Mathematics

Research article

Novel analysis of nonlinear dynamics of a fractional model for tuberculosis disease via the generalized Caputo fractional derivative operator (case study of Nigeria)

Saima Rashid[1,*], **Yolanda Guerrero Sánchez**[2,*], **Jagdev Singh**[3] **and Khadijah M Abualnaja**[4]

[1] Department of Mathematics, Government College University, Faisalabad, Pakistan

[2] Departamento de Anatomía y Psicobiología, Universidad de Murcia, 30100-Murcia

[3] Department of Mathematics, JECRC University, Jaipur 303905, Rajasthan, India

[4] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: saimarashid@gcuf.edu.pk, yolanda.guerreros@um.es.

Abstract: We propose a new mathematical framework of generalized fractional-order to investigate the tuberculosis model with treatment. Under the generalized Caputo fractional derivative notion, the system comprises a network of five nonlinear differential equations. Besides that, the equilibrium points, stability and basic reproductive number are calculated. The concerned derivative involves a power-law kernel and, very recently, it has been adapted for various applied problems. The existence findings for the fractional-order tuberculosis model are validated using the Banach and Leray-Schauder nonlinear alternative fixed point postulates. For the developed framework, we have generated various forms of Ulam's stability outcomes. To investigate the estimated response and nonlinear behaviour of the system under investigation, the efficient mathematical formulation known as the $\wp$-Laplace Adomian decomposition technique algorithm was implemented. It is important to mention that, with the exception of numerous contemporary discussions, spatial coherence was considered throughout the fractionalization procedure of the classical model. Simulation and comparison analysis yield more versatile outcomes than the existing techniques.

1. Introduction

TB (Tuberculosis) is one of the communicable diseases caused by bacteria in the respiratory systems of humans. According to the World Health Organization (WHO), almost 10 million individuals were diagnosed with tuberculosis in 2017 and 1.5 million others died from TB worldwide [1]. Experts are concerned that a global growth in the number of TB patients will endanger a significant proportion of individuals [2]. The lungs, the cerebellum, the endocrine system, the peripheral nerve mechanism, the vertebral column, and other tissues and organs may indeed be disrupted by this microorganism. Tuberculosis infection has been found in multiple cultures throughout history, including Mesopotamia, Persia, and Greece (see [3]). At this moment, one-third of the globe's infections are caused by tuberculosis, and the number of contagious people is growing at a pace of one every second [4]. In 2015, the aforesaid ailment was among the ten leading contributors of mortality globally, with around 10.4 million people affected. That year, 1.8 million people died from contagious illnesses, including 0.4 million people infected with hepatitis and tuberculosis . Major states (Bangladesh, Burma, Ghana, Burma, Ethiopia, and Namibia) accounted for 60 percent of tuberculosis infection worldwide [5]. Author [6] reported that tuberculosis and hepatitis are the leading factors of mortality globally, especially in Sub-Saharan Africa. Additionally, the HIV/AIDS outbreak poses a severe challenge to several governments around the globe. It is conclusive proof that vaccinations such as Bacillus Calmatte-Guerine (BCG) prevent kids globally from severe illness acquisition [7]. As a response, contemporary medication has lately been employed to discover and cure underlying tuberculosis in order to minimize the bacteria's tendency to transmission from collapsing, because only representatives of the contagious category may disseminate the infection to people.

Numerous processes and strategies are indeed being tried throughout the globe to address the source and prevent such maladies in the community. One of the most effective techniques is mathematical simulation, which enables us to comprehend the mechanisms of illness spread and propose methods for controlling illnesses in communities. The specified zone was formally established in 1927. Up to this point, a set of hypotheses have already been designed and examined (see [8–12]). In this approach, the following five frameworks for tuberculosis were developed in [13]:

$$\begin{cases} \frac{d\mathbf{M}(\varrho)}{d\varrho} = \theta\zeta - (\eta + \mu)\mathbf{M}(\varrho), \\ \frac{d\mathbf{N}(\varrho)}{d\varrho} = (1-\theta)\zeta + \eta\mathbf{M}(\varrho) - \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - \zeta\mathbf{N}(\varrho), \\ \frac{d\mathbf{O}(\varrho)}{d\varrho} = \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - (\varpi + \varphi + \mu)\mathbf{O}(\varrho), \\ \frac{d\mathbf{P}(\varrho)}{d\varrho} = \varphi\mathbf{O}(\varrho) - (\phi + \mu + \psi)\mathbf{P}(\varrho), \\ \frac{d\mathbf{Q}(\varrho)}{d\varrho} = \varpi\mathbf{O}(\varrho) + \phi\mathbf{P}(\varrho) - \mu\mathbf{Q}(\varrho). \end{cases} \quad (1.1)$$

