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Research article

Existence and uniqueness criteria for nonlinear quantum difference equations with p-Laplacian

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Abstract: Q-calculus plays an extremely important role in mathematics and physics, especially in quantum physics, spectral analysis and dynamical systems. In recent years, many scholars are committed to the research of nonlinear quantum difference equations. However, there are few works about the nonlinear $q-$difference equations with p-Laplacian. In this paper, we investigate the solvability for nonlinear second-order quantum difference equation boundary value problem with one-dimensional p-Laplacian via the Leray-Schauder nonlinear alternative and some standard fixed point theorems. The obtained theorems are well illustrated with the aid of two examples.

Keywords: quantum calculus; nonlinear quantum difference equations; p-Laplacian operator; Leray-Schauder nonlinear alternative; fixed point theorem
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1. Introduction

Quantum calculus initiated at the beginning of the twentieth century, is the calculus without the limits, see [1–5]. As well as known, many pratical problems can be reduced to the q-difference equations boundary value problems (BVPS), therefore the q-difference equations become a very interesting field in difference equations and gain widespread attention and considerable importance. In the last few decades, it has evolved into a multidisciplinary subject and plays an important role in several fields of physics and mathematics such as cosmic strings and black holes, conformal quantum mechanics, nuclear and high energy physics, combinations, orthogoral polynomials, basic hypergeometric functions and so on, see [6–8] and the reference therein.

At present, the research on solvability of linear q-difference equations has made great progress, see [3–5,9] and the reference therein. Relative to the linear q-difference equations, nonlinear q-difference equations can describe some phenomenon more accurately in the natural world. In 2010, M. EL-Shanhed and H. A. Hassan firstly studied the existence of positive solutions for nonlinear q-difference equations by Kransnoselskii's fixed point theorem in cone [10]. In the last years, many scholars have denoted to doing some research on this kind of BVPS, such as local or nonlocal problems for nonlinear q-difference equations, impulsive q-difference equations, fractional q-difference equations and so on, and obtained some results, see [11–24] and the reference therein. However, many aspects of theory for nonlinear q-difference equations BVPS need to be explored.

In 1945, Leibenson [25] obtained the following nonlinear diffusion equation when studying the one-dimensional polytropic turbulence problem of gas passing through a porous medium,

$$u_t = \frac{\partial}{\partial x}\Big(\frac{\partial u^m}{\partial x}\Big|\frac{\partial u^m}{\partial x}\Big|^{\mu-1}\Big), \quad m = n+1,$$

when $m > 1$, the equation is the porous medium equation [26], when $0 < m < 1$, the equation becomes the rapid diffusion equation, and when $m = 1$, it is the heat equation. However, when $m = 1$ and $\mu \neq 1$, the above equations often applied to the study of non-Newtonian fluids [27]. In view of the importance of such equations, they are abstracted into the following p-Laplacian equation

$$(\phi_p(u'))' = f(t,u),$$

where

$$(\phi_p(s)) = |s|^{p-2}s(s \neq 0), \quad \phi_p(0) = 0, \quad p > 1.$$

Obviously, when $p = 2$, the p-Laplacian equation degenerates into a classical second-order differential equation. In [28], Anderson et al. studied the existence of at least one positive solution for the time scale delta-nabla dynamic equation $(g(u^{\Delta}))^{\nabla} + c(t)f(u) = 0$, with boundary conditions, $u(a) - B_0(u^{\Delta}(v)) = 0$ and $u^{\Delta}(b) = 0$. Here, $g(z) = |z|^{p-2}z$ for $p > 1$, $v \in (a,b)$. In [29], Dogan considers the p-Laplacian dynamic equation on time scales

$$\begin{gathered}(\phi_p(u^{\Delta}(t)))^{\nabla} + g(t)f(t,u(t),u^{\Delta}) = 0, \quad t \in [0,T]_{\mathbb{T}},\\ u(0) - B_0(u^{\Delta}(v)) = 0, \quad u^{\Delta}(T) = 0,\end{gathered}$$

or

$$u^{\Delta}(0) = 0, \quad u(T) - B_1(u^{\Delta}(v)) = 0,$$

and obtained the existence of at least three positive solutions for the boundary-value problem by using the Avery and Peterson fixed-point theorem. Here, $\mathbb{T}$ is s a time scale, namely any closed subset of $\mathbb{R}$. Significantly, let $0 < q < 1, \mathbb{T} = I_q, T = 1$, then

$$x^{\Delta}(t) = \frac{x(t/q) - x(t)}{(1-q)(t/q)} = D_q x(t/q), \quad x^{\nabla}(t) = \frac{x(t) - x(qt)}{(1-q)t} = D_q x(t).$$

To the best of our knowledge, limited work has been done in the BVPs nonlinear q-difference equations with p-Laplacian operator. In particular, there is little research on uniqueness of solutions for BVPS of nonlinear q-difference equations with p-Laplacian. To fill this gap, we investigate the existence and uniqueness of solutions for nonlinear second-order q-difference equations with p-Laplacian operator

$$\begin{cases} D_q(\Phi_p(D_q x(t))) + a(t)f(t, x(t), D_q x(t)) = 0, \ \ t \in I_q, \\ \alpha x(0) - \beta D_q x(0) = 0, \ D_q x(1) = 0, \end{cases} \tag{1.1}$$

where $\Phi_p(s) = |s|^{p-2}s (p > 1)$ is an increasing function, $\Phi_p^{-1}(s) = \Phi_{\overline{p}}(s), \ \frac{1}{p} + \frac{1}{\overline{p}} = 1, \ f(t, x, y) \in C(I_q \times R^2, R), \ I_q = \{q^n : n \in N\} \bigcup \{0, 1\}, \ \alpha \neq 0, \ a(t) \in C(I_q, R^+), \ R^+ = [0, \infty)$.

