T\|\nabla^2 d\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac{3}{2}}\int_0^T t\|\nabla d_t\|_{H^1}^2dt + C\\
&\le C. \qquad (3.96)
\end{aligned}
$$

(3) It follows from (3.9), (3.55), (3.29), (3.36), (3.31), (3.78) and (3.79) that

$$
\begin{aligned}
\sup_{0\le t\le T}(t^{\frac{5}{2}}\|\nabla^2\theta\|_{L^2}^2) &\le C + C\sup_{0\le t\le T}(t^{\frac{3}{2}}\|\nabla u\|_{L^2}^2)\sup_{0\le t\le T}(t\|\nabla u\|_{L^2}^2)\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla^2 u\|_{L^2}^2)^{\frac{1}{2}}\sup_{0\le t\le T}(t^2\|\nabla^2 u\|_{L^2}^2)\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla^3 d\|_{L^2}^2)^{\frac{1}{2}}\sup_{0\le t\le T}(t^2\|\nabla^3 d\|_{L^2}^2)\\
&\quad + C\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac{1}{2}}\sup_{0\le t\le T}(t^2\|\nabla^3 d\|_{L^2}^2)\\
&\le C, \qquad (3.97)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T t^{\frac{3}{2}}\|\nabla^2\theta\|_{L^2}^2dt &\le C\int_0^T t^{\frac{3}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^2dt + C\sup_{0\le t\le T}(t^{\frac{3}{2}}\|\nabla\theta\|_{L^2}^2)\int_0^T\|\nabla u\|_{L^2}^2dt\\
&\quad + C\sup_{0\le t\le T}t^{\frac{5}{2}}\|\nabla^2 u\|_{L^2}^2\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2)dt
\end{aligned}
$$

$$
\begin{aligned}
&+C\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac12}C\sup_{0\le t\le T}(t^2\|\nabla^3 d\|_{L^2}^2)^{\frac12}\int_0^T\|\nabla^3 d\|_{L^2}^2dt\\
&+C\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)\sup_{0\le t\le T}(t\|\nabla^2 d\|_{L^2}^2)^{\frac12}\int_0^T\|\nabla^3 d\|_{L^2}^2dt\\
&\le C. \qquad (3.98)
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\int_0^T t^{\frac52}\|\nabla^2\theta\|_{L^6}^2dt &\le C\int_0^T t^{\frac52}(\|\rho\theta_t\|_{L^6}^2+\|\rho u\cdot\nabla\theta\|_{L^6}^2+\|\nabla u\|_{L^{12}}^4+\||\Delta d+|\nabla d|^2 d|\|_{L^{12}}^4)dt\\
&\le C\int_0^T t^{\frac52}(\|\nabla\theta_t\|_{L^2}^2+\|u\|_{L^\infty}^2\|\nabla\theta\|_{L^6}^2+\||\nabla^2 d||\nabla^3 d|\|_{L^2}^2+\||\nabla^2 d||\nabla d|^3\|_{L^2}^2)dt\\
&\le C\int_0^T t^{\frac52}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2\theta\|_{L^2}^2dt+C\int_0^T t^{\frac52}\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla^4 d\|_{L^2}^2dt\\
&\quad+C\int_0^T t^{\frac52}\|\nabla^2 d\|_{L^2}^5\|\nabla^3 d\|_{L^2}^3dt+C\\
&\le C\sup_{0\le t\le T}(t^2\|\nabla^2 u\|_{L^2}^2)^{\frac12}\int_0^T t^{\frac32}\|\nabla^2\theta\|_{L^2}^2dt+C\sup_{0\le t\le T}(t\|\nabla^3 d\|_{L^2}^2)^{\frac12}\int_0^T t^2\|\nabla^4 d\|_{L^2}^2dt\\
&\quad+C\sup_{0\le t\le T}\left\{(t\|\nabla^3 d\|_{L^2}^2)^{\frac12}(t\|\nabla^2 d\|_{L^2}^2)\right\}\int_0^T t\|\nabla^3 d\|_{L^2}^2dt+C\\
&\le C. \qquad (3.99)
\end{aligned}
$$

Thus, we completed the proof of the lemma. □

4. Proof of Thorem 1.1

Based on Lemma 2.1, there exists a $T_0>0$ such that the magneto-micropolar systems (1.1) and (1.2) have a unique local strong solution (ρ,u,θ,d,p) in $\mathbb{R}^3\times[0,T_0]$. To prove Theorem 1.1, it suffices to show that the local solution can be extended to be a global one. To do this, we assume from now that $\mathbb{K}_0\le\epsilon_0$ holds.

Set

$$T^*=\sup\{T\mid(\rho,u,\theta,d,p)\ \textit{is a strong solution on}\ [0,T]\}. \qquad (4.1)$$

We claim that

$$T^*=\infty. \qquad (4.2)$$

Otherwise, assume that $T^*<\infty$. By virtue of Lemmas 3.1 and 3.6 and Proposition 3.1, it holds that $(\rho,u,\theta,d,p)|_{t=T^*}$ satisfies (1.4) and (1.5). Thus, Lemma 2.1 implies that there exists some $T^{**}>T^*$, such that (ρ,u,θ,d,p) can be extended to a strong solution of (1.1) and (1.2) in $\mathbb{R}^3\times[0,T^{**})$, which contradicts (4.1). Hence, (4.2) holds.

5. Conclusions

In this study, we were concerned with an initial value problem related to non-isothermal incompressible nematic liquid crystal flows in $\mathbb{R}^3$. Using some time-weighted a priori estimates, we have proven the global existence of a strong solution provided that $\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right)\left(\|\nabla u_0\|_{L^2}^2 + \|\nabla^2 d_0\|_{L^2}^2\right)$ is suitably small. Furthermore, we have also obtained the large time behavior of the solutions.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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