The total population $\mathcal{N}(\varrho) = \mathbf{M}(\varrho) + \mathbf{N}(\varrho) + \mathbf{O}(\varrho) + \mathbf{P}(\varrho) + \mathbf{Q}(\varrho)$ has been classified into five groups according to the above-mentioned framework: Immunization group $\mathbf{M}$, susceptibility group $\mathbf{N}$, infected latent group $\mathbf{O}$, infectious group $\mathbf{P}$, and recovered group $\mathbf{Q}$. The following are the characteristics of the system under evaluation: The signified indicates the immunological component at conception ζ, η reflects the proportion of farrowing off the medication, the genetic mortality value is designated by the sign μ, β denotes the tuberculosis peristaltic speed, the therapeutic efficacy of contagious predisposition is designated by ϖ, φ is the proportion of collapse of innate tuberculosis into extremely contagious

tuberculosis, ϕ the effective remedy of contagious tuberculosis people, and the damage arising from the illness is represented by ψ.

In most cases, classical calculus does not adequately investigate the complexities of real-life scenarios in science and technology. Fractional calculus has received a lot of emphasis in recent generations in an attempt to address this weakness. We certainly recognize that scientists are progressively using fractional calculus for numerical techniques [14–26]. As a result, we explore the system in (1.1) using generalized Caputo fractional derivative as described in the following:

$$
\begin{cases}
({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{M})(\varrho) = \theta\zeta - (\eta + \mu)\mathbf{M}(\varrho), \\
({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{N})(\varrho) = (1 - \theta)\zeta + \eta\mathbf{M}(\varrho) - \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - \zeta\mathbf{N}(\varrho), \\
({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{O})(\varrho) = \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - (\varpi + \varphi + \mu)\mathbf{O}(\varrho), \\
({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{P})(\varrho) = \varphi\mathbf{O}(\varrho) - (\phi + \mu + \psi)\mathbf{P}(\varrho), \\
({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{Q})(\varrho) = \varpi\mathbf{O}(\varrho) + \phi\mathbf{P}(\varrho) - \mu\mathbf{Q}(\varrho).
\end{cases}
\tag{1.2}
$$

Model (1.2) is investigated under biologically viable initial settings:

$$
(\mathbf{M}(0), \mathbf{N}(0), \mathbf{O}(0), \mathbf{P}(0), \mathbf{Q}(0))^{\varrho} = (\mathbf{M}_0, \mathbf{N}_0, \mathbf{O}_0, \mathbf{P}_0, \mathbf{Q}_0)^{\varrho}.
$$

We investigate the system 1.1 proposed by [13] under the generalized Caputo fractional derivative [27] in light of the above-mentioned debate. The major goal of this study is to use well-known fixed point formalism like Banach's and Leray-Schauder nonlinear alternatives to investigate the existence and uniqueness of the fractional tuberculosis model described in (1.2). Furthermore, the stability analysis of the system is explored from the perspective of various stabilities, such as Ulam's, Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stable. Also, a novel algorithm approach with the aid of $\wp$-LADM to generate approximate fractional tuberculosis model solutions for different fractional derivative orders. Several observations on the suggested algorithm's convergence and stability are addressed. Additionally, experimental challenges are studied to demonstrate the suggested algorithm's efficacy, convenience, and characteristics.

The rest of this paper is organized as follows. In Section 2, we accomplish the description and formulation of the model. Section 3 deals with the disease-free and endemic equilibrium points and the corresponding global stability analysis. In Section 4, we establish the existence and uniqueness of the solution to the model via generalized Caputo fractional derivative operator. In the last section, we consider the analytical results of the fractional model by incorporating the modified Laplace Adomian decomposition into the model. Moreover, in this section, we perform a numerical simulation to verify the effect of the designed strategy for different values of fractional order and different compartments of the model.

2. Preliminaries

This part states certain formulae, concepts, and essential findings for generalized Caputo fractional derivative and related formulas that will be relevant throughout the study. For further information, see [28, 29].