Throughout this paper, we adopt the following assumptions.

(H_1) $f(t, x, y) \in C(I_q \times R^2, R)$ and there exists $g(t) \in C(I_q, R^+)$ such that

$$|f(t, x, y)| \leq g(t),$$

for $t \in I_q, \ (x, y) \in R^2$.

(H_2) $f(t, x, y) \in C(I_q \times R^2, R)$ and there exist $L_1(t), L_2(t) \in C(I_q, R^+)$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1(t)|x_1 - x_2| + L_2(t)|y_1 - y_2|,$$

for $t \in I_q, \ (x_1, y_1), (x_2, y_2) \in R^2$.

(H_3) $f(t, x, y) \in C(I_q \times R^2, R)$ and there exist two positive constants $L_1, \ L_2$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2|,$$

for $t \in I_q, \ (x_1, y_1), (x_2, y_2) \in R^2$.

(H_4) $f(t, x, y) \in C(I_q \times R^2, R)$ and there exist $L_1(t), L_2(t) \in L^1(I_q, R^+)$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1(t)|x_1 - x_2| + L_2(t)|y_1 - y_2|,$$

for $t \in I_q, \ (x_1, y_1), (x_2, y_2) \in R^2$.

(H_5) $f(t, x, y) \in C(I_q \times R^2, R)$ and there exists $h(t) \in C(I_q, R^+)$ and $h \neq 0$ such that

$$|f(t, x, y)| \geq h(t),$$

for $t \in I_q, \ (x, y) \in R^2$.

(H_6) $f(t, x, y) \in C(I_q \times R^2, R)$ and there exist three functions $g(t), h(t), r(t) \in L^1(I_q, R^+)$ such that

$$|f(t, x, y)| \leq g(t)\Phi_p(|x|) + h(t)\Phi_p(|y|) + r(t),$$

for $t \in I_q, \ (x, y) \in R^2$.

(H_7) $f(t, x, y) \in C(I_q \times R^2, R)$ and there exist three positive real numbers a, b, c such that

$$|f(t, x, y)| \leq a\Phi_p(|x|) + b\Phi_p(|y|) + c,$$

for $t \in I_q, \ (x, y) \in R^2$.

For convenience, we denote

$$M = \int_0^1 a(s)g(s)d_q s, \qquad m = \int_0^1 a(s)h(s)d_q s,$$

$$A = \int_0^1 a(s)d_q s, \qquad B = \int_0^1 a(s)[L_1(s) + L_2(s)]d_q s,$$
$$L = \max_{t\in I_q}\{L_1(t) + L_2(t)\}, \qquad K = \Phi_p^{-1}\Big(\int_0^1 a(s)\varphi_r(s)d_q s\Big).$$

We aim to obtain the existence and uniqueness of solutions for the BVP (1.1) via Banach's Contraction Mapping Principle, Leray-Schauder Nonlinear Alternative theorem and Schaefer's fixed point theorem. The organization of this paper is as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main results. In Section 3, we give and prove our main results. Finally, in Section 4, we give some examples to illustrate how the main results can be used in practice.

2. Preliminary results

In this section, we provide some definitions of q-calculus [9,30] and preliminary lemmas which are key tools for our main results.

Definition 2.1. For $0 < q < 1$, we define the q-derivative of a real-value function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t\to 0} D_q f(t).$$

Note that $\lim_{q\to 1^-} D_q f(t) = f'(t)$.

Definition 2.2. The higher order q-derivatives are defined inductively as

$$D_q^0 f(t) = f(t), \quad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in N.$$

Definition 2.3. The q-integral of a function f defined in the interval $[a,b]$ is given by

$$\int_a^x f(t)d_q t := \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n) - af(q^n a), \quad x \in [a,b],$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t)d_q t = \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n).$$

Then

$$\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t.$$

Notice that

$$\int_a^b D_q f(t)d_q t = f(b) - f(a).$$

Remark 2.4. In the limit $q \to 1^-$, the above results correspond to their counterparts in standard calculus.

