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Research article

Uniformly analytic solutions to a class of singular partial differential equations

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Abstract: We study the singular nonlinear partial differential equation $t\partial_t u = F(t, x, u, \partial_x u)$, where $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Under some growth conditions on the coefficients of the partial Taylor expansion of F, we construct the unique solution that is continuous in t and C^∞ in x.

Keywords: uniformly analytic functions; singular partial differential equation
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1. Introduction

Let $(t, x) \in \mathbb{R}\times\mathbb{R}^n$. Define D_R and B_R to be the open disk and open polydisk of radius R, respectively. Let T and R_1 be positive real numbers and Ω be an open subset of $\mathbb{R}^n$. Consider the equation

$$t\partial_t u = F\left(t, x, u, \partial_x u\right), \tag{1.1}$$

where $F(t, x, u, v)$ is a continuous function on $[0, T) \times \Omega \times D_{R_1} \times B_{R_1}$ and holomorphic in $D_{R_1} \times B_{R_1}$ for each fixed $(t, x) \in [0, T) \times \Omega$.

Equation (1.1) is a generalization to partial differential equations (PDEs) of the singular nonlinear ordinary differential equation (ODE) studied by Briot and Bouquet [4] in 1856. Gérard and Tahara [6] thoroughly studied this singular equation under the assumption that the right-hand side is holomorphic with respect to all its variables, and formulated existence and uniqueness theorems for both first order and higher order equations [6–8].

In 1973, Baouendi and Goulaouic [2] introduced singular linear PDEs that they consider to be natural generalizations to PDEs of Fuchsian ODEs. Their unique solvability result was extended by Lope [11, 12] by introducing the concept of weight functions. Although Baouendi and Goulaouic also considered nonlinear equations in [3], their formulation can be improved by using weight functions. This was done by Roque, Lope, and Tahara [13] on first order singular equations. In particular, they showed that under some growth conditions on the coefficients on the partial Taylor expansion of F, the unique solution of (1.1) also satisfies a growth order condition. This was extended by Bacani, Lope and Tahara [1] to the higher order case. A technique used in [1, 13] is by the method of Nishida and Nirenberg [15, 16] together with estimates of Nagumo type [14].

In 2016, Tolentino, Bacani and Tahara [19] considered the Cauchy problem for nonsingular, nonlinear PDEs and proved the unique existence of a C^∞-solution that is uniformly analytic. They used the method of majorants, formal norms, and the Fixed Point Theorem. Lope and Ona [17] also used formal norms in proving the unique solvability of a first order singular PDE of totally characteristic type.

In this paper, we consider the singular equation (1.1) using almost the same framework and machinery as in [19], thus obtaining a C^∞-solution of the equation considered in [13].

The remaining parts of the paper are outlined as follows. In the next section we will state our assumptions and the main result. In Section 3, we will discuss the preliminary concepts which will be essential throughout the paper together with some important estimates. In Section 4, we will prove a unique solvability result for a semilinear equation. The final sections of the paper include the proof of our main result and the references cited.

2. Statement of the main theorem

Denote $\partial_x = (\partial_{x_i})_{i=1}^n$ and for $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_n}^{\alpha_n}$.

Let $\mu(t) : [0,T] \to \mathbb{R}$ be a *weight function*; that is, a function which is continuous, nonnegative and increasing on $[0,T]$, with $\int_0^T \mu(s)/s\,dt < \infty$. Some common examples of weight functions are t^a and $(-\log t)^{-1-a}$ for some $a > 0$. Moreover, define $\theta : [0,T] \to \mathbb{R}$ as

$$\theta(t) := \int_0^t \frac{\mu(s)}{s} ds.$$

It follows that $\lim_{t\to 0}\mu(t) = 0$ and $\theta'(t) = \mu(t)/t$. Functions that are similarly defined are mentioned in [9, 18], where they have been referred to as *Dini functions*.

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^n$.

(1) A C^∞ function $f(x)$ is said to be *uniformly analytic* on Ω if there exist $C > 0$ and $h > 0$ such that

$$\sup_{x\in\Omega} |\partial_x^\alpha f(x)| \le Ch^{|\alpha|}|\alpha|! \tag{2.1}$$

for any $\alpha \in \mathbb{N}^n$. We denote the totality of uniformly analytic functions on Ω by $\mathcal{A}(\Omega)$.

(2) A function $u(t,x)$ said to belong to $C^0([0,T), \mathcal{A}(\Omega))$ if for all $\alpha \in \mathbb{N}^n$,

(i) $\partial_x^\alpha u(t,x) \in C^0([0,T) \times \Omega)$,

(ii) for any $0 < T_1 < T$, there exist $C_1 > 0$ and $h_1 > 0$ such that

$$\sup_{[0,T_1)\times\Omega} |\partial_x^\alpha u(t,x)| \le C_1 h_1^{|\alpha|}|\alpha|!. \tag{2.2}$$

Moreover, if ψ is a continuous increasing function on $[0,T]$ satisfying $\psi(0) = 0$, we say that $u(t,x) \in C^0([0,T), A(\Omega))$ is *of growth order* $O(\psi(t))$ if there exist $C, h > 0$ such that for all $\alpha \in \mathbb{N}^n$ and for all $t \in [0,T)$,

$$\sup_{x\in\Omega} |\partial_x^\alpha u(t,x)| \le C\psi(t)h^{|\alpha|}|\alpha|!. \tag{2.3}$$

(3) A function $g(t,x,z)$ is said to belong to $C^0([0,T),\mathcal{A}(\Omega),\mathcal{O}(B_R))$ if for all $(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^m$,

(i) $\partial_x^\alpha\partial_x^\beta g(t,x,0)\in C^0([0,T)\times\Omega)$,

(ii) $g(t,x,z)$ is holomorphic in B_R for each fixed $(t,x)\in[0,T)\times\Omega$,

(ii) for any $0<T_1<T$, there exist C_1 and $h_1>0$ such that

$$\sup_{[0,T_1)\times\Omega}|\partial_x^\alpha\partial_z^\beta g(t,x,0)|\le\frac{C_1}{R^{|\beta|}}h_1^{|\alpha|}|\alpha|!|\beta|!. \tag{2.4}$$

As $F(t,x,u,v)$ is holomorphic in $D_{R_1}\times B_{R_1}$, we can expand it as

$$F(t,x,u,v)=a(t,x)+\lambda(t,x)u+\sum_{i=1}^n b_i(t,x)v+f(t,x,u)+G(t,x,u,v), \tag{2.5}$$

where $a(t,x)=F(t,x,0,0)$, $\lambda(t,x)=\partial_uF(t,x,0,0)$, $b_i(t,x)=\partial_{v_i}F(t,x,0,0)$ $(i=1,2,\dots,n)$, $f(t,x,u)$ consists of the nonlinear terms of F which do not depend on v while $G(t,x,u,v)$ consists of the remaining nonlinear terms which depend on v.

Let $q\in[0,1]$ and $\mu(t)$ be a weight function. We shall study (1.1) under the following assumptions. First, $a(t,x)$, $\lambda(t,x)$ and $b_i(t,x)$ are in $C^0([0,T),\mathcal{A}(\Omega))$ for $i=1,2,\dots,n$ of growth orders $O(\mu(t)^q)$ and $O(\mu(t))$, respectively for each $i=1,2,\dots,n$. In particular, there exist $A,B,h>0$ such that for all $\alpha\in\mathbb{N}^n$ and $i=1,2,\dots,n$,

(A1) $\sup_{x\in\Omega}|\partial_x^\alpha a(t,x)|\le A\mu(t)^q h^{|\alpha|}|\alpha|!$

(A2) $\sup_{x\in\Omega}|\partial_x^\alpha b_i(t,x)|\le B\mu(t)h^{|\alpha|}|\alpha|!$.

We also have assumptions on $\lambda(t,x)$ and $f(t,x,u)$. Particularly, there exist A_1 and $c\in(0,1]$ such that for all $\alpha\in\mathbb{N}^n$ and $m\ge1$,

(A3) $\sup_{x\in\Omega}|\partial_x^\alpha\partial_u^m F(t,x,0,0)|\le\frac{A_1}{R_1^m}h^{|\alpha|}|\alpha|!m!$,

(A4) for all $x\in\Omega$, Re $\lambda(t,x)\le-2c$.

As we are interested in obtaining a local solution, we can assume throughout the paper that $R_1<1$ and $R_1<A_1$.

Finally, assume the following growth condition:

(A5) There exists $M>0$ such that for each $1\le i,j\le n$ and $t\in[0,T)$,

$$\sup_{\Omega\times D_{R_1}\times B_{R_1}}|\partial_x^\alpha\partial_{uv_i}^2F(t,x,u,v)|\le M\mu(t)^{1-q}h^{|\alpha|}|\alpha|!,$$

$$\sup_{\Omega\times D_{R_1}\times B_{R_1}}|\partial_x^\alpha\partial_{v_iv_j}^2F(t,x,u,v)|\le M\mu(t)^{1-q}h^{|\alpha|}|\alpha|!.$$

The main result is as follows.

