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Research article

Some operator mean inequalities for sector matrices

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Abstract: In this article, we obtain some operator mean inequalities of sectorial matrices involving operator monotone functions. Among other results, it is shown that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are such that $W(A), W(B) \subseteq S_\alpha$, $f, g, h \in \mathfrak{m}$ are such that $g'(1) = h'(1) = t$ for some $t \in (0, 1)$ and $0 < mI_n \leq \Re A, \Re B \leq MI_n$, then

$$\Re(\Phi(f(A))\sigma_h\Phi(f(B))) \leq \sec^4(\alpha)K\Re\Phi(f(A\sigma_g B)),$$

where M, m are scalars and $\mathfrak{m}$ is the collection of all operator monotone function $\varphi : (0, \infty) \to (0, \infty)$ satisfying $\varphi(1) = 1$. Moreover, we refine a norm inequality of sectorial matrices involving positive linear maps, which is a result of Bedrani, Kittaneh and Sababheh.

Keywords: sector matrices; operator monotone function; operator mean; numerical radius; positive linear maps
Mathematics Subject Classification: 15A45, 47A63

1. Introduction

Let $B(\mathcal{H})$ denote the C^* algebra of all bounded linear operators acting on a Hilbert space $\mathcal{H}$. When the dimension of $\mathcal{H}$ is finite, we identify $B(\mathcal{H})$ with $\mathbb{M}_n(\mathbb{C})$, denoting the set of $n \times n$ complex matrices. I denotes the identity operator in $B(\mathcal{H})$, while I_n denotes the identity matrix in $\mathbb{M}_n(\mathbb{C})$. For $A \in \mathbb{M}_n(\mathbb{C})$, the conjugate transpose of A is denoted by A^*, and the matrices $\Re A = \frac{1}{2}(A + A^*)$ and $\Im A = \frac{1}{2i}(A - A^*)$ are called the real part and imaginary part of A, respectively ([6, p.6] and [12, p.7]). Moreover, A is called accretive if $\Re A > 0$. For two Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we write $A \geq B$ (resp. $A > B$) if $A - B$ is positive semidefinite(resp. positive definite). A linear map $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is called positive if it maps positive semi-definite matrices in $\mathbb{M}_n(\mathbb{C})$ to positive semi-definite matrices in $\mathbb{M}_k(\mathbb{C})$ and is said to be unital if it maps the identity matrix in $\mathbb{M}_n(\mathbb{C})$ to the identity matrix in $\mathbb{M}_k(\mathbb{C})$. We reserve M, m for scalars.

The operator norm of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$\|A\| = \max\{|\langle Ax, y\rangle| : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1\}.$$

$A \in \mathbb{M}_n(\mathbb{C})$ is contractive if $\|A\| \leq 1$. Let $\|\cdot\|_u$ denote any unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$, which satisfies $\|UAV\|_u = \|A\|_u$ for any unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$ and all $A \in \mathbb{M}_n(\mathbb{C})$.

For $\alpha \in [0, \frac{\pi}{2})$, S_α denotes the sectorial region in the complex plane as follows:

$$S_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z)\tan\alpha\}.$$

If $W(A) \subseteq S_0$, then A is positive definite, and if $W(A), W(B) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A+B) \subseteq S_\alpha$, A is nonsingular and $\Re(A)$ is positive definite. Moreover, $W(A) \subseteq S_\alpha$ implies $W(X^*AX) \subseteq S_\alpha$ for any nonzero $n \times m$ matrix X, thus $W(A^{-1}) \subseteq S_\alpha$. Recently, Tan and Chen [20] also proved that for any positive linear map Φ, $W(A) \subseteq S_\alpha$ implies that $W(\Phi(A)) \subseteq S_\alpha$. Recent developments on sectorial matrices can be found in [3, 4, 9, 10, 16, 18, 22, 23].

The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

The numerical radius of A is defined by $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$. We note that if $A \geq 0$, then $\omega(A) = \|A\|$. The following inequality holds true

$$\omega(\Re A) \leq \omega(A) \leq \|A\| \tag{1.1}$$

for $A \in \mathbb{M}_n(\mathbb{C})$.