Definition 2.5. ([15]) $f : I_q \times R^2 \to R$ is called an S-Carathéodory function if and only if

(i) for each $(u,v)\in R^2, t\mapsto f(t,u,v)$ is measurable on I_q;

(ii) for a.e. $t\in I_q, (u,v)\mapsto f(t,u,v)$ is continuous on R^2;

(iii) for each $r>0$, there exists $\varphi_r(t)\in L^1(I_q,R^+)$ with $t\varphi_r(t)\in L^1(I_q,R^+)$ on I such that $\max\{|u|,|v|\}\leq r$ implies $|f(t,u,v)|\leq\varphi_r(t)$, for a.e. I, where $L^1(I_q,R^+)=\{u\in C_q:\int_0^1 u(t)d_qt$ exists$\}$, and normed by $\|u\|_{L^1}=\int_0^1|u(t)|d_qt$ for all $u\in L^1(I_q,R^+)$.

Theorem 2.6. (Nonlinear alternative for single value maps [31]) Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is $F(\overline{U})$ is a relatively compact subset of C) map. Then either

(i) F has a fixed point in $\overline{U}$, or

(ii)there is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.

Theorem 2.7. (Schaefer's fixed point theorem [32]) Let T be a continuous and compact mapping of a Banach space X into itself, such that the set $E=\{x\in X: x=\lambda Tx$ for some $0\leq\lambda\leq1\}$ is bounded. Then T has a fixed point.

Finally, we list below the following basic properties of the p-Laplacian operator which will be used in the sequel.

Lemma 2.8. ([33]) Assume $\Phi_p(s)$ is a p-Laplacian operator, then

$$\Phi_p(s+t)\leq\begin{cases}2^{p-1}(\Phi_p(s)+\Phi_p(t)), & p\geq2,\ \ s,t>0,\\ \Phi_p(s)+\Phi_p(t), & 1<p<2,\ \ s,t>0,\end{cases}$$

and

$$\Phi_p^{-1}(s+t)\leq\begin{cases}2^{\frac{1}{p-1}}(\Phi_p^{-1}(s)+\Phi_p^{-1}(t)), & p\geq2,\ \ s,t>0,\\ \Phi_p^{-1}(s)+\Phi_p^{-1}(t), & 1<p<2,\ \ s,t>0.\end{cases}$$

Lemma 2.9. ([34]) (1) If $1<p<2,\ uv>0$ and $|u|,\ |v|\geq m>0$, then

$$|\Phi_p(v)-\Phi_p(u)|\leq(p-1)m^{p-2}|v-u|.$$

(2) If $p>2$ and $|u|,\ |v|\leq M$, then

$$|\Phi_p(v)-\Phi_p(u)|\leq(p-1)M^{p-2}|v-u|.$$

3. Existence and uniqueness results

In this section, we establish some existence and uniqueness criteria for nonlinear $q-$difference equation (1.1).

Lemma 3.1. Let $y\in C[0,1]$, then the BVP

$$\begin{cases}D_q(\Phi_p(D_qx(t)))+y(t)=0,\ \ t\in I_q,\\ \alpha x(0)-\beta D_qx(0)=0,\ D_qx(1)=0\end{cases}\tag{3.1}$$

has a unique solution

$$x(t)=\frac{\beta}{\alpha}\Phi_p^{-1}\Big(\int_0^1 y(s)d_qs\Big)+\int_0^t\Phi_p^{-1}\Big(\int_s^1 y(\tau)d_q\tau\Big)d_qs.$$

Proof. Integrating the q-difference equation from t to 1, we get

$$\Phi_p(D_qx(1)) - \Phi_p(D_qx(t)) = -\int_t^1 y(s)d_qs.$$

By the BC $D_qx(1) = 0$, we have

$$D_qx(t) = \Phi_p^{-1}\Big(\int_t^1 y(s)d_qs\Big). \tag{3.2}$$

Integrate (3.2) from 0 to t, we have

$$x(t) = x(0) + \int_0^t \Phi_p^{-1}\Big(\int_s^1 y(\tau)d_q\tau\Big)d_qs. \tag{3.3}$$

Using the BC $\alpha x(0) - \beta D_qx(0) = 0$ in (3.3), we find that

$$x(0) = \frac{\beta}{\alpha}D_qx(0) = \frac{\beta}{\alpha}\Phi_p^{-1}\Big(\int_0^1 y(s)d_qs\Big).$$

Hence, the BVP (3.1) has a unique solution

$$x(t) = \frac{\beta}{\alpha}\Phi_p^{-1}\Big(\int_0^1 y(s)d_qs\Big) + \int_0^t \Phi_p^{-1}\Big(\int_s^1 y(\tau)d_q\tau\Big)d_qs.$$

This completes the proof.

We consider the Banach space $C_q = C(I_q, R)$ equipped with standard norm $\|x\| = \max\{\|x\|_\infty, \|D_qx\|_\infty\}$, and $\|\cdot\|_\infty = \sup\{\|\cdot\|, t \in I_q\},\ x \in C_q$.

Define an integral operator $T : C_q \to C_q$ by

$$\begin{aligned} Tx(t) = &\frac{\beta}{\alpha}\Phi_p^{-1}\Big(\int_0^1 a(s)f(s, x(s), D_qx(s))d_qs\Big) \\ &+ \int_0^t \Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau, x(\tau), D_qx(\tau))d_q\tau\Big)d_qs.\ \ t \in I_q,\ \ x \in C_q. \end{aligned} \tag{3.4}$$

Obviously, T is well defined and $x \in C_q$ is a solution of BVP (1.1) if and only if x is a fixed point of T.