Theorem 2.1. Suppose that (A1)–(A5) hold. If $q\in(0,1]$ and T is taken small enough, or if $q=0$ and T and A are taken small enough, then (1.1) has a solution in $C^0([0,T),\mathcal{A}(\Omega))$ of growth order $O(\mu(t)^q)$. Moreover, there exists a positive constant C^* such that following estimate holds for all $t\in[0,T)$:

$$\left\{\sup_{x\in\Omega}|u(t,x)|,\ \sup_{x\in\Omega}|\partial_xu(t,x)|\right\}\le AC^*\mu(t)^q. \tag{2.6}$$

In addition, if there is another solution $v(t, x)$ satisfying the same properties, then there exists $T^* \in (0, T)$ such that $u \equiv v$ in $[0, T^*] \times \Omega$.

The above result says that under some assumptions on the coefficients of F, our unique solution also satisfies some growth conditions. One important observation in this result is that our estimate for the solution depends on the estimate of the constant term in the Taylor expansion of F.

3. Preliminaries

In this section we introduce the majorants and formal norms that we will use throughout the paper. One can also refer to [17, 19] for these concepts.

3.1. Lax's majorant function

Let $f(\rho) = \sum_k f_k \rho^k$ and $g(\rho) = \sum_k g_k \rho^k$ be two formal power series with $g_k \geq 0$. We say that g majorizes f, denoted by $f \ll g$, provided $|f_k| \leq g_k$ for all k.

Throughout the paper, we will use a modification of Lax's majorant function in [10], which we will denote by $\phi(X)$ defined as the power series

$$\phi(X) = \frac{1}{4S} \sum_{k \in \mathbb{N}} \frac{X^k}{(k+1)^2}$$

where $S = \pi^2/6$. Some properties of $\phi(X)$ are stated below.

Lemma 3.1. The function $\phi(X)$ satisfies the following properties:

(1) $\phi(X)$ converges on $|X| < 1$;
(2) $\phi(X)^2 \ll \phi(X)$;
(3) For any $0 < \varepsilon < 1$, there exists $K_\varepsilon > 0$ such that

$$\frac{1}{1 - \varepsilon X} \ll K_\varepsilon \phi(X). \tag{3.1}$$

Note that if $0 < \varepsilon_0 < 1$, the constant K_{ε_0} will satisfy (3.1) for any $0 < \varepsilon \leq \varepsilon_0$. In particular, $\varepsilon_0 = 1/2$ will be used in the later discussions.

3.2. Formal norms

For an open set $\Omega \subseteq \mathbb{C}$, and a C^∞ function $f(x)$ on Ω, define the formal norm of f, denoted by $|||f|||_\rho$, as the formal power series given by

$$|||f|||_\rho = \sum_{\alpha \in \mathbb{N}^n} \frac{\sup_{x \in \Omega} |\partial_x^\alpha f(x)|}{|\alpha|!} \rho^{|\alpha|}. \tag{3.2}$$

In addition, for a function $u(t, x) \in C^0([0, T) \times \Omega)$ which is of class C^∞ in x, define its formal norm as

$$|||u(t)|||_\rho = \sum_{\alpha \in \mathbb{N}^n} \frac{\sup_{x \in \Omega} |\partial_x^\alpha u(t, x)|}{|\alpha|!} \rho^{|\alpha|}, \tag{3.3}$$

which is again a power series in ρ. The formal norm satisfies some majorant properties which are almost similar to usual norms.

Lemma 3.2 ([17, 19]). Suppose $p(x)$, $f(t,x)$, and $g(t,x)$ are continuous in all the variables and of class C^∞ in x and $z_1, z_2 \in \mathbb{C}$. Then the following hold:

(1) $p(x) \in \mathcal{A}(\Omega)$ if and only if $|||p|||_\rho \ll C/(1 - nh\rho)$ for some $C > 0$, where C and h are defined in (1) of Definition 2.1;
(2) $|||z_1 f(t) + z_2 g(t)|||_\rho \ll |z_1| \cdot |||f(t)|||_\rho + |z_2| \cdot |||g(t)|||_\rho$;
(3) $|||f(t)g(t)|||_\rho \ll |||f(t)|||_\rho |||g(t)|||_\rho$;
(4) For any $\alpha \in \mathbb{N}^n$, $\left|\left|\left|\partial_x^\alpha f(t)\right|\right|\right|_\rho \ll \partial_\rho^{|\alpha|} |||f(t)|||_\rho$.

Lastly, under certain conditions, we can compare formal norms of integrals.

Lemma 3.3. Let $H(t, x, \theta)$ be a function satisfying the following conditions for any $\alpha \in \mathbb{N}^n$:

(a) $\partial_x^\alpha H(t, \theta, x) \in C^0([0, T) \times [0, 1] \times \Omega)$;
(b) For any $0 < T_1 < T$, there are C_1, $h_1 > 0$ such that

$$\sup_{[0,T_1)\times[0,1]\times\Omega} |\partial_x^\alpha H(t, \theta, x)| \leq C_1 h_1^{|\alpha|} |\alpha|!.$$

Then for each $t \in [0, T)$,

$$\left|\left|\left| \int_0^1 H(t, x, \theta) d\theta \right|\right|\right|_\rho \ll \int_0^1 |||H(t, \theta)|||_\rho d\theta.$$

3.3. *The space $\mathscr{X}(T, R)$*

Definition 3.1. Let $T > 0$ and $R > 0$.

(1) A function $u(t, x)$ is said to belong in the space $\mathscr{X}(T, R)$ if
  (a) $\partial_x^\alpha u(t, x) \in C^0([0, T) \times \Omega)$ for any $\alpha \in \mathbb{N}^n$, and
  (b) there exists a positive constant A such that

$$|||u(t)|||_\rho \ll A\phi\left(\frac{\theta(t)}{\theta(T)} + \frac{\rho}{R}\right) \qquad \text{for any } t \in [0, T). \tag{3.4}$$

  Here, $\theta(t) = \int_0^t \mu(s)/s\, ds$.
(2) Let $\psi(t) : [0, \infty) \to \mathbb{R}$ be a continuous, increasing function satisfying $\psi(0) = 0$. A function $u(t, x) \in \mathscr{X}(T, R)$ is said to be of growth order $O(\psi(t))$ if the constant in (3.4) is of the form $A\psi(t)$.

The following lemma gives us a relationship between the spaces $\mathscr{X}(T, R)$ and $C^0([0, T), \mathcal{A}(\Omega))$.

Lemma 3.4. $\mathscr{X}(T, R)$ is a subspace of $C^0([0, T), \mathcal{A}(\Omega))$. Moreover, suppose that $u \in C^0([0, T), \mathcal{A}(\Omega))$ is of growth order $O(\mu(t)^q)$, i.e., for some $C, h > 0$ and $q \in [0, 1]$,

$$\sup_{x\in\Omega} |\partial_x^\alpha u(t, x)| \leq C\mu(t)^q h^{|\alpha|} |\alpha|! \qquad \text{for any } t \in [0, T). \tag{3.5}$$

If $0 < nhR < 1/2$, then $u \in \mathscr{X}(T, R)$ and is of growth order $O(\mu(t)^q)$. Furthermore,

$$|||u(t)|||_\rho \ll CK\mu(t)^q \phi\left(\frac{\theta(t)}{\theta(T)} + \frac{\rho}{R}\right) \qquad \text{for any } t \in [0, T), \tag{3.6}$$

for some $K > 1$.

Proof. The proof of the first claim may be found in [19] but will be reproduced here for completeness.

It can be easily shown that $\mathscr{X}(T,R)$ is a vector space over $\mathbb{R}$. Now, let $u(t,x) \in \mathscr{X}(T,R)$ and take $0 < T_1 < T$. Then $\varphi\left(\dfrac{\theta(T_1)}{\theta(T)} + \dfrac{z}{R}\right)$ is a holomorphic function on $\{z \in \mathbb{C} : |z| < R(1 - \theta(T_1)/\theta(T))\}$. By Cauchy's estimate, for any $k \in \mathbb{N}$,

$$|\varphi^{(k)}(\theta(T_1)/\theta(T))| \leq C_0 h^k k!,$$

for some C_0 and h, both positive. From the assumptions on $u(t,x)$, there exists $M > 0$ such that

$$|||u(t)|||_\rho \ll M\varphi\left(\frac{\theta(T_1)}{\theta(T)} + \frac{\rho}{R}\right)$$

for any $t \in [0, T_1)$. Taking the Taylor series of the right-hand side about $\rho = 0$ implies that the majorant relation is equivalent to

$$\sum_{\alpha\in\mathbb{N}^n} \frac{\sup_{x\in\Omega} |\partial_x^\alpha u(t,x)|}{|\alpha|!}\rho^{|\alpha|} \ll MC_0 \sum_{k=0}^{\infty} \left(\frac{h}{R}\right)^k \rho^k. \tag{3.7}$$

