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Research article

Notes on Hong's conjecture on nonsingularity of power LCM matrices

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Abstract: Let a, n be positive integers and $S = \{x_1, ..., x_n\}$ be a set of n distinct positive integers. The set S is said to be gcd (resp. lcm) closed if $\gcd(x_i, x_j) \in S$ (resp. $[x_i, x_j] \in S$) for all integers i, j with $1 \le i, j \le n$. We denote by (S^a) (resp. $[S^a]$) the $n \times n$ matrix having the ath power of the greatest common divisor (resp. the least common multiple) of x_i and x_j as its (i, j)-entry. In this paper, we mainly show that for any positive integer a with $a \ge 2$, the power LCM matrix $[S^a]$ defined on a certain class of gcd-closed (resp. lcm-closed) sets S is nonsingular. This provides evidences to a conjecture raised by Shaofang Hong in 2002.

Keywords: power LCM matrix; power GCD matrix; nonsingularity; gcd-closed set
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1. Introduction

Throughout, a, n stand for positive integers. Let $S = \{x_1, ..., x_n\}$ be a set of n distinct positive integers, where $x_1 < ... < x_n$. We denote by (x, y) (resp. $[x, y]$) the *greatest common divisor* (resp. *least common multiple*) of integers x and y. Let $(f((x_i, x_j)))$ (abbreviated by $(f(S))$) be the $n \times n$ matrix whose (i, j)-entry is that the arithmetic function f evaluates at (x_i, x_j). Let $(f([x_i, x_j]))$ (abbreviated by $(f[S])$) be the $n \times n$ matrix whose (i, j)-entry is that f evaluates at $[x_i, x_j]$. Let ξ_a be the arithmetic function defined by $\xi_a(x) = x^a$ for any positive integer x. The $n \times n$ matrix $(\xi_a((x_i, x_j)))$ (abbreviated by (S^a)) and $(\xi_a([x_i, x_j]))$ (abbreviated by $[S^a]$) are called *power GCD matrix* on S and *power LCM matrix* on S, respectively. When $a = 1$, they are simply called GCD matrix and LCM matrix. In 1875, Smith [15] showed that

$$\det(f((i, j))) = \prod_{k=1}^{n} (f * \mu)(k), \tag{1.1}$$

where μ is the Möbius function and $f * \mu$ is the Dirichlet convolution of the arithmetic function f and μ. Since then, lots of generalizations of Smith's determinant (1.1) and related results have been published (see, for example, [1–14,16–18]).

The study of power GCD matrices was initiated by Bourque and Ligh [2]. They showed that every power GCD matrix is positive definite. A set S is called *factor closed (FC)* if the conditions $x \in S$ and $d|x$ imply that $d \in S$. We say that the set S is *gcd (resp. lcm) closed* if $(x_i, x_j) \in S$ (resp. $[x_i, x_j] \in S$) for all integers i, j with $1 \leq i, j \leq n$. Evidently, any FC set is gcd closed but not conversely. We know [15] that (S) and $[S]$ are nonsingular when S is FC. In [1], Bourque and Ligh obtained a formula for the inverse of these matrices on FC sets. Furthermore, they conjectured that $[S]$ is nonsingular if S is gcd closed. In [12], Haukkanen, Wang and Sillanpää presented a counterexample to disprove the Bourque-Ligh conjecture [1].

Definition 1.1. *[4, 10] Let r be an integer with $1 \leq r \leq n-1$. The set S is 0-fold gcd closed if S is gcd closed. The set S is r-fold gcd closed if there is a divisor chain $R \subset S$ with $|R| = r$ such that* $\max(R) | \min(S \setminus R)$ *and the set $S \setminus R$ is gcd closed.*

Hong [4] proved the Bourque-Ligh conjecture if $n \leq 5$, and also if $n \geq 6$ and S is $(n-5)$-fold. He [6] proved this conjecture for $n \leq 7$ and disproved for $n \geq 8$. Further, he proved it under certain assumptions [5], and proposed [7] the following conjecture.