For two positive definite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ and $0 \leq t \leq 1$, the weighted geometric mean, weighted harmonic mean and weighted arithmetic mean are defined respectively as follows:

$$A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}},$$

$$A!_t B = ((1-t)A^{-1} + tB^{-1})^{-1},$$

$$A\nabla_t B = (1-t)A + tB.$$

In particular, when $t = \frac{1}{2}$, we denote the geometric mean, harmonic mean and arithmetic mean by $A\sharp B$, $A!B$ and $A\nabla B$, respectively. Another interesting operator mean is the Heron mean, which is defined by $F_t(A, B) = t(A\nabla B) + (1-t)(A\sharp B)$ for positive definite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ and $0 \leq t \leq 1$. The weighted arithmetic-geometric-harmonic mean inequalities states that

$$A!_t B \leq A\sharp_t B \leq A\nabla_t B. \tag{1.2}$$

For two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, Drury [9] defined the geometric mean of A and B as follows

$$A\sharp B = \left(\frac{2}{\pi}\int_0^\infty (tA + t^{-1}B)^{-1}\,\frac{dt}{t}\right)^{-1}. \tag{1.3}$$

This new geometric mean defined by (1.3) possesses some similar properties compared to the geometric mean of positive matrices. For instance, $A\sharp B = B\sharp A$, $(A\sharp B)^{-1} = A^{-1}\sharp B^{-1}$. Moreover, if $A, B \in \mathbb{M}_n(\mathbb{C})$ with $W(A), W(B) \subset S_\alpha$, then $W(A\sharp B) \subset S_\alpha$.

Later, Raissouli, Moslehian and Furuichi [19] defined the following weighted geometric mean of two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$A\sharp_\lambda B = \frac{\sin \lambda\pi}{\pi} \int_0^\infty t^{\lambda-1}(A^{-1} + tB^{-1})^{-1} \frac{dt}{t}, \tag{1.4}$$

where $\lambda \in [0, 1]$. If $\lambda = \frac{1}{2}$, then the formula (1.4) coincides with the formula (1.3).

We say a real valued continuous function $f : (0, \infty) \to (0, \infty)$ operator monotone (increasing) if for any two positive operators A, B, $A \geq B$ implies $f(A) \geq f(B)$. If $f(A) \leq f(B)$ whenever $A \geq B > 0$, we say f is operator monotone decreasing.

For the sake of convenience, we will need the following notation.

$$\mathfrak{m} = \{f(x), \text{ where } f : (0, \infty) \to (0, \infty) \text{ is an operator monotone function with } f(1) = 1\}.$$

Lately, Bedrani, Kittaneh and Sababheh [3] defined a more general operator mean for two accretive matrices $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$A\sigma_g B = \int_0^1 ((1 - s)A^{-1} + sB^{-1})^{-1} \, d\nu_g(s), \tag{1.5}$$

where g: $(0, \infty) \to (0, \infty)$ is an operator monotone function with $g(1) = 1$ and ν_g is the probability measure characterizing σ_g. We note that $!_t \leq \sigma_g \leq \nabla_t$ for positive matrices if $g \in \mathfrak{m}$ are such that $g'(1) = t$ for some $t \in (0, 1)$.

In the same paper, they also characterize the operator monotone function for an accretive matrix: let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive and $f \in \mathfrak{m}$,

$$f(A) = \int_0^1 ((1 - s)I + sA^{-1})^{-1} \, d\nu_f(s), \tag{1.6}$$

where ν_f is the probability measure satisfying $f(x) = \displaystyle\int_0^1 ((1 - s) + sx^{-1})^{-1} \, d\nu_f(s)$. This is because $A\sigma_g B = A^{\frac{1}{2}} g(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$ for accretive matrices A, B.

Ando [1] proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive definite, then for any positive linear map Φ,

$$\Phi(A\sigma_f B) \leq \Phi(A)\sigma_f \Phi(B). \tag{1.7}$$

Ando's formula (1.7) is known as a matrix Hölder inequality.

