Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 10286–10302.
DOI:10.3934/math.2022573
Received: 26 December 2021
Revised: 22 February 2022
Accepted: 02 March 2022
Published: 24 March 2022

Research article

Derivative of self-intersection local time for the sub-bifractional Brownian motion

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Abstract: Let $S^{H,K} = \{S_t^{H,K}, t \geq 0\}$ be the sub-bifractional Brownian motion (sbfBm) of dimension 1, with indices $H \in (0,1)$ and $K \in (0,1]$. We mainly consider the existence of the self-intersection local time and its derivative for the sbfBm. Moreover, we prove its derivative is Hölder continuous in space variable and time variable, respectively.

Keywords: sub-bifractional Brownian motion; self-intersection local time; derivative of self-intersection local time
Mathematics Subject Classification: 60G22, 60J55

1. Introduction and main results

Recently, El-Nouty and Journé [3] introduced the process $S^{H,K} = \{S_t^{H,K}, t \geq 0\}$ on the probability space (Ω, F, P) with indices $H \in (0,1)$ and $K \in (0,1]$, named the sub-bifractional Brownian motion (sbBm) and defined as follows:

$$S_t^{H,K} = \frac{1}{2^{(2-K)/2}}(B_t^{H,K} + B_{-t}^{H,K}), \tag{1.1}$$

where $\{B_t^{H,K}, t \in \mathbf{R}\}$ is a bifractional Brownian motion (bBm) with indices $H \in (0,1)$ and $K \in (0,1]$, namely,

$$B_t^{H,K} = \begin{cases} B_t^{H,K}(1), & \text{if } t \geq 0; \\ B_{-t}^{H,K}(2), & \text{if } t < 0. \end{cases} \tag{1.2}$$

and $B_t^{H,K}(1)$ and $B_{-t}^{H,K}(2)$ are independent bifractional Brownian motions in $[0, +\infty)$ with indices $H \in (0,1)$ and $K \in (0,1]$, where bifractional Brownian motion $\{B_t^{H,K}(1), t \geq 0\}$ is a centered Gaussian process, starting from zero, with covariance

$$\mathbf{E}[B_t^{H,K}(1)B_s^{H,K}(1)] = \frac{1}{2^K}\left[(t^{2H}+s^{2H})^K - |t-s|^{2HK}\right],$$

with $H \in (0,1)$ and $K \in (0,1]$.

Clearly, the sbBm is a centered Gaussian process such that $S_0^{H,K} = 0$, with probability 1, and $\mathrm{Var}(S_t^{H,K}) = (2^K - 2^{2HK-1})t^{2HK}$. Since $(2H-1)K-1 < K-1 \le 0$, it follows that $2HK-1 < K$. We can easily verify that $S^{H,K}$ is self-similar with index HK. When $K=1$, $S^{H,1}$ is the sub-fractional Brownian motion (sfBm). For more on sub-fractional Brownian motion, we can see Kuang and Xie [15, 16], Kuang and Liu [13, 14], Xie and Kuang [21] and so on. Straightforward computations show that for all $s, t \ge 0$,

$$\mathbf{E}(S_t^{H,K}S_s^{H,K}) = (t^{2H}+s^{2H})^K - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK} \tag{1.3}$$

and

$$C_1|t-s|^{2HK} \le \mathbf{E}[(S_t^{H,K} - S_s^{H,K})^2] \le C_2|t-s|^{2HK}, \tag{1.4}$$

where

$$C_1 = \min\{2^K-1, 2^K - 2^{2HK-1}\}, \quad C_2 = \max\{1, 2-2^{2HK-1}\}. \tag{1.5}$$

(See El-Nouty and Journé [3]). Kuang [11] investigated the collision local time of two independent sub-bifractional Brownian motions. Kuang and Li [12] obtained Berry-Esséen bounds and proved the almost sure central limit theorem for the quadratic variation of the sub-bifractional Brownian motion.

The self-intersection local time of fractional Brownian motion (fBm) $B^H = \{B_t^H, t \ge 0\}$ was first studied in Rosen [17], and formally defined by

$$\alpha_t(y) := \int_D \delta(B_s^H - B_r^H - y)drds,$$

where $D = \{(r,s) : 0 < r < s < t\}$ and δ is the Dirac delta function. It was further investigated in Hu [4] and Hu and Nualart [5]. Jiang and Wang [8] considered self-intersection local time and collision local time of bifractional Brownian motion. Chen et al. [2] studied renormalized self-intersection local time of bifractional Brownian motion.

The context of derivative for the self-intersection local time is organized as follows. Rosen [18] first studied the Brownian motion case, and Yan et al. [22] extended this to fractional Brownian motion. On this basis, Jung and Markowsky [9, 10] considered some in-depth results for derivative of the self-intersection local time of fractional Brownian motion. Jaramillo and Nualart [6, 7] studied asymptotic properties of the derivative of self-intersection local time of fractional Brownian motion and functional limit theorem for the self-intersection local time of the fractional Brownian motion, respectively. Yan and Yu [23] considered the multidimensional fractional Brownian motion case. Yu [24] investigated higher order derivative of self-intersection local time for fBm. Shi [19] investigated fractional smoothness of derivative of self-intersection local time for bi-fractional Brownian motion.

Moreover, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models, and such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fBm has been introduced. However, contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments. Thus, it seems interesting to study some extensions of fractional Brownian motion such as sbfBm.

In this paper, we will study the existence of the self-intersection local time and its derivative for the sub-bifractional Brownian motion $S^{H,K} = \{S_t^{H,K}, t \geq 0\}$. They are defined by, respectively,

$$\alpha_t(y) := \int_D \delta(S_s^{H,K} - S_r^{H,K} - y)drds \tag{1.6}$$

and

$$\alpha_t'(y) := -\int_D \delta'(S_s^{H,K} - S_r^{H,K} - y)drds, \tag{1.7}$$

where $D = \{(r, s) : 0 < r < s < t\}$.

Set

$$p_\epsilon(x) := \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} = \frac{1}{2\pi} \int_{\mathbf{R}} e^{ipx} e^{-\frac{\epsilon p^2}{2}} dp \tag{1.8}$$

and

$$p_\epsilon'(x) := \frac{dp_\epsilon(x)}{dx} = \frac{-x}{\sqrt{2\pi\epsilon^3}} e^{-\frac{x^2}{2\epsilon}} = \frac{i}{2\pi} \int_{\mathbf{R}} p e^{ipx} e^{-\frac{\epsilon p^2}{2}} dp. \tag{1.9}$$

Now we state our main results as follows.

Theorem 1.1. *Existence of self-intersection local time.*

Define:

$$\alpha_{t,\epsilon}(y) := \int_0^t \int_0^s p_\epsilon(S_s^{H,K} - S_r^{H,K} - y)drds, \quad \forall y \in \boldsymbol{R}, \tag{1.10}$$

for $H \in (0, 1)$ and $K \in (0, 1]$. Then $\alpha_{t,\epsilon}(y)$ converges in $L^2(\Omega, P)$ as $\epsilon \to 0$, we denote the limit by $\alpha_t(y)$.