First, we give the uniqueness result based on Banach's Contraction Mapping Principle.

Theorem 3.2. Suppose $1 < p \leq 2$, and $(H_1), (H_2)$ hold. Then the BVP (1.1) has a unique solution provided that $\Gamma < 1$, where $\Gamma = (\left|\frac{\beta}{\alpha}\right| + 1)\frac{1}{p-1}M^{\frac{2-p}{p-1}}LA$.

Proof. Let us set $\sup_{t\in I_q}|f(t, 0, 0)| = M_0$ and choose

$$r \geq \frac{\Gamma^*}{1-\delta},$$

where $\Gamma \leq \delta < 1$ and $\Gamma^* = (\left|\frac{\beta}{\alpha}\right| + 1)\Phi_p^{-1}(M_0 A)$. Now, we show that $TB_r \subset B_r$, where $B_r = \{x \in C_q : \|x\| \leq r\}$. For each $x \in B_r$, we have

$$
\begin{aligned}
|Tx(t)| &\leq \sup_{t\in I_q}\left|\frac{\beta}{\alpha}\Phi_p^{-1}\Big(\int_0^1 a(s)f(s,x(s),D_qx(s))d_qs\Big) + \int_0^t \Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau,x(\tau),D_qx(\tau))d_q\tau\Big)d_qs\right| \\
&\leq \sup_{t\in I_q}\Big(\left|\frac{\beta}{\alpha}\right|\Big|\Phi_p^{-1}\Big(\int_0^1 a(s)f(s,x(s),D_qx(s))d_qs\Big) - \Phi_p^{-1}\Big(\int_0^1 a(s)f(s,0,0)d_qs\Big) \\
&\qquad + \Phi_p^{-1}\Big(\int_0^1 a(s)f(s,0,0)d_qs\Big)\Big| \\
&\quad + \int_0^t \Big|\Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau,x(\tau),D_qx(\tau))d_q\tau\Big) - \Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau,0,0)d_q\tau\Big) \\
&\qquad + \Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau,0,0)d_q\tau\Big)\Big|d_qs\Big) \\
&\leq \left|\frac{\beta}{\alpha}\right|\left|\Phi_p^{-1}\Big(\int_0^1 a(s)f(s,x(s),D_qx(s))d_qs\Big) - \Phi_p^{-1}\Big(\int_0^1 a(s)f(s,0,0)d_qs\Big)\right| \\
&\quad + \left|\Phi_p^{-1}\Big(\int_0^1 a(\tau)f(\tau,x(\tau),D_qx(\tau))d_q\tau\Big) - \Phi_p^{-1}\Big(\int_0^1 a(\tau)f(\tau,0,0)d_q\tau\Big)\right| \\
&\quad + \left|\frac{\beta}{\alpha}\right|\Phi_p^{-1}\Big(\int_0^1 a(s)|f(s,0,0)|d_qs\Big) + \Phi_p^{-1}\Big(\int_0^1 a(\tau)|f(\tau,0,0)|d_q\tau\Big) \\
&\leq \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{p-2}{p-1}}\int_0^1 a(s)|f(s,x(s),D_qx(s)) - f(s,0,0)|d_qs) + \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\Phi_p^{-1}\Big(M_0\int_0^1 a(s)d_qs\Big) \\
&\leq \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{p-2}{p-1}}\int_0^1 a(s)(L_1(s)|x(s)| + L_2(s)|D_qx(s)|)d_qs + \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\Phi_p^{-1}(M_0A) \\
&\leq \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{p-2}{p-1}}LA\|x\| + \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\Phi_p^{-1}(M_0A) \\
&\leq \Gamma r + \Gamma^* \leq (\Lambda + 1 - \delta)r \leq r,
\end{aligned}
$$

and

$$
\begin{aligned}
|D_qTx(t)| &\leq \sup_{t\in I_q}\left|\Phi_p^{-1}\Big(\int_t^1 a(s)f(s,x(s),D_qx(s))d_qs\Big)\right| \\
&\leq \left|\Phi_p^{-1}\Big(\int_0^1 a(s)f(s,x(s),D_qx(s))d_qs\Big) - \Phi_p^{-1}\Big(\int_0^1 a(s)f(s,0,0)d_qs\Big)\right| \\
&\quad + \left|\Phi_p^{-1}\Big(\int_0^1 a(s)f(s,0,0)d_qs\Big)\right| \\
&\leq \frac{1}{p-1}M^{\frac{2-p}{p-1}}LA\|x\| + \Phi_p^{-1}(M_0A) \\
&< \Gamma\|x\| + \Gamma^* \leq r.
\end{aligned}
$$

Hence, we obtain that $\|Tx\| \leq r$, so $TB_r \subset B_r$.

