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Research article

On some generalized Raina-type fractional-order integral operators and related Chebyshev inequalities

Miguel Vivas-Cortez[1], **Pshtiwan O. Mohammed**[2], **Y. S. Hamed**[3], **Artion Kashuri**[4], **Jorge E. Hernández**[5] **and Jorge E. Macías-Díaz**[6,7,*]

1 Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Ecuador
2 Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Kurdistan Region, Iraq
3 Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia
4 Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora 9400, Albania
5 Departamento de Técnicas Cuantitativas, Universidad Centroccidental Lisandro Alvarado, Venezuela
6 Department of Mathematics and Didactics of Mathematics, Tallinn University, Estonia
7 Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Mexico

* **Correspondence:** Email: jemacias@correo.uaa.mx; Tel: +524499108400; Fax: +524499108400.

Abstract: In this work, we introduce generalized Raina fractional integral operators and derive Chebyshev-type inequalities involving these operators. In a first stage, we obtain Chebyshev-type inequalities for one product of functions. Then we extend those results to account for arbitrary products. Also, we establish some inequalities of the Chebyshev type for functions whose derivatives are bounded. In addition, we derive an estimate for the Chebyshev functional by applying the generalized Raina fractional integral operators. As corollaries of this study, some known results are recaptured from our general Chebyshev inequalities. The results of this work may prove useful in the theoretical analysis of numerical models to solve generalized Raina-type fractional-order integro-differential equations.

Keywords: Chebyshev inequality; generalized Raina integral operators; integral inequalities; fractional-order integrals; approximation techniques
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1. Introduction

Fractional calculus emerged as a generalization of conventional calculus. The classical differential and integral operators proposed by Leibniz more than three centuries ago have been extended to the fractional scenario in various different forms. In the way, various applications have been proposed to within mathematics, like Cauchy problems with Caputo Hadamard fractional derivatives [1], the synchronization for a class of fractional-order hyperchaotic systems [2] the inequality estimates for the boundedness of multilinear singular and fractional integral operators [3], the analysis of unstructured-mesh Galerkin finite element method for the two-dimensional multi-term time-space fractional Bloch-Torrey equations on irregular convex domains [4] and the design of numerically efficient and conservative model for a Riesz space-fractional Klein-Gordon-Zakharov system [5], among various other interesting problems [6].

It is worth pointing out that each fractional operator is fully characterized by its own special kernel, and they can be used in a range of particular problems. In this respect, the analysis on the uniqueness of solutions for fractional ordinary and partial differential equations may be performed by employing fractional integral inequalities, which obviously depend on the type of operator. The literature provides accounts of many applications potential applications, like the periodic orbit analysis for the delayed Filippov systems [7] the bifurcation of limit cycles at infinity in piecewise polynomial systems [8] and the number and stability of limit cycles for planar piecewise linear systems of node-saddle type [9]. Indeed, integral inequalities play a major role in the field of differential equations and applied mathematics. More well-known applications of integral inequalities are found in applied sciences, such as statistical problems, transform theory, numerical quadrature, and probability [10, 11].

In the last few years, many researchers have established various types of integral inequalities by employing different approaches. For example, there are reports on some new inequalities for (k, s)-fractional integrals [12], certain weighted integral inequalities involving the fractional hypergeometric operators [13] and some generalized integral inequalities for Hadamard fractional integrals [14]. Moreover, those results provide direct applications to other areas of mathematics, such as conformable fractional integral inequalities [15], including those of the Hermite-Hadamard type [16]. Moreover, there are reports in the literature on integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel [17] and Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals [18].

The so-called Chebyshev inequality is one important type of inequality which is related to the synchronous functions [19]. This inequality has been extensively studied in the literature. Indeed, these inequalities appear in the study of conformable κ-fractional integral operators [20] in the investigation of inequalities of the Grüss-type for conformable fractional integrals [21, 22], in Simpson's type integral inequalities for κ-fractional integrals [23], in generalized fractional conformable integrals [24], in the study of some fractional integral inequalities for the Katugampola integral operator [25], in the theory of Riemann-Liouville fractional operators [26], and in the context of generalized conformable fractional operators [27, 28]. Here, it is worth pointing out that the classical Chebyshev inequality reads as follows [19]:

$$\left(\frac{1}{d_2 - d_1}\int_{d_1}^{d_2}\psi_1(\mathbf{k})d\mathbf{k}\right)\left(\frac{1}{d_2 - d_1}\int_{d_1}^{d_2}\psi_2(\mathbf{k})d\mathbf{k}\right) \leq \frac{1}{d_2 - d_1}\int_{d_1}^{d_2}\psi_1(\mathbf{k})\psi_2(\mathbf{k})d\mathbf{k}. \tag{1.1}$$

Here, ψ_1 and ψ_2 are supposed to be synchronous and integrable functions on the interval $[d_1, d_2] \subseteq \mathbb{R}$. Recall that two functions are *synchronous* on $[d_1, d_2]$ if

$$(\psi_1(\mathbf{k}) - \psi_1(\mathbf{l}))(\psi_2(\mathbf{k}) - \psi_2(\mathbf{l})) \geq 0, \quad \forall\, \mathbf{k}, \mathbf{l} \in [d_1, d_2]. \tag{1.2}$$

It is worthwhile mentioning that Chebyshev's inequality (1.1) has been extended for functions whose derivatives belong to L_p spaces [29, 30] and a variant of Chebyshev's inequality was applied to obtain some inequalities for expectation and variance for cumulative distribution functions [31]. It is also important to mention that a new class of fractional derivatives and integrals (fractional conformable derivatives and integrals) were recently proposed in the literature [32, 33]. Later on, the fractional conformable integral operators were introduce [34], and various generalizations of the classical inequalities [35], including the Hadamard, Hermite-Hadamard and Opial inequalities [36], Grüss inequality [37], and the Ostrowski and Chebyshev inequalities [20]. Recently, some new fractional extensions of Chebyshev's inequality were derived using the generalized Katugampola integrals through the Polya-Szegö inequality [38].

In the present manuscript, we will present new Raina-type fractional integral operators, and we will devote our investigation to derive extensions of Chebyshev's inequality which consider the presents of those operators. As one of the most important results in this work, we will prove a version of Chebyshev's inequality for more than two functions. In the way, some Chebyshev-type inequalities will be proved for functions whose derivatives are bounded from below or from above. In addition, an estimate of the Chebyshev functional are proved using the new Raina-type fractional integral operators. Before closing this work we will establish similar inequalities for the Prabhakar-Salim fractional integrals. Overall, the present work proposes more general results than those in various existing reports available in the literature. To show this fact, we will notice that some known results from our general Chebyshev inequalities are obtained as particular cases.

2. Preliminaries

The present section is devoted to recall various definitions and results from the literature on fractional calculus, and to derive new properties which will be required in the sequel. Most of the standard definitions will be taken from [39, 40].