Theorem 1.2. *Existence of the derivative of self-intersection local time.*

Define:

$$\alpha_{t,\epsilon}'(y) := -\int_0^t \int_0^s p_\epsilon'(S_s^{H,K} - S_r^{H,K} - y)drds, \quad \forall y \in \boldsymbol{R}. \tag{1.11}$$

If $H \in (0, 1)$, $K \in (0, 1]$, and $0 < 2HK < 1$, then $\alpha_{t,\epsilon}'(y)$ converges in $L^2(\Omega, P)$ as $\epsilon \to 0$, we denote the limit by $\alpha_t'(y)$.

Theorem 1.3. *Let $n \geq 1$ be an arbitrary but fixed integer, $t, \tilde{t} \in [0, T]$ and as $0 < 2HK < 1$.*

(1) For any $\tau \in \left(0, min\left\{1, \frac{1}{HK} - 2\right\}\right)$, there exists a positive constant C, such that

$$\boldsymbol{E}\left[\left|\alpha_t'(y_1) - \alpha_t'(y_2)\right|^n\right] \leq C|y_1 - y_2|^{n\tau}, \tag{1.12}$$

where $y_1, y_2 \in \boldsymbol{R}$.

(2) For any $\gamma < 1 - 2HK$, there exists a positive constant C, such that

$$\boldsymbol{E}\left[\left|\alpha_t'(y) - \alpha_{\tilde{t}}'(y)\right|^n\right] \leq C|t - \tilde{t}|^{n\gamma}, \tag{1.13}$$

where $y \in \boldsymbol{R}$.

In what follows, we will use m to denote unspecified positive and finite constants whose value may be different in each occurrence.

2. Some useful lemmas

In this section, we give some useful lemmas in order to prove the Theorems 1.1–1.3.

Lemma 2.1. *For all constants* $0 < a < b$, $S^{H,K}$ *is strongly locally* φ*-nondeterministic on* $I = [a, b]$ *with* $\varphi(r) = r^{2HK}$. *That is, there exist positive constants* c_1 *and* r_0 *such that for all* $t \in I$ *and all* $0 < r \leq \min\{t, r_0\}$,

$$Var\{S_t^{H,K}|S_s^{H,K} : s \in I, r \leq |s-t| \leq r_0\} \geq c_1\varphi(r). \tag{2.1}$$

Proof. See Kuang [11]. □

From the local nondeterminism (see Berman [1], Xiao [20]), we have the following property: if $0 \leq t_1 < t_2 < \cdots < t_n < T$, then there is a constant $m > 0$ such that

$$\text{Var}\left(\sum_{i=2}^{n} u_i(S_{t_i}^{H,K} - S_{t_{i-1}}^{H,K})\right) \geq m \sum_{i=2}^{n} u_i^2|t_i - t_{i-1}|^{2HK}, \tag{2.2}$$

for any $u_i \in \mathbf{R}, i = 2, 3, \ldots, n$.

Lemma 2.2. *Let*

$$\lambda := Var\left(S_s^{H,K} - S_r^{H,K}\right), \rho := Var\left(S_{s'}^{H,K} - S_{r'}^{H,K}\right),$$

and

$$\mu := Cov\left(S_s^{H,K} - S_r^{H,K}, S_{s'}^{H,K} - S_{r'}^{H,K}\right).$$

Case 2.1. *If* $(r, s, r', s') \in D_1 := \{(r, s, r', s')|0 < r < r' < s < s' < t\}$, *denoting* $a = r' - r, b = s - r', c = s' - s$, *then we have*

(1)

$$C_1(a+b)^{2HK} \leq \lambda = \lambda_1 \leq C_2(a+b)^{2HK}, C_1(b+c)^{2HK} \leq \rho = \rho_1 \leq C_2(b+c)^{2HK}, \tag{2.3}$$

where C_1 *and* C_2 *are given by (1.5).*

(2) There exists a positive constant m, such that

$$\lambda_1\rho_1 - \mu_1^2 \geq m\left[(a+b)^{2HK}c^{2HK} + (b+c)^{2HK}a^{2HK}\right], \tag{2.4}$$

where $\mu = \mu_1$.

(3) When $0 < 2HK < 1$, *there exists a positive constant m, such that*

$$\mu = \mu_1 \leq m\left(a^{2HK} + b^{2HK} + c^{2HK}\right). \tag{2.5}$$

Case 2.2. *If* $(r, s, r', s') \in D_2 := \{(r, s, r', s')|0 < r < r' < s' < s < t\}$, *denoting* $a = r' - r, b = s' - r', c = s - s'$, *then we have*

(1)

$$C_1(a+b+c)^{2HK} \leq \lambda = \lambda_2 \leq C_2(a+b+c)^{2HK}, \;\; C_1b^{2HK} \leq \rho = \rho_2 \leq C_2b^{2HK}, \tag{2.6}$$

where C_1 *and* C_2 *are given by (1.5).*

(2) There exists a positive constant m, such that

$$\lambda_2\rho_2 - \mu_2^2 \geq mb^{2HK}\left(a^{2HK} + c^{2HK}\right), \tag{2.7}$$

where $\mu = \mu_2$.

(3) When $0 < 2HK < 1$, *there exists a positive constant m, such that*

$$\mu = \mu_2 \leq mb^{2HK}. \tag{2.8}$$

Case 2.3. *If* $(r, s, r', s') \in D_3 := \{(r, s, r', s')|0 < r < s < r' < s' < t\}$, *denoting* $a = s - r, b = r' - s, c = s' - r'$, *then we have*

(1)

$$C_1 a^{2HK} \leq \lambda = \lambda_3 \leq C_2 a^{2HK}, \ \ C_1 c^{2HK} \leq \rho = \rho_3 \leq C_2 c^{2HK}, \tag{2.9}$$

where C_1 *and* C_2 *are given by (1.5).*

(2) There exists a positive constant m, such that

$$\lambda_3\rho_3 - \mu_3^2 \geq ma^{2HK}c^{2HK}, \tag{2.10}$$

where $\mu = \mu_3$.

(3) When $0 < 2HK < 1$, *for* $\alpha, \beta > 0$ *with* $\alpha + \beta = 1$, *there exists a positive constant m, such that*

$$\mu = \mu_3 \leq mb^{2\alpha(HK-1)}(ac)^{\beta(HK-1)+1}. \tag{2.11}$$

Proof. The proof of this lemma is given in the Appendix since its proof is long. □

Lemma 2.3. *Let* $H \in (0, 1), K \in (0, 1]$, *as* $0 < 2HK < 1$, *we have*

$$\int_{D^2} \frac{\mu}{(\lambda\rho - \mu^2)^{3/2}} drdsdr'ds' < +\infty, \tag{2.12}$$

where $D^2 := \{(r, s, r', s')|0 < r < s < t, 0 < r' < s' < t\}$, *and* λ, ρ *and* μ *are given in Lemma 2.2.*

Proof. Since $D^2 \cap \{r < r'\} = D_1 \cup D_2 \cup D_3$, where D_1, D_2 and D_3 are given in Lemma 2.2, it is suffice to show that

$$\int_{D_i} \frac{\mu_i}{(\lambda_i\rho_i - \mu_i^2)^{3/2}} drdsdr'ds' < +\infty, \ \ i = 1, 2, 3.$$