Conjecture 1.1. *[7] There is a positive integer $k(a)$ depending only on a, such that $[S^a]$ is nonsingular if $n \leq k(a)$ and S is gcd closed. But for any integer $n \geq k(a)+1$, there is a gcd-closed set S such that $[S^a]$ is singular.*

Hong [7] noted that $k(a) \geq 7$ for any $a \geq 2$. He also [8] showed that $[S^a]$ is nonsingular if S is gcd closed and all its elements have at most two distinct prime factors. We denote by $\mathrm{lcm}(A)$ (resp. $\gcd(A)$) the least common multiple (resp. greatest common divisor) of all elements of A, where A is a set of finite distinct positive integers.

Definition 1.2. *[10] Let $m = \mathrm{lcm}(S)$. The reciprocal set of S is*

$$mS^{-1} = \left\{ \frac{m}{x_1}, ..., \frac{m}{x_n} \right\}.$$

Definition 1.3. *[14] Let r be an integer with $1 \leq r \leq n-1$. The set S is r-fold lcm closed if mS^{-1} is r-fold gcd closed.*

Definition 1.4. *[10, 14] The $n \times n$ matrix $(\frac{1}{S^a})$ having the reciprocal ath power of (x_i, x_j) as its (i, j)-entry is called reciprocal power GCD matrix defined on S. When $a = 1$, it is simply called reciprocal GCD matrix on S, and denoted by $(\frac{1}{S})$.*

Hong [10] proved that if $n \leq 7$ and S is gcd (resp. lcm) closed, then $[S]$ and $(\frac{1}{S})$ are nonsingular, which also holds if $n \geq 8$ and S is $(n-7)$-fold gcd (resp. lcm) closed. Hong, Shum and Sun [11] proved that $[S^a]$ is nonsingular if every element of a gcd-closed set S is of the form pqr, or p^2qr, or p^3qr except for the case $a = 1$ and $270, 520 \in S$, where p, q, r are distinct primes. They also showed that $[S^a]$ is nonsingular if S is a gcd-closed set satisfying $x_i < 180$ for all integers i with $1 \leq i \leq n$. Cao [3] developed Hong's method and proved that $k(a) \geq 8$ for all $a \geq 2$. Li [14] showed that if $n \leq 7$ and S is gcd (resp. lcm) closed, then $\left([x_i, x_j]^e\right)$ and $(\frac{1}{S^e})$ are nonsingular for any real number e with $e \geq 1$, and also if $n \geq 8$ and S is $(n-7)$-fold gcd (resp. lcm) closed. Wan, Hu and Tan [16] extended Hong's results [8] and [11] by showing that $[S^a]$ is nonsingular when S is gcd closed such that every element of S contains at most two distinct prime factors or is of the form $p^l qr$ with p, q, r being distinct primes and the positive integer l satisfying $1 \leq l \leq 4$ except for the case $a = 1$ and $270, 520, 810, 1040 \in S$. Let $x, y \in S$ with $x < y$. If $x|y$ and the conditions $x|d|y$ and $d \in S$ imply that $d \in \{x, y\}$, then we say that x is a *greatest-type divisor* of y in S. For $x \in S$, $G_S(x)$ stands for the set of all greatest-type divisors of x in S. The concept of greatest-type divisor was introduced by Hong and played a key role in his solution [6] of the Bourque-Ligh conjecture [1]. Korkee et al. [13] studied the invertibility of matrices in a more general matrix class, join matrices. Meanwhile, they provided a lattice-theoretic explanation for this solution of the Bourque-Ligh conjecture.

This paper is organized as follows. In Section 2, we supply some preliminary results that are needed in the proofs of the main results of this paper. Then in Section 3, we show that for a certain class of gcd-closed (resp. lcm-closed) sets S, $[S^a]$ is nonsingular. This provides evidences to Conjecture 1.1. In Section 4, we show that for a certain class of gcd-closed (resp. lcm-closed) sets S, $(\frac{1}{S^a})$ is nonsingular.