The famous Choi's inequality [5, p.41] states that if Φ is a positive unital linear map and $A > 0$, then

$$\Phi^t(A) \leq \Phi(A^t), \qquad t \in [-1, 0]. \tag{1.8}$$

$$\Phi^t(A) \geq \Phi(A^t), \qquad t \in [0, 1]. \tag{1.9}$$

A general situation of inequality (1.9) is the following one [1]:

$$\Phi(f(A)) \le f(\Phi(A)), \quad f \text{ is operator monotone.} \tag{1.10}$$

In a recent paper [13], The authors obtained some inequalities involving operator monotone (increasing) functions and operator monotone decreasing functions for positive operators: Let $A \in B(\mathcal{H})$ be such that $0 < mI \le A, B \le MI$, $!_t \le \sigma_h, \sigma_{h'} \le \nabla_t$ and $t \in [0,1]$. Then for every positive unital linear map Φ,

$$\Phi(f(A))\sigma_h\Phi(f(B)) \le K\Phi(f(A\sigma_{h'}B)), \tag{1.11}$$

$$g(\Phi(A\sigma_h B)) \le K(g(\Phi(A))\sigma_{h'}g(\Phi(B))), \tag{1.12}$$

$$(g(\Phi(A))\sigma_{h'}g(\Phi(B))) \le Kg(\Phi(A\sigma_h B)), \tag{1.13}$$

here the operator means $\sigma_h, \sigma_{h'}$ are defined for positive semidefinite matrices, $f : (0,\infty) \to (0,\infty)$ is operator monotone and $g : (0,\infty) \to (0,\infty)$ is operator monotone decreasing, K denotes the Kantorovich constant $K(\frac{M}{m}) = \frac{(M+m)^2}{4Mm}$ throughout the paper. Since (1.11)–(1.13) are inequalities for positive operator, whether we can obtain the accretive version of these inequalities partially triggers the motivation of this article.

From [2] we know that for a continuous nonnegative function f on $(0,\infty)$, f is operator monotone if and only if $\frac{1}{f}$(or f^{-1}) is operator monotone decreasing. Thus we can treat f^{-1} as operator monotone decreasing function when f is an operator monotone function.

In [3], the authors gave an comparison for sector matrices: Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$ and $0 < mI_n \le \Re(A), \Re(B) \le MI_n$. If $g, h \in \mathfrak{m}$ are such that $g'(1) = h'(1) = t$ for some $t \in (0,1)$, then for every positive unital linear map Φ,

$$\left\|\Phi(\Re(A\sigma_g B))\Phi^{-1}(\Re(A\sigma_h B))\right\| \le \sec^6(\alpha)K. \tag{1.14}$$

Very recently, the authors in [11] gave the definition of Heron mean of sector matrices $A, B \in \mathbb{M}_n(\mathbb{C})$(in particular, positive definite matrices): $F_t(A,B) = t(A\nabla B) + (1-t)(A\sharp B)$, $t \in [0,1]$. They also gave numerical radius inequalities for Heron mean of two sector matrices: Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in (0,1)$. Then

$$\cos^{2t+2}(\alpha)\omega(A\sharp B) \le \omega(F_t(A,B)) \le \sec^4(\alpha)(1 - t\sin^2(\alpha)\omega(A\nabla B). \tag{1.15}$$

In this paper, we intend to give some refinements of inequalities (1.11)–(1.15). Furthermore, we shall present more operator mean inequalities for sector matrices.

2. Main results

We begin this section with several lemmas which will be necessary for achieving our goals.

Lemma 2.1. *(see [3]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$. If $f \in \mathfrak{m}$, then*

$$f(\Re A) \le \Re(f(A)) \le \sec^2(\alpha)f(\Re A).$$

Lemma 2.2. *(see [3, 14, 19, 21]) Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $f \in \mathfrak{m}$. Then*

$$\Re A\sigma_f \Re B \leq \Re(A\sigma_f B) \leq \sec^2(\alpha)(\Re A\sigma_f \Re B).$$

Lemma 2.3. *(see [4]) Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then*

$$\cos(\alpha)\omega(A) \leq \omega(\Re A) \leq \omega(A).$$

The following lemma is a well-known result.

Lemma 2.4. *(see [13], Lemma 2.2) If $f : (0,\infty) \to (0,\infty)$ is operator monotone, then $f(\alpha t) \leq \alpha f(t)$ for $\alpha \geq 1$. The inequality is reversed when $0 \leq \alpha \leq 1$.*

Lemma 2.5. *(see [24]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$ and let $\|\cdot\|_u$ be any unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$. Then*

$$\cos(\alpha)\|A\|_u \leq \|\Re A\|_u \leq \|A\|_u.$$

Lemma 2.6. *(see [10, 15]) Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subseteq S_\alpha$. Then*

$$\Re(A^{-1}) \leq \Re^{-1}A \leq \sec^2(\alpha)\Re(A^{-1}).$$

Lemma 2.7. *(see [7]) Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2.$$

Theorem 2.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $0 < mI_n \leq \Re A, \Re B \leq MI_n$. If $f, g, h \in \mathfrak{m}$ are such that $g'(1) = h'(1) = t$ for some $t \in (0,1)$, then for every positive unital linear map Φ,*

$$\Re(\Phi(f(A))\sigma_h\Phi(f(B))) \leq \sec^4(\alpha)K\Re\Phi(f(A\sigma_g B)).$$