For $i = 1$, by (2.4) and (2.5), we obtain

$$\begin{aligned}
&\int_{D_1} \frac{\mu_1}{(\lambda_1\rho_1 - \mu_1^2)^{3/2}} drdsdr'ds' \\
&\leq m\int_{[0,t]^3} \frac{a^{2HK} + b^{2HK} + c^{2HK}}{[(a+b)^{2HK}c^{2HK} + (b+c)^{2HK}a^{2HK}]^{3/2}} dadbdc \\
&\leq m\int_{[0,t]^3} \frac{a^{2HK} + b^{2HK} + c^{2HK}}{(a+b)^{\frac{3HK}{2}}(b+c)^{\frac{3HK}{2}}(ac)^{\frac{3HK}{2}}} dadbdc \\
&\leq m\int_{[0,t]^3} \frac{a^{2HK}}{a^{\frac{3HK}{2}}b^{\frac{3HK}{2}}(ac)^{\frac{3HK}{2}}} dadbdc + m\int_{[0,t]^3} \frac{b^{2HK}}{b^{\frac{3HK}{2}}b^{\frac{3HK}{2}}(ac)^{\frac{3HK}{2}}} dadbdc \\
&\quad + m\int_{[0,t]^3} \frac{c^{2HK}}{b^{\frac{3HK}{2}}c^{\frac{3HK}{2}}(ac)^{\frac{3HK}{2}}} dadbdc
\end{aligned}$$

$$= m\int_{[0,t]^3} \frac{1}{a^{HK} b^{\frac{3HK}{2}} c^{\frac{3HK}{2}}} dadbdc + m\int_{[0,t]^3} \frac{1}{b^{HK} a^{\frac{3HK}{2}} c^{\frac{3HK}{2}}} dadbdc$$
$$+m\int_{[0,t]^3} \frac{1}{c^{HK} a^{\frac{3HK}{2}} b^{\frac{3HK}{2}}} dadbdc$$
$$< +\infty,$$

since $0 < 2HK < 1$.

For $i = 2$, by (2.7) and (2.8), we obtain

$$\int_{D_2} \frac{\mu_2}{(\lambda_2\rho_2 - \mu_2^2)^{3/2}} drdsdr^{'}ds^{'}$$
$$\leq m\int_{[0,t]^3} \frac{b^{2HK}}{\left[b^{2HK}(a^{2HK} + c^{2HK})\right]^{3/2}} dadbdc$$
$$\leq m\int_{[0,t]^3} \frac{1}{b^{HK} a^{\frac{3HK}{2}} c^{\frac{3HK}{2}}} dadbdc$$
$$< +\infty,$$

since $0 < 2HK < 1$.

For $i = 3$, by (2.10) and (2.11), we obtain

$$\int_{D_3} \frac{\mu_3}{(\lambda_3\rho_3 - \mu_3^2)^{3/2}} drdsdr^{'}ds^{'}$$
$$\leq m\int_{[0,t]^3} \frac{b^{2\alpha(HK-1)}(ac)^{\beta(HK-1)+1}}{\left[(a^{2HK}c^{2HK})\right]^{3/2}} dadbdc$$
$$= m\int_{[0,t]^3} \frac{1}{b^{2\alpha(1-HK)}(ac)^{\beta(1-HK)+3HK-1}} dadbdc.$$

Since $0 < 2HK < 1$, then $2(1-HK) = 2-2HK > 1$. We first choose $\alpha > 0$, such that $2\alpha(1-HK) < 1$, and we have

$$\beta(1 - HK) + 3HK - 1 = (1 - \alpha)(1 - HK) + 3HK - 1$$
$$= 2HK - \alpha(1 - HK)$$
$$< 1,$$

which imply

$$\int_{D_3} \frac{\mu_3}{(\lambda_3\rho_3 - \mu_3^2)^{3/2}} drdsdr^{'}ds^{'} < +\infty.$$

The proof of Lemma 2.3 is now complete. □

3. Proofs of theorems

In this section, we will prove Theorems 1.1–1.3.

3.1. Proof of Theorem 1.1.

Proof. We prove the theorem in two steps.
Step 1. Show that for each $\epsilon > 0, \alpha_{t,\epsilon}(y) \in L^2(\Omega, P)$. In fact, by (1.8) and (1.10), we have

$$\begin{aligned}\alpha_{t,\epsilon}(y) &= \int_0^t \int_0^s p_\epsilon(S_s^{H,K} - S_r^{H,K} - y)drds \\ &= \frac{1}{2\pi}\int_0^t \int_0^s \int_{\mathbf{R}} e^{i\xi(S_s^{H,K} - S_r^{H,K} - y)} e^{-\frac{\epsilon\xi^2}{2}} d\xi drds.\end{aligned}$$

Let $D^2 := \{(r, s, r', s')|0 < r < s < t, 0 < r' < s' < t\}$. Then,

$$\begin{aligned}\mathbf{E}\left(\left|\alpha_{t,\epsilon}(y)\right|^2\right) &= \mathbf{E}\left[\frac{1}{4\pi^2}\int_{D^2}\int_{\mathbf{R}^2} \exp\left(i\xi(S_s^{H,K} - S_r^{H,K} - y) + i\eta(S_{s'}^{H,K} - S_{r'}^{H,K} - y)\right)\right. \\ &\qquad \left.\cdot \exp\left(-\frac{\epsilon(\xi^2+\eta^2)}{2}\right) d\xi d\eta drdsdr'ds'\right] \\ &\le \frac{1}{4\pi^2}\int_{D^2}\int_{\mathbf{R}^2} \mathbf{E}\left[\exp\left(i\xi(S_s^{H,K} - S_r^{H,K} - y) + i\eta(S_{s'}^{H,K} - S_{r'}^{H,K} - y)\right)\right] d\xi d\eta drdsdr'ds' \\ &= \frac{1}{4\pi^2}\int_{D^2}\int_{\mathbf{R}^2} \exp\left[-\frac{1}{2}\mathrm{Var}\left(\xi(S_s^{H,K} - S_r^{H,K} - y) + \eta(S_{s'}^{H,K} - S_{r'}^{H,K} - y)\right)\right] d\xi d\eta drdsdr'ds' \\ &= \frac{1}{4\pi^2}\int_{D^2}\int_{\mathbf{R}^2} \exp\left[-\frac{1}{2}\mathrm{Var}\left(\xi(S_s^{H,K} - S_r^{H,K}) + \eta(S_{s'}^{H,K} - S_{r'}^{H,K})\right)\right] d\xi d\eta drdsdr'ds' \\ &= \frac{1}{2\pi^2}\left(\int_{D_1} + \int_{D_2} + \int_{D_3}\right)\int_{\mathbf{R}^2} \exp\left[-\frac{1}{2}\mathrm{Var}\left(\xi(S_s^{H,K} - S_r^{H,K}) + \eta(S_{s'}^{H,K} - S_{r'}^{H,K})\right)\right] d\xi d\eta drdsdr'ds' \\ &:= \frac{1}{2\pi^2}\left(A_1 + A_2 + A_3\right),\end{aligned}$$

where

$$\begin{aligned}D_1 &= \{(r, s, r', s')|0 < r < r' < s < s' < t\}, \\ D_2 &= \{(r, s, r', s')|0 < r < r' < s' < s < t\}, \\ D_3 &= \{(r, s, r', s')|0 < r < s < r' < s' < t\}.\end{aligned}$$