As

$$\sum_{\alpha\in\mathbb{N}^n} \frac{\sup_{x\in\Omega} |\partial_x^\alpha u(t,x)|}{|\alpha|!}\rho^{|\alpha|} = \sum_{k=0}^{\infty}\Bigg[\sum_{|\alpha|=k} \frac{\sup_{x\in\Omega} |\partial_x^\alpha u(t,x)|}{|\alpha|!}\Bigg]\rho^k, \tag{3.8}$$

then the majorant relation implies that for all $\alpha \in \mathbb{N}^n$,

$$\sup_{x\in\Omega} |\partial_x^\alpha u(t,x)| \leq MC_0 (h/R)^{|\alpha|}|\alpha|!.$$

The above inequality holds for all $t \in [0, T_1)$. Hence, it follows that

$$\sup_{[0,T_1)\times\Omega} |\partial_x^\alpha u(t,x)| \leq MC_0 (h/R)^{|\alpha|}|\alpha|!.$$

For the second claim, using (1) of Lemma 3.2 we have

$$|||u(t)|||_\rho \ll \frac{C\mu(t)^q}{1 - nh\rho} \ll CK\mu(t)^q\phi\left(\frac{\rho}{R}\right) \ll CK\mu(t)^q\phi\left(\frac{\theta(t)}{\theta(T)} + \frac{\rho}{R}\right).$$

Here, $K = K(R)$ is the constant obtained when (3) of Lemma 3.1 is applied. □

As was stated in a previous remark, if we limit $0 < nhR < 1/2$, we can find $K > 0$, independent of R, such that (3.6) holds. Henceforth, for brevity, the arguments for ϕ and μ will be omitted and will be written only if necessary.

4. Some estimates

In this section, we will discuss the solvability of the equation $\mathcal{L}u = g(t,x)$ in $\mathscr{X}(T,R)$, where the operator $\mathcal{L}$, dependent on λ_0, is defined as

$$\mathcal{L} = t\partial_t - \lambda_0(t,x).$$

Here, $\lambda_0(t,x)$ is a function such that for some $\Lambda_0 > 1$ and $c_0 \in (0,1]$, (B1) for all $\alpha \in \mathbb{N}^n$, $\sup_{[0,T)\times\Omega} |\partial_x^\alpha \lambda_0(t,x)| \leq \Lambda_0 h^{|\alpha|}|\alpha|!$

(B2) for all $(t,x) \in [0,T)\times\Omega$, Re $\lambda_0(t,x) \leq -2c_0 < 0$.

The next lemma will give estimates which will be helpful in our later computations.

Lemma 4.1. Set $H(\lambda_0;t,\tau,x) = \exp\big(\int_\tau^t \lambda_0(s,x)s^{-1}ds\big)$. Then for all $\alpha \in \mathbb{N}^n$, we have

$$\sup_{x\in\Omega}\left|\partial_x^\alpha H(\lambda_0;t,\tau,x)\right| \leq \left(\frac{\tau}{t}\right)^{c_0}\left(\frac{4\Lambda_0 hn}{c_0}\right)^{|\alpha|}|\alpha|!. \tag{4.1}$$

Proof. Take $f(x) = e^x$ and $g(t,\tau,x) = \int_\tau^t \lambda_0(s,x)s^{-1}ds$. By the multivariate Faá di Bruno formula [5], we have

$$\partial_x^\alpha H(\lambda_0;t,\tau,x) = \sum_{1\leq r\leq m} f^{(r)}[g(t,\tau,x)]\sum_{p(\alpha,r)}\alpha!\prod_{j=1}^m \frac{[\partial_x^{\boldsymbol{\ell}_j} g]^{k_j}}{(k_j!)(\boldsymbol{\ell}_j!)^{k_j}},$$

where $m = |\alpha|$, $\alpha! = \alpha_1!\cdot\alpha_2!\ldots\alpha_m!$ and $p(\alpha,r) = \Big\{(k_1,\ldots,k_m),(\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m)$: For some $1\leq s\leq m, k_i = 0, \boldsymbol{\ell}_i = 0$ for $1\leq i\leq m-s$, $k_i > 0$ for $m-s+1\leq i\leq m$ and $0 \prec \boldsymbol{\ell}_{m-s+1} \prec \ldots \prec \boldsymbol{\ell}_m$ with $\sum_{i=1}^m k_i = r$, $\sum_{i=1}^m k_i\boldsymbol{\ell}_i = \alpha\Big\}$.

Here, we write $\alpha \prec \beta$ if for $\alpha = (\alpha_1,\ldots,\alpha_m)$ and $\beta = (\beta_1,\ldots,\beta_m)$, one of the following holds:

(1) $|\alpha| < |\beta|$,
(2) $|\alpha| = |\beta|$ and $\alpha_1 < \beta_1$, or
(3) $|\alpha| = |\beta|$, $\alpha_1 = \beta_1,\ldots,\alpha_k = \beta_k$ and $\alpha_{k+1} < \beta_{k+1}$ for $1\leq k < m$.

One important note to also take into consideration is that

$$\sum_{i=1}^m k_i\boldsymbol{\ell}_i = \alpha \quad \text{implies} \quad \sum_{i=1}^m k_i|\boldsymbol{\ell}_i| = m.$$

Under our assumptions on $\lambda_0(t,x)$ we have that for any $x\in\Omega$,

i. $|\partial_x^\alpha g(t,\tau,x)| \leq \Lambda_0 h^{|\alpha|}|\alpha|!(-\ln(\tau/t))$, for any $\alpha \in \mathbb{N}^n$,
ii. $\left|f^{(k)}(g(t,\tau,x))\right| \leq (\tau/t)^{2c_0}$, for any $k \in \mathbb{N}$.

Using these estimates, we have

$$\begin{aligned}|\partial_x^\alpha H(\lambda_0;t,\tau,x)| &\leq \sum_{1\leq r\leq m}\left(\frac{\tau}{t}\right)^{2c_0}\sum_{p(\alpha,r)}\alpha!\prod_{j=1}^m\frac{(\Lambda_0 h^{|\boldsymbol{\ell}_j|}|\boldsymbol{\ell}_j|!(-\ln(\tau/t)))^{k_j}}{(k_j!)(\boldsymbol{\ell}_j!)^{k_j}}\\ &\leq \left(\frac{\tau}{t}\right)^{c_0} h^m|\alpha|!\sum_{1\leq r\leq m}\Lambda_0^r\sum_{p(\alpha,r)}\prod_{j=1}^m\frac{(|\boldsymbol{\ell}_j|!)^{k_j}\left((\tau/t)^{c_0}(-\ln(\tau/t))^{k_j}\right)}{(k_j!)(\boldsymbol{\ell}_j!)^{k_j}}\\ &\leq \left(\frac{\tau}{t}\right)^{c_0} h^m|\alpha|!\sum_{1\leq r\leq m}\left(\frac{\Lambda_0}{c_0}\right)^r\sum_{p(\alpha,r)}\prod_{j=1}^m\left(\frac{|\boldsymbol{\ell}_j|!}{(\boldsymbol{\ell}_j!)}\right)^{k_j}.\end{aligned}$$

Here, we used the fact that $|y^{c_0}(-\ln y)| \leq (k/ec_0)^k \leq 1/c_0^k\cdot k!$, for any $y\in(0,1)$ and $k\in\mathbb{N}$. By a consequence of the multinomial theorem, for each j,

$$\frac{|\boldsymbol{\ell}_j|!}{(\boldsymbol{\ell}_j!)} = \frac{|\boldsymbol{\ell}_j|!}{\ell_{1,j}!\ell_{2,j}!\ldots\ell_{n,j}!} = \binom{|\boldsymbol{\ell}_j|!}{\ell_{1,j},\ell_{2,j}\ldots,\ell_{n,j}} \leq n^{|\boldsymbol{\ell}_j|}.$$

Therefore, for any $x \in \Omega$ with the assumption that $\Lambda_0 > 1$,

$$
\begin{aligned}
|\partial_x^\alpha H(\lambda_0; t, \tau, x)| &\le \left(\frac{\tau}{t}\right)^{c_0} \left(\frac{\Lambda_0 hn}{c_0}\right)^m |\alpha|! \sum_{1\le r\le m} \binom{m+r-1}{r} \\
&= \left(\frac{\tau}{t}\right)^{c_0} \left(\frac{\Lambda_0 hn}{c_0}\right)^m |\alpha|! \cdot \binom{2m}{m} \\
&\le \left(\frac{\tau}{t}\right)^{c_0} \left(\frac{4\Lambda_0 hn}{c_0}\right)^m |\alpha|!.
\end{aligned}
$$

□

Lemma 4.2. If $g(t,x) \in C^0((0,T], \mathcal{A}(\Omega))$ is of growth order $O(\mu(t)^q)$, then the equation $\mathcal{L}w = g(t,x)$ has the unique solution

$$
w(t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\frac{ds}{s}\right) g(\tau,x)\frac{d\tau}{\tau}. \tag{4.2}
$$

Moreover, if there exists $A > 0$ for which $\sup_{x\in\Omega} |\partial_x^\alpha g(t,x)| \le A\mu(t)^q h^{|\alpha|}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, and if $R > 0$ is chosen small enough such that

$$
nhR\left(1 + \frac{4\Lambda_0 n}{c_0}\right) < \frac{1}{2}, \tag{4.3}
$$

then for any $t \in [0,T)$,

$$
|||w(t)|||_\rho \ll \frac{K}{c_0} A\mu(t)^q \phi\left(\frac{\theta(t)}{\theta(T)} + \frac{\rho}{R}\right), \tag{4.4}
$$

where $K = K_{1/2}$ is the constant obtained in the application of (3) of Lemma 3.1.