Definition 2.1. Let $\imath > 0$ be a real number and suppose that $\psi : [d_1, d_2] \to \mathbb{R}$ is a function such that $\psi \in L_1[d_1, d_2]$. Then the standard *left and right Riemann-Liouville fractional integral operators* of order $\imath$ are defined, respectively, by

$$\left(\mathcal{I}^{\imath}_{d_1+}\psi\right)(\mathbf{k}) = \frac{1}{\Gamma(\imath)} \int_{d_1}^{\mathbf{k}} (\mathbf{k} - \ell)^{\imath - 1} \psi(\ell) d\ell, \quad (\mathbf{k} > d_1), \tag{2.1}$$

$$\left(\mathcal{I}^{\imath}_{d_2-}\psi\right)(\mathbf{k}) = \frac{1}{\Gamma(\imath)} \int_{\mathbf{k}}^{d_2} (\ell - \mathbf{k})^{\imath - 1} \psi(\ell) d\ell, \quad (\mathbf{k} < d_2). \tag{2.2}$$

In turn, if we agree that $n = \lfloor \imath \rfloor + 1$, then the *left and right Riemann-Liouville fractional derivatives* are given by

$$D^{\imath}_{d_1+}\psi(\ell) = \frac{\mathrm{d}^n}{\mathrm{d}\mathbf{k}^n} \mathcal{I}^{n-\imath}_{d_1+}\psi(\ell), \tag{2.3}$$

$$D^{\iota}_{d_2-}\psi(\ell) = \frac{\mathrm{d}^n}{\mathrm{d}\mathbf{k}^n}\mathcal{I}^{n-\iota}_{d_2-}\psi(\ell). \tag{2.4}$$

Definition 2.2 (Raina [41])**.** Let $\sigma = (\sigma(i))_{i=1}^{\infty}$ be a bounded real sequence. The *modified Mittag-Leffler* function with parameters $\varrho, \rho > 0$ and σ at $\mathbf{k} \in \mathbb{C}$ is given by

$$\mathcal{F}^{\sigma}_{\rho,\varrho}(\mathbf{k}) = \sum_{i=0}^{\infty} \frac{\sigma(i)}{\Gamma(\rho i + \varrho)}\mathbf{k}^i. \tag{2.5}$$

The following definition introduces the Raina fractional operators. It is important to notice that the those operators are obtained from (2.1) and (2.2) by replacing the Mittag-Leffler kernel with the modified Mittag-Leffler function with parameters $\varrho, \rho > 0$ and σ. Meantime, the Raina fractional model is proposed in [41] as one important models of fractional calculus and this is defined by integral similar to but with a modified Mittag-Leffler function in the kernel. This is defined by the following definition.

Definition 2.3 (Agarwal [42])**.** Suppose that $\varrho, \rho > 0$, and let $w \in \mathbb{R}$. For any function $\psi \in L_1[d_1, d_2]$, the *left and right Raina fractional integral operators* of ψ are defined by the following integral transforms, respectively:

$$\left(\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}\psi\right)(\mathbf{k}) = \int_{d_1}^{\mathbf{k}} (\mathbf{k}-\ell)^{\varrho-1}\,\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\mathbf{k}-\ell)^{\rho}\right]\psi(\ell)d\ell, \quad (\mathbf{k} > d_1), \tag{2.6}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\varrho,d_2-;w}\psi\right)(\mathbf{k}) = \int_{\mathbf{k}}^{d_2} (\ell-\mathbf{k})^{\varrho-1}\,\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell-\mathbf{k})^{\rho}\right]\psi(\ell)d\ell, \quad (\mathbf{k} < d_2). \tag{2.7}$$

Various known particular scenarios are obtained depending on specific expressions of ρ, ϱ and σ. For example, if $\rho = 1$, $\varrho = 0$ and $\sigma(i) = ((\imath)_i(\beta)_i/(\gamma)_i)$ for $i \in \mathbb{N} \cup \{0\}$ in the formula (2.5), then we get the classical hypergeometric function, which is given by

$$\mathcal{F}^{\sigma}_{\rho,\varrho}(\mathbf{k}) = F(\imath, \beta; \gamma; \mathbf{k}) = \sum_{i=0}^{\infty} \frac{(\imath)_i(\beta)_i}{(\gamma)_i i!}\mathbf{k}^i. \tag{2.8}$$

Here, $\imath$, β and γ are arbitrary complex or real parameters with the property that $\gamma \neq 0, -1, -2, \ldots$, and $(d_1)_i$ is the Pochhammer symbol defined by

$$(d_1)_i = \frac{\Gamma(d_1 + i)}{\Gamma(d_1)} = d_1(d_1+1)\cdots(d_1+i-1), \quad i \in \mathbb{N} \cup \{0\}. \tag{2.9}$$

On the other hand, if $\sigma(i) = (1, 1, 1, \ldots)$ with $\rho = \imath > 0$ and $\varrho = 1$ in (2.5), then we get the standard Mittag-Leffler function, which is given by

$$E_{\imath}(\mathbf{k}) = \sum_{i=0}^{\infty} \frac{1}{\Gamma(\imath i + 1)}\mathbf{k}^i. \tag{2.10}$$

Extensions of the Mittag-Leffler function are obtained from (2.5) when $\sigma(i) = \frac{(\eta)_i}{i!}$, $\rho = \imath$ and $\varrho = \beta$, see [43], and when $\sigma(i) = \frac{(\gamma)_{qi}}{(\delta)_{pi}}$, $\rho = \imath$, and $\varrho = \beta$, see [44].

Let $\sigma = (\sigma(i))_{i=1}^{\infty}$ be a bounded real sequence and $w \in \mathbb{R}$, then the following lemma holds.

Lemma 2.4. *The integral operators* $\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}\psi$ *and* $\mathcal{J}^{\sigma}_{\rho,\varrho,d_2-;w}\psi$ *are bounded on* $L_1[d_1,d_2]$*, if* $\mathfrak{M} := \mathcal{F}^{|\sigma|}_{\rho,\varrho+1}[|w|(d_2-d_1)^{\rho}] < \infty$*. In fact, for* $\psi \in L_1[d_1,d_2]$*, we have*

$$\max\left\{\left\|\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}\psi\right\|_1, \left\|\mathcal{J}^{\sigma}_{\rho,\varrho,d_2-;w}\psi\right\|_1\right\} \leq \mathfrak{M}\|\psi\|_1, \tag{2.11}$$

where $\|\cdot\|_p$ *is the usual norm in* L_p*.* □

Let us define $\mathfrak{G} := \mathcal{F}^{\sigma}_{\rho,\varrho+1}[w(d_2-d_1)^{\rho}] < \infty$. Taking $\sigma(i) \geq 0$, where $i \in \mathbb{N}\cup\{0\}$ and $w \geq 0$, in above lemma, we get the following result:

$$\max\left\{\left\|\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}\psi\right\|_1, \left\|\mathcal{J}^{\sigma}_{\rho,\varrho,d_2-;w}\psi\right\|_1\right\} \leq \mathfrak{G}\|\psi\|_1. \tag{2.12}$$

Raina's fractional operators are important due to their level of generality. More precisely, by specifying the coefficient $\sigma(i)$, we can obtain many useful fractional integral operators. For example, if $\varrho = \iota$, $\sigma(0) = 1$, $\sigma(i) = 0$ for each $i \in \mathbb{N}$ and $w = 0$ in Definition 2.3, then we readily retrieve the left and right Riemann-Liouville fractional integrals. On the other hand, if $\sigma(i) = \frac{(\eta)_i}{i!}$, $\rho = \iota$, $\varrho = \beta$ and $d_1 = 0$ in Definition 2.3, we obtain the well-known Prabhakar fractional integral in [43], which is given by

$$^{P}I^{\iota,\beta,\eta,w}_{0+}\psi(\mathbf{k}) = \int_0^{\mathbf{k}} (\mathbf{k}-\ell)^{\beta-1}E^{\eta}_{\iota,\beta}[w(\mathbf{k}-\ell)^{\iota}]\psi(\ell)d\ell. \tag{2.13}$$

Finally, when $\sigma(i) = \frac{(\gamma)_{qi}}{(\delta)_{pi}}$, $\rho = \iota$ and $\varrho = \beta$, we obtain the well-known Salim fractional integral [44]:

$$^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,d_1+}\psi(\mathbf{k}) = \int_{d_1}^{\mathbf{k}} (\mathbf{k}-\ell)^{\beta-1}E^{\gamma,\delta,p}_{\iota,\beta,q}[w(\mathbf{k}-\ell)^{\iota}]\psi(\ell)d\ell, \tag{2.14}$$

where $p, q > 0$ and $q < \iota + p$.