Denote $M = \mathrm{Var}\left(\xi(S_s^{H,K} - S_r^{H,K}) + \eta(S_{s'}^{H,K} - S_{r'}^{H,K})\right)$, by (2.2), we obtain
(1) if $(r, s, r', s') \in D_1$, then

$$\begin{aligned}M &= \mathrm{Var}\left(\xi(S_{r'}^{H,K} - S_r^{H,K}) + (\xi+\eta)(S_s^{H,K} - S_{r'}^{H,K}) + \eta(S_{s'}^{H,K} - S_s^{H,K})\right) \\ &\ge m\left[\xi^2(r'-r)^{2HK} + (\xi+\eta)^2(s-r')^{2HK} + \eta^2(s'-s)^{2HK}\right] \\ &\ge m\left[\xi^2(r'-r)^{2HK} + (\xi+\eta)^2(s-r')^{2HK}\right].\end{aligned}$$

Thus,

$$A_1 = \int_{D_1}\int_{\mathbf{R}^2} \exp\left(-\frac{M}{2}\right) d\xi d\eta drdsdr'ds'$$

$$\leq \int_{D_1}\int_{\mathbf{R}^2} \exp\left(-\frac{m\left[\xi^2(r^{'}-r)^{2HK}+(\xi+\eta)^2(s-r^{'})^{2HK}\right]}{2}\right)d\xi d\eta drdsdr^{'}ds^{'}$$

$$= \frac{2\pi}{m}\int_{D_1}(r^{'}-r)^{-HK}(s-r^{'})^{-HK}drdsdr^{'}ds^{'}$$

$$< +\infty,$$

since $0 < HK < 1$.

(2) if $(r, s, r^{'}, s^{'}) \in D_2$, then

$$M = \mathrm{Var}\left(\xi(S^{H,K}_{r^{'}} - S^{H,K}_{r}) + (\xi+\eta)(S^{H,K}_{s^{'}} - S^{H,K}_{r^{'}}) + \xi(S^{H,K}_{s} - S^{H,K}_{s^{'}})\right)$$

$$\geq m\left[\xi^2(r^{'}-r)^{2HK}+(\xi+\eta)^2(s^{'}-r^{'})^{2HK}+\xi^2(s-s^{'})^{2HK}\right]$$

$$\geq m\left[\xi^2(r^{'}-r)^{2HK}+(\xi+\eta)^2(s^{'}-r^{'})^{2HK}\right].$$

Thus,

$$A_2 = \int_{D_2}\int_{\mathbf{R}^2}\exp\left(-\frac{M}{2}\right)d\xi d\eta drdsdr^{'}ds^{'}$$

$$\leq \int_{D_2}\int_{\mathbf{R}^2} \exp\left(-\frac{m\left[\xi^2(r^{'}-r)^{2HK}+(\xi+\eta)^2(s^{'}-r^{'})^{2HK}\right]}{2}\right)d\xi d\eta drdsdr^{'}ds^{'}$$

$$= \frac{2\pi}{m}\int_{D_2}(r^{'}-r)^{-HK}(s^{'}-r^{'})^{-HK}drdsdr^{'}ds^{'}$$

$$< +\infty,$$

since $0 < HK < 1$.

(3) if $(r, s, r^{'}, s^{'}) \in D_3$, then

$$M = \mathrm{Var}\left(\xi(S^{H,K}_{s} - S^{H,K}_{r}) + \eta(S^{H,K}_{s^{'}} - S^{H,K}_{r^{'}})\right) \geq m\left[\xi^2(s-r)^{2HK}+\eta^2(s^{'}-r^{'})^{2HK}\right].$$

Thus,

$$A_3 = \int_{D_3}\int_{\mathbf{R}^2}\exp\left(-\frac{M}{2}\right)d\xi d\eta drdsdr^{'}ds^{'}$$

$$\leq \int_{D_3}\int_{\mathbf{R}^2} \exp\left(-\frac{m\left[\xi^2(s-r)^{2HK}+\eta^2(s^{'}-r^{'})^{2HK}\right]}{2}\right)d\xi d\eta drdsdr^{'}ds^{'}$$

$$= \frac{2\pi}{m}\int_{D_3}(s-r)^{-HK}(s^{'}-r^{'})^{-HK}drdsdr^{'}ds^{'}$$

$$< +\infty,$$

since $0 < HK < 1$. Hence, for any $H \in (0, 1)$ and $K \in (0, 1]$, we have

$$\mathbf{E}\left(\left|\alpha_{t,\epsilon}(y)\right|^2\right) < +\infty.$$

Step 2. Show that $\{\alpha_{t,\epsilon}(y), \epsilon > 0\}$ is a Cauchy sequence in $L^2(\Omega, P)$. Since the proof of Step 2 is similar to that of Theorem 3.1 in Jiang and Wang [8], we omitted the details.

Therefore we obtain $\lim_{\epsilon\to 0}\alpha_{t,\epsilon}(y)$ exists in L^2 and the theorem follows. □

3.2. *Proof of Theorem 1.2.*

Proof. By (1.9) and (1.11), we get

$$\begin{aligned}\mathbf{E}\left[\alpha^{'}_{t,\epsilon}(y)\right]^2 &= \mathbf{E}\left[-\frac{i}{2\pi}\int_0^t\int_0^s\int_{\mathbf{R}}\xi e^{i\xi(S^{H,K}_s-S^{H,K}_r-y)}e^{-\frac{\epsilon\xi^2}{2}}d\xi drds\right]^2\\
&= -\frac{1}{4\pi^2}\int_{D^2}\int_{\mathbf{R}^2}\xi\eta e^{-\frac{1}{2}\mathrm{Var}\left(\xi(S^{H,K}_s-S^{H,K}_r-y)+\eta(S^{H,K}_{s'}-S^{H,K}_{r'}-y)\right)}e^{-\frac{\epsilon(\xi^2+\eta^2)}{2}}d\xi d\eta drdsdr^{'}ds^{'}\\
&= -\frac{1}{4\pi^2}\int_{D^2}\int_{\mathbf{R}^2}\xi\eta e^{-\frac{1}{2}\mathrm{Var}\left(\xi(S^{H,K}_s-S^{H,K}_r)+\eta(S^{H,K}_{s'}-S^{H,K}_{r'})\right)}e^{-\frac{\epsilon(\xi^2+\eta^2)}{2}}d\xi d\eta drdsdr^{'}ds^{'}\\
&= -\frac{1}{4\pi^2}\int_{D^2}\int_{\mathbf{R}^2}\xi\eta e^{-\frac{\lambda+\epsilon}{2}\xi^2-\frac{\rho+\epsilon}{2}\eta^2-\xi\eta\mu}d\xi d\eta drdsdr^{'}ds^{'}\\
&= \frac{1}{2\pi}\int_{D^2}\frac{\mu}{[(\lambda+\epsilon)(\rho+\epsilon)-\mu^2]^{3/2}}drdsdr^{'}ds^{'},\end{aligned}$$

where λ, ρ and μ are given in Lemma 2.2,

By Lemma 2.3, as $0 < 2HK < 1$, we have

$$\lim_{\epsilon\to 0}\mathbf{E}\left[\alpha^{'}_{t,\epsilon}(y)\right]^2 < +\infty,$$

which implies $\alpha^{'}_{t,\epsilon}(y)\in L^2(\Omega, P)$ for each $\epsilon > 0$.