Definition 2.1. ([27]) For $\vartheta, \wp > 0$, then the generalized fractional integral of the mapping f_1 is denoted by $^{\wp}\mathcal{I}_{a_1^+}^{\vartheta}$ and expressed as

$$(^{\wp}\mathcal{I}_{a_1^+}^{\vartheta} f_1)(\varrho) = \frac{1}{\Gamma(\vartheta)} \int_{a_1}^{\varrho} \left(\frac{\varrho^{\wp} - \mathbf{s}^{\wp}}{\wp}\right)^{\vartheta-1} f_1(\mathbf{s}) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}, \quad \varrho > a_1 \geq 0, \tag{2.1}$$

and $\Gamma(z) = \int_{0}^{+\infty} \exp(-\mathbf{s})\mathbf{s}^{z-1} d\mathbf{s}$ is the Euler-Gamma function.

Definition 2.2. ([27]) For $\vartheta \in (0, 1], \wp > 0$, then the generalized fractional derivative of a continuous mapping $f_1 : [0, +\infty] \mapsto \mathcal{R}$ is denoted by $^{\wp}\mathcal{D}_{a_1^+}^{\vartheta}$ is expressed as

$$(^{\wp}\mathcal{D}_{a_1^+}^{\vartheta} f_1)(\varrho) = \frac{1}{\Gamma(1 - \vartheta)} \int_{a_1}^{\varrho} \left(\frac{\varrho^{\wp} - \mathbf{s}^{\wp}}{\wp}\right)^{-\vartheta} f_1(\mathbf{s}) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}, \quad \varrho > a_1 \geq 0. \tag{2.2}$$

Definition 2.3. ([27]) For $\vartheta \in (0, 1], \wp > 0$, then the generalized Caputo fractional derivative of the continuous mapping $f_1 : [0, +\infty] \mapsto \mathcal{R}$ is denoted by $^{C\wp}\mathcal{D}_{a_1^+}^{\vartheta}$ and expressed as

$$(^{C\wp}\mathcal{D}_{a_1^+}^{\vartheta} f_1)(\varrho) = \frac{1}{\Gamma(n - \vartheta)} \int_{a_1}^{\varrho} \left(\frac{\varrho^{\wp} - \mathbf{s}^{\wp}}{\wp}\right)^{-\vartheta} (\phi^n f_1)(\mathbf{s}) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}, \tag{2.3}$$

where $\varrho > a_1 \geq 0$ and $\phi = \varrho^{1-\phi} \frac{d}{d\varrho}$.

Definition 2.4. ([30]) The $\wp$-Laplace transform of a continuous mapping $f : [0, +\infty] \mapsto \mathcal{R}$ is described as

$$\mathcal{L}_{\wp}\{f_1(\varrho)\}(\mathbf{s}) = \int_{0}^{\infty} \exp\left(-\mathbf{s}\frac{\varrho^{\wp}}{\wp}\right) f_1(\varrho) d\varrho, \quad \mathfrak{R}(\mathbf{s}) > 0. \tag{2.4}$$

The $\wp$-Laplace transform form of the generalized Caputo fractional derivative of a continuous mapping f_1 is presented by [30]:

$$\mathcal{L}_{\wp}\{(^{C\wp}\mathcal{D}_{a_1^+}^{\vartheta} f_1)(\varrho)\} = \mathbf{s}^{\vartheta} \mathcal{L}_{\wp}\{f_1(\varrho)\} - \sum_{\kappa=0}^{\mathbf{n}-1} \mathbf{s}^{\vartheta-\kappa-1} (^{\wp}\mathcal{I}^{\vartheta} \phi^n f_1)(0). \tag{2.5}$$

Now we present a significant result, which is known as the Banach fixed point theorem, and it will be useful for our next results.

Throughout this investigation, we symbolize Banach space by $\mathbf{B}_s$ and fixed point by (f_p).

Lemma 2.5. ([31]) Assume that a $\mathbf{B}_s$ of χ, and also, there be a nonempty subset Δ which is closed in χ. If there be a contraction map $\Upsilon : \Delta \mapsto \Delta$, then, Υ has a f_p in Δ.

Our next result is the well-known Leray-Schauder nonlinear alternative (LSNA), see [32].

Lemma 2.6. *([32]) Assume that a* $\mathbf{B}_s$ *of* χ *and* $\tilde{C} \subseteq \chi$ *assume to be closed and convex. Also, suppose that an open subset of* $\tilde{C}$ *is* $\tilde{V}$ *and* $0 \in \tilde{V}$. *Let there be a continuous compact map* $\Delta : \underline{V} \mapsto \tilde{C}$, *then either*
(i) Δ *has* f_p *in* $\underline{V}$
or
(ii) there is $a \in \partial V$ *(the frontier of* $\tilde{V}$ *in* $\tilde{C}$*) and* $\nu \in (0, 1)$ *with* $x_1 = \nu \Delta x_1$.