The new Raina fractional integral operators are introduced next.

Definition 2.5. Let d_1 and d_2 be real numbers satisfying $0 \leq d_1 < d_2$, and let $\psi \in L_1[d_1,d_2]$ be any function. The *generalized left and right Raina fractional integral operator* on $\psi(\ell)$ associated to the parameters ϱ, $\rho > 0$, $w \in \mathbb{R}$, $\vartheta \in (0,1]$, $\varpi \geq 0$, and σ any bounded arbitrary sequence of real (or complex) numbers, are respectively defined by the following integral transforms:

$$\begin{aligned}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}\psi\right)(\mathbf{k}) = &\int_{d_1}^{\mathbf{k}} \left(\frac{\mathbf{k}^{\vartheta+\varpi}-\ell^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\ell^{\vartheta+\varpi-1}\\ &\times\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\mathbf{k}^{\vartheta+\varpi}-\ell^{\vartheta+\varpi})^{\rho}\right]\psi(\ell)d\ell, \quad (\mathbf{k} > d_1),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_2-;w}\psi\right)(\mathbf{k}) = &\int_{\mathbf{k}}^{d_2} \left(\frac{\ell^{\vartheta+\varpi}-\mathbf{k}^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\ell^{\vartheta+\varpi-1}\\ &\times\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-\mathbf{k}^{\vartheta+\varpi})^{\rho}\right]\psi(\ell)d\ell, \quad (\mathbf{k} < d_2).\end{aligned} \tag{2.16}$$

It is obvious from Definition 2.5 that ${}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}\psi$ and ${}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_2-;w}\psi$ are linear operators, and we claim that they are bounded integral operators on $L_1[d_1,d_2]$, if

$$\mathfrak{M}_1 := \frac{d_2^{\vartheta+\varpi-1}}{(\vartheta+\varpi)^{\rho-1}}\mathcal{F}^{|\sigma|}_{\rho,\varrho+1}\left[|w|\left(d_2^{\vartheta+\varpi}-d_1^{\vartheta+\varpi}\right)^{\rho}\right] < \infty. \tag{2.17}$$

Indeed, for $\psi \in L_1[d_1, d_2]$, we readily obtain that

$$\max\left\{\left\|{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}\psi\right\|_1, \left\|{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_2-;w}\psi\right\|_1\right\} \leq \mathfrak{M}_1 \|\psi\|_1 . \tag{2.18}$$

The relevance of the present study lies in the fact that generalized forms of existing inequalities will be provided [45]. To that end, it is important to notice that different expressions of the sequence σ. As example, if $\rho = 1$, $w = 1$, $\sigma(0) = 1$, and $\sigma(i) = 0$ for each $i \in \mathbb{N}$, we readily obtain the left and right generalized fractional conformable integral operators [24, 27], which are respectively defined by

$$\left({}^{\vartheta}_{\varpi}\mathcal{I}^{\varrho}_{d_1+}\psi\right)(\mathbf{k}) = \frac{1}{\Gamma(\varrho)}\int_{d_1}^{\mathbf{k}}\left(\frac{\mathbf{k}^{\vartheta+\varpi} - \ell^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\ell^{\vartheta+\varpi-1}\psi(\ell)d\ell, \quad (\mathbf{k} > d_1), \tag{2.19}$$

$$\left({}^{\vartheta}_{\varpi}\mathcal{I}^{\varrho}_{d_2-}\psi\right)(\mathbf{k}) = \frac{1}{\Gamma(\varrho)}\int_{\mathbf{k}}^{d_2}\left(\frac{\ell^{\vartheta+\varpi} - \mathbf{k}^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\ell^{\vartheta+\varpi-1}\psi(\ell)d\ell, \quad (\mathbf{k} < d_2). \tag{2.20}$$

If, in addition, $\varpi = 0$, then we readily obtain the Katugampola conformable left and right fractional integral operators, which are respectively defined by

$$\left({}^{\vartheta}\mathcal{I}^{\varrho}_{d_1+}\psi\right)(\mathbf{k}) = \frac{1}{\Gamma(\varrho)}\int_{d_1}^{\mathbf{k}}\left(\frac{\mathbf{k}^{\vartheta} - \ell^{\vartheta}}{\vartheta}\right)^{\varrho-1}\ell^{\vartheta-1}\psi(\ell)d\ell, \quad (\mathbf{k} > d_1, d_1 > 0), \tag{2.21}$$

$$\left({}^{\vartheta}\mathcal{I}^{\varrho}_{d_2-}\psi\right)(\mathbf{k}) = \frac{1}{\Gamma(\varrho)}\int_{\mathbf{k}}^{d_2}\left(\frac{\ell^{\vartheta} - \mathbf{k}^{\vartheta}}{\vartheta}\right)^{\varrho-1}\ell^{\vartheta-1}\psi(\ell)d\ell, \quad (\mathbf{k} < d_2, d_2 > 0). \tag{2.22}$$

With the parametric choice $\vartheta = 1$, it is easy to verify that left and right Riemann-Liouville fractional integrals are obtained from (2.15) and (2.16), respectively. Similarly, if $\vartheta = 1$, $\varpi = 0$, $\sigma(i) = \frac{(\eta)_i}{i!}$, $\rho = \imath$ and $\varrho = \beta$, we readily obtain the Prabhakar fractional integral (2.13). Also, if we let $\sigma(i) = \frac{(\gamma)_{qi}}{(\delta)_{pi}}$, $\rho = \imath$ and $\varrho = \beta$, then we readily retrieve the Salim fractional integral (2.14).

3. Main results

For the remainder of this work, we suppose in general that $\{\sigma(n)\}_{n\in\mathbb{N}_0}$ is a sequence of non-negative real numbers and $w \geq 0$.

