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Research article

The research of (G,w)-Chaos and G-Lipschitz shadowing property

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Abstract: In this paper, we introduce the concepts of $(G,w)-$ Chaos and $G-$ Lipschitz shadowing property. We study the dynamical properties of $(G,w)-$ Chaos in the inverse limit space under group action. In addition, we study the dynamical properties of $G-$ Lipschitz shadowing property respectively under topological $G-$ conjugate and iterative systems. The following conclusions are obtained. (1) Let $(X_f, \bar{G}, \bar{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action. If the self-map f is $(G,w)-$ chaotic, the shift map σ is $(G,w)-$ chaotic; (2) Let (X,d) be a metric $G-$ space and f be topologically $G-$ conjugate to g. Then the map f has $G-$ Lipschitz shadowing property if and only if the map g has $G-$ Lipschitz shadowing property. (3) Let (X,d) be a metric $G-$ space and f be an equivariant Lipschitz map from X to X. Then for any positive integer $k \geq 2$, the map f has the $G-$ Lipschitz shadowing property if and only if the iterative map f^k has the $G-$ Lipschitz shadowing property. These results enrich the theory of topological $G-$ conjugate, iterative system and the inverse limit space under group action.

Keywords: inverse limit space; topological $G-$ conjugate; iterative map; $(G,w)-$ Chaos; $G-$ Lipschitz shadowing property
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1. Introduction

Chaos and shadowing property are important concepts in dynamical systems. Many scholars have studied their dynamical properties. See [1–16] for relevant results. For example, Li [1] proved that if the self-map f is $w-$chaotic, then the shift map σ is $w-$chaotic; Shah, Das and Das [2] proved that if a uniformly continuous self-map of a uniform locally compact Hausdorff space has topological weak specification property, then it admits a topologically distributionally scrambled set of type 3; Kostic [3] introduced two different notions of disjoint distributional chaos for sequences of continuous linear operators in Frechet spaces; Wang and Liu [4] generalized the notion of the ergodic shadowing property to the iterated function systems and proved some related theorems; Wang and Zeng [5] studied the relationship between average shadowing property and $\underline{q}-$average shadowing property; for any $k \neq 0$ and $f \in H(X)$, Li [6] proposed that f is chaotic if and only if f^k is chaotic; Li [7] proved that a chaotic semi-flow θ on a manifold M in the sense of Devaney with some assumptions is an expanding semi-flow; Li and Zhou [8] presented that if a continuous Lyapunov stable map f from a compact metric space X into itself is topologically transitive and has the asymptotic average shadowing property, then X is consisting of one point; Li [9] proved that if the set of all periodic points of $\phi\times\theta$ is dense in $X\times Y$, then $\phi\times\theta$ is chaotic.

According to the concept of $w-$Chaos [1], we introduced the definition of $(G,w)-$Chaos. Let (X,d) be a metric $G-$space and f be a continuous map from X to X. We say that S is an $(G,w)-$scrambled set if for any $x,y\in S$ with $x\neq y$, the following three conditions are satisfied: (1) $w_G(x,f)-w_G(y,f)$ is uncountable; (2) $w_G(x,f)\cap w_G(y,f)$ is nonempty; (3) $w_G(x,f)\not\subset P_G(f)$. The map f is said to be $(G,w)-$chaotic if there exists an uncountable $(G,w)-$scrambled set. Then we proved that if the self-map is $(G,w)-$chaotic, the shift map σ is $(G,w)-$chaotic in the inverse limit space under action group which generalizes the conclusion of $w-$Chaos given by Li [1].

Let (X,d) be a metric $G-$space and f be a continuous map from X to X. The map f has $G-$Lipschitz shadowing property if there exists positive constant $L>0$ and $\delta_0>0$ such that for any $0<\delta<\delta_0$ and $(G,\delta)-$pseudo orbit $\{x_i\}_{i\geq 0}$ of f there exists a point $x\in X$ such that the sequence $\{x_i\}_{i=0}^{\infty}$ is $(G,L\delta)-$shadowed by the point x [17]. Finally, the dynamical properties of $G-$Lipschitz shadowing property are studied under topological $G-$conjugate and iterative systems. We derive that (1) If f is topologically $G-$conjugate to g, then the map f has $G-$Lipschitz shadowing property if and only if the map g has $G-$Lipschitz shadowing property. (2) For any positive integer $k\geq 2$, the map f has the $G-$Lipschitz shadowing property if and only if the iterative map f^k has the $G-$Lipschitz shadowing property. These results enrich the theory of topological $G-$conjugate and iterative system.

Next, I will give the proof of above three conclusions in sections 2–4.

2. (G,w)-Chaos in the inverse limit space under group action

Definition 2.1. [18] Let (X,d) be a metric $G-$space, G be a topological group and φ be a continuous map from $G\times X$ to X. The $(X,G,\varphi) or X$ is called to be a metric $G-$space if the following conditions are satisfied:

(1) $\varphi(e,x)=x$ for all $x\in X$ where e is the identity of G;

(2) $\varphi(g_1,\varphi(g_2,x))=\varphi(g_1g_2,x)$ where for all $x\in X$ and all $g_1, g_2\in G$.