2. Preliminary results

If f is an arithmetic function and m is any positive integer, we denote by $1/f$ the arithmetic function defined as follows:

$$\frac{1}{f}(m) = \begin{cases} 0 & \text{if } f(m) = 0, \\ \dfrac{1}{f(m)} & \text{otherwise.} \end{cases}$$

First, we need a result which gives the formula for the determinant of the power LCM matrix on a gcd-closed set.

Lemma 2.1. *[8, Lemma 2.1] If S is gcd closed, then*

$$\det[S^a] = \prod_{k=1}^{n} x_k^{2a} \alpha_{a,k}, \tag{2.1}$$

where

$$\alpha_{a,k} = \sum_{\substack{d|x_k \\ d\nmid x_t, x_t<x_k}} \left(\frac{1}{\xi_a} * \mu\right)(d) \tag{2.2}$$

throughout this paper.

Lemma 2.2. *Let n be a positive integer. Then*

$$\sum_{d|n} \left(\frac{1}{\xi_a} * \mu\right)(d) = n^{-a}.$$

Proof. The result follows immediately from [7, Lemma 7] applied to $f = 1/\xi_a$. □

Lemma 2.3. *If $k = 1$, then $\alpha_{a,k} = x_1^{-a}$.*

Proof. Lemma 2.3 follows immediately from Lemma 2.2. $\square$

We also need Hong's reduction formulas.

Lemma 2.4. *[8, Lemma 2.5] If S is gcd closed and $k \geq 2$, then*

$$\alpha_{a,k} = x_k^{-a} + \sum_{t=1}^{k-1}(-1)^t \sum_{1\leq i_1<\cdots<i_t\leq k-1} (x_k, x_{i_1}, ..., x_{i_t})^{-a}. \tag{2.3}$$

Lemma 2.5. *If S is gcd closed and $k \geq 3$, then*

$$\alpha_{a,k} = x_k^{-a} + \sum_{t=1}^{k-2}(-1)^t \sum_{2\leq i_1<\cdots<i_t\leq k-1} (x_k, x_{i_1}, ..., x_{i_t})^{-a}. \tag{2.4}$$

Proof. Since S is gcd closed, $(x_k, x_1, x_{i_1}, ..., x_{i_t}) = x_1$ for $2 \leq i_1 < \cdots < i_t \leq k-1$. By (2.3), one has

$$\begin{aligned}
\alpha_{a,k} =& x_k^{-a} - \sum_{1\leq i\leq k-1}(x_k, x_i)^{-a} + \sum_{1\leq i<j\leq k-1}(x_k, x_i, x_j)^{-a} \\
&+ \cdots + (-1)^t \sum_{1\leq i_1<\cdots<i_t\leq k-1} (x_k, x_{i_1}, ..., x_{i_t})^{-a} + \cdots + (-1)^{k-1}(x_k, x_1, ..., x_{k-1})^{-a} \\
=& -(x_k, x_1)^{-a} + \sum_{2\leq i_1\leq k-1}(x_k, x_1, x_{i_1})^{-a} + \cdots + (-1)^t \sum_{2\leq i_1<\cdots<i_{t-1}\leq k-1}(x_k, x_1, x_{i_1}, ..., x_{i_{t-1}})^{-a} \\
&+ \cdots + (-1)^{k-2} \sum_{2\leq i_1<\cdots<i_{k-3}\leq k-1}(x_k, x_1, x_{i_1}, ..., x_{i_{k-3}})^{-a} + (-1)^{k-1}(x_k, x_1, ..., x_{k-1})^{-a} \\
&+ x_k^{-a} - \sum_{2\leq i\leq k-1}(x_k, x_i)^{-a} + \sum_{2\leq i<j\leq k-1}(x_k, x_i, x_j)^{-a} \\
&+ \cdots + (-1)^t \sum_{2\leq i_1<\cdots<i_t\leq k-1}(x_k, x_{i_1}, ..., x_{i_t})^{-a} + \cdots + (-1)^{k-2}(x_k, x_2, ..., x_{k-1})^{-a} \\
=&(-1)\sum_{w=0}^{k-2}\binom{k-2}{w}(-1)^w x_1^{-a} + x_k^{-a} + \sum_{t=1}^{k-2}(-1)^t \sum_{2\leq i_1<\cdots<i_t\leq k-1}(x_k, x_{i_1}, ..., x_{i_t})^{-a} \\
=&(-1)\times 0 + x_k^{-a} + \sum_{t=1}^{k-2}(-1)^t \sum_{2\leq i_1<\cdots<i_t\leq k-1}(x_k, x_{i_1}, ..., x_{i_t})^{-a} \\
=& x_k^{-a} + \sum_{t=1}^{k-2}(-1)^t \sum_{2\leq i_1<\cdots<i_t\leq k-1}(x_k, x_{i_1}, ..., x_{i_t})^{-a}
\end{aligned}$$