Theorem 3.1. *Let $\varrho, \rho > 0$, $\vartheta \in (0, 1]$ and $\varpi \geq 0$, and suppose that ψ_1 and ψ_2 are two synchronous functions defined on $[0, \infty)$. For all $\ell > 0$, we have*

$$\begin{aligned} {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1\psi_2)(\ell) \geq & \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}} \\ & \times {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1)(\ell)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_2)(\ell). \end{aligned} \tag{3.1}$$

Proof. Since the functions ψ_1 and ψ_2 are synchronous on $[0, \infty)$, then for all $r, s \geq 0$, it follows that $(\psi_1(r) - \psi_1(s))(\psi_2(r) - \psi_2(s)) \geq 0$. As a consequence,

$$\psi_1(r)\psi_2(r) + \psi_1(s)\psi_2(s) \geq \psi_1(r)\psi_2(s) + \psi_1(s)\psi_2(r). \tag{3.2}$$

Multiplying now both sides of (3.2) by $\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]$ and letting $r \in (0,\ell)$, we may obtain readily that

$$\begin{aligned}
&\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(r)\psi_2(r)\\
&\quad+\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(s)\psi_2(s)\\
&\geq\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(r)\psi_2(s)\\
&\quad+\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(s)\psi_2(r).
\end{aligned}\tag{3.3}$$

Integrating over $r \in (0,\ell)$ yields

$$\begin{aligned}
&\int_0^{\ell}\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(r)\psi_2(r)dr\\
&\quad+\int_0^{\ell}\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(s)\psi_2(s)dr\\
&\geq\int_0^{\ell}\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(r)\psi_2(s)dr\\
&\quad+\int_0^{\ell}\left(\frac{\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} r^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-r^{\vartheta+\varpi})^{\rho}\right]\psi_1(s)\psi_2(r)dr.
\end{aligned}\tag{3.4}$$

From here, it is easy to check that

$$\begin{aligned}
&{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1\psi_2)(\ell)+\psi_1(s)\psi_2(s)\frac{\left(\ell^{\vartheta+\varpi}\right)^{\varrho}}{(\vartheta+\varpi)^{\varrho}}\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right]\\
&\qquad\geq\psi_2(s)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1)(\ell)+\psi_1(s)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_2)(\ell).
\end{aligned}\tag{3.5}$$

Multiplying this last inequality by $\left(\frac{\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} s^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi})^{\rho}\right]$ on both sides, letting $s \in (0,\ell)$. Integrate then over all $s \in (0,\ell)$, we obtain that

$$\begin{aligned}
&{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1\psi_2)(\ell)\int_0^{\ell}\left(\frac{\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} s^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi})^{\rho}\right]ds\\
&+\frac{(\ell^{\vartheta+\varpi})^{\varrho}}{(\vartheta+\varpi)^{\varrho}}\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right]\int_0^{\ell}\left[\left(\frac{\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} s^{\vartheta+\varpi-1}\right.\\
&\left.\times\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi})^{\rho}\right]\psi_1(s)\psi_2(s)\right]ds\\
&\geq\sum_{j=1}^{2}{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_j)(\ell)\int_0^{\ell}\left[\left(\frac{\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1} s^{\vartheta+\varpi-1}\right.\\
&\left.\times\,\mathcal{F}^{\sigma}_{\rho,\varrho}\left[w(\ell^{\vartheta+\varpi}-s^{\vartheta+\varpi})^{\rho}\right]\psi_j(s)\right]ds.
\end{aligned}\tag{3.6}$$

Rewriting this expression and simplifying, we readily reach the conclusion. □

Corollary 3.2. *Let $\varrho, \rho > 0$, $\vartheta \in (0,1]$ and $\varpi \geq 0$, and suppose that ψ_1 and ψ_2 are two synchronous functions defined on $[d_1, \infty)$ where $d_1 > 0$. For all $\ell > d_1$, we have*

$$\begin{aligned} {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}(\psi_1\psi_2)(\ell) \geq & \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}} \\ & \times {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}(\psi_1)(\ell) {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,d_1+;w}(\psi_2)(\ell). \end{aligned} \tag{3.7}$$

Proof. The proof is similar to that of the previous result. □

Some particular consequences can be derived from Theorem 3.1 for the right Raina conformable fractional integral operator. For example, if $\varrho = \iota$, $\sigma(0) = 1$, $\sigma(i) = 0$ for each $i \in \mathbb{N}$, $w = 1$, $\varpi = 0$ and $\vartheta = 1$ in Theorem 3.1, we obtain that

$$\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\ell) \geq \frac{\Gamma(\iota+1)}{\ell^{\iota}}\mathcal{I}^{\iota}_{0+}(\psi_1)(\ell)\mathcal{I}^{\iota}_{0+}(\psi_2)(\ell), \tag{3.8}$$

which is an inequality investigated in [46, Theorem 3.1]. If, in addition, $\varrho = \iota$, $\sigma(0) = 1$ and $w = 0$, it follows that

$$\mathcal{I}^{\iota}_{d_1+}(\psi_1\psi_2)(\ell) \geq \frac{\Gamma(\iota+1)}{\ell^{\iota}}\mathcal{I}^{\iota}_{d_1+}(\psi_1)(\ell)\mathcal{I}^{\iota}_{d_1+}(\psi_2)(\ell). \tag{3.9}$$

On the other hand, $\varrho = \iota$, $\sigma(0) = 1$, $\sigma(i) = 0$ for each $i \in \mathbb{N}$ and $w = 1$ yield

$$ {}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\ell) \geq \frac{\Gamma(\iota+1)(\vartheta+\varpi)^{\iota}}{(\ell^{\vartheta+\varpi})^{\iota}} {}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\psi_1)(\ell) {}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\psi_2)(\ell), \tag{3.10}$$

which was established in [24, Theorem 2.1]. Finally, if we let $\varpi = 0$, then we retrieve the Chebyshev inequality for the Katugampola fractional integral, which reads

$$ {}^{\vartheta}\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\ell) \geq \frac{\Gamma(\iota+1)(\vartheta)^{\iota}}{\ell^{\vartheta\iota}} {}^{\vartheta}\mathcal{I}^{\iota}_{0+}(\psi_1)(\ell) {}^{\vartheta}\mathcal{I}^{\iota}_{0+}(\psi_2)(\ell). \tag{3.11}$$

In this point of our discussion, it is useful to note that the Chebyshev inequality for the Katugampola fractional integral operator reported in [25] is wrong. Actually, the error results from the integral calculation of the term ${}^{\vartheta}I^{\iota}_0(1)$. In fact, the correct expression for this term results from

$$ {}^{\vartheta}I^{\iota}_0(1) = \frac{\vartheta^{1-\iota}}{\Gamma(\iota)}\int_0^x \frac{\tau^{\vartheta-1}}{(x^{\vartheta}-\tau^{\vartheta})^{1-\iota}}d\tau = \frac{\vartheta^{1-\iota}}{\vartheta\Gamma(\iota)}\int_0^{x^{\vartheta}} u^{\iota-1}du = \frac{\vartheta^{-\iota}}{\Gamma(\iota+1)}x^{\vartheta\iota}. \tag{3.12}$$

As we pointed out in our work using suitable parameter values of the Raina conformable fractional integral operator and the conclusion of Theorem 3.1, we can express that result in terms of the Prabhakar and Salim fractional integrals. More concretely, if $\vartheta = 1$, $\varpi = 0$, $\sigma(i) = \frac{(\eta)_i}{i!}$, $\rho = \iota$ and $\varrho = \beta$ in Theorem 3.1, we derive the following inequality for the Prabhakar fractional integral:

$$ {}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_1\psi_2)(\ell) \geq \frac{\ell^{-\beta}}{E^{\eta}_{\iota,\beta+1}(\ell)} {}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_1)(\ell) {}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_2)(\ell). \tag{3.13}$$

On the other hand, if we agree that $\vartheta = 1$, $\varpi = 0$, $\sigma(i) = \frac{(\gamma)_{qi}}{(\delta)_{pi}}$, $\rho = \iota$ and $\varrho = \beta$ in Theorem 3.1, we get the corresponding inequality for the Salim fractional integral, which reads

$$ {}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}(\psi_1\psi_2)(\ell) \geq \frac{\ell^{-\beta}}{E^{\gamma,\delta,q,}_{\iota,\beta+1,p}(\ell)} {}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}\psi_1(\ell) {}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}\psi_2(\ell). \tag{3.14}$$

Moreover, taking $\varpi = 0$ and $\vartheta = 1$ in Theorem 3.1, then we obtain [28, Theorem 2] for the classical Raina fractional integral.