Similar to Step 2 of the proof of Theorem 1.1, we can prove that $\{\alpha^{'}_{t,\epsilon}(y), \epsilon > 0\}$ is a Cauchy sequence in $L^2(\Omega, P)$. Here we omitted the details. Therefore the proof of Theorem 1.2 is completed. □

3.3. *Proof of Theorem 1.3.*

Proof. (1) It is enough to prove that

$$\mathbf{E}\left[\left|\alpha^{'}_{t,\epsilon}(y_1)-\alpha^{'}_{t,\epsilon}(y_2)\right|^n\right]\le m|y_1-y_2|^{n\tau},$$

holds for every $t\in[0,T]$ and $n\ge 1$.

Since

$$\left|\alpha^{'}_{t,\epsilon}(y_1)-\alpha^{'}_{t,\epsilon}(y_2)\right| = \left|-\frac{i}{2\pi}\int_0^t\int_0^s\int_{\mathbf{R}}\xi e^{i\xi\left(S^{H,K}_s-S^{H,K}_r\right)}e^{-\frac{\epsilon\xi^2}{2}}\left(e^{-i\xi y_1}-e^{-i\xi y_2}\right)d\xi drds\right|, \tag{3.1}$$

and

$$\left|e^{-i\xi b}-e^{-i\xi a}\right|\le m|\xi|^\tau|b-a|^\tau,\quad \text{for any } \tau\in(0,1). \tag{3.2}$$

Hence

$$\begin{aligned}&\mathbf{E}\left[\left|\alpha^{'}_{t,\epsilon}(y_1)-\alpha^{'}_{t,\epsilon}(y_2)\right|^n\right]\\
&= \frac{1}{(2\pi)^n}\left|\int_{D^n}\int_{\mathbf{R}^n}\mathbf{E}\prod_{j=1}^n\xi_j e^{i\xi_j\left(S^{H,K}_{s_j}-S^{H,K}_{r_j}\right)}\left(e^{-i\xi_j y_1}-e^{-i\xi_j y_2}\right)e^{-\frac{\epsilon\xi_j^2}{2}}\prod_{j=1}^n d\xi_j dr_j ds_j\right|\\
&\le m|y_1-y_2|^{n\tau}\int_{D^n}\int_{\mathbf{R}^n}\left|\mathbf{E}\prod_{j=1}^n e^{i\xi_j\left(S^{H,K}_{s_j}-S^{H,K}_{r_j}\right)}\right|\prod_{j=1}^n|\xi_j|^{1+\tau}d\xi_j dr_j ds_j,\end{aligned} \tag{3.3}$$

for any $\tau \in (0,1)$.

Let us first deal with the product insider the expectation. Using the method of sample configuration as in Jung and Markowsky [10]. Fix such an ordering and let $l_1 \le l_2 \le \cdots \le l_{2n}$ be a relabeling of the set $\{r_1, s_1, r_2, s_2, \cdots, r_n, s_n\}$. We obtain

$$\prod_{j=1}^{n} e^{i\xi_j\left(S_{s_j}^{H,K}-S_{r_j}^{H,K}\right)} = \prod_{j=1}^{2n-1} e^{iu_j\left(S_{l_{j+1}}^{H,K}-S_{l_j}^{H,K}\right)}, \tag{3.4}$$

where the u_j's are properly chosen linearly combinations of the ξ_j's to make (3.4) an equality. By local non-deterministic property for $S_t^{H,K}$, we have

$$\left|\mathbf{E}\prod_{j=1}^{n} e^{i\xi_j\left(S_{s_j}^{H,K}-S_{r_j}^{H,K}\right)}\right| \le e^{-m\sum_{j=1}^{2n-1} u_j^2(l_{j+1}-l_j)^{2HK}}. \tag{3.5}$$

Fixing ξ_j, we choose j_1 to be the smallest value such that u_{j_1} contains ξ_j as a term and then choose j_2 to be the smallest value strictly larger than j_1 such that u_{j_2} does not contain ξ_j as a term. Thus,

$$\begin{aligned}|\xi_j|^{1+\tau} &= \left|u_{j_1} - u_{j_1-1}\right|^{\frac{1+\tau}{2}} \left|u_{j_2-1} - u_{j_2}\right|^{\frac{1+\tau}{2}} \\ &\le m\left(\left|u_{j_1}\right|^{\frac{1+\tau}{2}} + \left|u_{j_1-1}\right|^{\frac{1+\tau}{2}}\right)\left(\left|u_{j_2}\right|^{\frac{1+\tau}{2}} + \left|u_{j_2-1}\right|^{\frac{1+\tau}{2}}\right).\end{aligned} \tag{3.6}$$

Let $a_j = l_{j+1} - l_j$ (with $l_0 = 0$),by (3.3), (3.5) and (3.6), we deduce

$$\begin{aligned}\mathbf{E}\left[\left|\alpha'_{t,\epsilon}(y_1) - \alpha'_{t,\epsilon}(y_2)\right|^n\right] &\le m\int_{[0,t]^{2n}}\int_{\mathbf{R}^n} e^{-m\sum_{j=1}^{2n-1} u_j^2 a_j^{2HK}} \prod_{j=1}^{2n}\left(\left|u_j\right|^{\frac{1+\tau}{2}} + \left|u_{j-1}\right|^{\frac{1+\tau}{2}}\right) d\xi dl \\ &\le m\int_{\mathbf{R}^n} \frac{\prod_{j=1}^{2n}\left(\left|u_j\right|^{\frac{1+\tau}{2}} + \left|u_{j-1}\right|^{\frac{1+\tau}{2}}\right)}{\prod_{j=1}^{2n-1}\left(1 + |u_j|^{\frac{1}{HK}}\right)} d\xi,\end{aligned} \tag{3.7}$$

where $d\xi = d\xi_1 d\xi_2\cdots d\xi_n$, $dl = dl_1 dl_2\cdots dl_{2n}$, $u_0 = u_{2n} = 0$, and we use the inequality

$$\int_0^t e^{-m|u|^2 a^{2HK}} da \le \frac{m}{1 + |u|^{\frac{1}{HK}}}. \tag{3.8}$$

Expanding the product in the numerator of (3.7) gives us the sum of a number of terms of the form

$$\prod_{j=1}^{2n-1} \left|u_j\right|^{\frac{(1+\tau)m_j}{2}},$$

where $m_j = 0, 1$ or 2 (terms containing $|u_0|$ or $|u_{2n}|$ are equal to 0). Thus, it is suffice to show that

$$\int_{\mathbf{R}^n} \frac{1}{\prod_{j=1}^{2n-1}\left(1 + |u_j|^{\frac{1}{HK} - \frac{(1+\tau)m_j}{2}}\right)} d\xi_1 d\xi_2\cdots d\xi_n \tag{3.9}$$

is finite. We make a linear transformation changing this into an integral with respect to variables $u_{k_1}, u_{k_2}, \cdots, u_{k_n} \in \{u_1, \cdots, u_{2n-1}\}$ which span $\{\xi_1, \xi_2, \cdots, \xi_n\}$ in order to bound (3.9) by

$$m \int_{\mathbf{R}^n} \frac{1}{\prod_{j=1}^{2n-1} \left(1 + |u_{k_j}|^{\frac{1}{HK} - \frac{(1+\tau)m_j}{2}}\right)} du_{k_1} du_{k_2} \cdots du_{k_n}$$

$$\leq m \int_{\mathbf{R}^n} \frac{1}{\prod_{j=1}^{2n-1} \left(1 + |u_{k_j}|^{\frac{1}{HK} - (1+\tau)}\right)} du_{k_1} du_{k_2} \cdots du_{k_n}.$$

This is finite if we choose τ such that $\frac{1}{HK} - (1+\tau) > 1$, namely $\tau < \frac{1}{HK} - 2$.