as expected. This concludes the proof of Lemma 2.5. $\square$

Lemma 2.6. *If S is gcd closed, then*

$$\alpha_{a,k} = \sum_{J\subset G_S(x_k)} \frac{(-1)^{|J|}}{(\gcd(J \cup x_k))^a}. \tag{2.5}$$

Proof. This follows immediately from [9, Theorem 1.2] applied to $f = 1/\xi_a$. □

Lemma 2.7. *[3, Theorem 5.1] Let a and n be positive integers with $a \geq 2$ and $n \leq 8$. If S is gcd closed, then $[S^a]$ is nonsingular.*

Lemma 2.8. *[10, Lemma 2.2] The set S is lcm closed if and only if mS^{-1} is gcd closed.*

3. Nonsingularity of power LCM matrices

As usual, for any nonzero integer c and prime number p, $\nu_p(c)$ is the greatest nonnegative integer ν such that p^ν divides c. Then $\nu_p(c) \geq 1$ if and only if p divides c.

Theorem 3.1. *Let S be gcd closed. If there are primes $p_3, ..., p_n$ (not necessarily distinct) such that*

$$\nu_{p_k}(x_k) > \max_{2\leq i\leq k-1}\{\nu_{p_k}(x_i)\}, \tag{3.1}$$

then $[S^a]$ is nonsingular.

Proof. By Lemma 2.3, we have $\alpha_{a,1} = x_1^{-a}$. By Lemma 2.4, one has

$$\alpha_{a,2} = x_2^{-a} - (x_1, x_2)^{-a} = x_2^{-a} - x_1^{-a}$$

since $(x_2, x_1) = x_1$. Thus $\alpha_{a,1} \neq 0$ and $\alpha_{a,2} \neq 0$. Now let $k \geq 3$. For any $2 \leq i_1 < \cdots < i_t \leq k-1$ $(1 \leq t \leq k-2)$, it follows from (3.1) that

$$\nu_{p_k}(x_k, x_{i_1}, ..., x_{i_t}) = \min\{\nu_{p_k}(x_k), \nu_{p_k}(x_{i_1}), ..., \nu_{p_k}(x_{i_t})\} < \nu_{p_k}(x_k).$$

Therefore,

$$\frac{x_k}{(x_k, x_{i_1}, ..., x_{i_t})} \equiv 0 \pmod{p_k}. \tag{3.2}$$

Multiplying both sides of (2.4) by x_k^a, by (3.2), one has $x_k^a\alpha_{a,k} \equiv 1 \pmod{p_k}$. Thus, $\alpha_{a,k} \neq 0$ $(k \geq 3)$. So $\alpha_{a,1}\alpha_{a,2}\cdots\alpha_{a,n} \neq 0$. It can be deduced from Lemma 2.1 that $[S^a]$ is nonsingular. □

Corollary 3.1. *Let S be gcd closed. If $x_i|x_j$ for all integers i, j with $1 \leq i < j \leq n$, then $[S^a]$ is nonsingular.*