Now, for $x, y \in C_q$ and for each $t \in I_q$, we have

$$
\begin{aligned}
&|Tx(t) - Ty(t)| \\
\le& \sup_{t\in I_q} \left|\frac{\beta}{\alpha}\Big(\Phi_p^{-1}\Big(\int_0^1 a(s)f(s, x(s), D_q x(s))d_q s\Big) - \Phi_p^{-1}\Big(\int_0^1 a(s)f(s, y(s), D_q y(s))d_q s\Big)\Big)\right. \\
&\left. + \Big(\int_0^t \Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau, x(\tau), D_q x(\tau))d_q\tau\Big)d_q s - \int_0^t \Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau, y(\tau), D_q y(\tau))d_q\tau\Big)d_q s\Big)\right| \\
\le& \left|\frac{\beta}{\alpha}\right|\left|\Phi_p^{-1}\Big(\int_0^1 a(s)f(s, x(s), D_q x(s))d_q s\Big) - \Phi_p^{-1}\Big(\int_0^1 a(s)f(s, y(s), D_q y(s))d_q s\Big)\right| \\
&+ \int_0^1 \left|\Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau, x(\tau), D_q x(\tau))d_q\tau\Big) - \Phi_p^{-1}\Big(\int_s^1 a(\tau)f(\tau, y(\tau), D_q y(\tau))d_q\tau\Big)\right| d_q s \\
\le& \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{2-p}{p-1}}\int_0^1 a(s)|f(s, x(s), D_q x(s)) - f(s, y(s), D_q y(s))|d_q s \\
\le& \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{2-p}{p-1}}\int_0^1 a(s)[L_1(s)|x(s) - y(s)| + L_2(s)|D_q x(s) - D_q y(s)|]d_q s \\
\le& \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{2-p}{p-1}}LA\|x - y\| \le \Gamma\|x - y\| < \|x - y\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_q Tx(t) - D_q Ty(t)| \\
\le& \sup_{t\in I_q}\left|\Phi_p^{-1}\Big(\int_t^1 a(s)f(s, x(s), D_q x(s))d_q s\Big) - \Phi_p^{-1}\Big(\int_t^1 a(s)f(s, y(s), D_q y(s))d_q s\Big)\right| \\
\le& \left|\Phi_p^{-1}\Big(\int_0^1 a(s)f(s, x(s), D_q x(s))d_q s\Big) - \Phi_p^{-1}\Big(\int_0^1 a(s)f(s, y(s), D_q y(s))d_q s\Big)\right| \\
\le& \frac{1}{p-1}M^{\frac{2-p}{p-1}}\int_0^1 a(s)|f(s, x(s), D_q x(s)) - f(s, y(s), D_q y(s))|d_q s \\
\le& \frac{1}{p-1}M^{\frac{2-p}{p-1}}LA\|x - y\| \le \Gamma\|x - y\| < \|x - y\|.
\end{aligned}
$$

Therefore, we obtain that $\|Tx-Ty\| < \|x-y\|$, so T is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof of Theorem 3.2.

Corollary 3.3. Suppose $1 < p \le 2$, and $(H_1), (H_3)$ hold. Then the BVP(1.1) has a unique solution provided that $\Gamma_1 < 1$, where $\Gamma_1 = \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{2-p}{p-1}}(L_1 + L_2)A$.

Corollary 3.4. Suppose $1 < p \le 2$, and $(H_1), (H_4)$ hold. Then the BVP(1.1) has a unique solution provided that $\Gamma_2 < 1$, where $\Gamma_2 = \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}M^{\frac{2-p}{p-1}}B$.

Theorem 3.5. Suppose $p > 2$, and $(H_2), (H_5)$ hold. Then the BVP(1.1) has a unique solution provided that $\Gamma_3 < 1$, where $\Gamma_3 = \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}m^{\frac{2-p}{p-1}}LA$.

Proof. It is similar to the proof of Theorem 3.1.

Corollary 3.6. Suppose $p > 2$, and $(H_3), (H_5)$ hold. Then the BVP(1.1) has a unique solution provided that $\Gamma_4 < 1$, where $\Gamma_4 = \Big(\left|\frac{\beta}{\alpha}\right| + 1\Big)\frac{1}{p-1}m^{\frac{2-p}{p-1}}(L_1 + L_2)A$.

Corollary 3.7. Suppose $p > 2$, and $(H_4), (H_5)$ hold. Then the BVP(1.1) has a unique solution provided that $\Gamma_5 < 1$, where $\Gamma_5 = \left(\left|\frac{\beta}{\alpha}\right| + 1\right)\frac{1}{p-1}m^{\frac{2-p}{p-1}}B$.

The next existence result is based on Leray-Schauder nonlinear alternative theorem.

Lemma 3.8. Let $f : I \times R^2 \to R$ be an S-Carathéodory function. Then $T : C_q \to C_q$ is completely continuous.

Proof. The proof consists of several steps.