Proof. Set $F(t,\tau,x) = H(t,\tau,x)g(\tau,x)$, where $H(t,\tau,x) = H(\lambda_0; t,\tau,x)$ is the function defined in Lemma 4.1. By (1) of Lemma 3.2 and Lemma 4.1, we have for all $\tau \in [0,t]$,

$$
\begin{aligned}
|||H(t,\tau)|||_\rho &\ll \left(\frac{\tau}{t}\right)^{c_0} \frac{1}{1 - n(4\Lambda_0 hn/c_0)\rho}, \\
|||g(\tau)|||_\rho &\ll \frac{A\mu(\tau)^q}{1 - nh\rho}.
\end{aligned}
$$

As $(1-x)^{-1}(1-y)^{-1} \ll (1-x-y)^{-1}$, for all $\tau \in [0,t]$,

$$
|||F(t,\tau)|||_\rho \ll \left(\frac{\tau}{t}\right)^{c_0} \frac{A\mu(\tau)^q}{1 - (nhR)\left(1 + \frac{4\Lambda_0 n}{c_0}\right)(\rho/R)} \ll \left(\frac{\tau}{t}\right)^{c_0} \cdot AK\mu(\tau)^q \phi. \tag{4.5}
$$

Moreover, if we expand ϕ around $\rho = 0$, then the above majorant relation implies that for all $\alpha \in \mathbb{N}^n$

$$
\sum_{|\alpha|=k} \sup_{x\in\Omega} |\partial_x^\alpha F(t,\tau,x)| \le \frac{AK}{R^k}\left(\frac{\tau}{t}\right)^{c_0} \mu(\tau)^q \phi^{(|\alpha|)}(\theta(t)/\theta(T)).
$$

Therefore, by the increasing property of μ, θ and $\phi^{(k)}$, we obtain

$$
|||w(t)|||_\rho = \sum_{\alpha\in\mathbb{N}^n} \frac{\sup_{x\in\Omega}\left|\partial_x^k\left(\int_0^t F(t,\tau,x)\frac{d\tau}{\tau}\right)\right|}{k!}\rho^{|\alpha|}
$$

$$\ll \sum_{k\in\mathbb{N}} \left[\sum_{|\alpha|=k} \sup_{x\in\Omega} \int_0^t \left| \partial_x^\alpha F(t,\tau,x) \frac{d\tau}{\tau} \right| \right] \frac{\rho^k}{k!}$$
$$\ll \sum_{k\in\mathbb{N}} \left(\int_0^t \frac{AK}{R^k} \mu(\tau)^q \left(\frac{\tau}{t}\right)^{c_0} \phi^{(k)} \left(\frac{\theta(\tau)}{\theta(T)}\right) \frac{d\tau}{\tau} \right) \frac{\rho^k}{k!}$$
$$\ll \frac{K}{c_0} A\mu(t)^q \phi \left(\frac{\theta(t)}{\theta(T)} + \frac{\rho}{R} \right).$$

□

Remark 4.1. If instead of a Cauchy type estimate, $g(t,x)$ satisfies the majorant relation $|||g(t)|||_\rho \ll A\mu^q\phi$, then $|||F(t,\tau)|||_\rho$ can be estimated instead as follows:

$$|||F(t,\tau)|||_\rho \ll \left(\frac{\tau}{t}\right)^{c_0} A\mu^q\phi \cdot \frac{1}{1-n(4\Lambda_0 hn)\rho} \ll \left(\frac{\tau}{t}\right)^{c_0} AK\mu^q\phi,$$

which can use the same constant $K = K_{1/2}$. Thus, applying the same arguments, we have the similar result $|||w(t)|||_\rho \ll KAc_0^{-1}\mu^q\phi$ on $[0,T)$.

The next lemma will aid in finding majorants for the derivative of the solution of $\mathcal{L}w = g(t,x)$, where the right-hand is of growth order $O(\mu(t)^{1+q})$.

Lemma 4.3. Let $g(t,x) \in \mathscr{X}(T,R)$ be of growth order $O(\mu^{1+q})$ and let $A > 0$ for which $|||g(t)|||_\rho \ll A\mu^{1+q}\phi$. If $\mathcal{L}w = g(t,x)$ and $R > 0$ is chosen small enough such that (4.3) holds, then for all $t \in [0,T)$ and $i = 1,\dots,n$,

$$|||w(t)|||_\rho \ll \frac{K}{c_0} A\mu^{1+q}\phi \quad \text{and} \quad \left|\left|\left|\partial_{x_i} w(t)\right|\right|\right|_\rho \ll \frac{AK\theta(T)}{R}\mu^q\phi.$$

Proof. The proof for the estimate $|||w(t)|||_\rho$ is the same as in Remark 4.1 and so we only prove the second majorant relation. Let $H(t,\tau,x) = H(\lambda_0;t,\tau,x)$. By the same arguments in Remark 4.1 to estimate $|||H(t,\tau)g(\tau)|||_\rho$, we can differentiate under the integral sign, and together with (4.2), we obtain

$$\left|\left|\left|\partial_{x_i} w(t)\right|\right|\right|_\rho \ll \sum_{k\in\mathbb{N}} \left[\int_0^t \sum_{|\alpha|=k} \sup_{x\in\Omega} \left| \partial_x^\alpha \partial_{x_i} \left(H(t,\tau,x) \cdot \frac{g(\tau,x)}{\tau} \right) \right| d\tau \right] \frac{\rho^k}{k!}.$$

Note that the integrand is exactly the coefficient of $\rho^k/k!$ in $\left|\left|\left|\partial_{x_i}\left(H(t,\tau)\cdot g(\tau)/\tau\right)\right|\right|\right|_\rho$.

Again, by Lemma 3.2 (1), we have for all $\tau \in [0,t]$,

$$|||H(t,\tau)|||_\rho \ll \left(\frac{\tau}{t}\right)^{c_0} \frac{1}{1-n(4\Lambda_0 hn/c_0)\rho}.$$

Thus, under assumption (4.3),

$$\left|\left|\left| H(t,\tau) \cdot \frac{g(\tau)}{\tau} \right|\right|\right|_\rho \ll \left(\frac{\tau}{t}\right)^{c_0} K \cdot \frac{A\mu(\tau)^{1+q}}{\tau} \phi \left(\frac{\theta(\tau)}{\theta(T)} + \frac{\rho}{R} \right).$$

Since $(\tau/t)^{c_0} < 1$ and $\theta'(\tau) = \mu(\tau)/\tau$, by (4) of Lemma 3.2,

$$\left|\left|\left| \partial_{x_i} \left(H(t,\tau) \cdot \frac{g(\tau)}{\tau} \right) \right|\right|\right|_\rho \ll \frac{AK}{R} \mu(\tau)^q \theta'(\tau) \phi' \left(\frac{\theta(\tau)}{\theta(T)} + \frac{\rho}{R} \right). \tag{4.6}$$

Expanding ϕ' at $\rho = 0$, we see that

$$\phi'\left(\frac{\theta(t)}{\theta(T)} + \frac{\rho}{R}\right) = \sum_{k\in\mathbb{N}} \left(\frac{\phi^{(k+1)}(\theta(\tau)/\theta(T))}{k!}\right)\left(\frac{\rho}{R}\right)^k.$$

Hence, (4.6) implies that

$$\sum_{|\alpha|=k} \sup_{x\in\Omega} \left|\partial_x^\alpha \partial_{x_i}\left(H(t,\tau,x)\cdot\frac{g(\tau,x)}{\tau}\right)\right| \le \frac{AK}{R^{k+1}}\cdot\mu(\tau)^q\theta'(\tau)\phi^{(k+1)}(\theta(\tau)/\theta(T)).$$

Therefore, we obtain

$$\begin{aligned}\left|\left|\left|\partial_{x_i} w(t)\right|\right|\right|_\rho &\ll \frac{AK}{R}\mu(t)^q \sum_{k\in\mathbb{N}} \int_0^t \frac{\theta'(\tau)\phi^{(k+1)}(\theta(\tau)/\theta(T))}{k!}d\tau\left(\frac{\rho}{R}\right)^k \\ &\ll \frac{AK\theta(T)}{R}\mu^q\phi.\end{aligned}$$

□

5. Semilinear estimates

In this section we first solve a semilinear equation which is simpler as in (1.1). This will aid us in the proof of Theorem 2.1.