Proof. We have the following chain of inequalities

$$\begin{aligned}
\Re(\Phi(f(A))\sigma_h\Phi(f(B))) &\leq \sec^2(\alpha)\Re\Phi(f(A))\sigma_h\Re\Phi(f(B)) \quad \text{(by Lemma 2.2)}\\
&\leq \sec^4(\alpha)\Phi(f(\Re A))\sigma_h\Phi(f(\Re B)) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^4(\alpha)K\Phi(f(\Re A\sigma_g\Re B)) \quad \text{(by inequality (1.11))}\\
&\leq \sec^4(\alpha)K\Phi(f(\Re(A\sigma_g B))) \quad \text{(by Lemma 2.2)}\\
&\leq \sec^4(\alpha)K\Re\Phi(f(A\sigma_g B)), \quad \text{(by Lemma 2.1)}
\end{aligned}$$

which completes the proof.

Note that when $A, B \geq 0$ in Theorem 2.1, we get inequality (1.11).

Theorem 2.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $0 < mI_n \leq \Re A, \Re B \leq MI_n$. If $f, g \in \mathfrak{m}$ are such that $g'(1) = t$ for some $t \in (0,1)$, then*

$$\frac{\|f(A)\sigma_g f(B)\|_u}{\|A\sigma_g B\|_u} \leq \sec^5(\alpha)K\left\|\frac{f(A\sigma_g B)}{A\sigma_g B}\right\|_u.$$

Proof. Compute

$$\begin{aligned}
\frac{\|f(A)\sigma_g f(B)\|_u}{\|A\sigma_g B\|_u} &\leq \sec(\alpha)\frac{\|\Re(f(A)\sigma_g f(B))\|_u}{\|A\sigma_g B\|_u} \quad \text{(by Lemma 2.5)}\\
&\leq \sec^5(\alpha)K\frac{\|\Re(f(A\sigma_g B))\|_u}{\|A\sigma_g B\|_u} \quad \text{(by Theorem 2.1)}\\
&\leq \sec^5(\alpha)K\frac{\|f(A\sigma_g B)\|_u}{\|A\sigma_g B\|_u} \quad \text{(by Lemma 2.5)}\\
&\leq \sec^5(\alpha)K\left\|\frac{f(A\sigma_g B)}{A\sigma_g B}\right\|_u,
\end{aligned}$$

which completes the proof.

Theorem 2.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $0 < mI_n \leq \Re A, \Re B \leq MI_n$. If $f, g, h \in \mathfrak{m}$ are such that $g'(1) = h'(1) = t$ for some $t \in (0,1)$, then for every positive unital linear map Φ,*

$$\Re f^{-1}\left(\Phi(A\sigma_g B)\right) \leq \sec^4(\alpha)K\Re(f^{-1}(\Phi(A))\sigma_h f^{-1}(\Phi(B))).$$

Proof. We have

$$\begin{aligned}
\Re f^{-1}\left(\Phi(A\sigma_g B)\right) &\leq (\Re f(\Phi(A\tau_t B)))^{-1} \quad \text{(by Lemma 2.6)}\\
&\leq \left(f(\Re(\Phi(A\sigma_g B))\right)^{-1} \quad \text{(by Lemma 2.1)}\\
&\leq \left(f(\Phi(\Re A\sigma_g \Re B))\right)^{-1} \quad \text{(by Lemma 2.2)}\\
&\leq Kf^{-1}(\Phi(\Re A))\sigma_h f^{-1}(\Phi(\Re B)) \quad \text{(by inequality (1.12))}\\
&\leq \sec^2(\alpha)K\Re^{-1} f(\Phi(A))\sigma_h \Re^{-1} f(\Phi(B)) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^4(\alpha)K\Re f^{-1}(\Phi(A))\sigma_h \Re f^{-1}(\Phi(B)) \quad \text{(by Lemma 2.6)}\\
&\leq \sec^4(\alpha)K\Re(f^{-1}(\Phi(A))\sigma_h f^{-1}(\Phi(B))), \quad \text{(by Lemma 2.2)}
\end{aligned}$$

which completes the proof.