Proof. We show that there are primes $p_3, ..., p_n$ satisfying (3.1). Let $3 \leq k \leq n$. For any prime p, we have $\nu_p(x_k) \geq \nu_p(x_{k-1})$ since $x_{k-1}|x_k$. Thus there exists a prime p_k such that $\nu_{p_k}(x_k) > \nu_{p_k}(x_{k-1})$ since $x_{k-1} < x_k$. Moreover, $x_2|...|x_{k-1}$ implies that

$$\nu_{p_k}(x_2) \leq \nu_{p_k}(x_3) \leq \cdots \leq \nu_{p_k}(x_{k-1}).$$

Hence $\max_{2\leq i\leq k-1}\{\nu_{p_k}(x_i)\} = \nu_{p_k}(x_{k-1})$. Then (3.1) holds. The statement is true from Theorem 3.1. □

Corollary 3.2. *Let S be gcd closed. If $x_2/x_1, ..., x_n/x_1$ are pairwise relatively prime, then $[S^a]$ is nonsingular.*

Proof. Let $3 \leq k \leq n$. By the hypothesis, x_k/x_1 is prime to x_i/x_1 for $2 \leq i \leq k-1$. Recall that $x_k/x_1 > x_i/x_1$ $(2 \leq i \leq k-1)$. Let p_k be any prime factor of x_k/x_1. Then we know that $p_k \nmid x_i/x_1$ $(2 \leq i \leq k-1)$. Thus, $\nu_{p_k}(x_i/x_1) = 0$ $(2 \leq i \leq k-1)$. Therefore

$$\max_{2\leq i\leq k-1}\left\{\nu_{p_k}\left(\frac{x_i}{x_1}\right)\right\} = 0.$$

So

$$\max_{2\leq i\leq k-1}\{\nu_{p_k}(x_i)\} = \nu_{p_k}(x_1).$$

Since $p_k|x_k/x_1$, one has

$$\nu_{p_k}\left(\frac{x_k}{x_1}\right) \geq 1.$$

This yields that

$$\nu_{p_k}(x_k) \geq \nu_{p_k}(x_1) + 1 > \nu_{p_k}(x_1).$$

Then (3.1) holds and the proof of Corollary 3.2 is complete. □

Theorem 3.2. *Let S be gcd closed and $F_k = \{x \in S\,|x < x_k$ and $x|x_k\}$. If*

$$x_k > \mathrm{lcm}(F_k), \tag{3.3}$$

for $k = 2, ..., n$, then $[S^a]$ is nonsingular.

Proof. By (2.3), we have

$$\alpha_{a,k} = \frac{1}{x_k^a} + \frac{t}{y^a}, \tag{3.4}$$

where t is an integer and $y = \mathrm{lcm}(F_k)$. Then $y|x_k$. On the other hand, (3.3) yields $y < x_k$. Then there exists a prime factor p of x_k such that $\nu_p(y) < \nu_p(x_k)$. So we obtain that

$$\nu_p\left(\frac{x_k}{y}\right) > 0. \tag{3.5}$$

Multiplying by x_k^a on both sides of (3.4), by (3.5), we have $\alpha_{a,k}x_k^a \equiv 1 \pmod p$. Therefore, $\alpha_{a,k} \neq 0$. So $\alpha_{a,1}\alpha_{a,2}\cdots\alpha_{a,n} \neq 0$. Thus the statement is true from Lemma 2.1. □

Corollary 3.3. *Let S be gcd closed and $H_k = \{x_i|1 \leq i \leq k-1\}$. If*

$$x_k > \mathrm{lcm}(H_k), \tag{3.6}$$

for $k = 2, ..., n$, then $[S^a]$ is nonsingular.

Proof. Let F_k be defined as in Theorem 3.2. Since $F_k \subseteq H_k$, one has

$$\mathrm{lcm}(H_k) \geq \mathrm{lcm}(F_k). \tag{3.7}$$

Equations (3.6) and (3.7) yield (3.3). Then by Theorem 3.2, we know that $[S^a]$ is nonsingular. □

Corollary 3.4. *Let S be gcd closed. If there is at most one x_{i_k} $(2 \leq i_k \leq k-1)$ such that $x_{i_k}|x_k$, then $[S^a]$ is nonsingular.*

Proof. Let F_k be defined as in Theorem 3.2. If there does not exist any integer i_k with $2 \leq i_k \leq k-1$ such that $x_{i_k}|x_k$, then we have $\text{lcm}(F_k) = x_1$. So (3.3) holds. It then follows from Theorem 3.2 that $[S^a]$ is nonsingular. The statement is true for this case.