Consider the semilinear equation

$$(t\partial_t - \lambda(t,x))u(t,x) = a(t,x) + f(t,x,u), \tag{5.1}$$

where $a(t,x)$ can be in either $C^0([0,T),\mathcal{A}(\Omega))$ or $\mathscr{X}(T,R)$, both of growth order $O(\mu^q)$. Moreover, $\lambda(t,x)$ and $f(t,x,u)$ are the same as in Eq (2.5). As $f(t,x,u)$ is holomorphic in u for each fixed t and x, we can expand it as follows:

$$f(t,x,u) = \sum_{m\ge 2} \frac{\partial_u^m F(t,x,0,0)}{m!}u^m.$$

We have the following result.

Proposition 5.1. Let $q \in (0,1]$ and suppose $\lambda(t,x)$ and $f(t,x,u)$ satisfy (A3) and (A4). If $A,h > 0$ for which $a(t,x)$ satisfies either

(C1) $\sup_{x\in\Omega}|\partial_x^\alpha a(t,x)| \le A\mu^q h^{|\alpha|}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$, or
(C2) $|||a(t)|||_\rho \ll A\mu^q\phi$ on $t \in [0,T)$,

then $T,R > 0$ can be chosen small enough such that (5.1) has a unique solution in $\mathscr{X}(T,R)$ of growth order $O(\mu(t)^q)$ which satisfies the following estimate:

$$|||u(t)|||_\rho \ll \frac{2AK}{c}\mu^q\phi \quad \text{for all } t \in [0,T). \tag{5.2}$$

To prove this result, we first have the following lemma that will deal with the collection of nonlinear terms $f(t,x,u)$.

Lemma 5.2. If $|||u(t)|||_\rho \ll J\mu\phi$ for all $t \in [0, T)$, and R and T satisfy $0 < R < R_1$, $0 < 2nhR < 1/2$ and $J\mu(T)^q < R_1/2$, then there exists $C_1 > 1$, independent of T and c such that for all $i = 1, 2, \ldots, n$ and $t \in [0, T)$,

(1) $|||f(t, u)|||_\rho \ll C_1 J^2\mu^{2q}\phi$,
(2) $|||\partial_u f(t, u)|||_\rho \ll C_1 J\mu^q\phi$.
(3) $\left|\left|\left|\partial_{x_i} f(t, u)\right|\right|\right|_\rho \ll C_1 J^2\mu^{2q}\phi$.

Proof. We will only estimate (2) and (3) as the proof for (1) is similar.

Since R is chosen such that $0 < 2nhR < 1/2$, by assumption (A3) and Lemma 3.4 we see that $\left|\left|\left|\partial_u^m F(t, 0, 0)\right|\right|\right|_\rho \ll \frac{A_1 K m!}{R_1^m}\phi$. Thus, using the Taylor series expansion of f at $u = 0$, we get

$$\begin{aligned}|||\partial_u f(t, u)|||_\rho &\ll \sum_{m\geq 2} \left|\left|\left|\partial_u^m F(t, 0, 0)\right|\right|\right|_\rho \frac{|||u|||_\rho^{m-1}}{(m-1)!}\\ &\ll \frac{A_1 K}{R_1^2} J\mu^q\phi \sum_{m\geq 0}(m+2)\left(\frac{J\mu^q}{R_1}\right)^m\\ &\ll \frac{6A_1K}{R_1^2} J\mu^q\phi.\end{aligned}$$

To prove (3), note that by (A3),

$$\sup_{[0,T)\times\Omega} |\partial_x^\alpha \partial_{x_i}\partial_u^m F(t, x, 0, 0)| \leq \frac{A_1 h^{|\alpha|+1}(|\alpha|+1)!m!}{R_1^m},$$

and so

$$\begin{aligned}\left|\left|\left|\partial_{x_i}\partial_u^m F(t, 0, 0)\right|\right|\right|_\rho &\ll \sum_{\alpha\in\mathbb{N}^n} \frac{A_1 h^{|\alpha|+1}(|\alpha|+1)m!}{R_1^m}\rho^{|\alpha|}\\ &= \frac{A_1 h m!}{R_1^m}\sum_{\alpha\in\mathbb{N}^n}(h\rho)^{|\alpha|}(|\alpha|+1).\end{aligned}$$

Using the fact that $2^n \geq n+1$ and $(1-x)^{-n} \ll (1-nx)^{-1}$ we get

$$\left|\left|\left|\partial_{x_i}\partial_u^m F(t, 0, 0)\right|\right|\right|_\rho \ll \frac{A_1 h m!}{R_1^m}\cdot\frac{1}{1-2nh\rho} \ll \frac{A_1 K h m!}{R_1^m}\phi.$$

Thus,

$$\begin{aligned}\left|\left|\left|\partial_{x_i} f(t, u)\right|\right|\right|_\rho &\ll \sum_{m\geq 2} \left|\left|\left|\partial_{x_i}\partial_u^m F(t, 0, 0)\right|\right|\right|_\rho \frac{|||u|||_\rho^m}{m!}\\ &\ll \frac{A_1 h K}{R_1^2}(J\mu^q)^2\phi\sum_{m\geq 0}(1/2)^{m-2}\\ &= \frac{2A_1 h K}{R_1^2} J^q\mu^{2q}\phi.\end{aligned}$$

The claim is proved by taking $C_1 > 1$ to be the maximum of all the obtained constants. □

Remark 5.1. Throughout the proof and the proceeding section we will make use of the following observation. If a function $f(t, x)$ satisfies $|||w(t)|||_\rho \ll M\phi$ for all $[0, T) \times \Omega$, then $|w(t, x)| \leq M$. This is because the majorant relation implies that

$$\sum_{k\in\mathbb{N}}\Big[\sum_{|\alpha|=k} \frac{\sup_{x\in\Omega}|\partial_x^\alpha w(t,x)|}{k!}\Big]\rho^k \ll M\sum_{k\in\mathbb{N}} \frac{\phi^{(k)}(\theta(t)/\theta(T))}{k!}\left(\frac{\rho}{R}\right)^k$$

from which the result is obtained by comparing coefficients at $k = 0$ and the fact that $\phi(1) < 1$.

We now finally present the proof of Proposition 5.1.

Proof of Proposition 5.1. We will solve (5.1) by successive approximations. Define the approximate solutions as follows:

$$\begin{aligned}
\mathcal{L}u_0(t,x) &= a(t,x),\\
\mathcal{L}u_1(t,x) &= a(t,x) + f(t,x,u_0),\\
&\vdots\\
\mathcal{L}u_n(t,x) &= a(t,x) + f(t,x,u_{n-1}).
\end{aligned}$$

Also define $d_n(t, x) = u_n(t, x) - u_{n-1}(t, x)$ where $u_{-1}(t, x) \equiv 0$. We will prove that if we choose R and T small enough such that

$$nhR\left(1 + \frac{4A_1 n}{R_1 c}\right) < \frac{1}{2} \tag{5.3}$$

$$\frac{4AK^2}{c}\mu(T)^q < \min\left\{\frac{c}{C_1}, \frac{R_1}{2}\right\}, \tag{5.4}$$

where C_1 is the constant obtained in the application of Lemma 5.2, then u_n and d_n satisfy

$$|||d_n(t)|||_\rho \ll \frac{AK}{c}\left(\frac{1}{2}\right)^n \mu^q\phi \quad \text{and} \quad |||u_n(t)|||_\rho \ll \frac{2AK}{c}\mu^q\phi, \tag{5.5}$$

for all $n \in \mathbb{N}$ and $t \in [0, T)$. We take note that it is enough to prove the claim by proving only the estimates for d_n since for any $n \geq 0$, $u_n(t, x) = \sum_{j=0}^n d_j(t, x)$.

Suppose $a(t, x)$ satisfies (C1). Since $\lambda(t, x)$ satisfies (B1) and (B2) with $\Lambda_0 = A_1/R_1$ and $c_0 = c$. Thus, under assumption (5.3) and Lemma 4.2 we have

$$|||d_0(t)|||_\rho = |||u_0(t)|||_\rho \ll \frac{AK}{c}\mu^q\phi$$

on $t \in [0, T)$. If $a(t, x)$ satisfies (C2), then we instead use Remark 4.1. This proves the case when $k = 0$.