Note that when $A, B \geq 0$ in Theorem 2.3, we get inequality (1.12).

Theorem 2.4. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $0 < mI_n \leq \Re A, \Re B \leq MI_n$. If $f, g, h \in \mathfrak{m}$ are such that $g'(1) = h'(1) = t$ for some $t \in (0,1)$, then for every positive unital linear map Φ,*

$$\Re(f^{-1}(\Phi(A))\sigma_h f^{-1}(\Phi(B))) \leq \sec^8(\alpha)K\Re f^{-1}\left(\Phi(A\sigma_g B)\right).$$

Proof. Compute

$$\begin{aligned}
\Re(f^{-1}(\Phi(A))\sigma_h f^{-1}(\Phi(B))) &\leq \sec^2(\alpha)\Re(f^{-1}(\Phi(A)))\sigma_h \Re(f^{-1}(\Phi(B))) \quad \text{(by Lemma 2.2)}\\
&\leq \sec^2(\alpha)\Re^{-1}(f(\Phi(A)))\sigma_h \Re^{-1}(f(\Phi(B))) \quad \text{(by Lemma 2.6)}\\
&\leq \sec^2(\alpha)f^{-1}(\Re(\Phi(A)))\sigma_h f^{-1}(\Re(\Phi(B))) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^2(\alpha)Kf^{-1}(\Phi(\Re A\sigma_g \Re B)) \quad \text{(by inequality (1.13))}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sec^2(\alpha)Kf^{-1}(\cos^2(\alpha)\Phi(\Re(A\sigma_g B))) \quad \text{(by Lemma 2.2)}\\
&\leq \sec^4(\alpha)Kf^{-1}(\Phi(\Re(A\sigma_g B))) \quad \text{(by Lemma 2.4)}\\
&\leq \sec^6(\alpha)K\Re^{-1}(f(\Phi(A\sigma_g B))) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^8(\alpha)K\Re f^{-1}\left(\Phi(A\sigma_g B)\right), \quad \text{(by Lemma 2.6)}
\end{aligned}
$$

which completes the proof.

Note that when $A, B \geq 0$ in Theorem 2.4, we get inequality (1.13).

Theorem 2.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$ and $f \in \mathfrak{m}$. Then for any positive unital linear map Φ,*

$$f(\Phi(A\sharp A^*)) \geq \cos^2(\alpha)\Re\left(\Phi f(A)\right).$$

Proof. Compute

$$
\begin{aligned}
f(\Phi(A\sharp A^*)) &= f(\Phi(\Re(A\sharp A^*)))\\
&\geq \Phi f(\Re(A\sharp A^*)) \quad \text{(by inequality (1.10))}\\
&\geq \Phi f(\Re A\sharp \Re A^*) \quad \text{(by Lemma 2.2)}\\
&= \Phi f(\Re A)\\
&\geq \cos^2(\alpha)\Re\left(\Phi f(A)\right), \quad \text{(by Lemma 2.1)}
\end{aligned}
$$

which completes the proof.

Corollary 2.1. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive. Then*

$$A\sharp A^* \geq \Re A.$$

Corollary 2.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be contractive. Then*

$$I_n - A^*A \leq (I_n - A^*)(I_n + A)\sharp(I_n + A^*)(I_n - A).$$

In particular, if $A = U$ is unitary, then $0 \leq (U - U^*)\sharp(U^* - U)$.

In [17], the authors obtained that $(I_n - A^*B)\sharp(I_n - B^*A) \geq (I_n - A^*A)\sharp(I_n - B^*B)$ for contractions $A, B \in \mathbb{M}_n(\mathbb{C})$. Imposing Φ on both sides implies $\Phi((I_n - A^*B)\sharp(I_n - B^*A)) \geq \Phi((I_n - A^*A)\sharp(I_n - B^*B))$. We note that a stronger result holds $\Phi((I_n - A^*B)\sharp(I_n - B^*A)) \geq \Phi(I_n - A^*A)\sharp\Phi(I_n - B^*B)$.