If X is compact, then X is also said to be a compact metric $G-$space. For the convenience, $\varphi(g,x)$

is usually abbreviated as gx .

Definition 2.2. [19] Let (X,d) be a metric space and f be a continuous map from X to X . We say that X_f is the inverse limit spaces of X if we write

$$X_f = \{(x_0, x_1, x_2 \cdots) : f(x_{i+1}) = x_i, \forall i \geq 0\},$$

Where let $\underleftarrow{\lim}(X, f)$ denoted by the inverse limit spaces X_f .

The metric $\overline{d}$ in X_f is defined by

$$\overline{d}(\overline{x}, \overline{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

where $\overline{x} = (x_0, x_1, x_2 \cdots) \in X_f$ and $\overline{y} = (y_0, y_1, y_2 \cdots) \in X_f$.

The shift map $\sigma : X_f \to X_f$ is defined by

$$\sigma(\overline{x}) = (f(x_0), x_0, x_1 \cdots).$$

For every $i \geq 0$ the projection map $\pi_i : X_f \to X$ is defined by

$$\pi_i(\overline{x}) = x_i .$$

Thus $(X_f, \overline{d})$ is compact metric space and the shift map σ is homeomorphism.

Definition 2.3. [19] Let (X,d) be a metric $G-$space and f be equivariant map from X to X . Write

$$\overline{G} = \{(g, g, g \cdots) : g \in G\} \text{ and } G_\infty = \prod_{i=0}^{\infty} G_i ,$$

where $G_i = G, i \geq 0$.

The map $\varphi : \overline{G} \times X_f \to X_f$ is defined by

$$\varphi(\overline{g}, \overline{x}) = \overline{g} \cdot \overline{x} = (gx_0, gx_1, gx_2 \cdots),$$

where $\overline{g} = (g, g, g \cdots) \in \overline{G}$ and $\overline{x} = (x_0, x_1, x_2 \cdots) \in X_f$.

Then $(X_f, \overline{G}, \overline{d}, \sigma)$ is a metric $G-$space.

Let (X, G, d, f) and $(X_f, \overline{G}, \overline{d}, \sigma)$ are shown as above. The space $(X_f, \overline{G}, \overline{d}, \sigma)$ is called to be the inverse limit spaces of (X, G, d, f) under group action.

Definition 2.4. [20] Let (X,d) be a metric $G-$space and f be a continuous map from X to X .We say that the map f is an equivariant map if we have $f(px) = pf(x)$ for all $x \in X$ and all $p \in G$.

Definition 2.5. [21] Let (X,d) be a metric $G-$space and f be a continuous map from X to X . The point x is called to be an $G-$periodic point if there exists positive integer m and $g \in G$ such that $gf^m(x) = x$. Denoted by $P_G(f)$ the $G-$periodic point set of f .

Definition 2.6. [4] Let (X,d) be a metric $G-$space and f be a continuous map from X to X . The point y is said to be an $G-$limit point of the point x if there exists $\{n_i\} \subset N_+$ and $\{g_i\} \subset G$ such that $\lim_{i \to \infty} g_i f^{n_i}(x) = y$. Denoted by $w_G(x, f)$ the $G-$limit point set of the point x .

Definition 2.7. [21] Let (X,d) be a metric space and f be a continuous map from X to X . If $f(A) \subset A$ then we say that the set A is invariant to the map f .

Definition 2.8. [1] Let (X,d) be a metric space and f be a continuous map from X to X . We say that S is an $w-$ scrambled set if for any $x, y \in S$ with $x \neq y$ the following three conditions are satisfied:

(1) $w(x, f) - w(y, f)$ is uncountable;

(2) $w(x, f) \cap w(y, f)$ is not empty;

(3) $w(x, f) \not\subset P(f)$.

We say that the map f is $w-$ chaotic if there exists an uncountable $w-$ scrambled set.

Remark 2.9. According to the definition of $w-$Chaos in metric space, we give the concept of $(G, w)-$ Chaos in metric $G-$space.

Definition 2.1. [1] Let (X, d) be a metric $G-$space and f be a continuous map from X to X. We say that S is an $(G, w)-$ scrambled set if for any $x, y \in S$ with $x \neq y$ the following three conditions are satisfied:

(1) $w_G(x, f) - w_G(y, f)$ is uncountable;

(2) $w_G(x, f) \cap w_G(y, f)$ is not empty;

(3) $w_G(x, f) \not\subset P_G(f)$.

We say that the map f is $(G, w)-$ chaotic if there exists an uncountable $(G, w)-$ scrambled set.

Definition 2.11. [22] Let (X, d) be a metric space and f be a continuous map from X to X. The set A is said to be a minimal set if for any $x \in A$ we have $\overline{orb(x, f)} = A$.