(2) It is enough to prove that

$$\mathbf{E}\left[\left|\alpha'_{t,\epsilon}(y) - \alpha'_{\tilde{t},\epsilon}(y)\right|^n\right] \leq m|t - \tilde{t}|^{n\gamma},$$

holds for $t, \tilde{t} \in [0, T]$, every $y \in \mathbf{R}$ and $n \geq 1$.

Since

$$\left|\alpha'_{t,\epsilon}(y) - \alpha'_{\tilde{t},\epsilon}(y)\right| = \frac{1}{2\pi} \left| \int_t^{\tilde{t}} \int_0^s \int_{\mathbf{R}} \xi e^{i\xi(S_s^{H,K} - S_r^{H,K} - y)} e^{-\frac{\epsilon \xi^2}{2}} d\xi dr ds \right|.$$

Hence,

$$\begin{aligned}
&\mathbf{E}\left[\left|\alpha'_{t,\epsilon}(y) - \alpha'_{\tilde{t},\epsilon}(y)\right|^n\right] \\
&\leq m \int_{[t,\tilde{t}]^n} \int_{[0,s_1]\times\cdots\times[0,s_n]} \int_{\mathbf{R}^n} \prod_{j=1}^n |\xi_j| \mathbf{E}\left[\prod_{j=1}^n e^{i\xi_j (S_{s_j}^{H,K} - S_{r_j}^{H,K})}\right] \prod_{j=1}^n d\xi_j dr_j ds_j \\
&\leq m \int_{\tilde{D}^n} \prod_{j=1}^n I_{[t,\tilde{t}]}(s_j) \int_{\mathbf{R}^n} \prod_{j=1}^n |\xi_j| \mathbf{E}\left[\prod_{j=1}^n e^{i\xi_j (S_{s_j}^{H,K} - S_{r_j}^{H,K})}\right] \prod_{j=1}^n d\xi_j dr_j ds_j \\
&\leq m|t - \tilde{t}|^{n\gamma} \left\{ \int_{\tilde{D}^n} \left[\int_{\mathbf{R}^n} \prod_{j=1}^n |\xi_j| \mathbf{E}\left[\prod_{j=1}^n e^{i\xi_j (S_{s_j}^{H,K} - S_{r_j}^{H,K})}\right] \prod_{j=1}^n d\xi_j \right]^{\frac{1}{1-\gamma}} \prod_{j=1}^n dr_j ds_j \right\}^{1-\gamma} \\
&=: m|t - \tilde{t}|^{n\gamma} \Lambda,
\end{aligned}$$

where $\tilde{D} = \{(r, s) : 0 < r < s < \tilde{t}\}$ and we use the Hölder's inequality in the last inequality with $\gamma < 1 - 2HK$.

Using the similar method as in the proof of (1) in Theorem 1.3, Λ is bounded by

$$\left[\int_{E^n} \prod_{j=1}^n (l_{j+1} - l_j)^{-\frac{HK}{1-\gamma}} \prod_{j=1}^{2n-1} (l_{j+1} - l_j)^{-\frac{HKm_j}{2(1-\gamma)}} dl_1 dl_2 \cdots dl_{2n} \right]^{1-\gamma},$$

where $E^n = \{0 < l_1 < \cdots < l_{2n} < t\}$.

Since $1 - \gamma > 2HK$, there exists a constant $m > 0$, such that

$$\mathbf{E}\left[\left|\alpha'_{t,\epsilon}(y) - \alpha'_{\tilde{t},\epsilon}(y)\right|^n\right] \leq m|t - \tilde{t}|^{n\gamma}.$$

Thus, we finished the proof of Theorem 1.3. □

4. Conclusions

We mainly study the existence of the self-intersection local time and its derivative for the sub-bifractional Brownian motion. Moreover, we prove its derivative is Hölder continuous in space variable and time variable, respectively. In the future, we will consider its higher order derivative.

Acknowledgments

Research supported by the Natural Science Foundation of Hunan Province under Grant 2021JJ30233.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. S. Berman, Local nondeterminism and local times of Gaussian processes, *Bull. Amer. Math. Soc.*, **79** (1973), 475–477.
2. Z. Chen, L. Sang, X. Hao, Renormalized self-intersection local time of bifractional Brownian motion, *J. Inequal. Appl.*, **2018** (2018), 326. http://dx.doi.org/10.1186/s13660-018-1916-3
3. C. El-Nouty, J. Journé, The sub-bifractional Brownian motion, *Stud. Sci. Math. Hung.*, **50** (2013), 67–121. http://dx.doi.org/10.1556/SScMath.50.2013.1.1231
4. Y. Hu, Self-intersection local time of fractional Brownian motions-via chaos expansion, *J. Math. Kyoto Univ.*, **41** (2001), 233–250. http://dx.doi.org/10.1215/kjm/1250517630
5. Y. Hu, D. Nualart, Renormalized self-intersection local time for fractional Brownian motion, *Ann. Probab.*, **33** (2005), 948–983. http://dx.doi.org/10.2307/3481716
6. A. Jaramillo, D. Nualart, Asymptotic properties of the derivative of self-intersection local time of fractional Brownian motion, *Stoch. Proc. Appl.*, **127** (2017), 669–700. http://dx.doi.org/10.1016/j.spa.2016.06.023
7. A. Jaramillo, D. Nualart, Functional limit theorem for the self-intersection local time of the fractional Brownian motion, *Ann. Inst. H. Poincaré Probab. Statist.*, **55** (2019), 480–527. http://dx.doi.org/10.1214/18-AIHP889
8. Y. Jiang, Y. Wang, Self-intersection local times and collision local times of bifractional Brownian motions, *Sci. China Ser. A-Math.*, **52** (2009), 1905–1919. http://dx.doi.org/10.1007/s11425-009-0081-z
9. P. Jung, G. Markowsky, On the Tanaka formula for the derivative of self-intersection local time of fractional Brownian motion, *Stoch. Proc. Appl.*, **124** (2014), 3846–3868. http://dx.doi.org/10.1016/j.spa.2014.07.001

10. P. Jung, G. Markowsky, Hölder continuity and occupation-time formulas for fBm self-intersection local time and its derivative, *J. Theor. Probab.*, **28** (2015), 299–312. http://dx.doi.org/10.1007/s10959-012-0474-8