Theorem 2.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subseteq S_\alpha$ and $f \in \mathfrak{m}$. Then for any positive unital linear map Φ,*

$$f^{-1}(\Phi(A\sharp A^*)) \leq \sec^4(\alpha)\Re(\Phi f^{-1}(A)).$$

Proof. We have

$$
\begin{aligned}
f^{-1}(\Phi(A\sharp A^*)) &\leq \Phi^{-1}(f(A\sharp A^*)) \quad \text{(by inequality (1.10))}\\
&\leq \Phi(f^{-1}(A\sharp A^*)) \quad \text{(by inequality (1.8))}\\
&\leq \Phi(f^{-1}(\Re A)) \quad \text{(by Corollary 2.1)}\\
&\leq \sec^2(\alpha)\Phi(\Re^{-1} f(A)) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^4(\alpha)\Re(\Phi f^{-1}(A)), \quad \text{(by Lemma 2.6)}
\end{aligned}
$$

which completes the proof.

Theorem 2.7. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $0 < mI_n \leq \Re A, \Re B \leq MI_n$. If $f \in \mathfrak{m}$ and $t \in (0,1)$, then for every positive unital linear map Φ,*

$$\Re(\Phi(f(A!_t B))) \leq \sec^4(\alpha)\Re(f(\Phi(A))!_t f(\Phi(B))).$$

Proof. Compute

$$\begin{aligned}
\Re(\Phi(f(A!_t B))) &\leq \sec^2(\alpha)\Phi f(\Re(A!_t B)) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^2(\alpha)\Phi f(\sec^2(\alpha)\Re A!_t \Re B) \quad \text{(by Lemma 2.2)}\\
&\leq \sec^4(\alpha)\Phi f(\Re A!_t \Re B) \quad \text{(by Lemma 2.4)}\\
&\leq \sec^4(\alpha) f(\Phi(\Re A!_t \Re B)) \quad \text{(by inequality (1.10))}\\
&\leq \sec^4(\alpha) f(\Phi(\Re A)!_t \Phi(\Re B)) \quad \text{(by inequality (1.7))}\\
&= \sec^4(\alpha) f(\Re(\Phi(A))!_t \Re(\Phi(B)))\\
&\leq \sec^4(\alpha) f(\Re(\Phi(A)))!_t f(\Re(\Phi(B))) \quad \text{(by Theorem 4 in [8])}\\
&\leq \sec^4(\alpha)\Re f(\Phi(A))!_t \Re f(\Phi(B)) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^4(\alpha)\Re(f(\Phi(A))!_t f(\Phi(B))), \quad \text{(by Lemma 2.2)}
\end{aligned}$$

which completes the proof.

Theorem 2.8. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $0 < mI_n \leq \Re A, \Re B \leq MI_n$. If $f \in \mathfrak{m}$ and $t \in (0,1)$, then*

$$\Re f^{-1}(A\nabla_t B) \leq \sec^4(\alpha)\Re(f^{-1}(A)!_t f^{-1}(B)).$$

Proof. Compute

$$\begin{aligned}
\Re f^{-1}(A\nabla_t B) &\leq (\Re f(A\nabla_t B))^{-1} \quad \text{(by Lemma 2.6)}\\
&\leq f^{-1}(\Re A\nabla_t \Re B) \quad \text{(by Lemma 2.1)}\\
&\leq f^{-1}(\Re A)!_t f^{-1}(\Re B) \quad \text{(by Remark 2.6 in [2])}\\
&\leq \sec^2(\alpha)\Re^{-1}(f(A))!_t \Re^{-1}(f(B)) \quad \text{(by Lemma 2.1)}\\
&\leq \sec^4(\alpha)\Re f^{-1}(A)!_t \Re f^{-1}(B) \quad \text{(by Lemma 2.6)}\\
&\leq \sec^4(\alpha)\Re(f^{-1}(A)!_t f^{-1}(B)), \quad \text{(by Lemma 2.2)}
\end{aligned}$$

which completes the proof.