Lemma 2.12. [21] Let $(X_f, \overline{G}, \overline{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action, f be an equivariant homeomorphism map from X to X and $\overline{x} = (x_0, x_1, x_2 \cdots) \in X_f$. Then we have

$$w_G(\overline{x}, \sigma) = \underleftarrow{\lim}(w_G(x_0, f), f).$$

Lemma 2.13. [21] Let (X, d) be a metric $G-$space, f be an equivariant homeomorphism map from X to X and $x \in X$. Then we have that $w_G(x, f)$ is closed and

$$f(w_G(x, f)) = w_G(x, f).$$

Lemma 2.14. [21] Let $(X_f, \overline{G}, \overline{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action and f be an equivariant homeomorphism map from X to X. Then we have

$$P_G(\sigma) = \underleftarrow{\lim}(P_G(f), f).$$

Theorem 2.15. Let $(X_f, \overline{G}, \overline{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action and f be an equivariant homeomorphism map from X to X. If the self-map f is $(G, w)-$ chaotic, then the shift map σ is $(G, w)-$ chaotic.

Proof. Suppose that the self-map f is $(G, w)-$ chaotic. Then there exists an uncountable $(G, w)-$ scrambled set S. Write

$$D' = \{\overline{x} : \overline{x} \in \pi_0^{-1}(x), x \in S\}.$$

Thus D' is an uncountable set in X_f. Next, we will show that D' is an $(G, w)-$ scrambled set in X_f. Let

$$\overline{x} = (x_0, x_1, x_2 \cdots) \in D' \text{ and } \overline{y} = (y_0, y_1, y_2 \cdots) \in D' \text{ with } \overline{x} \neq \overline{y}.$$

According to that f is an homeomorphism map, the point x_0 and y_0 are two different points in S. By the definition of $(G, w)-$ scrambled set S, we have the following three conditions:

(1) $w_G(x_0, f) - w_G(y_0, f)$ is uncountable;

(2) $w_G(x_0, f) \cap w_G(y_0, f)$ is not empty;

(3) $w_G(x_0, f) \not\subset P_G(f)$.
Firstly, we will show that $w_G(\overline{x}, \sigma) - w_G(\overline{y}, \sigma)$ is uncountable. Let

$$s_0 \in w_G(x_0, f) - w_G(y_0, f).$$

By Lemmas 2.12 and 2.13, we have that

$$w_G(\overline{x}, \sigma) = \underleftarrow{\lim}(w_G(x_0, f), f).$$

$$f(w_G(x, f)) = w_G(x, f).$$

Hence there exists $\overline{s} \in w_G(\overline{x}, \sigma)$ such that $\pi_0(\overline{s}) = s_0$. If $\overline{s} \in w_G(\overline{y}, \sigma)$ then there exists positive integer sequence $\{n_i\}_{i=0}^{\infty}$ and $\overline{g}_i = (g_i, g_i, g_i \cdots) \in \overline{G}$ such that

$$\lim_{i\to\infty} \overline{g}_i \sigma^{n_i}(\overline{y}) = \overline{s}.$$

Thus $\lim_{i\to\infty} g_i f^{n_i}(y_0) = s_0$. So $s_0 \in w_G(y_0, f)$ which is absurd. Hence $\overline{s} \notin w_G(\overline{y}, \sigma)$. Thus, we have that

$$\overline{s} \in w_G(\overline{x}, \sigma) - w_G(\overline{y}, \sigma).$$

That is,

$$s_0 = \pi_0(\overline{s}) \in \pi_0(w_G(\overline{x}, \sigma) - w_G(\overline{y}, \sigma))$$

Then we can get that

$$w_G(x_0, f) - w_G(y_0, f) \subset \pi_0(w_G(\overline{x}, \sigma) - w_G(\overline{y}, \sigma)).$$

According to that the set $w_G(x, f) - w_G(y, f)$ is uncountable, we can get that the set

$$\pi_0(w_G(\overline{x}, \sigma) - w_G(\overline{y}, \sigma))$$

is uncountable. Hence the set $w_G(\overline{x}, \sigma) - w_G(\overline{y}, \sigma)$ is uncountable.
Secondly, we will show that $w_G(\overline{x}, \sigma) \bigcap w_G(\overline{y}, \sigma)$ is not empty. By Lemma 2.12, we have that

$$w_G(\overline{x}, \sigma) \bigcap w_G(\overline{y}, \sigma) = \underleftarrow{\lim}(w_G(x_0, f), f) \bigcap \underleftarrow{\lim}(w_G(x_0, f), f)$$
$$= \underleftarrow{\lim}(w_G(x_0, f) \bigcap w_G(x_0, f), f).$$

Since $w_G(x_0, f) \bigcap w_G(x_0, f)$ is a nonempty closed invariant subset and X is compact metric space, there exists a minimal set M in $w_G(x_0, f) \bigcap w_G(x_0, f)$. Hence, we can get that

$$\varnothing \neq \underleftarrow{\lim}(M, f) \subset \underleftarrow{\lim}(w_G(x_0, f) \bigcap w_G(x_0, f), f).$$

So, we have that

$$w_G(\overline{x}, \sigma) \bigcap w_G(\overline{y}, \sigma) \neq \varnothing.$$