3. TB-model under generalized Caputo fractional derivative sense

Here, we investigate the **TB**-model infection via the generalized Caputo fractional derivative [27] and the interpretation of all variables is explained in the preceding sections.

$$\begin{cases} ({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{M})(\varrho) = \Lambda_1(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}), \\ ({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{N})(\varrho) = \Lambda_2(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}), \\ ({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{O})(\varrho) = \Lambda_3(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}), \\ ({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{P})(\varrho) = \Lambda_4(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}), \\ ({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{Q}(\varrho) = \Lambda_5(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}), \end{cases} \tag{3.1}$$

where ${}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}(.)$ signifies the generalized Caputo fractional derivative of order ϑ with $\vartheta \in (0, 1]$ and $\wp > 0$. In the preceding setup, we streamline the paradigm (1.2) for ease of explanation.

Also, the nonlinear mappings $\Lambda_1 - \Lambda_5$ are presented by

$$\begin{cases} \Lambda_1(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) = \theta\zeta - (\eta + \mu)\mathbf{M}(\varrho), \\ \Lambda_2(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) = (1 - \theta)\zeta + \eta\mathbf{M}(\varrho) - \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - \zeta\mathbf{N}(\varrho), \\ \Lambda_3(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) = \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - (\varpi + \varphi + \mu)\mathbf{O}(\varrho), \\ \Lambda_4(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) = \varphi\mathbf{O}(\varrho) - (\phi + \mu + \psi)\mathbf{P}(\varrho), \\ \Lambda_5(\varrho, \mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) = \varpi\mathbf{O}(\varrho) + \phi\mathbf{P}(\varrho) - \mu\mathbf{Q}(\varrho), \end{cases} \tag{3.2}$$

having initial conditions $(\mathbf{M}(0), \mathbf{N}(0), \mathbf{O}(0), \mathbf{P}(0), \mathbf{Q}(0))^\varrho = (\mathbf{M}_0, \mathbf{N}_0, \mathbf{O}_0, \mathbf{P}_0, \mathbf{Q}_0)^\varrho$.

3.1. Equilibrium points and stability analysis

Next we state the epidemiologically feasible (positivity and boundedness) region of this investigation in Theorem 3.1 and illustrate that the region is positively invariant and bounded.

Theorem 3.1. *The epidemiologically feasible region of TB model* (1.2) *is presented by*

$$\Psi =: \left\{ (\mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) \in \mathbb{R}_+^5 : 0 \le \mathbf{M} + \mathbf{N} + \mathbf{O} + \mathbf{P} + \mathbf{Q} \le \mathcal{N} \le \frac{\theta\zeta}{\eta + \mu} \right\}. \tag{3.3}$$

The existence and uniqueness of the solution of model (1.2) are now proved, and it remains to show that the set Ψ defined in (3.3) is positively invariant. The following lemma will be used for the proof of Theorem 3.1.

Lemma 3.2. *(Generalized mean value theorem) ([33]) Suppose $g_1(\mathbf{x}) \in \mathbb{C}[\mathbf{a}, \mathbf{b}]$ and let $^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathcal{G}(\mathbf{x}) \in \mathbb{C}[\mathbf{a}, \mathbf{b}]$, when $0 < \vartheta \le 1$. Then we have $\mathcal{G}(\mathbf{x}) = \mathcal{G}(\mathbf{a}) + \frac{1}{\Gamma(\vartheta)} {}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathcal{G}(\xi)(\frac{\mathbf{x}^{\wp}-\mathbf{a}^{\wp}}{\wp})^{\vartheta}, \ \wp > 0$, when $0 \le \xi \le \mathbf{x}, \ \forall \mathbf{x} \in (\mathbf{a}, \mathbf{b}]$.*

Clearly, by utilizing by Lemma (3.2), if $\mathcal{G}(\mathbf{x}) \in [0, \mathbf{b}]$, $^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathcal{G}(\mathbf{x}) \in (0, \mathbf{b}]$ and $^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathcal{G}(\mathbf{x}) \ge 0, \ \forall \mathbf{x} \in (0, \mathbf{b}]$ when $\vartheta \in (0, 1]$, then the function $\mathcal{G}(\mathbf{x})$ is n non-decreasing and if $^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathcal{G}(\mathbf{x}) \le 0, \ \forall \mathbf{x} \in (0, \mathbf{b}]$, then the mapping $\mathcal{G}(\mathbf{x})$ is non-increasing $\forall \mathbf{x} \in [0, \mathbf{b}]$.