Theorem 2.9. *Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$ and $0 < mI_n \leq \Re(A), \Re(B) \leq MI_n$. If $g, h \in \mathfrak{m}$ are such that $g'(1) = h'(1) = t$ for some $t \in (0,1)$, then for every positive unital linear map Φ,*

$$\left\|\Phi(\Re(A\sigma_g B))\Phi^{-1}(\Re(A\sigma_h B))\right\| \leq \sec^2(\alpha)K. \tag{2.1}$$

Proof. From $0 < mI_n \leq \Re(A), \Re(B) \leq MI_n$ we have

$$(1-t)(M - \Re(A))(m - \Re A)A^{-1} \leq 0,$$

which is equivalent to

$$(1-t)\mathfrak{R}(A)+(1-t)Mm\mathfrak{R}^{-1}(A)\leq(1-t)(M+m)I_n. \tag{2.2}$$

Similarly, we have

$$t\mathfrak{R}(B)+tMm\mathfrak{R}^{-1}(B)\leq t(M+m)I_n. \tag{2.3}$$

Summing up inequalities (2.2) and (2.3), we get

$$\mathfrak{R}\left(A\nabla_t B\right)+Mm(\mathfrak{R}^{-1}(A)\nabla_t\mathfrak{R}^{-1}(B))\leq(M+m)I_n. \tag{2.4}$$

By computation, we have

$$\begin{aligned}
&\left\|\sec^2(\alpha)Mm\Phi(\mathfrak{R}(A\sigma_g B))\Phi^{-1}(\mathfrak{R}(A\sigma_h B))\right\|\\
&\leq\frac{1}{4}\left\|\Phi(\mathfrak{R}(A\sigma_g B))+\sec^2(\alpha)Mm\Phi^{-1}(\mathfrak{R}(A\sigma_h B))\right\|^2 \quad \text{(by Lemma 2.7)}\\
&\leq\frac{1}{4}\left\|\Phi(\mathfrak{R}(A\sigma_g B))+\sec^2(\alpha)Mm\Phi(\mathfrak{R}^{-1}(A\sigma_h B))\right\|^2 \quad \text{(by inequality (1.8))}\\
&\leq\frac{1}{4}\left\|\Phi(\mathfrak{R}(A\sigma_g B))+\sec^2(\alpha)Mm\Phi((\mathfrak{R}A\sigma_h\mathfrak{R}B)^{-1})\right\|^2 \quad \text{(by Lemma 2.2)}\\
&\leq\frac{1}{4}\left\|\Phi(\mathfrak{R}(A\sigma_g B))+\sec^2(\alpha)Mm\Phi((\mathfrak{R}A!_t\mathfrak{R}B)^{-1})\right\|^2\\
&=\frac{1}{4}\left\|\Phi(\mathfrak{R}(A\sigma_g B))+\sec^2(\alpha)Mm\Phi(\mathfrak{R}^{-1}A\nabla_t\mathfrak{R}^{-1}B)\right\|^2\\
&\leq\frac{1}{4}\left\|\sec^2(\alpha)\Phi(\mathfrak{R}(A\nabla_t B))+\sec^2(\alpha)Mm\Phi(\mathfrak{R}^{-1}A\nabla_t\mathfrak{R}^{-1}B)\right\|^2 \quad \text{(by Theorem 5.2 in [3])}\\
&=\frac{1}{4}\left\|\sec^2(\alpha)\Phi(\mathfrak{R}(A\nabla_t B)+Mm(\mathfrak{R}^{-1}A\nabla_t\mathfrak{R}^{-1}B))\right\|^2\\
&\leq\frac{1}{4}\sec^4(\alpha)(M+m)^2. \quad \text{(by inequality (2.4))}
\end{aligned}$$

That is,

$$\left\|\Phi(\mathfrak{R}(A\sigma_g B))\Phi^{-1}(\mathfrak{R}(A\sigma_h B))\right\|\leq\sec^2(\alpha)K.$$

This completes the proof.

We remark that Theorem 2.9 is an improvement of inequality (1.14).

Theorem 2.10. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be such that $W(A),W(B)\subseteq S_\alpha$ and $t\in(0,1)$. Then for every positive unital linear map Φ,*

$$\mathfrak{R}F_t^{-1}\left(\Phi(A),\Phi(B)\right)\leq\sec^2(\alpha)\mathfrak{R}F_t\left(\Phi(A^{-1}),\Phi(B^{-1})\right).$$