If there exists exactly one x_{i_k} $(2 \leq i_k \leq k-1)$ such that $x_{i_k}|x_k$, then we have $\text{lcm}(F_k) = x_{i_k}$. The condition $x_{i_k} < x_k$ yields (3.3). The corollary follows immediately from Theorem 3.2 as expected. The statement is true for this case. □

Theorem 3.3. *Let a and n be positive integers with $a \geq 2$ and $n \geq 9$. If S is $(n-8)$-fold gcd closed, then $[S^a]$ is nonsingular.*

Proof. First of all, any $(n-8)$-fold gcd-closed set S must satisfy that $x_1|...|x_{n-7}$ and the set $\{x_{n-7}, ..., x_n\}$ is gcd closed. Since $G_S(x_k) = \{x_{k-1}\}$ for all $2 \leq k \leq n-7$, by (2.5), we have $\alpha_{a,k} = x_k^{-a} - x_{k-1}^{-a}$. Hence $\alpha_{a,k} \neq 0$ for all integers k with $1 \leq k \leq n-7$. Now let $n-6 \leq k \leq n$. One has $x_{n-7}|x_k$ since $\{x_{n-7}, ..., x_n\}$ is gcd closed. So $G_S(x_k)$ equals the set of greatest-type divisors of x_k in the set $S_k := \{x_{n-7}, ..., x_k\}$. Thus by (2.5) we have

$$\alpha_{a,k} = \sum_{J \subset G_S(x_k)} \frac{(-1)^{|J|}}{(\gcd(J \cup \{x_k\}))^a} = \sum_{J \subset G_{S_k}(x_k)} \frac{(-1)^{|J|}}{(\gcd(J \cup \{x_k\}))^a}. \tag{3.8}$$

Note that $2 \leq k-n+8 \leq 8$ since $n-6 \leq k \leq n$. So $|S_k| \leq 8$. Since S_k is gcd closed, it follows immediately from Lemma 2.7 that $[S_k^a]$ is nonsingular. So $\det[S_k^a] \neq 0$. one can easily deduce from Lemma 2.1 that

$$\alpha_{a,k} = \sum_{J \subset G_{S_k}(x_k)} \frac{(-1)^{|J|}}{(\gcd(J \cup \{x_k\}))^a} \neq 0. \tag{3.9}$$

By (3.8) and (3.9), we obtain that $\alpha_{a,k} \neq 0$ for all integers k with $n-6 \leq k \leq n$. So $\alpha_{a,1}\alpha_{a,2} \cdots \alpha_{a,n} \neq 0$. Then by Lemma 2.1, we derive that $[S^a]$ is nonsingular as desired.

This finishes the proof of Theorem 3.3. □

In the following theorem, we will show that $[S^a]$ defined on a certain class of lcm-closed sets is nonsingular.

Theorem 3.4. *Let a and n be positive integers with $a \geq 2$. Then each of the following is true:*

(i). *If $n \leq 8$ and S is lcm closed, then $[S^a]$ is nonsingular.*

(ii). *If $n \geq 9$ and S is $(n-8)$-fold lcm closed, then $[S^a]$ is nonsingular.*

Proof. Let m be defined as in Definition 1.2. Since

$$[x_i, x_j] = \frac{x_i x_j}{(x_i, x_j)} = \frac{1}{m} \cdot x_i \cdot \frac{m}{(x_i, x_j)} \cdot x_j = \frac{1}{m} \cdot x_i \cdot \left[\frac{m}{x_i}, \frac{m}{x_j}\right] \cdot x_j,$$

we have

$$[S^a] = \frac{1}{m^a} W[(mS^{-1})^a]W,$$

where $W = \mathrm{diag}(x_1^a, ..., x_n^a)$. So

$$\det[S^a] = \det[\left(mS^{-1}\right)^a]\prod_{k=1}^{n}\left(\frac{x_k^2}{m}\right)^a.$$

Therefore, to show the desired result, it suffices to prove that $\det[(mS^{-1})^a] \neq 0$.