Theorem 3.3. *Let $\varrho, \rho > 0$, $\vartheta \in (0,1]$ and $\varpi \geq 0$, and suppose that $\{\psi_i\}_{i=1}^n$ are increasing positive functions defined on $[0,\infty)$. Then, for any $\ell > 0$, we have*

$$\begin{aligned} &{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}\left(\prod_{i=1}^{n}\psi_i\right)(\ell) \\ &\quad \geq \left(\frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}_{\rho,\varrho+1}^{\sigma}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\right)^{n-1}\prod_{i=1}^{n}{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_i)(\ell). \end{aligned} \tag{3.15}$$

Proof. The proof will use mathematical induction. In the case that $n = 1$, is it easy to check that ${}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_1)(\ell) \geq {}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_1)(\ell)$, for all $\ell > 0$. Now, let $n \geq 2$ and suppose that the conclusion is satisfied for $n-1$ in place of n. Since $\{\psi_i\}_{i=1}^n$ are all increasing positive functions, then $\prod_{i=1}^n \psi_i$ is also an increasing function. We can apply now Theorem 3.1 with $\psi_1{}^{\star} = \prod_{i=1}^{n-1}\psi_i$ and $\psi_2{}^{\star} = \psi_n$ along with the induction hypothesis to obtain

$$\begin{aligned} {}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}\left(\prod_{i=1}^{n}\psi_i\right)(\ell) &= {}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_1^{\star}\psi_2^{\star})(\ell) \\ &\geq \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}_{\rho,\varrho+1}^{\sigma}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\,{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_1{}^{\star})(\ell)\,{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_2{}^{\star})(\ell) \\ &= \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}_{\rho,\varrho+1}^{\sigma}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\,{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}\left(\prod_{i=1}^{n-1}\psi_i\right)(\ell)\,{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_n)(\ell) \\ &\geq \left(\frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}_{\rho,\varrho+1}^{\sigma}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\right)\left(\frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}_{\rho,\varrho+1}^{\sigma}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\right)^{(n-1)-1} \\ &\quad \times \prod_{i=1}^{n-1}{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_i)(\ell)\,{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_n)(\ell) \\ &= \left(\frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}_{\rho,\varrho+1}^{\sigma}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\right)^{n-1}\prod_{i=1}^{n}{}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\varrho,0+;w}^{\sigma}(\psi_i)(\ell). \end{aligned} \tag{3.16}$$

The conclusion of this result follows now from mathematical induction. □

One may readily check that the inequality

$$\mathcal{I}_{0+}^{\iota}\left(\prod_{i=1}^{n}\psi_i\right)(\ell) \geq \left(\frac{\Gamma(\iota+1)}{\ell^{\iota}}\right)^{n-1}\prod_{i=1}^{n}\mathcal{I}_{0+}^{\iota}(\psi_i)(\ell), \tag{3.17}$$

results from Theorem 3.3 when $\varrho = \iota, \sigma(0) = 1, w = 1, \vartheta = 1$ and $\varpi = 0$. This inequality was obtained in [46, Theorem 3.3]. On the other hand, if $\varrho = \iota$, $\sigma(0) = 1$ and $w = 1$, then we obtain [24, Theorem 2.3]. Meanwhile, if $\varpi = 0$, then we reach [25, Theorem 2.3] for conformable fractional integrals. In similar fashion, we can deduce the following inequalities related to Prabhakar and Salim fractional integrals, respectively:

$$ {}^{P}I^{\iota,\beta,\eta,w}_{0+}\left(\prod_{i=1}^{n}\psi_i\right)(\ell) \geq \left(\frac{\ell^{-\beta}}{E^{\eta}_{\iota,\beta+1}(\ell)}\right)^{n-1}\prod_{i=1}^{n}{}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_i)(\ell), \tag{3.18} $$

and

$$ {}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}\left(\prod_{i=1}^{n}\psi_i\right)(\ell) \geq \left(\frac{\ell^{-\beta}}{E^{\gamma,\delta,q}_{\iota,\beta+1,p}(\ell)}\right)^{n-1}\prod_{i=1}^{n}{}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}(\psi_i)(\ell). \tag{3.19} $$

Also, [24, Theorem 2.3] and [25, Theorem 2.3] can be obtained from (3.15). Moreover, if we let $\varpi = 0$ and $\vartheta = 1$ (which concern the classical Raina fractional integral), then Theorem 3.3 reduces to [28, Theorem 4].

Theorem 3.4. *Let $\varrho, \rho > 0$, $\vartheta \in (0, 1]$ and $\varpi \geq 0$. Assume that ψ_1 and ψ_2 are two functions such that ψ_1 is increasing and ψ_2 is differentiable. If $m := \inf_{\ell\geq 0}\psi_2{}'(\ell)$ exists in the real numbers and $\phi(\ell) = \ell^{\vartheta+\varpi}$, then*

$$ \begin{aligned} &{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1\psi_2)(\ell) \\ &\geq \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1)(\ell)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_2)(\ell) \\ &\quad - \frac{m\left(\ell^{\vartheta+\varpi}\right)\mathcal{F}^{\sigma}_{\rho,\varrho+2}\left[w\left(\ell^{\vartheta+\varpi}\right)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right]}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1)(\ell) \\ &\quad + m\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\phi\psi_1)(\ell), \end{aligned} \tag{3.20} $$

holds true in the following cases:

(1) $\vartheta + \varpi = 1$, *and for all* $\ell > 0$;
(2) $\vartheta + \varpi > 1$, *and for all* $\ell \in [0, \ell_0]$;
(3) $\vartheta + \varpi < 1$, *and for all* $\ell \geq \ell_0$,

where $\ell_0 = (\vartheta + \varpi)^{\frac{1}{1-(\vartheta+\varpi)}}$.

Proof. Define the function $h(\ell) := \psi_2(\ell) - m\phi(\ell)$, where $\phi(\ell) = \ell^{\vartheta+\varpi}$. One can easily verify that h is increasing and differentiable function according to the above cases. Moreover, notice that

$$ {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1 h)(\ell) = {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1\psi_2)(\ell) - m\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\phi\psi_1)(\ell). \tag{3.21} $$

Using then Theorem 3.1, it is possible to check that

$$\begin{aligned}
&{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1 h\right)(\ell)\\
&\geq \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(h\right)(\ell)\\
&= \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell)\\
&\quad\times\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_2\right)(\ell)-m\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\phi)(\ell)\right)\\
&\geq \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_2\right)(\ell)\\
&\quad-\frac{m\left(\ell^{\vartheta+\varpi}\right)\mathcal{F}^{\sigma}_{\rho,\varrho+2}\left[w\left(\ell^{\vartheta+\varpi}\right)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right]}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell)\\
&\quad+m\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\psi_1\right)(\ell).
\end{aligned}\tag{3.22}$$

This completes the proof of this result. □

The conclusion of the following results is reached using the same arguments as in Theorem 3.4, and letting $h(\ell) := \psi_2(\ell) - M\phi(\ell)$.