For the case $k = 1$, observe that by Lemma 5.2,

$$|||f(t,x,u_0)|||_\rho \ll C_1\left(\frac{AK}{c}\right)^2 \mu^{2q}\phi.$$

Since $d_1(t, x)$ satisfies $\mathcal{L}d_1(t, x) = f(t, x, u_0)$, by Remark 4.1,

$$|||d_1(t)|||_\rho \ll \frac{AK}{c}\left(\frac{AC_1K^2}{c^2}\mu(T)^q\right)\mu^q\phi.$$

Thus, if (5.4) holds, then $d_1(t,x)$ satisfies $|||d_1(t)|||_\rho \ll \frac{AK}{2c}\mu^q\phi$ which is what we wanted. For $k \geq 2$, we proceed by induction. Observe that for $k \geq 1$, d_{k+1} satisfies $\mathcal{L}d_{k+1} = f(t,x,u_k) - f(t,x,u_{k-1})$. Now,

$$f(t,x,u_n) - f(t,x,u_{n-1}) = d_n(t,x) \times \int_0^1 \partial_u f(t,x,\zeta d_n + u_{n-1})d\zeta.$$

For any $\zeta \in [0,1]$, $|||\zeta d_n + u_{n-1}|||_\rho \ll \frac{3AK}{c}\mu^q\phi$. Hence, by the choice of T and Lemmas 3.3 and 5.2, we have

$$\begin{aligned} |||f(t,x,u_k) - f(t,x,u_{k-1})|||_\rho &\ll \frac{AK}{c}\cdot\left(\frac{1}{2}\right)^n \mu^q\phi\cdot\left(\frac{3AC_1K}{c}\right)\mu^q\phi \\ &\ll A\left(\frac{1}{2}\right)^n\left(\frac{3AC_1K^2}{c^2}\mu(T)^q\right)\mu^q\phi \\ &\ll A\left(\frac{1}{2}\right)^{n+1}\mu^q\phi. \end{aligned}$$

By an application of Lemma 4.2, we finally conclude that

$$|||d_{n+1}(t)|||_\rho \ll \frac{AK}{c}\left(\frac{1}{2}\right)^{n+1}\mu^q\phi \qquad \text{on } t \in [0,T).$$

The estimate for $|||u_{k+1}(t)|||_\rho$ follows from (2) of Lemma 3.2. This proves the existence of a solution $u(t,x) \in C^0([0,T),\mathcal{A}(\Omega))$.

To prove uniqueness, suppose that u and v are two distinct solutions of (5.1) and set $w(t,x) = u(t,x) - v(t,x)$. Then, $w(t,x)$ satisfies the equation $\mathcal{L}w = f(t,x,u) - f(t,x,v)$ and consequently satisfies

$$\left(t\partial_t - \lambda(t,x) - \int_0^1 \partial_u f(t,x,\zeta v + (1-\zeta)u)d\zeta\right)w(t,x) = 0. \tag{5.6}$$

Since for any $\zeta \in [0,1]$, $|||\zeta v + (1-\zeta)u|||_\rho \ll \frac{4AK}{c}\mu^q\phi$, by Lemma 5.2 and Remark 5.1,

$$|\partial_u f(t,x,\zeta v + (1-\zeta)u)| \leq \frac{4AC_1K}{c}\mu(T)^q < \frac{c}{2} < c.$$

This means that Re $(\lambda + \int_0^1 \partial_u f(t,x,\zeta v + (1-\zeta)u)d\zeta) < -c < 0$ and the operator on the left-hand side of (5.6) is invertible. This will imply that $w = 0$. □

6. Proof of main result

To begin this section, we first prove an estimate involving the function $G(t,x,u,\partial_x u)$. Recall that G is the collection of nonlinear terms in the expansion of F which contain $\partial_x u$.

Lemma 6.1. Suppose that for any $t \in [0,T)$,

$$\left\{|||u(t)|||_\rho, |||v(t)|||_\rho\right\} \ll J\mu^q\phi.$$

Then there exists $C_3 > 0$ such that for $1 \leq i \leq n$ and for all $(t,u,v) \in [0,T) \times D_{R_1} \times B_{R_1}$,

$$\left\{|||\partial_u G(t,u,v)|||_\rho, \left|\left|\left|\partial_{v_i} G(t,u,v)\right|\right|\right|_\rho\right\} \ll C_3 J\mu\phi.$$

Proof. Recall that $G(t, x, u, v)$ has the expansion

$$G(t, x, u, v) = \sum_{i+|j|\geq 2, |j|\geq 1} \partial_u^i \partial_v^j F(t, x, 0, 0) u^i v^j.$$

Note that for all $1 \leq i \leq n$, $\partial^2_{uv_i} F = \partial^2_{uv_i} G$ and $\partial^2_{v_i v_j} F = \partial^2_{v_i v_j} G$. Thus, by assumption (A5) and Lemma 3.4,

$$\left\{ \left|\left|\left| \partial^2_{uv_i} G(t, u, v) \right|\right|\right|_\rho, \left|\left|\left| \partial^2_{v_i v_j} G(t, u, v) \right|\right|\right|_\rho \right\} \ll MK\mu^{1-q}\phi.$$

for any $(t, u, v) \in [0, T) \times D_{R_1} \times B_{R_1}$ and for $1 \leq i, j \leq n$.

Since the lowest power of v in the expansion of $\partial_u G$ is 1, then $\partial_u G(t, x, u, 0) = 0$. Therefore,

$$\begin{aligned} \partial_u G(t, x, u, v) &= \partial_u G(t, x, u, v) - \partial_u G(t, x, u, 0) \\ &= \sum_{i=1}^n v_i(t, x) \int_0^1 \partial_{uv_i} G(t, x, u, sv) ds \end{aligned}$$

and so

$$\begin{aligned} |||\partial_u G(t, u, v)|||_\rho &\ll \sum_{i=1}^n |||v_i(t)|||_\rho \int_0^1 \left|\left|\left| \partial_{uv_i} G(t, u, sv) \right|\right|\right|_\rho ds \\ &\ll nJ\mu^q\phi \cdot MK\mu^{1-q}\phi \\ &\ll nJMK\mu\phi. \end{aligned}$$

An estimate for $\left|\left|\left| \partial_{v_i} G(t, u, v) \right|\right|\right|_\rho$ can similarly be obtained. Taking C_3 to be the maximum of the obtained constants, the claim is proved. □

Lastly, we present the following lemma which will be used to deal with operators involving derivatives.

Lemma 6.2. Let $\widetilde{\lambda}(v; t, x) = \lambda(t, x) + \partial_u f(t, x, v)$. Suppose that λ and f satisfy (A3) and (A4). If $|||v(t)|||_\rho \ll J_1\phi$ such that

$$J_1 < \min\{c/C_1, R_1/2\},$$

then (B1) and (B2) are satisfied with $\Lambda_0 = 5A_1/R_1$ and $c_0 = c/2$.

Proof. By (2) of Lemma 5.2, we have $|||\partial_u f(t, v)|||_\rho \ll C_1 J_1 \phi$. Consequently, by Remark 5.1, $|\partial_u f(t, v)| < C_1 J_1 < c$ which implies that $\operatorname{Re} \widetilde{\lambda}(v; t, x) \leq -c$ for any $(t, x) \in [0, T) \times \Omega$. Moreover, for any $(t, x) \in [0, T) \times \Omega$ and $\alpha \in \mathbb{N}^n$,

$$\begin{aligned} |\partial_x^\alpha \partial_u f(t, x, v)| &\leq \sum_{m\geq 2} \frac{|\partial_x^\alpha \partial_u^m F(t, x, 0, 0)|}{(m-1)!} |v|^{m-1} \\ &\leq \sum_{m\geq 2} \frac{A_1 h^{|\alpha|} |\alpha|! m!}{R_1^m (m-1)!} \left(\frac{R_1}{2}\right)^{m-1} \\ &= \frac{A_1}{2R_1} h^{|\alpha|} |\alpha|! \sum_{m\geq 0} (m+2)(1/2)^m \end{aligned}$$

$$= \frac{3A_1}{R_1} h^{|\alpha|} |\alpha|!.$$

Thus, $\sup_{[0,T)\times\Omega} |\partial_x^\alpha \widetilde{\lambda}(t,x)| \leq 4A_1 R_1^{-1} h^{|\alpha|} |\alpha|!.$ □

Remark 6.1. If we replace $\partial_u f(t,x,v)$ in the definition of $\widetilde{\lambda}$ with $\int_0^1 \partial_u f(t,x,v) d\zeta$, where $v = v(\zeta,t,x)$, then the result will still hold if $|||v(\zeta,t)|||_\rho \ll J_1\phi$ for any $\zeta \in [0,1]$. This result follows from Lemma 3.3.

We require the following to hold:

$$nhR\left(1 + \frac{8A_1 n}{R_1 c}\right) < 1/2, \tag{6.1}$$

$$\frac{4AKQ}{c}\mu(T)^q < \min\left\{\frac{c}{C_1}, \frac{R_1}{2}\right\}, \tag{6.2}$$

$$P\eta(T,R) < 1/2, \tag{6.3}$$

where $\eta(T,R) = \max\{\mu(T), \theta(T)/R\}$ and P, Q are constants such that

$$Q = C_2 + \frac{2KC_2}{c} + K \qquad \text{and} \qquad P = nBK \cdot \frac{4K}{c} + 16C_3(n+1)\frac{AK^2Q}{c^2}. \tag{6.4}$$

Here, C_2 is chosen sufficiently large such that for all $t \in [0,T)$ and $i = 1,2,\ldots,n$,

$$\left|\left|\left|\partial_{x_i} a(t)\right|\right|\right|_\rho \ll AC_2\mu^q\phi \quad \text{and} \quad \left|\left|\left|\partial_{x_i} \lambda(t)\right|\right|\right|_\rho \ll C_2\phi.$$

We note that condition (6.1) will enable us to use Proposition 5.1 and Lemma 4.2. Condition (6.2) ensures that the approximate solutions are within the domain of definition of both f and G and would allow us to apply Lemma 6.2. On the other hand, (6.3) guarantees the convergence of both $\{u_k(t,x)\}$ and $\{\partial_{x_i} u_k(t,x)\}$. Note that we must choose R small enough so that (6.1) will be satisfied.