Finally, we will show $w_G(\overline{x}, \sigma) \not\subset P_G(\sigma)$. Suppose $w_G(\overline{x}, \sigma) \subset P_G(\sigma)$. By Lemmas 2.12 and 2.14, we have that

$$w_G(\overline{x}, \sigma) = \underleftarrow{\lim}(w_G(x_0, f), f).$$

$$P_G(\sigma) = \varprojlim(P_G(f), f).$$

Hence, we can get that

$$\varprojlim(w_G(x_0, f), f) \subset \varprojlim(P_G(f), f).$$

So, we have that

$$\pi_0(\varprojlim(w_G(x_0, f), f)) \subset \pi_0(\varprojlim(P_G(f), f)).$$

That is

$$w_G(x_0, f) \subset P_G(f).$$

Thus, the assumption is absurd. So $w_G(\overline{x}, \sigma) \not\subset P_G(\sigma)$. Hence the set D' is an uncountable $(G, w)-$ scrambled set in X_f. So, the shift map σ is $(G, w)-$ chaotic. Thus, we complete the proof. □

3. G-Lipschitz shadowing property under topological G-conjugate

Definition 3.1. [16] Let (X, d) be a metric space and f be a continuous map from X to X. The map f is said to be a Lipschitz map if there exists a positive constant L such that for all $x, y \in X$ implies

$$d(f(x), f(y)) \le Ld(x, y).$$

Definition 3.2. [17] Let (X, d) be a metric $G-$space and f be a continuous map from X to X. The map f has $G-$Lipschitz shadowing property if there exists positive constant L and δ_0 such that for any $0 < \delta < \delta_0$ and $(G, \delta)-$pseudo orbit $\{x_i\}_{i\ge 0}$ of f there exists a point $x \in X$ such that the sequence $\{x_i\}_{i=0}^{\infty}$ is $(G, L\delta)-$shadowed by the point x.

Definition 3.3. [18] Let (X, d) and (Y, d) be a metric $G-$space, f be a continuous map from X to X and g be a continuous map from Y to Y. We say that f is topological $G-$conjugate to g about $h: X \to Y$ if $h \circ f = g \circ h$ and h is an equivariant homeomorphism map from X to Y.

Theorem 3.4. Let (X, d) be metric $G-$ space, (Y, d) be metric $G-$ space and f be topologically $G-$conjugate to g about the map $h: X \to Y$. If h is a Lipschitz map with Lipschitz constant L_1 from X to Y and h^{-1} is a Lipschitz map with Lipschitz constant L_2 from Y to X, then the map f has $G-$ Lipschitz shadowing property if and only if the map g has $G-$Lipschitz shadowing property.

Proof. Suppose that the map f has the $G-$Lipschitz shadowing property. Then there exists positive constant $L_0 > 0$ and $\varepsilon_0 > 0$ such that for any $0 < \delta < \varepsilon_0$ and $(G, \delta)-$pseudo orbit $\{x_i\}_{i\ge 0}$ of f there exists a point $x \in X$ such that the sequence $\{x_i\}_{i\ge 0}$ is $(G, L_0\delta)-$shadowed by the point x. Let

$$L_3 = L_0 L_1 L_2 \text{ and } \varepsilon_1 = \frac{\varepsilon_0}{L_2}.$$

For any $0 < \eta < \varepsilon_1$, let $\{y_i\}_{i=0}^{+\infty}$ be $(G, \eta)-$pseudo orbit of g. Then for any $i \ge 0$ there exists $t_i \in G$ satisfying

$$d(t_i g(y_i), y_{i+1}) < \eta.$$

According to that h^{-1} is a Lipschitz map with Lipschitz constant L_2, we can get that

$$d(h^{-1}(t_i g(y_i)), h^{-1}(y_{i+1})) \le L_2 d(t_i g(y_i), y_{i+1}) < L_2\eta .$$

According to that h is an equivalent map and $h \circ f = g \circ h$, for any $i \ge 0$ we have that

$$d(t_i f(h^{-1}(y_i)), h^{-1}(y_{i+1})) < L_2\eta < \varepsilon_0 .$$

Thus $h^{-1}(y_i)$ is $(G, L_2\eta)-$ pseudo orbit of f . According to that f has the $G-$Lipschitz shadowing property, there exists $x \in X$ such that for any nonnegative integer $i \ge 0$ there exists $p_i \in G$ satisfying

$$d(f^i(x), p_i h^{-1}(y_i)) < L_0 L_2 \eta.$$

Since h is a Lipschitz map with Lipschitz constant L_1, we can obtain that

$$d(h(f^i(x)), h(p_i h^{-1}(y_i))) \le L_1 d(f^i(x), p_i h^{-1}(y_i)) < L_0 L_1 L_2 \eta.$$

According to that h is an equivalent map and $h \circ f = g \circ h$, for any $i \ge 0$ we have that

$$d(g^i(h(z), p_i y_i) < L_0 L_1 L_2 \eta = L_3 \eta.$$

Hence the map g has $G-$Lipschitz shadowing property.

The method is the same as above and the proof is omitted here. Thus, we complete the proof. □

4. G-Lipschitz shadowing property of iterative map

Definition 4.1. [23] Let (X,d) be a metric $G-$ space. The metric d is said to be invariant to the topological group G provided that $d(x,y) = d(gx, gy)$ for all $x, y \in X$ and $g \in G$.