To show that Ψ is positively invariant, by means of Lemma 3.2, we have

$$
\begin{cases}
\left({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{M} \right)\big|_{\mathbf{M=0}} = \theta\zeta, \\
\left({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{N} \right)\big|_{\mathbf{N=0}} = (1-\theta)\zeta + \eta\mathbf{M}(\varrho), \\
\left({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{O} \right)\big|_{\mathbf{O=0}} = \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho), \\
\left({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{P} \right)\big|_{\mathbf{P=0}} = \varphi\mathbf{O}(\varrho), \\
\left({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathbf{Q} \right)\big|_{\mathbf{Q=0}} = \varpi\mathbf{O}(\varrho) + \phi\mathbf{P}(\varrho).
\end{cases}
\tag{3.4}
$$

It follows from (3.4) that each of the solution (1.2) is non-negative and remains in $\mathbb{R}_+^5$, and so the set Ψ described in (3.3) is positively invariant for the system (1.2).

Ultimately, to construct the boundedness of the solution of the fractional model (1.2), taking into consideration that all the parameters are positive, we continue by adding all equations of the model that presents

$$
{}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathcal{N}(\varrho) = \zeta - \mu\mathcal{N}(\varrho) - \zeta\mathbf{N} - \mu(\mathbf{O} + \mathbf{Q} + \mathbf{P}) - \psi\mathbf{P} \le \zeta - \mu\mathcal{N}(\varrho).
\tag{3.5}
$$

Applying Laplace transform leads to

$$
\mathcal{L}_{\wp}\left({}^{C\wp}\mathcal{D}_{0^+}^{\vartheta}\mathcal{N}(\varrho) + \mu\mathcal{N}(\varrho) \right) \le \mathcal{L}_{\wp}(\zeta)
$$

$$
\mathbf{s}^{\wp}\mathcal{L}_{\wp}(\mathcal{N}) - \mathbf{s}^{\wp-1}\mathcal{N}(0) \le \frac{\zeta}{\mathbf{s}}
$$

$$
\mathcal{L}_{\wp}(\mathcal{N}) \le \left(\frac{\zeta}{\mathbf{s}^{\wp+1}} + \frac{1}{\mathbf{s}}\mathcal{N}(0) \right).
\tag{3.6}
$$

Applying the inverse transform, the solution is presented by

$$
\mathcal{N}(\varrho) = \zeta - \int_{a_1}^{\varrho} \left(\frac{\varrho^{\wp} - \mathbf{s}^{\wp}}{\wp} \right)^{-\vartheta} (\phi^n f_1)(\mathbf{s}) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} + \mathcal{N}(0)\left(\frac{\varrho^{\wp} - \mathbf{s}^{\wp}}{\wp} \right)^{-\vartheta},
\tag{3.7}
$$

it is not difficult to observe that $\mathcal{N}(\varrho) \mapsto \zeta$ as $\varrho \mapsto \infty$. Hence (3.3) is the biologically feasible region of system (1.2).

3.2. The disease-free equilibrium point

The disease-free equilibrium of system (1.2) is given by $\mathcal{N}_0 = \left(\frac{\theta\zeta}{\eta+\mu}, \frac{(\eta+\mu)(1-\theta)\zeta+eta\theta\zeta}{\mu(\eta+\mu)}, 0, 0, 0 \right)$ as the disease free equilibrium state

$$
\mathbf{M}^* = \frac{\theta\zeta}{\eta + \mu}, \qquad \mathbf{N}^* = \frac{(\varpi + \varphi + \mu)(\phi + \mu + \psi)}{\beta\varphi},
$$

$$\mathbf{O}^* = \frac{(\phi + \mu + \psi)(\zeta\beta\varphi(\mu + \eta) - \beta\varphi\mu\theta\zeta - \mu(\mu + \eta)(\varpi + \varphi + \mu)(\phi + \mu + \psi))}{\beta\varphi(\mu + \eta)(\mu(\phi + \mu + \psi) + \varpi(\mu + \psi) + \varpi(\phi + \mu + \psi) + \varphi\phi)},$$

$$\mathbf{P}^* = \frac{(\zeta\beta\varphi(\mu + \eta) - \beta\varphi\mu\theta\zeta - \mu(\mu + \eta)(\varpi + \psi + \mu)(\phi + \mu + \psi))\varphi}{\beta\varphi(\mu + \eta)(\mu(\phi + \mu + \psi) + \varpi(\mu + \psi) + \varpi(\phi + \mu + \psi) + \varphi\phi)},$$

$$\mathbf{Q}^* = \frac{(\varpi(\phi + \mu + \psi) + \varphi\phi)(\zeta\beta\varphi(\mu + \eta) - \beta\varphi\mu\theta\zeta - \mu(\mu + \eta)(\varpi + \varphi + \mu)(\psi + \mu + \phi))}{\mu\beta\varphi(\mu + \eta)(\mu(\phi + \mu + \psi) + \varpi(\mu + \psi) + \varpi(\phi + \mu + \psi) + \varphi\phi)}$$

$$(3.8)$$

as the endemic equilibrium state.