Proof. We have the following chain of inequalities

$$\begin{aligned}
\mathfrak{R}F_t^{-1}\left(\Phi(A),\Phi(B)\right) &= \mathfrak{R}(t\Phi(A)\nabla\Phi(B)+(1-t)\Phi(A)\sharp\Phi(B))^{-1}\\
&\leq \mathfrak{R}^{-1}(t\Phi(A)\nabla\Phi(B)+(1-t)\Phi(A)\sharp\Phi(B)) \quad \text{(by Lemma 2.6)}
\end{aligned}$$

$$
\begin{aligned}
&= (t\Phi(\Re A)\nabla\Phi(\Re B) + (1-t)\Re(\Phi(A)\sharp\Phi(B)))^{-1} \\
&\le t(\Phi(\Re A)\nabla\Phi(\Re B))^{-1} + (1-t)\Re^{-1}(\Phi(A)\sharp\Phi(B)) \\
&\le t\Phi((\Re A\nabla\Re B)^{-1}) + (1-t)(\Re\Phi(A)\sharp\Re\Phi(B))^{-1} \quad \text{(by Lemma 2.2 and (1.8))} \\
&= t\Phi((\Re A\nabla\Re B)^{-1}) + (1-t)\Phi^{-1}(\Re A)\sharp\Phi^{-1}(\Re B) \\
&\le t\Phi((\Re A\nabla\Re B)^{-1}) + (1-t)\Phi(\Re^{-1}A)\sharp\Phi(\Re^{-1}B) \quad \text{(by (1.8))} \\
&\le t\Phi((\Re A\nabla\Re B)^{-1}) + (1-t)\sec^2(\alpha)\Phi(\Re A^{-1})\sharp\Phi(\Re B^{-1}) \quad \text{(by Lemma 2.6)} \\
&\le t\Phi(\Re^{-1}A\nabla\Re^{-1}B) + (1-t)\sec^2(\alpha)\Re(\Phi(A^{-1})\sharp\Phi(B^{-1})) \quad \text{(by Lemma 2.2)} \\
&\le t\sec^2(\alpha)\Phi(\Re(A^{-1})\nabla\Re(B^{-1})) + (1-t)\sec^2(\alpha)\Re(\Phi(A^{-1})\sharp\Phi(B^{-1})) \quad \text{(by Lemma 2.6)} \\
&= t\sec^2(\alpha)\Re(\Phi(A^{-1})\nabla\Phi(B^{-1})) + (1-t)\sec^2(\alpha)\Re(\Phi(A^{-1})\sharp\Phi(B^{-1})) \\
&= \sec^2(\alpha)\Re F_t\left(\Phi(A^{-1}),\Phi(B^{-1})\right),
\end{aligned}
$$

which completes the proof.

We note that by letting $\Phi(X) = X$ for every $X \in \mathbb{M}_n(\mathbb{C})$ in Theorem 2.10, we get the right hand side of inequalities in Theorem 3 in [11].

Theorem 2.11. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A), W(B) \subseteq S_\alpha$ and $t \in (0,1)$. Then*

$$\cos^{2t+1}(\alpha)\omega(A\sharp B) \le \omega(F_t(A,B)) \le \sec^3(\alpha)(1 - t\sin^2(\alpha)\omega(A\nabla B).$$

Proof. Compute

$$
\begin{aligned}
\omega(A\sharp B) &\le \|A\sharp B\| \quad \text{(by inequality (1.1))} \\
&\le \sec(\alpha)\|\Re(A\sharp B)\| \quad \text{(by Lemma 2.5)} \\
&\le \sec^{2t+1}(\alpha)\|\Re F_t(A,B)\| \quad \text{(by Theorem 1 in [11])} \\
&= \sec^{2t+1}(\alpha)\omega(\Re F_t(A,B)) \\
&\le \sec^{2t+1}(\alpha)\omega(F_t(A,B)) \quad \text{(by inequality (1.1))}
\end{aligned}
$$

and

$$
\begin{aligned}
\omega(F_t(A,B)) &\le \|F_t(A,B)\| \quad \text{(by inequality (1.1))} \\
&\le \sec(\alpha)\|\Re(F_t(A,B))\| \quad \text{(by Lemma 2.5)} \\
&\le \sec^3(\alpha)(1 - t\sin^2(\alpha))\|\Re(A\nabla B)\| \quad \text{(by Theorem 1 in [11])} \\
&= \sec^3(\alpha)(1 - t\sin^2(\alpha))\omega(\Re(A\nabla B)) \\
&\le \sec^3(\alpha)(1 - t\sin^2(\alpha))\omega(A\nabla B), \quad \text{(by inequality (1.1))}
\end{aligned}
$$

which completes the proof.

We remark that Theorem 2.11 is an improvement of inequality (1.15) when taking the bound on both sides into consideration.

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Conflict of interest

The author declares no conflict of interest in this paper.

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