11. N. Kuang, On the collision local time of sub-bifractional Brownian motions (Chinese), *Advances in Mathematics (China)*, **48** (2019), 627–640. http://dx.doi.org/10.11845/sxjz.2018023b

12. N. Kuang, Y. Li, Berry-Esséen bounds and almost sure CLT for the quadratic variation of the sub-bifractional Brownian motion, *Commun. Stat.-Simul. C.*, in press. http://dx.doi.org/10.1080/03610918.2020.1740265

13. N. Kuang, B. Liu, Parameter estimations for the sub-fractional Brownian motion with drift at discrete observation, *Braz. J. Probab. Stat.*, **29** (2015), 778–789. http://dx.doi.org/10.1214/14-BJPS246

14. N. Kuang, B. Liu, Least squares estimator for α-sub-fractional bridges, *Stat. Papers*, **59** (2018), 893–912. http://dx.doi.org/10.1007/s00362-016-0795-2

15. N. Kuang, H. Xie, Maximum likelihood estimator for the sub-fractional Brownian motion approximated by a random walk, *Ann. Inst. Stat. Math.*, **67** (2015), 75–91. http://dx.doi.org/10.1007/s10463-013-0439-4

16. N. Kuang, H. Xie, Asymptotic behavior of weighted cubic variation of sub-fractional brownian motion, *Commun. Stat.-Simul. C.*, **46** (2017), 215–229. http://dx.doi.org/10.1080/03610918.2014.957849

17. J. Rosen, The intersection local time of fractional Brownian motion in the plane, *J. Multivariate Anal.*, **23** (1987), 37–46. http://dx.doi.org/10.1016/0047-259x(87)90176-x

18. J. Rosen, Derivatives of self-intersection local time, In: *Lecture notes in mathematics*, Berlin: Springer, 2005, 263–281. http://dx.doi.org/10.1007/978-3-540-31449-3_18

19. Q. Shi, Fractional smoothness of derivative of self-intersection local times with respect to bi-fractional Brownian motion, *Syst. Control Lett.*, **138** (2020), 104627. http://dx.doi.org/10.1016/j.sysconle.2020.104627

20. Y. Xiao, Strong local nondeterminism and the sample path properties of Gaussian random fields. In: *Asymptotic theory in probability and statistics with applications*, Beijing: Higher Education Press, 2007, 136–176.

21. H. Xie, N. Kuang, Least squares type estimations for discretely observed nonergodic Gaussian Ornstein-Uhlenbeck processes of the second kind, *AIMS Mathematics*, **7** (2022), 1095–1114. http://dx.doi.org/10.3934/math.2022065

22. L. Yan, X. Yang, Y. Lu, p-variation of an integral functional driven by fractional Brownian motion, *Stat. Probabil. Lett.*, **78** (2008), 1148–1157. http://dx.doi.org/10.1016/j.spl.2007.11.008

23. L. Yan, X. Yu, Derivative for self-intersection local time of multidimensional fractional Brownian motion, *Stochastics*, **87** (2015), 966–999. http://dx.doi.org/10.1080/17442508.2015.1019883

24. Q. Yu, Higher order derivative of self-intersection local time for fractional Brownian motion, *J. Theor. Probab.*, **34** (2021), 1749–1774. http://dx.doi.org/10.1007/s10959-021-01093-6

Appendix

In this section we prove Lemma 2.2.

Proof. By (1.4), we obtain (2.3), (2.6) and (2.9) easily. Since the proofs of (2.4), (2.7) and (2.10) are similar to those of Hu [4]and Hu and Nualart [5], we omitted the details. Thus we only need to prove (2.5), (2.8) and (2.11).

In order to prove (2.5), let $e = r$, by (1.3), we have

$$\mu = \left[(s^{2H} + (s^{'})^{2H})^K - (s^{2H} + (r^{'})^{2H})^K - (r^{2H} + (s^{'})^{2H})^K + (r^{2H} + (r^{'})^{2H})^K\right]$$
$$+\frac{1}{2}\left[(s + r^{'})^{2HK} - (s + s^{'})^{2HK} + (r + s^{'})^{2HK} - (r + r^{'})^{2HK}\right]$$
$$+\frac{1}{2}\left[|s - r^{'}|^{2HK} - |s - s^{'}|^{2HK} + |r - s^{'}|^{2HK} - |r - r^{'}|^{2HK}\right]. \tag{A.1}$$

Hence,

$$\mu_1 = \Big\{\left[(e + a + b)^{2H} + (e + a + b + c)^{2H}\right]^K - \left[(e + a + b)^{2H} + (e + a)^{2H}\right]^K$$
$$-\left[e^{2H} + (e + a + b + c)^{2H}\right]^K + \left[e^{2H} + (e + a)^{2H}\right]^K\Big\}$$
$$+\frac{1}{2}\left[(2e + 2a + b)^{2HK} - (2e + 2a + 2b + c)^{2HK} + (2e + a + b + c)^{2HK} - (2e + a)^{2HK}\right]$$
$$+\frac{1}{2}\left[b^{2HK} - c^{2HK} + (a + b + c)^{2HK} - a^{2HK}\right]$$
$$:= \Delta_{1,1} + \Delta_{1,2} + \Delta_{1,3}.$$

For $\Delta_{1,1}$, we obtain

$$\Delta_{1,1} = \int_e^{e+a+b} d\Big\{\left[x^{2H} + (e + a + b + c)^{2H}\right]^K - \left[x^{2H} + (e + a)^{2H}\right]^K\Big\}$$
$$= 2HK\int_e^{e+a+b} x^{2H-1}\Big\{\left[x^{2H} + (e + a + b + c)^{2H}\right]^{K-1} - \left[x^{2H} + (e + a)^{2H}\right]^{K-1}\Big\}dx$$
$$\leq 0, \tag{A.2}$$

since $0 < K \leq 1$.

For $\Delta_{1,2}$, we get

$$\Delta_{1,2} = \frac{1}{2}\left[(2e + 2a + b)^{2HK} - (2e + 2a + 2b + c)^{2HK} + (2e + a + b + c)^{2HK} - (2e + a)^{2HK}\right]$$
$$\leq \frac{1}{2}\left[(2e + a + b + c)^{2HK} - (2e + a)^{2HK}\right]$$
$$\leq \frac{1}{2}(b + c)^{2HK}$$

$$
\begin{aligned}
&= \frac{1}{2}(b^2 + 2bc + c^2)^{HK} \\
&\le \frac{1}{2}[2(b^2 + c^2)]^{HK} \\
&\le 2^{HK-1}(b^{2HK} + c^{2HK}), \qquad (A.3)
\end{aligned}
$$

since $0 < 2HK < 1$, where we use the inequality $x^\alpha - y^\alpha \le |x - y|^\alpha$ for any $x > 0, y > 0, 0 < \alpha < 1$.

For $\Delta_{1,3}$, we deduce,

$$
\begin{aligned}
\Delta_{1,3} &= \frac{1}{2}\left[(a + b + c)^{2HK} + b^{2HK} - a^{2HK} - c^{2HK}\right] \\
&= \frac{1}{2}\left[(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)^{HK} + b^{2HK} - a^{2HK} - c^{2HK}\right] \\
&\le \frac{1}{2}\left[(3a^2 + 3b^2 + 3c^2)^{HK} + b^{2HK} - a^{2HK} - c^{2HK}\right] \\
&\le \frac{3^{HK} - 1}{2}\left(a^{2HK} + c^{2HK}\right) + \frac{3^{HK} + 1}{2} b^{2HK} \\
&\le \frac{3^{HK} + 1}{2}\left(a^{2HK} + b^{2HK} + c^{2HK}\right). \qquad (A.4)
\end{aligned}
$$

Therefore (2.5) holds from (A.2)–(A.4).