3.3. The basic reproductive number $\mathcal{R}_0$

In order to evaluate the basic reproduction number, we consider only the infectious classes of the model. Let $\mathcal{V} = (L, I)^t$, with the aid of the proposed system, we write

$$\frac{d\mathcal{V}}{d\varrho} = \mathcal{F} - \mathcal{V} = \begin{bmatrix} \beta L I \\ 0 \end{bmatrix} - \begin{bmatrix} (\varpi + \varphi + \mu)L \\ -\varphi L + (\phi + \mu + \varpi)I \end{bmatrix}.$$

The Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ are given by

$$\mathbb{J} = \begin{bmatrix} 0 & \beta I^0 \\ 0 & 0 \end{bmatrix} \quad and \quad \mathbb{V} = \begin{bmatrix} \varpi + \varphi + \mu & 0 \\ -\varphi & (\phi + \mu + \varpi)I \end{bmatrix}.$$

The inverse matrix of $\mathbb{V}$ is given by

$$\mathbb{V}^{-1} = \begin{bmatrix} \frac{1}{\varpi + \varphi + \mu} & 0 \\ \varpi & \frac{1}{\phi + \mu + \varpi} \end{bmatrix}.$$

Hence, the next generation matrix $\mathbb{J}\mathbb{V}^{-1}$ is calculated a

$$\mathbb{J}\mathbb{V}^{-1} = \begin{bmatrix} \frac{\beta I^0 \varpi}{(\varpi + \varphi + \mu)(\varpi + \varphi + \psi)} & \frac{\beta I^0}{\varpi + \varphi + \mu} \\ 0 & 0 \end{bmatrix}. \tag{3.9}$$

The spectral radius of the next generation matrix (3.9) gives the threshold quantity $\mathcal{R}_0$ [34]. Thus

$$\mathcal{R}_0 = \frac{\beta\varpi(\eta + \mu - \mu\theta)\varpi}{(\eta + \mu)(\varpi + \varphi + \mu)(\varpi + \psi + \varphi)}.$$

This quantity plays the key role in stability analysis and in finding conditions for the said purpose.

Theorem 3.3. *For $\mathcal{R}_0$, the system* (1.2) *has a unique equilibrium point $\mathcal{N}^* = (\mathbf{M}^*, \mathbf{N}^*, \mathbf{O}^*, \mathbf{P}^*, \mathbf{Q}^*)$ given by* (3.8). *The global stability of the endemic equilibrium point is proved in Theorem 3.4 by utilizing the Lyapunov function method.*

Theorem 3.4. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium point $\mathcal{N}^*$ of system* (1.2) *is globally asymptotically stable in the region Ψ.*

Proof. Define a Lyapunov function candidate by

$$\mho(\mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) = \frac{1}{2}\big((\mathbf{M} - \mathbf{M}^*) + (\mathbf{N} - \mathbf{N}^*) + (\mathbf{O} - \mathbf{O}^*) + (\mathbf{P} - \mathbf{p}^*) + (\mathbf{Q} - \mathbf{Q}^*)\big)^2. \qquad (3.10)$$

Then $\mho(\mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) \geq 0$ and $\mho(\mathbf{M}^*, \mathbf{N}^*, \mathbf{O}^*, \mathbf{P}^*, \mathbf{Q}^*) = 0$. Also, $\frac{d\mho}{d\varrho} = [(\mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q}) - (\mathbf{M}^*, \mathbf{N}^*, \mathbf{O}^*, \mathbf{P}^*, \mathbf{Q}^*)]\frac{d\mathcal{N}}{d\varrho}$.