Corollary 3.5. *Let $\varrho, \rho > 0$, $\vartheta \in (0,1]$ and $\varpi \geq 0$. Suppose that ψ_1, ψ_2 are two functions such that ψ_1 is decreasing and ψ_2 is differentiable. If $M := \sup_{\ell\geq 0}\psi_2{}'(\ell)$ exists in the real numbers and if ϕ is defined as in Theorem 3.4, then*

$$\begin{aligned}
&{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\psi_2\right)(\ell)\\
&\geq \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_2\right)(\ell)\\
&\quad-\frac{M\left(\ell^{\vartheta+\varpi}\right)\mathcal{F}^{\sigma}_{\rho,\varrho+2}\left[w\left(\ell^{\vartheta+\varpi}\right)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right]}\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell)\\
&\quad+M\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\psi_1\right)(\ell),
\end{aligned}\tag{3.23}$$

is satisfied for the following cases:

(1) $\vartheta + \varpi = 1$, *and for all* $\ell > 0$;
(2) $\vartheta + \varpi > 1$, *and for all* $\ell \geq \ell_0$;
(3) $\vartheta + \varpi < 1$, *and for all* $\ell \in [0, \ell_0]$,

where $\ell_0 = (\vartheta+\varpi)^{\frac{1}{1-(\vartheta+\varpi)}}$. □

Corollary 3.6. *Let $\varrho, \rho > 0$, $\vartheta \in (0,1]$ and $\varpi \geq 0$. Let ψ_1 and ψ_2 be functions such that ψ_1 is increasing, and ψ_1 and ψ_2 are differentiable. If $m_1 := \inf_{\ell\geq 0}\psi_1{}'(\ell)$ and $m_2 := \inf_{\ell\geq 0}\psi_2{}'(\ell)$ exist in the real numbers* *and ϕ is defined as in Theorem 3.4, then*

$$\begin{aligned}
&{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\psi_2\right)(\ell) - m_1{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\,\psi_2\right)(\ell) \\
&\quad - m_2{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\psi_1\right)(\ell) + m_1m_2{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi^2\right)(\ell) \\
&\geq \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\Big({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell){}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_2\right)(\ell) \\
&\quad - \sum_{j=1}^{2} m_j\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\right)(\ell){}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_j\right)(\ell) \\
&\quad + m_1m_2\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\right)(\ell)\right)^2\Big),
\end{aligned} \tag{3.24}$$

is satisfied for the following cases:

(1) $\vartheta + \varpi = 1$, *and for all* $\ell > 0$;
(2) $\vartheta + \varpi > 1$, *and for all* $\ell \in [0, \ell_0]$;
(3) $\vartheta + \varpi < 1$, *and for all* $\ell \geq \ell_0$,

where $\ell_0 = (\vartheta + \varpi)^{\frac{1}{1-(\vartheta+\varpi)}}$.

Proof. The proof is similar to that of Theorem 3.4, setting $h_1(\ell) = \psi_2(\ell) - m_1\phi(\ell)$ and $h_2(\ell) = \psi_2(\ell) - m_2\phi(\ell)$. □

Corollary 3.7. *Let* $\varrho, \rho > 0$, $\vartheta \in (0, 1]$ *and* $\varpi \geq 0$, *and let* ψ_1, ψ_2 *be such that* ψ_1 *is increasing, and* ψ_1 *and* ψ_2 *are differentiable. Let* $M_1 := \sup_{\ell\geq 0}\psi_1{}'(\ell)$ *and* $M_2 := \sup_{\ell\geq 0}\psi_2{}'(\ell)$ *be real numbers, and let* ϕ *be defined as in Theorem 3.4. Then*

$$\begin{aligned}
&{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\psi_2\right)(\ell) - M_1{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\psi_2\right)(\ell) \\
&\quad - M_2{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\psi_1\right)(\ell) + M_1M_2{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi^2\right)(\ell) \\
&\geq \frac{(\vartheta+\varpi)^{\varrho}}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\ell^{\vartheta+\varpi})^{\rho}\right](\ell^{\vartheta+\varpi})^{\varrho}}\Big({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_1\right)(\ell){}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_2\right)(\ell) \\
&\quad - \sum_{j=1}^{2} M_j\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\right)(\ell){}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\psi_j\right)(\ell) \\
&\quad + M_1M_2\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}\left(\phi\right)(\ell)\right)^2\Big),
\end{aligned} \tag{3.25}$$

is satisfied for the following cases:

(1) $\vartheta + \varpi = 1$, *and for all* $\ell > 0$;
(2) $\vartheta + \varpi > 1$, *and for all* $\ell \in [0, \ell_0]$;
(3) $\vartheta + \varpi < 1$, *and for all* $\ell \geq \ell_0$,

where $\ell_0 = (\vartheta + \varpi)^{\frac{1}{1-(\vartheta+\varpi)}}$. □

4. Some consequences

Beforehand, we would like to point out that various well-known inequalities result as particular cases of Theorem 3.3 and Corollaries 3.5–3.7. More concretely, Chebyshev inequalities for Riemann-Liouville, Prabhakar, Salim and other conformable fractional integral operators are obtained from that result.

More precisely, notice that if $\varrho = \iota$, $\sigma(0) = 1$, $\sigma(i) = 0$ for each $i \in \mathbb{N}$, $w = 1$, $\vartheta = 1$ and $\varpi = 0$, then we reach the Riemann-Liouville fractional integral inequality

$$\begin{aligned}\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\ell) \geq{}& \frac{\Gamma(\iota+1)}{\ell^{\iota}}\mathcal{I}^{\iota}_{0+}(\psi_1)(\ell)\mathcal{I}^{\sigma}_{0+}(\psi_2)(\ell)\\ &- \frac{m\ell}{\iota+1}\mathcal{I}^{\sigma}_{0+}(\psi_1)(\ell) + m\mathcal{I}^{\iota}_{0+}(\ell\psi_1)(\ell)\end{aligned} \tag{4.1}$$

(it is worth to point out here that Theorem 3.4 in reference [46] is wrong due to an erroneous application of Theorem 3.1 in the same reference). In the case that $\varrho = \iota$, $\sigma(0) = 1$, $\sigma(i) = 0$ for all $i \in \mathbb{N}$, we readily obtain

$$\begin{aligned}{}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\ell) \geq{}& \frac{\Gamma(\iota+1)(\vartheta+\varpi)^{\iota}}{(\ell^{\vartheta+\varpi})^{\iota}}{}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\psi_1)(\ell){}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\psi_2)(\ell)\\ &- \frac{m\left(\ell^{\vartheta+\varpi}\right)}{\iota+1}{}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\psi_1)(\ell) + m{}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(\phi\psi_1)(\ell),\end{aligned} \tag{4.2}$$

which was derived in [24, Theorem 2.4]. In turn, if $\varrho = \iota$, $\sigma(0) = 1$, $\sigma(i) = 0$ for each $i \in \mathbb{N}$ and $\varpi = 0$, then we have the following inequality for the Katugampola fractional integral operator:

$$\begin{aligned}{}^{\vartheta}\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\ell) \geq{}& \frac{\Gamma(\iota+1)(\vartheta)^{\iota}}{\ell^{\vartheta\iota}}{}^{\vartheta}\mathcal{I}^{\sigma}_{0+}(\psi_1)(\ell){}^{\vartheta}\mathcal{I}^{\sigma}_{0+}(\psi_2)(\ell)\\ &- \frac{m\left(\ell^{\vartheta}\right)}{\iota+1}{}^{\vartheta}\mathcal{I}^{\sigma}_{0+}(\psi_1)(\ell) + m{}^{\vartheta}\mathcal{I}^{\sigma}_{0+}(\phi\psi_1)(\ell).\end{aligned} \tag{4.3}$$