Definition 4.2. [24] Let G be topological group. G is said to be commutative provided that $p \cdot g = g \cdot p$ for all $p, g \in G$.

Theorem 4.3. Let (X,d) be a compact metric $G-$space, $f: X \to X$ be an equivalent Lipschitz map with Lipschitz constant L and the metric d be invariant to the topological group G where G is commutative. Then the map f has the $G-$Lipschitz shadowing property if and only if for any positive integer $k \ge 2$ the iterative map f^k has the $G-$Lipschitz shadowing property.

Proof. Suppose that the map f has the $G-$Lipschitz shadowing property. Then there exists positive constant $L_0 > 0$ and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $(G,\varepsilon)-$pseudo orbit $\{x_i\}_{i\ge 0}$ of f there exists a point $x \in X$ such that the sequence $\{x_i\}_{i\ge 0}$ is $(G, L_0\varepsilon)-$ shadowed by the point x . Let $\{y_i\}_{i=0}^{+\infty}$ be $(G,\varepsilon)-$ pseudo orbit of f^k and $x_{ki+j} = f^j(y_i)$ where $i \ge 0$ and $0 \le j \le k-1$. Thus $\{x_i\}_{i=0}^{\infty}$ is $(G,\varepsilon)-$pseudo orbit of f . According to that f has the $G-$Lipschitz shadowing property, there exists $x \in X$ such that for any nonnegative integer $i \ge 0$ there exists $g_i \in G$ satisfying

$$d(f^i(x), g_i x_i) < L_0\varepsilon .$$

Hence for any $i \ge 0$ we have that

$$d(f^{ki}(x), g_{ki} x_{ki}) < L_0\varepsilon .$$

That is,

$$d((f^k)^i(x), g_{ki}y_i) < L_0\varepsilon .$$

So, the iterative map f^k has the $G-$Lipschitz shadowing property.

Suppose that the iterative map f^k has the $G-$Lipschitz shadowing property. Then there exists $L_1 > 0$ and $\varepsilon_1 > 0$ such that for any $0 < \delta < \varepsilon_1$ and any $(G,\delta)-$pseudo orbit $\{x_i\}_{i\geq 0}$ of f^k there exists a point $z \in X$ such that the sequence $\{x_i\}_{i\geq 0}$ is $(G, L_1\delta)-$shadowed by the point z .

Case1. When $L \geq 1$. Write

$$L_2 = L^{k-1} + L^{k-2} + \cdots + L + 1 .$$

For any $0 < \eta < \frac{\varepsilon_1}{L_2}$,let $\{x_i\}_{i=0}^{\infty}$ be $(G,\eta)-$ pseudo orbit $\{x_i\}_{i\geq 0}$ of f .Then for any $i \geq 0$ there exists $t_i \in G$ satisfying

$$d(t_i f(x_i), x_{i+1}) < \eta . \tag{1}$$

Hence for any $i \geq 0$ we have that

$$d(t_{ki} f(x_{ki}), x_{ki+1}) < \eta .$$

$$d(t_{ki+1} f(x_{ki+1}), x_{ki+2}) < \eta .$$

$$d(t_{ki+2} f(x_{ki+2}), x_{ki+3}) < \eta .$$

$$\cdots\cdots$$

$$d(t_{ki+k-2} f(x_{ki+k-2}), x_{ki+k-1}) < \eta .$$

$$d(t_{ki+k-1} f(x_{ki+k-1}), x_{ki+k}) < \eta .$$

According to that f is an equivalent Lipschitz map with Lipschitz constant L , we can get that

$$d(t_{ki} f^k(x_{ki}), f^{k-1}(x_{ki+1})) < L^{k-1}\eta .$$

$$d(t_{ki+1} f^{k-1}(x_{ki+1}), f^{k-2}(x_{ki+2})) < L^{k-2}\eta .$$

$$d(t_{ki+2} f^{k-2}(x_{ki+2}), f^{k-3}(x_{ki+3})) < L^{k-3}\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+k-2} f^2(x_{ki+k-2}), f(x_{ki+k-1})) < L\eta .$$

$$d(t_{ki+k-1} f(x_{ki+k-1}), x_{ki+k}) < \eta .$$

Since d is invariant to the topological group G and G is commutative, we can obtain that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+k-1} f^k(x_{ki}), t_{ki+1}t_{ki+2}\cdots t_{ki+k-1} f^{k-1}(x_{ki+1})) < L^{k-1}\eta .$$

$$d(t_{ki+1}t_{ki+2}\cdots t_{ki+k-1} f^{k-1}(x_{ki+1}), t_{ki+2}\cdots t_{ki+k-1} f^{k-2}(x_{ki+2})) < L^{k-2}\eta .$$

$$d(t_{ki+2}t_{ki+3}\cdots t_{ki+k-1} f^{k-2}(x_{ki+2}), t_{ki+3}\cdots t_{ki+k-1} f^{k-3}(x_{ki+3})) < L^{k-3}\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+k-2}t_{ki+k-1}f^2(x_{ki+k-2}),t_{ki+k-1}f(x_{ki+k-1}))<L\eta\ .$$

$$d(t_{ki+k-1}f(x_{ki+k-1}),x_{ki+k})<\eta\ .$$

Let $y_i = x_{ki}, i \geq 0$. Thus, for any $i \geq 0$ we have that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^k(y_i),y_{i+1})=d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^k(x_{ki}),x_{ki+k})<$$