Since $\mathbf{M}^*, \mathbf{N}^*, \mathbf{O}^*, \mathbf{P}^*, \mathbf{Q}^* = \frac{\zeta}{\delta}$ and $\frac{d\mathcal{N}}{d\varrho} = \zeta - \mu\mathcal{N}(\varrho) - \zeta\mathbf{N} - \mu(\mathbf{O} + \mathbf{Q} + \mathbf{P}) - \psi\mathbf{P}$, we have

$$\frac{d\mho}{d\varrho} = (\zeta - \mu\mathcal{N})(\zeta - \mu\mathcal{N}(\varrho) - \zeta\mathbf{N} - \mu(\mathbf{O} + \mathbf{Q} + \mathbf{P}) - \psi\mathbf{P}) \leq 0.$$

Note that at the endemic equilibrium point, we have $\mathcal{N} \leq \zeta/\delta$. Hence, it follows that $\frac{d\mho}{d\varrho} \leq 0$ and $\frac{d\mho}{d\varrho} = 0$ if and only if $\mathbf{M} = \mathbf{M}^*$, $\mathbf{N} = \mathbf{N}^*$, $\mathbf{O} = \mathbf{O}^*$, $\mathbf{P} = \mathbf{P}^*$, $\mathbf{Q} = \mathbf{Q}^*$. Therefore the largest closed and bounded invariant set in $\{\mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q} \in \Psi\}$ is the set $\{\mathcal{N}^* : \mathcal{N}^* = (\mathbf{M}^*, \mathbf{N}^*, \mathbf{O}^*, \mathbf{P}^*, \mathbf{Q}^*)\}$. By LaSalle's invariance principle the unique equilibrium point $\mathcal{N}^*$ is globally asymptotically stable when $\mathcal{R}_0 > 1$ in the region Ψ. $\qquad \square$

3.4. *Local stability analysis*

In this section, we analyze the local stability of the abstaining-free equilibrium and the abstaining equilibrium.

Theorem 3.5. *The abstaining-free equilibrium $\mathcal{N}_0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$, whereas $\mathcal{N}_0$ is unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix at $\mathcal{N}_0$ is given by

$$\mathcal{J} = \begin{bmatrix} -(\eta + \mu) & 0 & 0 & 0 & 0 \\ \eta & -(\beta\mathbf{N} + \zeta) & 0 & -\beta\mathbf{N} & 0 \\ 0 & \beta\mathbf{P} & -(\varpi + \varphi + \mu) & \beta\mathbf{N} & 0 \\ 0 & 0 & \varphi & -(\phi + \mu + \psi) & 0 \\ 0 & 0 & \varpi & \phi & -\mu \end{bmatrix}. \qquad (3.11)$$

It follows that $\mathcal{J}$ are

$$\det(\mathcal{J} - \lambda\mathcal{I}) = 0.$$

$$\det\begin{bmatrix} -(\eta + \mu + \lambda) & 0 & 0 & 0 & 0 \\ \eta & -(\beta\mathbf{N} + \zeta + \lambda) & 0 & -\beta\mathbf{N} & 0 \\ 0 & \beta\mathbf{P} & -(\varpi + \varphi + \mu + \lambda) & \beta\mathbf{N} & 0 \\ 0 & 0 & \varphi & -(\phi + \mu + \psi + \lambda) & 0 \\ 0 & 0 & \varpi & \phi & -(\mu + \lambda) \end{bmatrix} = 0. \qquad (3.12)$$

At the disease free equilibrium state $\mathcal{N}_0 = \left(\frac{\theta\zeta}{\eta+\mu}, \frac{(\eta+\mu)(1-\theta)\zeta+eta\theta\zeta}{\mu(\eta+\mu)}, 0, 0, 0\right)$. Hence, evaluating the determinant and plugging 0 for $\mathbf{P}$ in (3.12) yields:

$$(\eta + \mu + \lambda)(\mu + \lambda)\big\{ - (\varpi + \varphi + \mu + \lambda)(\phi + \mu + \psi + \lambda) + \beta\mathbf{N}\varphi\big\} = 0.$$

Therefore, eigenvalues of the characteristic equation of $\lambda_1 = \lambda_2 = -\mu$ and $\lambda_3 = -(\eta + \mu)(\mathcal{R}_0 - 1)$. Therefore, all the eigenvalues of the characteristic equation are negative if $\mathcal{R}_0 < 1$. Thus, $|Arg(\lambda_i)| = \pi > (\vartheta\pi/2)$ for $i = 1, 2, 3$. Hence, the equilibrium point $\mathcal{N}_0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

Now, we study the local stability of the abstaining equilibrium $\mathcal{N}^*$.