Finally, if we let $\vartheta = 1$ and $\varpi = 0$ in Theorem 3.4, then we obtain [28, Theorem 5]:

$$\begin{aligned}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1\psi_2)(\ell) \geq{}& \frac{1}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}[w(\ell)^{\rho}](\ell)^{\varrho}}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1)(\ell)\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_2)(\ell)\\ &- \frac{m(\ell)\mathcal{F}^{\sigma}_{\rho,\varrho+2}[w(\ell)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\varrho+1}[w(\ell)^{\rho}]}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1)(\ell)\\ &+ m\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\ell\psi_1)(\ell).\end{aligned} \tag{4.4}$$

As pointed out previously, it is easy to derive versions of the inequality (3.20) for the Prabhakar and Salim fractional integrals, which read as

$$\begin{aligned}{}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_1\psi_2)(\ell) \geq{}& \frac{\ell^{-\beta}}{E^{\eta}_{\iota,\beta+1}(\ell)}{}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_1)(\ell){}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_2)(\ell)\\ &- \frac{m\ell E^{\eta}_{\iota,\beta+2}(\ell)}{E^{\eta}_{\iota,\beta+1}(\ell)}{}^{P}I^{\iota,\beta,\eta,w}_{0+}(\psi_1)(\ell) + m{}^{P}I^{\iota,\beta,\eta,w}_{0+}(\ell\psi_1)(\ell),\end{aligned} \tag{4.5}$$

and

$$
{}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}\left(\psi_1\psi_2\right)(\ell) \geq \frac{\ell^{-\beta}}{E^{\gamma,\delta,q}_{\iota,\beta+1,p}(\ell)}\,{}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}\left(\psi_1\right)(\ell)\,{}^{P}I^{\iota,\beta,\eta,w}_{0+}\left(\psi_2\right)(\ell)
- \frac{m\ell E^{\gamma,\delta,q}_{\iota,\beta+2,p}(\ell)}{E^{\gamma,\delta,q}_{\iota,\beta+1,p}(\ell)}\,{}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}\left(\psi_1\right)(\ell)
+ m\,{}^{R}I^{\gamma,\delta,q}_{\iota,\beta,w,p,0+}\left(\ell\psi_1\right)(\ell). \tag{4.6}
$$

Similarly, appropriate choices of the parameters in the conformable Raina fractional integral we can reduce the conclusions of Corollaries 3.5–3.7 to some known inequalities.

Theorem 4.1. *Let $\varrho,\rho > 0$, $w \in \mathbb{R}$, $\vartheta \in (0,1]$ and $\varpi \geq 0$. Suppose that h is a positive function on $[0,\infty)$, and that ψ_1 and ψ_2 are differentiable functions on $[0,\infty)$. If $\psi_1' \in L_r[0,\infty), \psi_2' \in L_s[0,\infty)$ with $r > 1$ and $\frac{1}{r}+\frac{1}{s} = 1$, then*

$$
\left|{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1\psi_2)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k}) - {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_2)(\mathbf{k})\right|
\leq \frac{\|\psi_1'\|_r\|\psi_2'\|_s\mathbf{k}}{2}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k})\right)^2. \tag{4.7}
$$

Proof. Let h, ψ_1 and ψ_2 be as in the hypotheses of this theorem. For each $\tau,\nu \in (0,\mathbf{k})$ and $\mathbf{k} > 0$ we define the function $H(\tau,\nu) := (\psi_1(\tau) - \psi_1(\nu))(\psi_2(\tau) - \psi_2(\nu))$. Multiplying this identity by $\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\tau^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi} - \tau^{\vartheta+\varpi})^{\rho}]h(\tau)$ with $\tau \in (0,\mathbf{k})$, and integrating then over $(0,\mathbf{k})$, we obtain

$$
\int_0^{\mathbf{k}}\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\tau^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi} - \tau^{\vartheta+\varpi})^{\rho}]h(\tau)H(\tau,\nu)d\tau
= {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1\psi_2)(\mathbf{k}) - \psi_1(\nu)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_2)(\mathbf{k})
- \psi_2(\nu)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1)(\mathbf{k}) + \psi_1(\nu)\psi_2(\nu)\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k}). \tag{4.8}
$$

Multiplying (4.8) by $\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\nu^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi})^{\rho}]h(\nu)$ with $\nu \in (0,\mathbf{k})$, and integrating then over $(0,\mathbf{k})$ yields

$$
\int_0^{\mathbf{k}}\int_0^{\mathbf{k}}\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\nu^{\vartheta+\varpi-1}\tau^{\vartheta+\varpi-1}
\times\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi} - \nu^{\vartheta+\varpi})^{\rho}]\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi} - \tau^{\vartheta+\varpi})^{\rho}]h(\nu)h(\tau)H(\tau,\nu)d\tau d\nu
= {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1\psi_2)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k})
- {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_2)(\mathbf{k})
- {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_2)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1)(\mathbf{k})
+ {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1\psi_2)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k})
= 2\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1\psi_2)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k})
- {}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1)(\mathbf{k})\,{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_2)(\mathbf{k})\right). \tag{4.9}
$$

On the other hand, rewriting H as a double integral and using Hölder's inequality for double integrals we obtain

$$\begin{aligned}|H(\tau,\nu)| &= \left|\int_\tau^\nu \int_\tau^\nu \psi_1{}'(u)\psi_2{}'(v)dudv\right| \\ &\le \left|\int_\tau^\nu \int_\tau^\nu |\psi_1{}'(u)|^r\, dudv\right|^{1/r} \left|\int_\tau^\nu \int_\tau^\nu |\psi_2{}'(u)|^s\, dudv\right|^{1/s} \\ &= |\tau-\nu| \left|\int_\tau^\nu |\psi_1{}'(u)|^r\, du\right|^{1/r} \left|\int_\tau^\nu |\psi_2{}'(v)|^s\, dv\right|^{1/s}.\end{aligned} \tag{4.10}$$