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^k(x_{ki}),t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^{k-1}(x_{ki+1}))+$$

$$d(t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^{k-1}(x_{ki+1}),t_{ki+2}\cdots t_{ki+k-1}f^{k-2}(x_{ki+2}))+$$

$$d(t_{ki+2}t_{ki+3}\cdots t_{ki+k-1}f^{k-2}(x_{ki+2}),t_{ki+3}\cdots t_{ki+k-1}f^{k-3}(x_{ki+3}))+$$

$$\cdots\cdots$$

$$d(t_{ki+i-2}t_{ki+k-1}f^2(x_{ki+k-2}),t_{ki+k-1}f(x_{ki+k-1}))+$$

$$d(t_{ki+k-1}f(x_{ki+k-1}),x_{ki+k})<$$

$$L^{k-1}\eta+L^{k-2}\eta+L^{k-3}\eta+L^{k-4}\eta+\cdots+L\eta+\eta=$$

$$(L^{k-1}+L^{k-2}+\cdots+L+1)\eta=L_2\eta<\varepsilon_1\ .$$

Hence $\{y_i\}_{i=0}^{\infty}$ is $(G, L_2\eta)-$pseudo orbit $\{x_i\}_{i\geq 0}$ of f^k. According to that f^k has the $G-$Lipschitz shadowing property, there exists $z \in X$ such that for any nonnegative integer $i \geq 0$ there exists $p_i \in G$ satisfying

$$d((f^k)^i(z),p_iy_i)<L_1L_2\eta\ .$$

That is, for any $i \geq 0$, we have that

$$d(f^{ki}(z),p_ix_{ki})<L_1L_2\eta\ . \tag{2}$$

By Eq (1) for any $0 \leq j \leq k-1$ and $i \geq 0$ we can get that

$$d(t_{ki}f(x_{ki}),x_{ki+1})<\eta\ .$$

$$d(t_{ki+1}f(x_{ki+1}),x_{ki+2})<\eta\ .$$

$$d(t_{ki+2}f(x_{ki+2}),x_{ki+3})<\eta\ .$$

$$\cdots\cdots$$

$$d(t_{ki+j-2}f(x_{ki+j-2}),x_{ki+j-1})<\eta\ .$$

$$d(t_{ki+j-1}f(x_{ki+j-1}),x_{ki+j})<\eta\ .$$

According to that f is an equivalent Lipschitz map with Lipschitz constant L, we can get that

$$d(t_{ki}f^j(x_{ki}),f^{j-1}(x_{ki+1}))<L^{j-1}\eta\ .$$

$$d(t_{ki+1}f^{j-1}(x_{ki+1}), f^{j-2}(x_{ki+2})) < L^{j-2}\eta .$$

$$d(t_{ki+2}f^{j-2}(x_{ki+2}), f^{j-3}(x_{ki+3})) < L^{j-3}\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+j-2}f^{2}(x_{ki+j-2}), x_{ki+j-1}) < L\eta .$$

$$d(t_{ki+j-1}f(x_{ki+j-1}), x_{ki+j}) < \eta .$$

Since d is invariant to the topological group G and G is commutative, we can obtain that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki}), t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1})) < L^{j-1}\eta .$$

$$d(t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1}), t_{ki+2}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2})) < L^{j-2}\eta .$$

$$d(t_{ki+2}t_{ki+3}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2}), t_{ki+3}\cdots t_{ki+j-1}f^{j-3}(x_{ki+3})) < L^{j-3}\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+j-2}t_{ki+j-1}f^{2}(x_{ki+j-2}), t_{ki+j-1}f(x_{ki+j-1})) < L\eta .$$

$$d(t_{ki+j-1}f(x_{ki+j-1}), x_{ki+j}) < \eta .$$

By Eq (2) and according to that f is an equivalent Lipschitz map with Lipschitz constant L and d is invariant to the topological group, we can obtain that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}p_i^{-1}f^{ki+j}(z), t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki})) < L^{j}L_1L_2\eta .$$

Hence, we have that

$$\begin{gathered}
d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}p_i^{-1}f^{ki+j}(z), x_{ki+j}) < \\
d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}p_i^{-1}f^{ki+j}(z), t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki})) + \\
d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki}), t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1})) + \\
d(t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1}), t_{ki+2}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2})) + \\
d(t_{ki+2}t_{ki+3}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2}), t_{ki+3}\cdots t_{ki+j-1}f^{j-3}(x_{ki+3})) + \\
\cdots\cdots \\
d(t_{ki+j-2}t_{ki+j-1}f^{2}(x_{ki+j-2}), t_{ki+i-1}f(x_{ki+j-1})) + \\
d(t_{ki+j-1}f(x_{ki+j-1}), x_{ki+j}) < \\
L^{j}L_1L_2\eta + L^{j-1}\eta + L^{j-2}\eta + L^{j-3}\eta + \cdots + L\eta + \eta < \\
(L^{k}L_1L_2 + KL^{K})\eta < (L_1L_2 + K)L^{K}\eta .
\end{gathered}$$

That is,

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+i-1}p_i^{-1}f^{ki+j}(z),x_{ki+j})<(L_1L_2+K)L^K\eta . \qquad (3)$$

By Eqs (2) and (3) for any $i\geq 0$ there exists $s_i\in G$ satisfying

$$d(f^i(z),s_ix_i)<(L_1L_2+K)L^K\eta .$$

So, when $L\geq 1$, the map f has the $G-$Lipschitz shadowing property.