In order to prove (2.8), let $e = r$, by (1.3) and $(A.1)$, we have

$$
\begin{aligned}
\mu_2 = &\Big\{\left[(e + a + b + c)^{2H} + (e + a + b)^{2H}\right]^K - \left[(e + a + b + c)^{2H} + (e + a)^{2H}\right]^K \\
&- \left[e^{2H} + (e + a + b)^{2H}\right]^K + \left[e^{2H} + (e + a)^{2H}\right]^K\Big\} \\
&+ \frac{1}{2}\left[(2e + 2a + b + c)^{2HK} - (2e + 2a + 2b + c)^{2HK} + (2e + a + b)^{2HK} - (2e + a)^{2HK}\right] \\
&+ \frac{1}{2}\left[(b + c)^{2HK} - c^{2HK} + (a + b)^{2HK} - a^{2HK}\right] \\
:= &\ \Delta_{2,1} + \Delta_{2,2} + \Delta_{2,3}.
\end{aligned}
$$

For $\Delta_{2,1}$, we obtain

$$
\begin{aligned}
\Delta_{2,1} &= \int_e^{e+a+b+c} d\left\{\left[x^{2H} + (e + a + b)^{2H}\right]^K - \left[x^{2H} + (e + a)^{2H}\right]^K\right\} \\
&= 2HK \int_e^{e+a+b+c} x^{2H-1}\left\{\left[x^{2H} + (e + a + b)^{2H}\right]^{K-1} - \left[x^{2H} + (e + a)^{2H}\right]^{K-1}\right\} dx \\
&\le 0, \qquad (A.5)
\end{aligned}
$$

since $0 < K \le 1$.

For $\Delta_{2,2}$, we get

$$
\Delta_{2,2} = \frac{1}{2}\left[(2e + 2a + b + c)^{2HK} - (2e + 2a + 2b + c)^{2HK} + (2e + a + b)^{2HK} - (2e + a)^{2HK}\right]
$$

$$\leq \frac{1}{2}\left[(2e+a+b)^{2HK}-(2e+a)^{2HK}\right]$$
$$\leq \frac{1}{2}b^{2HK}, \qquad (A.6)$$

since $0 < 2HK < 1$.

For $\Delta_{2,3}$, we deduce,

$$\Delta_{2,3} = \frac{1}{2}\left[(a+b)^{2HK}-a^{2HK}+(b+c)^{2HK}-c^{2HK}\right]$$
$$\leq \frac{1}{2}\left(b^{2HK}+b^{2HK}\right)$$
$$= b^{2HK}, \qquad (A.7)$$

since $0 < 2HK < 1$. Therefore (2.8) holds from (A.5)–(A.7).

In order to prove (2.11), let $e = r$, by (1.3) and $(A.1)$, we have

$$\mu_3 = \Big\{\left[(e+a)^{2H}+(e+a+b+c)^{2H}\right]^K-\left[(e+a)^{2H}+(e+a+b)^{2H}\right]^K$$
$$-\left[e^{2H}+(e+a+b+c)^{2H}\right]^K+\left[e^{2H}+(e+a+b)^{2H}\right]^K\Big\}$$
$$+\frac{1}{2}\left[(2e+2a+b)^{2HK}-(2e+2a+b+c)^{2HK}+(2e+a+b+c)^{2HK}-(2e+a+b)^{2HK}\right]$$
$$+\frac{1}{2}\left[b^{2HK}-(b+c)^{2HK}+(a+b+c)^{2HK}-(a+b)^{2HK}\right]$$
$$:= \Delta_{3,1}+\Delta_{3,2}+\Delta_{3,3}.$$

For $\Delta_{3,1}$, we obtain

$$\Delta_{3,1} = \int_e^{e+a} d\left\{\left[x^{2H}+(e+a+b+c)^{2H}\right]^K-\left[x^{2H}+(e+a+b)^{2H}\right]^K\right\}$$
$$= 2HK\int_e^{e+a} x^{2H-1}\left\{\left[x^{2H}+(e+a+b+c)^{2H}\right]^{K-1}-\left[x^{2H}+(e+a+b)^{2H}\right]^{K-1}\right\}dx$$
$$\leq 0, \qquad (A.8)$$

since $0 < K \leq 1$.

For $\Delta_{3,2}$, and for $\alpha, \beta > 0$ with $\alpha+\beta = 1$, we get

$$\Delta_{3,2} = -\frac{1}{2}\int_1^2 d\left[(2e+b+c+ua)^{2HK}-(2e+b+ua)^{2HK}\right]$$
$$= -HKa\int_1^2\left[(2e+b+c+ua)^{2HK-1}-(2e+b+ua)^{2HK-1}\right]du$$
$$= -HKa\int_1^2\left[\int_0^1 d(2e+b+ua+vc)^{2HK-1}\right]du$$

$$
\begin{aligned}
&= -HK(2HK-1)ac\int_1^2\int_0^1 (2e+b+ua+vc)^{2HK-2}dvdu\\
&= HK(1-2HK)ac\int_1^2\int_0^1 (2e+b+ua+vc)^{2HK-2}dvdu\\
&\le HK(1-2HK)ac\int_1^2\int_0^1 (b+ua+vc)^{2HK-2}dvdu\\
&\le HK(1-2HK)ac\int_1^2\int_0^1 \left[b^{\alpha}(ua+vc)^{\beta}\right]^{2HK-2}dvdu\\
&\le mac\int_1^2\int_0^1 \left[b^{\alpha}(ua)^{\frac{\beta}{2}}(vc)^{\frac{\beta}{2}}\right]^{2HK-2}dvdu\\
&\le mb^{2\alpha(HK-1)}(ac)^{\beta(HK-1)+1},
\end{aligned}
\tag{A.9}
$$

since $0 < 2HK < 1$.

For $\Delta_{3,3}$, we deduce,

$$
\begin{aligned}
\Delta_{3,3} &= \frac{1}{2}\left[(a+b+c)^{2HK}-(b+c)^{2HK}-(a+b)^{2HK}+b^{2HK}\right]\\
&= \frac{1}{2}\int_0^1 d\left[(b+a+vc)^{2HK}-(b+vc)^{2HK}\right]\\
&= HKc\int_0^1 \left[(b+a+vc)^{2HK-1}-(b+vc)^{2HK-1}\right]dv\\
&= HKc\int_0^1 \left[\int_0^1 d(b+ua+vc)^{2HK-1}\right]dv\\
&= HK(2HK-1)ac\int_0^1\int_0^1 (b+ua+vc)^{2HK-2}dudv\\
&\le 0,
\end{aligned}
\tag{A.10}
$$

since $0 < 2HK < 1$. Therefore (2.11) holds from (A.8)–(A.10). Thus we have finished the proof of Lemma 2.2. □