Suppose that $0 < |\tau-\nu| < \mathbf{k}$ is satisfied. Using (4.10) in (4.9), applying Hölder's inequality and using that $\psi_1' \in L_r[0,\infty)$, $\psi_2' \in L_s[0,\infty)$ and $\frac{1}{r}+\frac{1}{s}=1$, it follows that

$$\begin{aligned}&2\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1\psi_2)(\mathbf{k}){}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k})\ -{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_1)(\mathbf{k}){}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h\psi_2)(\mathbf{k})\right) \\ &\le \int_0^{\mathbf{k}}\int_0^{\mathbf{k}} \left(\frac{\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\nu^{\vartheta+\varpi-1}\tau^{\vartheta+\varpi-1} \\ &\quad\times|\tau-\nu|\,h(\nu)h(\tau)\left|\int_\tau^\nu |\psi_1{}'(u)|^r\,du\right|^{1/r}\left|\int_\tau^\nu |\psi_2{}'(v)|^s\,dv\right|^{1/s} d\tau d\nu \\ &\le \left[\int_0^{\mathbf{k}}\int_0^{\mathbf{k}} \left(\frac{\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\nu^{\vartheta+\varpi-1}\tau^{\vartheta+\varpi-1}\right. \\ &\quad\times\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi})^{\rho}]\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi})^{\rho}] \\ &\quad\left.\times|\tau-\nu|\,h(\nu)h(\tau)\left|\int_\tau^\nu|\psi_1{}'(u)|^r\,du\right|d\tau d\nu\right]^{1/r} \\ &\quad\times\left[\int_0^{\mathbf{k}}\int_0^{\mathbf{k}} \left(\frac{\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\nu^{\vartheta+\varpi-1}\tau^{\vartheta+\varpi-1}\right. \\ &\quad\times\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi})^{\rho}]\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi})^{\rho}] \\ &\quad\left.\times|\tau-\nu|\,h(\nu)h(\tau)\left|\int_\tau^\nu|\psi_2{}'(u)|^s\,du\right|d\tau d\nu\right]^{1/s} \\ &\le \|\psi_1{}'\|_r\|\psi_2{}'\|_s\left[\int_0^{\mathbf{k}}\int_0^{\mathbf{k}}\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\right. \\ &\quad\times\left(\frac{\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\varrho-1}\nu^{\vartheta+\varpi-1}\tau^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi}-\nu^{\vartheta+\varpi})^{\rho}] \\ &\quad\left.\times\mathcal{F}^{\sigma}_{\rho,\varrho}[w(\mathbf{k}^{\vartheta+\varpi}-\tau^{\vartheta+\varpi})^{\rho}]\,|\tau-\nu|\,h(\nu)h(\tau)d\tau d\nu\right] \\ &= \|\psi_1{}'\|_r\|\psi_2{}'\|_s\mathbf{k}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(h)(\mathbf{k})\right)^2,\end{aligned} \tag{4.11}$$

which completes the proof. □

Corollary 4.2. *Let $\varrho,\rho > 0$, $w \in \mathbb{R}$, $\vartheta \in (0,1]$ and $\varpi \ge 0$, and suppose that ψ_1 and ψ_2 are two differentiable functions on $[0,\infty)$. If $\psi_1' \in L_r[0,\infty)$, $\psi_2' \in L_s[0,\infty)$, $r > 1$ and $\frac{1}{r}+\frac{1}{s}=1$, then the* *following inequality is satisfied:*

$$\begin{aligned}&\left|{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1\psi_2)(\mathbf{k})-\frac{1}{\left(\mathcal{F}^{\sigma}_{\rho,\varrho+1}(\mathbf{k})\right)}{}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_1)(\mathbf{k}){}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\varrho,0+;w}(\psi_2)(\mathbf{k})\right|\\&\quad\leq\frac{1}{2}\|\psi_1'\|_r\|\psi_2'\|_s\frac{\mathbf{k}^{2\varrho(\vartheta+\varpi)+1}}{(\vartheta+\varpi)^{2(\varrho-1)}}\left(\mathcal{F}^{\sigma}_{\rho,\varrho+1}\left[w(\mathbf{k}^{\vartheta+\varpi})\right]\right)^2.\end{aligned}\tag{4.12}$$

Proof. The proof is reached by applying Theorem 4.1 with $h\equiv 1$. □

Before closing this section, it is worth pointing out that some well-known particular inequalities are readily obtained for specific values of the parameters in Theorem 4.1. For example, if $\varrho=\iota$, $\sigma(0)=1$, $\sigma(i)=0$ for each $i\in\mathbb{N}$, $w=0$, $\vartheta=1$ and $\varpi=0$, then we obtain the following inequality for the Riemann-Liouville fractional integral operator:

$$|\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\mathbf{k})\mathcal{I}^{\iota}_{0+}(h)(\mathbf{k})-\mathcal{I}^{\iota}_{0+}\psi_1(\mathbf{k})\mathcal{I}^{\iota}_{0+}(\psi_2)(\mathbf{k})|\leq\frac{1}{2}\|\psi_1'\|_r\|\psi_2'\|_s\mathbf{k}\left(\mathcal{I}^{\iota}_{0+}(h)(\mathbf{k})\right)^2.\tag{4.13}$$

This inequality was derived in [26, Theorem 3.1]. If, in addition, $h\equiv 1$, then we reach

$$\left|\frac{\mathbf{k}^{\iota}}{\Gamma(\iota+1)}\mathcal{I}^{\iota}_{0+}(\psi_1\psi_2)(\mathbf{k})-\mathcal{I}^{\iota}_{0+}\psi_1(\mathbf{k})\mathcal{I}^{\iota}_{0+}(\psi_2)(\mathbf{k})\right|\leq\frac{1}{2}\|\psi_1'\|_r\|\psi_2'\|_s\frac{\mathbf{k}^{2\iota+1}}{\Gamma^2(\iota+1)},\tag{4.14}$$

which was obtained in [26, Corollary 3.3]. On the other hand, letting $\varrho=\iota$, $\sigma(0)=1$, $\sigma(i)=0$ for each $i\in\mathbb{N}$ and $w=1$ yields

$$\begin{aligned}&\left|{}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(h\psi_1\psi_2)(\mathbf{k}){}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(h)(\mathbf{k})-{}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(h\psi_1)(\mathbf{k}){}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(h\psi_2)(\mathbf{k})\right|\\&\quad\leq\frac{1}{2}\|\psi_1'\|_r\|\psi_2'\|_s\mathbf{k}\left({}^{\vartheta}_{\varpi}\mathcal{I}^{\iota}_{0+}(h)(\mathbf{k})\right)^2,\end{aligned}\tag{4.15}$$

and if $\varpi=0$, we obtain an inequality corresponding to the Katugampola fractional integral. Meanwhile, if $\sigma(i)=\frac{(\eta)_i}{i!}$, $\rho=\iota$, $\varrho=\beta$, $\vartheta=1$ and $\varpi=0$ in Theorem 4.1, we obtain the inequality associated to the Prabhakar fractional integral operator. Finally, when $\sigma(i)=\frac{(\gamma)_{qi}}{(\delta)_{pi}}$, $\rho=\iota$, $\varrho=\beta$, $\vartheta=1$, and $\varpi=0$ in Theorem 4.1, then we readily obtain the an inequality in terms of the Salim fractional integral.

5. Conclusions

In this work, we proposed a generalization of the Raina fractional integral operators, and established a generalized form the Chebyshev inequality. Various generalizations of this inequality were obtained, considering various scenarios. More precisely, we established generalizations of the Chebyshev inequality for the product of an arbitrary finite number of functions, and for pairs of functions whose derivatives are bounded from below or from above. In addition, an estimate for the Chebyshev functional was established using the generalized Raina fractional integral operators. Some particular forms of the Chebyshev inequality for the Prabhakar and Salim fractional integral operators were retrieved from our main results. In that sense, the fractional integral operators presented in this work yield more general results.

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Conflict of interest

The authors declare no potential conflicts of interest in this work.

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