(i) T maps bounded sets into bounded sets in C_q. Let $B_r = \{x \in C_q : \|x\| \leq r\}$ be a bounded set in C_q and $x \in B_r$. Then we have

$$\begin{aligned}
|Tx(t)| &\leq \left|\frac{\beta}{\alpha}\right|\Phi_p^{-1}\Big(\int_0^1 a(s)|f(s, x(s), D_qx(s))|d_qs\Big) + \int_0^t \Phi_p^{-1}\Big(\int_s^1 a(\tau)|f(\tau, x(\tau), D_qx(\tau))|d_q\tau\Big)d_qs \\
&\leq \left|\frac{\beta}{\alpha}\right|\Phi_p^{-1}\Big(\int_0^1 a(s)\varphi_r(s)d_qs\Big) + \int_0^1 \Phi_p^{-1}\Big(\int_s^1 (a(\tau)\varphi_r(\tau))d_q\tau\Big)d_qs \\
&\leq \left(\left|\frac{\beta}{\alpha}\right| + 1\right)\Phi_p^{-1}\Big(\int_0^1 a(s)\varphi_r(s)d_qs\Big),
\end{aligned}$$

and

$$\begin{aligned}
|D_qTx(t)| &\leq \Phi_p^{-1}\Big(\int_0^1 a(s)|f(s, x(s), D_qx(s))|d_qs\Big) \\
&\leq \Phi_p^{-1}\Big(\int_0^1 a(s)\varphi_r(s)d_qs\Big).
\end{aligned}$$

Thus $\|Tx\| \leq \max\{\|Tx\|_\infty, \|D_qTx\|_\infty\} \leq \left(\left|\frac{\beta}{\alpha}\right| + 1\right)K$.

(ii) T maps bounded sets into equicontinuous sets of C_q.

Let $t_1, t_2 \in I_q, t_1 < t_2$, and B_r be a bounded set of C_q as before. Then for $x \in B_r$, we have

$$\begin{aligned}
|Tx(t_1) - Tx(t_2)| &= \left|\int_{t_1}^{t_2} \Phi_p^{-1}\Big(\int_s^1 a(\tau)|f(\tau, x(\tau), D_qx(\tau))d_q\tau\Big)d_qs\right| \\
&\leq \int_{t_1}^{t_2} \Phi_p^{-1}\Big(\int_s^1 a(\tau)\varphi_r(\tau)d_q\tau\Big)d_qs \\
&\leq \int_{t_1}^{t_2} \Phi_p^{-1}\Big(\int_0^1 a(\tau)\varphi_r(\tau)d_q\tau\Big)d_qs \\
&\leq |t_2 - t_1|\Phi_p^{-1}\Big(\int_0^1 a(\tau)\varphi_r(\tau)d_q\tau\Big) \\
&= |t_2 - t_1|K \to 0, \qquad (t_2 - t_1 \to 0),
\end{aligned}$$

and by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&|D_qTx(t_2) - D_qTx(t_1)| \\
= &\left|\Phi_p^{-1}\Big(\int_{t_2}^1 a(s)f(s, x(s), D_qx(s))d_qs\Big) - \Phi_p^{-1}\Big(\int_{t_1}^1 a(s)f(s, x(s), D_qx(s))d_qs\Big)\right| \to 0, \ (t_2 - t_1 \to 0).
\end{aligned}$$

As a consequence of the Arzel$\acute{a}$-Ascoli theorem, we can conclude that $T : C_q \to C_q$ is completely continuous. This proof is be completed.

Theorem 3.9. Let $f : I \times R^2 \to R$ be an S-Carathéodory function. Suppose further that there exists a real number $N > 0$ such that

$$\frac{|\alpha|N}{(|\alpha| + |\beta|)K} > 1$$

holds, where

$$K = \Phi_p^{-1}\Big(\int_0^1 a(s)\varphi_r(s)d_q s\Big) \neq 0.$$

Then the BVP (1.1) has at least one solution.
Proof. In view of Lemma 3.7, we obtain that $T : C_q \to C_q$ is completely continuous. Let $\lambda \in (0, 1)$, and $x = \lambda Tx$. Then, for $t \in I_q$, we have

$$\begin{aligned}
|x(t)| &= |\lambda Tx(t)| \\
&\leq \Big|\frac{\beta}{\alpha}\Big|\Phi_p^{-1}\Big(\int_0^1 a(s)|f(s, x(s), D_q x(s))|d_q s\Big) + \int_0^t \Phi_p^{-1}\Big(\int_s^1 a(\tau)|f(\tau, x(\tau), D_q x(\tau))|d_q\tau\Big)d_q s \\
&\leq \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)\Phi_p^{-1}\Big(\int_0^1 a(s)\varphi_r(s)d_q s\Big) = \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)K,
\end{aligned}$$

and

$$\begin{aligned}
|D_q Tx(t)| = |\lambda D_q Tx(t)| &\leq \int_0^t \Phi_p^{-1}\Big(\int_s^1 a(\tau)|f(\tau, x(\tau), D_q x(\tau))|d_q\tau\Big)d_q s \\
&\leq \int_0^1 \Phi_p^{-1}\Big(\int_s^1 a(\tau)\varphi_r(\tau)d_q\tau\Big)d_q s \\
&\leq \Phi_p^{-1}\Big(\int_0^1 a(s)\varphi_r(s)d_q s\Big) = K.
\end{aligned}$$

Hence, consequently

$$\frac{\|x\|}{\Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)K} \leq 1.$$

Therefore, there exists $N > 0$ such that $\|x\| \neq N$. Let us set $U = \{x \in C_q : \|x\| < N\}$. Note that the operator $T : \overline{U} \to C_q$ is completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Tx$ for some $\lambda \in (0, 1)$. Consequently, by Theorem 2.6, we deduce that T has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof.