If $q \in (0,1]$, we have to choose a small $T > 0$ so that (6.2) and (6.3) hold. If $q = 0$, we must also have to choose $A > 0$ to be small as to satisfy (6.2). Finally, observe also that as $K < Q$, (6.2) implies (5.4) so that we can impose this condition in the proof of Proposition 5.1.

6.1. Proof of existence

We again use the method of successive approximations. Define the approximate solutions as follows:

$$\mathcal{L}u_0 = a(t,x) + f(t,x,u_0), \tag{6.5}$$

$$\mathcal{L}u_1 = a(t,x) + \sum_{i=1}^n b_i(t,x)\partial_{x_i} u_0 + f(t,x,u_1) + G\left(t,x,u_0,\partial_x u_0\right), \tag{6.6}$$

$$\vdots$$

$$\mathcal{L}u_n = a(t,x) + \sum_{i=1}^n b_i(t,x)\partial_{x_i} u_{n-1} + f(t,x,u_n) + G\left(t,x,u_{n-1},\partial_x u_{n-1}\right). \tag{6.7}$$

Furthermore, set $d_n := u_n - u_{n-1}$, with $u_{-1}(t,x) \equiv 0$. Again, it suffices to prove the convergence of the partial sums of d_i.

By Proposition 5.1, (6.5) has a unique solution $u_0(t,x) \in C^0([0,T), \mathcal{A}(\Omega))$ that satisfies the estimate

$$|||d_0(t)|||_\rho = |||u_0(t)|||_\rho \ll \frac{2AK}{c}\mu^q\phi \qquad \text{on } t \in [0,T).$$

To find an estimate for $\left|\left|\left|\partial_{x_i} d_0(t)\right|\right|\right|_\rho$, we differentiate (6.5) with respect to x_i which is given by

$$(t\partial_t - \lambda(t,x) - \partial_u f(t,x,u_0))\,\partial_{x_i} u_0 = \partial_{x_i} a(t,x) + u_0 \partial_{x_i}\lambda(t,x) + \partial_{x_i} f(t,x,u_0).$$

We first estimate the right-hand side. By (3) of Lemma 5.2 and (6.2), we get

$$\left|\left|\left|\partial_{x_i} f(t,u_0)\right|\right|\right|_\rho \ll C_1 \left(\frac{2AK}{c}\mu^q\right)^2 \phi \ll AK\mu^q\phi.$$

This means that the formal norm of the right-hand side is majorized by

$$\left(AC_2 + \frac{2AKC_2}{c} + AK\right)\mu^q\phi \ll AQ\mu^q\phi,$$

where Q is the constant described in (6.4).

Under condition (6.1) and Lemma 6.2, we can apply Lemma 4.2 with $c_0 = c/2$ to have

$$\left|\left|\left|\partial_{x_i} u_0\right|\right|\right|_\rho \ll \frac{2}{c} AKQ\mu^q\phi.$$

Thus, for the case $k = 0$, we have the following estimates:

$$\left\{|||u_0(t)|||_\rho, \left|\left|\left|\partial_{x_i} u_0(t)\right|\right|\right|_\rho\right\} \ll \frac{2AKQ}{c}\mu^q\phi. \tag{6.8}$$

For the case $k \geq 1$, recall that u_k satisfies

$$\mathcal{L}u_k = a(t,x) + f(t,x,u_k) + \Phi(u_{k-1}), \tag{6.9}$$

where

$$\Phi(w) = \sum_{i=1}^{n} b_i(t,x)\partial_{x_i} w + G(t,x,w,\partial_x w).$$

With that, we have the following result for all $k \geq 1$.

Proposition 6.3. The following hold for $k \geq 1$ and for all $1 \leq i \leq n$ and for all $t \in [0,T)$:

$$\left\{|||d_k|||_\rho(t), \left|\left|\left|\partial_{x_i} d_k(t)\right|\right|\right|_\rho\right\} \ll \frac{2AKQ}{c}\left(\frac{1}{2}\right)^k \mu^q\phi,$$
$$\left\{|||u_k|||_\rho(t), \left|\left|\left|\partial_{x_i} u_k(t)\right|\right|\right|_\rho\right\} \ll \frac{4AKQ}{c}\mu^q\phi,$$

on $[0,T) \times \Omega$.

Proof. We start with the case $k = 1$. Recall that by assumption (A2), we have

$$\sup_{x\in\Omega}\left|\partial_x^\alpha b_i(t,x)\right| \leq B\mu(t)h^{|\alpha|}|\alpha|!.$$

Thus, by Lemma 3.4, $|||b_i(t)|||_\rho \ll BK\mu\phi$ for each $i = 1, 2, \dots, n$ and for $t \in [0, T)$,

$$\left|\left|\left|\sum_{i=1}^n b_i(t)\partial_{x_i}u_0\right|\right|\right|_\rho \ll nBK \cdot \frac{2AKQ}{c}\mu^{1+q}\phi.$$

Also, as $G(t, x, u_0, \partial_x u_0) = G(t, x, u_0, \partial_x u_0) - G(t, x, 0, 0)$, by Lemma 6.1,

$$\begin{aligned} |||G(t, u_0, \partial_x u_0)|||_\rho &\ll |||u_0|||_\rho \int_0^1 |||\partial_u G(t, \zeta u, \zeta\partial_x u_0)|||_\rho d\zeta \\ &\quad + \sum_{i=1}^n \left|\left|\left|\partial_{x_i}u_0\right|\right|\right|_\rho \int_0^1 \left|\left|\left|\partial_{v_i}G(t, x, \zeta u, \zeta\partial_x u_0)\right|\right|\right|_\rho d\zeta \\ &\ll C_3(n+1)\left(\frac{2AKQ}{c}\right)^2 \mu^{1+q}\phi. \end{aligned}$$

Therefore, $|||\Phi(u_0)|||_\rho \ll APQ\mu^{1+q}\phi$, where P is the constant defined in (6.4). Since $K < Q$, by (6.3),

$$|||a(t) + \Phi(u_0)|||_\rho \ll (AK + AQ)\mu^q\phi \ll 2AQ\mu^q\phi.$$

Applying Proposition 5.1, we have

$$|||u_1(t)|||_\rho \ll \frac{4AKQ}{c}\mu^q\phi \qquad \text{on } t \in [0, T).$$

Note that $d_1 = u_1 - u_0$ satisfies the equation

$$(t\partial_t - \lambda(t, x))d_1 = f(t, x, u_1) - f(t, x, u_0) + \Phi(u_0).$$

Expressing the difference $f(t, x, u_1) - f(t, x, u_0)$ as an integral and transferring this expression to the left-hand side, the above equation becomes

$$\left(t\partial_t - \lambda(t, x) - \int_0^1 \partial_u f(t, x, \zeta u_1 + (1-\zeta)u_0)d\zeta\right)d_1 = \Phi(u_0).$$

Since $|||\zeta u_1 + (1-\zeta)u_0|||_\rho \ll \frac{8AKQ}{c}\mu^q\phi$ for any $\zeta \in [0, 1]$, then under condition (6.1) and Lemma 6.2, we can apply Lemma 4.2 with $c_0 = c/2$ to have

$$|||d_1(t)|||_\rho = |||u_1 - u_0|||_\rho \ll \frac{2AKPQ}{c}\mu^{1+q}\phi.$$

Moreover, since $\Phi(u_0)$ is of growth order $O(\mu^{1+q})$, we can apply Lemma 4.3 to obtain

$$\left|\left|\left|\partial_{x_i}d_1(t)\right|\right|\right|_\rho \ll \frac{2AKPQ}{R}\theta(T)\mu^q\phi.$$

Thus under condition (6.3), it is shown that

$$\left\{|||d_1(t)|||_\rho, \left|\left|\left|\partial_{x_i} d_1(t)\right|\right|\right|_\rho\right\} \ll \frac{2AKQ}{c}\left(\frac{1}{2}\right)\mu^q\phi \qquad \text{on } t \in [0,T), \tag{6.10}$$

$$\left\{|||u_1(t)|||_\rho, \left|\left|\left|\partial_{x_i} u_1(t)\right|\right|\right|_\rho\right\} \ll \frac{4AKQ}{c}\mu^q\phi \qquad \text{on } t \in [0,T) \tag{6.11}$$

which concludes the case $k = 1$.