Case2. When $0<L<1$. For any $0<\eta<\frac{\varepsilon_1}{k}$, let $\{x_i\}_{i=0}^{\infty}$ be $(G,\eta)-$pseudo orbit $\{x_i\}_{i\geq 0}$ of f. Then for any $i\geq 0$ there exists $t_i\in G$ satisfying

$$d(t_if(x_i),x_{i+1})<\eta . \qquad (4)$$

Hence, for any $i\geq 0$ we have that

$$d(t_{ki}f(x_{ki}),x_{ki+1})<\eta .$$

$$d(t_{ki+1}f(x_{ki+1}),x_{ki+2})<\eta .$$

$$d(t_{ki+2}f(x_{ki+2}),x_{ki+3})<\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+k-2}f(x_{ki+2}),x_{ki+k-1})<\eta .$$

$$d(t_{ki+k-1}f(x_{ki+k-1}),x_{ki+k})<\eta .$$

According to that f is an equivalent Lipschitz map with Lipschitz constant L, we can get that

$$d(t_{ki}f^k(x_{ki}),f^{k-1}(x_{ki+1}))<L^{k-1}\eta .$$

$$d(t_{ki+1}f^{k-1}(x_{ki+1}),f^{k-2}(x_{ki+2}))<L^{k-2}\eta .$$

$$d(t_{ki+2}f^{k-2}(x_{ki+2}),f^{k-3}(x_{ki+3}))<L^{k-3}\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+k-2}f^2(x_{ki+k-2}),t_{ki+k-1}f(x_{ki+k-1}))<L\eta .$$

$$d(t_{ki+k-1}f(x_{ki+k-1}),x_{ki+k})<\eta .$$

Since d is invariant to the topological group G and G is commutative, we can obtain that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^k(x_{ki}),t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^{k-1}(x_{ki+1}))<L^{k-1}\eta .$$

$$d(t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^{k-1}(x_{ki+1}),t_{ki+2}\cdots t_{ki+k-1}f^{k-2}(x_{ki+2}))<L^{k-2}\eta .$$

$$d(t_{ki+2}t_{ki+3}\cdots t_{ki+k-1}f^{k-2}(x_{ki+2}),t_{ki+3}\cdots t_{ki+k-1}f^{k-3}(x_{ki+3}))<L^{k-3}\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+k-2}t_{ki+k-1}f^2(x_{ki+k-2}),t_{ki+k-1}f(x_{ki+k-1}))<L\eta .$$

$$d(t_{ki+k-1}f(x_{ki+k-1}),x_{ki+k})<\eta .$$

Write $y_i = x_{ki}$ where $i \geq 0$. We have that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+i-1}f^k(y_i),y_{i+1}) = d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^k(x_{ki}),x_{ki+k}) <$$

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^k(x_{ki}),t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^{k-1}(x_{ki+1}))+$$

$$d(t_{ki+1}t_{ki+2}\cdots t_{ki+k-1}f^{k-1}(x_{ki+1}),t_{ki+2}\cdots t_{ki+k-1}f^{k-2}(x_{ki+2}))+$$

$$d(t_{ki+2}t_{ki+3}\cdots t_{ki+k-1}f^{k-2}(x_{ki+2}),t_{ki+3}\cdots t_{ki+k-1}f^{k-3}(x_{ki+3}))+$$

$$\cdots\cdots$$

$$d(t_{ki+k-2}t_{ki+k-1}f^2(x_{ki+2}),t_{ki+k-1}f(x_{ki+k-1}))+$$

$$d(t_{ki+k-1}f(x_{ki+k-1}),x_{ki+k})<$$

$$L^{k-1}\eta + L^{k-2}\eta + L^{k-3}\eta + L^{k-4}\eta + \cdots + L\eta + \eta < k\eta .$$