Finally, we give the existence result is based on Schaefer's fixed point theorem.
Theorem 3.10. Assume that (H_6) holds. Suppose further $\Lambda < 1$ holds, then BVP(1.1) has at least one solution, where $\Lambda = \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)2^{\frac{1}{p-1}}\Phi_p^{-1}(\int_0^1 a(s)[g(s) + h(s)]d_q s)$.
Proof. It is easy to know that T is completely continuous. Next, we show that the set

$E = \{x \in C_q : x = \lambda Tx$ for $0 \leq \lambda \leq 1\}$ is bounded, which is independent of λ. Let $x \in E$, then $x(t) = \lambda Tx(t)$ for $0 \leq \lambda \leq 1$. By (H_6), for each $t \in I_q$, we have

$$\begin{aligned}
|x(t)| &= |\lambda Tx(t)| \leq |Tx(t)| \\
&\leq \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)\Phi_p^{-1}\Big(\int_0^1 a(s)|f(s, x(s), D_q x(s))| d_q s\Big) \\
&\leq \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)2^{\frac{1}{p-1}}\Phi_p^{-1}\Big(\int_0^1 a(s)[g(s) + h(s)] d_q s\Big) \cdot \|x\| + \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)2^{\frac{1}{p-1}}\Phi_p^{-1}\Big(\int_0^1 a(s)r(s) d_q s\Big) \\
&\leq \Lambda\|x\| + \Lambda^*,
\end{aligned}$$

and

$$\begin{aligned}
|D_q x(t)| &= |\lambda D_q Tx(t)| \\
&\leq |D_q Tx(t)| \\
&\leq \Phi_p^{-1}\Big(\int_0^1 a(s)|f(s, x(s), D_q x(s))| d_q s\Big) \\
&\leq 2^{q-1}\Phi_p^{-1}\Big(\int_0^1 a(s)[g(s) + h(s)] d_q s\Big) \cdot \|x\| + 2^{q-1}\Phi_p^{-1}\Big(\int_0^1 a(s)r(s) d_q s\Big) \\
&\leq \Lambda\|x\| + \Lambda^*,
\end{aligned}$$

where $\Lambda^* = \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)2^{p-1}\Phi_p^{-1}\Big(\int_0^1 a(s)r(s) d_q s\Big)$.
Therefore,

$$\|x\| \leq \Lambda\|x\| + \Lambda^*.$$

Hence,

$$\|x\| \leq \frac{\Lambda^*}{1 - \Lambda} := M.$$

This shows that the set E is bounded. By Theorem 2.7, we obtain that BVP(1.1) has at least one solution. This proof is completed.

Corollary 3.11. Assume that (H_7) holds. In addition, if $\Lambda_1 < 1$ holds, where $\Lambda_1 = \Big(\Big|\frac{\beta}{\alpha}\Big| + 1\Big)2^{\frac{1}{p-1}}\Phi_p^{-1}((a + b)A)$. Then the BVP(1.1) has at least one solution.

4. Applications

In this section, we give two examples to illustrated our results.

Example 4.1. Consider the following q-difference equation BVP:

$$\begin{cases} D_q(\Phi_p(D_q x(t))) + \frac{1}{\sqrt{t}}[\frac{1}{12}(1 + \sqrt{t})\sin(x(t)) + \frac{1}{6}\arctan(D_q x(t))] = 0, & t \in I_q, \\ x(0) - \frac{1}{3}D_q x(0) = 0, \quad D_q x(1) = 0. \end{cases} \tag{4.1}$$

Here, $f(t, x(t), D_q x(t)) = \frac{1}{12}(1 + \sqrt{t})\sin(x(t)) + \frac{1}{6}\arctan(D_q x(t))$, $p = \frac{3}{2}$, $q = \frac{1}{4}$, $\alpha = 1$, $\beta = \frac{1}{3}$, $a(t) = \frac{1}{\sqrt{t}}$, $g(t) = \frac{1}{12} + \frac{\pi}{12} + \frac{1}{12}\sqrt{t}$.

Obviously, $\forall t \in I_q,\ (x_1, y_1),\ (x_2, y_2) \in R^2$, we have

$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \frac{1}{6}|x_1 - x_2| + \frac{1}{6}|y_1 - y_2|$, and $|f(t, x, y)| \leq g(t)$.

Then, $L_1 = L_2 = \frac{1}{6}$, $A = \int_0^1 a(s)d_qs = \frac{3}{2}$, $M = \int_0^1 a(s)g(s)d_qs = \frac{1+\pi}{12} \cdot \frac{3}{2} + \frac{1}{12} = \frac{5}{24} + \frac{3\pi}{24} = \frac{5+3\pi}{24}$, and

$\Gamma_1 = (\frac{\beta}{\alpha} + 1)\frac{1}{p-1}M^{\frac{2-p}{p-1}}(L_1 + L_2)A = \frac{4}{3} \cdot 2 \cdot \frac{5+3\pi}{24} \cdot \frac{1}{3} \cdot \frac{3}{2} = \frac{5+3\pi}{18} < 1.$

By Corollary 3.3, we obtain that BVP (4.1) has a unique solution.