First, we prove part (i). By Lemma 2.8, we know that the reciprocal set mS^{-1} is gcd closed. Lemma 2.7 tells us that $[(mS^{-1})^a]$ is nonsingular. So $\det[(mS^{-1})^a] \neq 0$. This completes the proof of part (i).

Consequently, we prove part (ii). Since S is $(n-8)$-fold lcm closed, by Definition 1.3, we know that mS^{-1} is $(n-8)$-fold gcd closed. Theorem 3.3 tells us that $[(mS^{-1})^a]$ is nonsingular. So $\det[(mS^{-1})^a] \neq 0$. This finishes the proof of part (ii). □

4. Nonsingularity of reciprocal power GCD matrices

In this section, we will show that $(\frac{1}{S^a})$ defined on a certain class of gcd-closed (resp. lcm-closed) sets is nonsingular.

Theorem 4.1. *Let a and n be positive integers with $a \geq 2$. Then each of the following is true:*

(i). *If $n \leq 8$ and S is gcd closed, then $(\frac{1}{S^a})$ is nonsingular.*

(ii). *If $n \geq 9$ and S is $(n-8)$-fold gcd closed, then $(\frac{1}{S^a})$ is nonsingular.*

Proof. Since

$$\frac{1}{(x_i, x_j)^a} = \frac{[x_i, x_j]^a}{x_i^a x_j^a},$$

we have

$$(\frac{1}{S^a}) = T[S^a]T,$$

where

$$T = \mathrm{diag}\left(\frac{1}{x_1^a}, ..., \frac{1}{x_n^a}\right).$$

So

$$\det(\frac{1}{S^a}) = \det[S^a]\prod_{k=1}^{n} x_k^{-2a}. \tag{4.1}$$

By Lemma 2.1 and (4.1), we get that

$$\det(\frac{1}{S^a}) = \prod_{k=1}^{n} \alpha_{a,k}.$$

So we only need to prove that $\prod_{k=1}^{n} \alpha_{a,k} \neq 0$.

We prove part (i) as follows. Lemma 2.7 tells us that $[S^a]$ is nonsingular. By Lemma 2.1, we know that $\prod_{k=1}^{n} \alpha_{a,k} \neq 0$. This concludes the proof of part (i).

Now we prove part (ii). Since S is $(n-8)$-fold gcd closed, Theorem 3.3 tells us that $[S^a]$ is nonsingular. By Lemma 2.1, we know that $\prod_{k=1}^{n} \alpha_{a,k} \neq 0$. This completes the proof of part (ii). □

Theorem 4.2. *Let a and n be positive integers with $a \geq 2$. Then each of the following is true:*

(i). *If $n \leq 8$ and S is lcm closed, then $(\frac{1}{S^a})$ is nonsingular.*

(ii). *If $n \geq 9$ and S is $(n-8)$-fold lcm closed, then $(\frac{1}{S^a})$ is nonsingular.*

Proof. Let m be defined as in Definition 1.2. Since

$$\frac{1}{(x_i, x_j)} = \frac{1}{m} \cdot \frac{m}{(x_i, x_j)} = \frac{1}{m} \cdot \left[\frac{m}{x_i}, \frac{m}{x_j}\right],$$

one deduces that

$$(\frac{1}{S^a}) = \frac{1}{m^a}[(mS^{-1})^a].$$

So

$$\det(\frac{1}{S^a}) = \frac{1}{m^{na}} \det[(mS^{-1})^a]. \tag{4.2}$$

Thus, it is sufficient to prove the truth of the claim by stating that $\det[mS^{-1})^a] \neq 0$. By the proof of Theorem 3.4, we know that the claim is true.

This finishes the proof of Theorem 4.2. □

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Conflict of interest

We declare that we have no conflict of interest.

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