Hence $\{y_i\}_{i=0}^{\infty}$ is $(G,k\eta)-$pseudo orbit $\{x_i\}_{i\geq 0}$ of f^k . According to that f^k has the $G-$Lipschitz shadowing property, there exists $z \in X$ such that for any nonnegative integer $i \geq 0$ there exists $p_i \in G$ satisfying

$$d((f^k)^i(z),p_iy_i) < L_1k\eta .$$

That is, for any $i \geq 0$, we have that

$$d(f^{ki}(z),p_ix_{ki}) < L_1k\eta . \tag{5}$$

By Eq (4), for any $0 \leq j \leq k-1$ and $i \geq 0$ we can get that

$$d(t_{ki}f(x_{ki}),x_{ki+1})<\eta .$$

$$d(t_{ki+1}f(x_{ki+1}),x_{ki+2})<\eta .$$

$$d(t_{ki+2}f(x_{ki+2}),x_{ki+3})<\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+j-2}f(x_{ki+j-2}),x_{ki+j-1})<\eta .$$

$$d(t_{ki+j-1}f(x_{ki+j-1}),x_{ki+j})<\eta .$$

According to that f is an equivalent Lipschitz map with Lipschitz constant L , we can get that

$$d(t_{ki}f^j(x_{ki}),f^{j-1}(x_{ki+1}))<L^{j-1}\eta .$$

$$d(t_{ki+1}f^{j-1}(x_{ki+1}),f^{j-2}(x_{ki+2}))<L^{j-2}\eta .$$

$$d(t_{ki+2}f^{j-2}(x_{ki+2}),f^{j-3}(x_{ki+3}))<L^{j-3}\eta .$$

$$\cdots\cdots$$

$$d(t_{ki+j-2}f^2(x_{ki+j-2}),t_{ki+j-1}f(x_{ki+j-1}))<L\eta\ .$$

$$d(t_{ki+j-1}f(x_{ki+j-1}),x_{ki+j})<\eta\ .$$

Since d is invariant to the topological group G and G is commutative, we can obtain that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki}),t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1}))<L^{j-1}\eta\ .$$

$$d(t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1}),t_{ki+2}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2}))<L^{j-2}\eta\ .$$

$$d(t_{ki+2}t_{ki+3}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2}),t_{ki+3}\cdots t_{ki+j-1}f^{j-3}(x_{ki+3}))<L^{j-3}\eta\ .$$

……

$$d(t_{ki+j-2}t_{ki+j-1}f^2(x_{ki+j-2}),t_{ki+j-1}f(x_{ki+j-1}))<L\eta\ .$$

$$d(t_{ki+j-1}f(x_{ki+j-1}),x_{ki+j})<\eta\ .$$

By Eq (5) and according to that f is an equivalent Lipschitz map with Lipschitz constant L and d is invariant to the topological group, we can obtain that

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}p_i^{-1}f^{ki+j}(z),t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki}))<L^{j}L_1k\eta\ .$$

Thus, we have that

$$\begin{gathered}
d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}p_i^{-1}f^{ki+j}(z),x_{ki+j})< \\
d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}p_i^{-1}f^{ki+j}(z),t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki}))+ \\
d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j}(x_{ki}),t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1}))+ \\
d(t_{ki+1}t_{ki+2}\cdots t_{ki+j-1}f^{j-1}(x_{ki+1}),t_{ki+2}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2}))+ \\
d(t_{ki+2}t_{ki+3}\cdots t_{ki+j-1}f^{j-2}(x_{ki+2}),t_{ki+3}\cdots t_{ki+j-1}f^{j-3}(x_{ki+3}))+ \\
\cdots\cdots \\
d(t_{ki+j-2}t_{ki+j-1}f^2(x_{ki+j-2}),t_{ki+j-1}f(x_{ki+j-1}))+ \\
d(t_{ki+j-1}f(x_{ki+j-1}),x_{ki+j})< \\
L^{j}L_1k\eta+L^{j-1}\eta+L^{j-2}\eta+L^{j-3}\eta+\cdots+L\eta+\eta<L_1k\eta+k\eta=k(L_1+1)\eta\ .
\end{gathered}$$

That is,

$$d(t_{ki}t_{ki+1}t_{ki+2}\cdots t_{ki+i-1}p_i^{-1}f^{ki+j}(z),x_{ki+j})<k(L_1+1)\eta\ . \tag{6}$$

By Eqs (5) and (6) for any $i\geq 0$ there exists $s_i\in G$ satisfying

$$d(f^{i}(z),s_ix_i)<k(L_1+1)\eta\ .$$

Hence, when $0 < L < 1$, the map f has the $G-$ Lipschitz shadowing property. Thus, we complete the proof. □

5. Conclusions

Firstly, we study the dynamical properties of $(G, w)-$ Chaos in the inverse limit space under group action in the paper. We obtained that the self-map f is $(G, w)-$ chaotic, the shift map σ is $(G, w)-$ chaotic. The conclusion generalizes the corresponding results of $w-$ Chaos given in Li [1]. Secondly, the dynamical properties of $G-$ Lipschitz shadowing property are studied under topological $G-$ conjugate and iterative systems. The following conclusions are obtained. (1) If f is topologically $G-$ conjugate to g, then the map f has $G-$ Lipschitz shadowing property if and only if the map g has $G-$ Lipschitz shadowing property. (2) For any positive integer $k \geq 2$, the map f has the $G-$ Lipschitz shadowing property if and only if the iterative map f^k has the $G-$ Lipschitz shadowing property. These results enrich the theory of topological $G-$ conjugate and iterative system. It provided the theoretical basis and scientific foundation for the application of various shadowing property in computational mathematics and biological mathematics.

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Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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