Example 4.2. Consider the following q-difference equation BVP:

$$\begin{cases} D_q(\Phi_p(D_qx(t))) + \frac{e^{-qt}-e^{-t}}{t}[\frac{1}{8}\sin(|x(t)|) + \frac{1}{8}\ln(1 + |D_qx(t)|)] = 0, & t \in I_q, \\ x(0) - D_qx(0) = 0, \quad D_qx(1) = 0. \end{cases} \tag{4.2}$$

Here, $a(t) = \frac{e^{-qt}-e^{-t}}{t}$, $f(t, x(t), D_qx(t)) = \frac{1}{8}\sin(|x(t)|) + \frac{1}{8}\ln(1 + |D_qx(t)|) + 1,\ p = 3,\ q = \frac{1}{2},\ \alpha = \beta = 1.$
Obviously, $\forall t \in I_q,\ (x, y) \in {R_+}^2$, we have

$|f(t, x, y)| \leq \frac{1}{8}|x(t)| + \frac{1}{8}|D_qx(t)| + 1.$

Then $a = b = \frac{1}{8}$, $A = \int_0^1 a(s)d_qs = \frac{1}{2}(1 - e^{-1})$, and $\Lambda_1 = \sqrt{1 - e^{-1}} < 1$. By Corollary 3.11, we obtain that BVP (4.2) has at least one solution.

5. Concluding remarks and observations

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are known to have widespread applications, in particular, in several areas of combinatorial analysis and number theory such as (for example) the theory of partitions. In fact, q-calculus is widely used in many fields, such as finite vector spaces, Lie theory, particle physics, non-linear electric circuit theory, mechanical engineering, cosmology and so on. We now choose to indicate some obvious connection between the classical q-analysis and the so-called (p, q)-analysis. Srivastava et al. have observed the fact that the results for the q-analogues, which we have considered in this article for $0 < q < 1$, can easily (and possibly trivially) be translated into the corresponding results for the (p, q)-analogues (with $0 < q < p \leq 1$) by applying some obvious parametric and argument variations, the additional parameter p being redundant, see Page 340 of the work [35] and Pages 1511–1512 of the work [36].

The so-called (p, q)-number $[n]_{p,q}$ is given (for $0 < q < p \leq 1$) by

$$[n]_{p,q} := \begin{cases} \frac{p^n-q^n}{p-q}, & (n \in \{1, 2, 3, \cdots\}) \\ 0, & (n = 0) \end{cases}$$

$$=: p^{n-1}[n]_{\frac{q}{p}}, \tag{5.1}$$

where, for the classical q-number $[n]_q$, we have

$$[n]_q := \frac{1 - q^n}{1 - q} = p^{1-n}(\frac{p^n - (pq)^n}{p - (pq)}) = p^{1-n}[n]_{p,pq}. \tag{5.2}$$

Furthermore, the so-called (p, q)-derivative or the so-called (p, q)-difference of a suitably constrained function $f(z)$ is denoted by $(D_{p,q}f)(z)$ and defined, in a given subset of $\mathbb{C}$,by

$$(D_{p,q}f)(z) = \begin{cases} \frac{f(pz)-f(qz)}{(p-q)z}, & (z \in \mathbb{C}\{0\}; 0 < q < p \leq 1) \\ f'(0), & (z = 0; 0 < q < p \leq 1) \end{cases} \tag{5.3}$$

Thus, clearly, we have the following connection with the familiar q-derivative or the q-difference $(D_{p,q}f)(z)$ given by the definition 5.3:

$$(D_{p,q}f)(z) = (D_{\frac{p}{q}}f)(pz) \;\; and \;\; (D_q f)(z) = (D_{p,pq}f)\left(\frac{z}{p}\right) \;\; (z \in \mathbb{C}; 0 < q < p \leq 1). \tag{5.4}$$

From the Eqs (5.1)–(5.4), we can find that the subject for $0 < q < 1$ can easily (and possibly trivially) be translated into the corresponding (p, q)-analogues (with $0 < q < p \leq 1$) by applying some straightforward parametric and argument variations of the types indicated above, the additional parameter p being redundant. Therefore, it is worthwhile to draw the attention of the interested readers toward.

In the last few decades, quantum difference equation has evolved into a multidisciplinary subject and plays an important role in several fields of physics and mathematics. In this paper, we obtain existence and uniqueness of solutions for nonlinear second-order quantum difference equation BVPs with p-Laplacian by means of the Leray-Schauder nonlinear alternative and some standard fixed point theorems. The results are novel, enrich the theories for q-difference equations with p-Laplacian, and provide the theoretical guarantee for the application of fractional q-difference equations in every field. In the future, we will use bifurcation theory, critical point theory, variational method, and other methods to continue our works in this area.

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Conflict of interest

The authors declare that they have no competing interests.

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