Now, assume the claim holds for $k = 1, 2, \dots, m$. We will show that the claim still holds for $k = m + 1$. Recall that u_{m+1} satisfies

$$(t\partial_t - \lambda(t,x))u_{m+1} = a(t,x) + f(t,x,u_{m+1}) + \Phi(u_m).$$

Using similar arguments, we can show that

$$\left|\left|\left|\sum_{i=1}^n b_i(t)\partial_{x_i} u_m(t)\right|\right|\right|_\rho \ll nBK \cdot \frac{4AKQ}{c}\mu^{1+q}\phi,$$

$$|||G(t, u_m, \partial_x u_m)|||_\rho \ll C_3(n+1)\left(\frac{4AKQ}{c}\right)^2 \mu^{1+q}\phi.$$

By (6.3), we have $|||a(t) + \Phi(u_m)|||_\rho \ll (AK+APQ\mu)\mu^q\phi \ll 2AQ\mu^q\phi$. Consequently, by Proposition 5.1, $u_{m+1}(t,x)$ satisfies

$$|||u_{m+1}(t)|||_\rho \ll \frac{4AKQ}{c}\mu^q\phi \qquad \text{on } t \in [0,T).$$

In addition, $d_{m+1} = u_{m+1} - u_m$ satisfies

$$\left(t\partial_t - \lambda(t,x) - \int_0^1 \partial_u f(t,x,\zeta u_{m+1} + (1-\zeta)u_m)d\zeta\right)d_{m+1} = \Phi(u_m) - \Phi(u_{m-1}). \tag{6.12}$$

Let

$$H_m = G(t,x,u_m,\partial_x u_m) - G(t,x,u_{m-1},\partial_x u_{m-1}).$$

Since for any $\zeta \in [0,1]$, and $i = 1, \dots, n$,

$$\left\{|||\zeta u_m + (1-\zeta)u_{m-1}|||_\rho, \left|\left|\left|\zeta\partial_{x_i} u_m + (1-\zeta)\partial_{x_i} u_{m-1}\right|\right|\right|_\rho\right\} \ll \frac{4AKQ}{c}\mu^q\phi,$$

then

$$\begin{aligned}|||H_m|||_\rho &\ll |||d_m(t)|||_\rho \int_0^1 \left|\left|\left|\partial_u G(t, \zeta u_m + (1-\zeta)u_{m-1}, \zeta\partial_{x_i} u_m + (1-\zeta)\partial_{x_i} u_{m-1})\right|\right|\right|_\rho d\zeta \\ &\quad + \sum_{i=1}^n \left|\left|\left|\partial_{x_i} d_m\right|\right|\right|_\rho \int_0^1 \left|\left|\left|\partial_{v_i} G(t, \zeta u_m + (1-\zeta)u_{m-1}, \zeta\partial_{x_i} u_m + (1-\zeta)\partial_{x_i} u_{m-1})\right|\right|\right|_\rho d\zeta \\ &\ll 8C_3(n+1)\left(\frac{AKQ}{c}\right)^2\left(\frac{1}{2}\right)^m \mu^{1+q}\phi \qquad \text{on } t \in [0,T).\end{aligned}$$

Hence, we conclude that

$$|||\Phi(u_m) - \Phi(u_{m-1})|||_\rho \ll \frac{APQ}{c}\left(\frac{1}{2}\right)^m \mu^{1+q}\phi \qquad \text{on } t \in [0, T). \tag{6.13}$$

Since $|||\zeta u_{m+1} + (1-\zeta)u_m|||_\rho \ll \frac{8AKQ}{c}\mu^q\phi$ for any $\zeta \in [0,1]$, then under condition (6.1) and Lemma 6.2, we can apply Lemmas 4.2 and 4.3 with $c_0 = c/2$ to prove our claim. □

Furthermore, by Remark 5.1, this proves the existence of a solution $u(t,x)$ in $C^0([0,T), \mathcal{A}(\Omega))$ of growth order $O(\mu(t)^q)$.

6.2. *Proof of uniqueness*

Suppose u and v are two distinct solutions to (1.1). By our recently proved result, we can assume they satisfy the following estimate:

$$\left\{|||u(t)|||_\rho, \left|\left|\left|\partial_{x_i}u(t)\right|\right|\right|_\rho, |||v(t)|||_\rho, \left|\left|\left|\partial_{x_i}v(t)\right|\right|\right|_\rho\right\} \ll \frac{4AKQ}{c}\mu^q\phi \quad \text{on } [0,T). \tag{6.14}$$

Set $w = u - v$. Under assumption (6.14), we have

$$\left\{|||w(t)|||_\rho, \left|\left|\left|\partial_{x_i}w(t)\right|\right|\right|_\rho\right\} \ll \frac{8AKQ}{c}\mu^q\phi. \tag{6.15}$$

Choose $T^* \in (0, T]$ such that the following holds on $[0, T^*)$:

$$\frac{8AC_1KQ}{c}\mu(T^*) \le \left\{\frac{c}{C_1}, \frac{R_1}{2}\right\}. \tag{6.16}$$

Moreover, by our choice of T and $T^* \in [0,T)$, it follows that $P\eta(T^*, R) < 1/2$. Again, (6.16) implies that $w(t,x)$ is in our domain of definition of f and G and will allow us to use Lemma 6.2. We have

$$\left(t\partial_t - \lambda(t,x) - \int_0^1 \partial_u f(t,x,\zeta u + (1-\zeta)v)d\zeta\right)w(t,x) = \Phi(u) - \Phi(v). \tag{6.17}$$

Since $\left|\left|\left|\partial_{x_i}w(t)\right|\right|\right|_\rho \ll \left|\left|\left|\partial_{x_i}u(t)\right|\right|\right|_\rho + \left|\left|\left|\partial_{x_i}v(t)\right|\right|\right|_\rho$, we have

$$\left|\left|\left|\sum_{i=1}^n b_i(t)\partial_{x_i}w(t)\right|\right|\right|_\rho \ll nBK \cdot \frac{8AKQ}{c}\mu^{1+q}\phi \qquad \text{on } t \in [0, T^*).$$

Moreover, since for all $\zeta \in [0,1]$,

$$\left\{|||\zeta u(t) + (1-\zeta)v(t)|||_\rho, \left|\left|\left|\zeta\partial_{x_i}u(t) + (1-\zeta)\partial_{x_i}v(t)\right|\right|\right|_\rho\right\} \ll \frac{4AKQ}{c}\mu^q\phi, \tag{6.18}$$

by (6.14), together with (6.1), we have

$$|||G(t,u,\partial_x u) - G(t,x,v,\partial_x v)|||_\rho \ll \frac{64A^2C_3K^2Q^2}{c^2}(n+1)\mu^{1+q}\phi.$$

Consequently, we have

$$\begin{aligned}|||\Phi(u)-\Phi(v)|||_\rho &\ll \sum_{i=1}^{n}|||b_i(t)|||_\rho\left|\left|\left|\partial_{x_i}w(t)\right|\right|\right|_\rho+|||G(t,u,\partial_x u)-G(t,v,\partial_x v)|||_\rho\\ &\ll 4AQ\left(nBK\cdot\frac{2K}{c}+16C_3(n+1)\frac{AK^2Q}{c^2}\right)\mu^{1+q}\phi\\ &\ll 4APQ\mu^{1+q}\phi.\end{aligned}$$

Since (6.18) holds for any $\zeta\in[0,1]$ together with (6.16), the operator on the left-hand side of Eq (6.17) satisfies Lemma 6.2. Hence, we conclude that

$$\left\{|||w(t)|||_\rho,\left|\left|\left|\partial_{x_i}w(t)\right|\right|\right|_\rho\right\}\ll\frac{8AKQ}{c}\left(\frac{1}{2}\right)\mu^q\phi\qquad\text{on } t\in[0,T^*).\tag{6.19}$$

Repeating our arguments, but replacing (6.14) with (6.19), it can be shown that

$$|||\Phi(u)-\Phi(v)|||_\rho\ll 4APQ\cdot(1/2)\mu^{1+q}\phi$$

and $\left\{|||w(t)|||_\rho,\left|\left|\left|\partial_{x_i}w(t)\right|\right|\right|_\rho\right\}\ll 4AKQ/c\cdot(1/2)^2\mu^q\phi$. Proceeding inductively, we can show that $|||w(t)|||_\rho$ and $\left|\left|\left|\partial_{x_i}w(t)\right|\right|\right|_\rho$ satisfy the following estimate for any $k\geq 0$:

$$\left\{|||w(t)|||_\rho,\left|\left|\left|\partial_{x_i}w(t)\right|\right|\right|_\rho\right\}\ll\frac{8AKQ}{c}\left(\frac{1}{2}\right)^k\mu^q\phi\qquad\text{on } t\in[0,T^*).\tag{6.20}$$

As the above estimate holds for all $k\in\mathbb{N}$, if we let $k\to\infty$, we conclude that $|||w(t)|||_\rho=0$ and so $u=v$ on $[0,T^*)\times\Omega$. This proves that the solution is unique, which completes the proof of Theorem 2.1.

7. Conclusions

In this paper, we proved an existence and uniqueness theorem for a class of first order nonlinear singular partial differential equations. The obtained solution is continuous in the 'time' variable and uniformly analytic in the 'space' variable, and satisfies the same growth order as the inhomogeneous term. The proof made use of formal norms and Lax's majorant function.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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