The Jacobian matrix at $\mathcal{N}^*$ is given by

$$\mathcal{J}_{\mathcal{N}^*} = \begin{bmatrix} -(\eta + \mu) & 0 & 0 & 0 & 0 \\ \eta & -\beta\mathbf{P}^* - \zeta & 0 & \beta\mathbf{N}^* & 0 \\ 0 & \beta\mathbf{N}^* & -(\varpi + \varphi + \mu) & \beta\mathbf{N}^* & 0 \\ 0 & 0 & \varphi & -(\phi + \mu + \psi) & 0 \\ 0 & 0 & \varpi & \phi & -\mu \end{bmatrix}. \tag{3.13}$$

Its characteristic equation is

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{3.14}$$

$$c_1 = (\eta + \mu), \quad c_2 = (\beta + \zeta)(\varpi + \varphi + \mu)(\mathcal{R}_0 - 1), \quad c_3 = (\eta + \mu)(\phi + \mu + \psi)(\mathcal{R}_0 - 1),$$
$$c_1c_2 - c_3 = \big((\eta + \mu)(\beta + \zeta)(\varpi + \varphi + \mu) - (\eta + \mu)(\phi + \mu + \psi)\big)(\mathcal{R}_0 - 1). \tag{3.15}$$

If $\mathcal{R}_0 > 1$, then $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_1c_2 > c_3$. So, the Routh-Hurwitz conditions are satisfied. Let $\mathcal{D}(U)$ denote the discriminant of the polynomial $U(\lambda)$ given by (3.14), then

$$\mathcal{D}(U) = -\begin{vmatrix} 1 & c_1 & c_2 & c_3 & 0 \\ 0 & 1 & c_1 & c_2 & c_3 \\ 3 & 2c_1 & c_2 & 0 & 0 \\ 0 & 3 & 2c_1 & c_2 & 0 \\ 0 & 0 & 3 & 2c_1 & c_2 \end{vmatrix}$$
$$= 4c_1^3 c_3 - c_1^2 c_2^2 - 18c_1 c_2 c_3 + 4c_2^3 + 27c_3^2. \tag{3.16}$$

$\square$

From [35], we have the following theorem.

Theorem 3.6. *We assume that $\mathcal{R}_0 > 1$:*
(1) If $\mathcal{D}(U) > 0$ and $0 < \vartheta < 1$ along with $\wp = 1$, then $\mathcal{N}^*$ is locally asymptotically stable.
(2) If $\mathcal{D}(U) > 0$ and $\vartheta < 2/3$ along with $\wp = 1$, then $\mathcal{N}^*$ is locally asymptotically stable.

4. Existence and uniqueness consequences

This portion explores the existence and uniqueness of elucidations to the provided framework (1.2) considering the fixed point theorems approach.

Surmising that $\mathbb{G} = \mathbb{C}([0, Q], \mathcal{R})$ represents the $\mathbf{B}_s$ containing continuous mappings from $[0, Q]$ to $\mathcal{R}$ represented by the norm as

$$|\Omega| = \sup_{\varrho \in [0,Q]} |\Omega(\varrho)|, \quad where \quad |\Omega(\varrho)| = |\mathbf{M}| + |\mathbf{N}| + |\mathbf{O}| + |\mathbf{P}| + |\mathbf{Q}|,$$

and $\mathbf{M}, \mathbf{N}, \mathbf{O}, \mathbf{P}, \mathbf{Q} \in \mathbb{G}$. By the virtue of (2.5), the system (1.2) can be expressed as the initial value problem (IVP)

$$
\begin{cases}
({}^{C\wp}\mathbf{D}^{\vartheta}_{0^+}\Omega)(\varrho) = \Lambda(\varrho, \Omega(\varrho)), \\
\Omega(0) = \Omega_0 \geq 0,
\end{cases}
\tag{4.1}
$$

which is analogous to the integral equation of Volterra type

$$
\Omega(\varrho) = \Omega_0 + \frac{1}{\Gamma(\vartheta)} \int_0^{\varrho} \left(\frac{\varrho^{\wp} - \mathbf{s}^{\wp}}{\wp} \right)^{\vartheta-1} \Lambda(\mathbf{s}, \Omega(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}},
\tag{4.2}
$$

where $\Omega(\varrho) = (\mathbf{M}(\varrho), \mathbf{N}(\varrho), \mathbf{O}(\varrho), \mathbf{P}(\varrho), \mathbf{Q}(\varrho))^Q$ for $J = 1, ..., 5$.

Utilizing the fact of (4.2), an operator $\Theta : \mathbb{G} \mapsto \mathbb{G}$ stated by

$$
(\Theta\Omega)(\varrho) = \Omega_0 + \frac{1}{\Gamma(\vartheta)} \int_0^{\varrho} \left(\frac{\varrho^{\wp} - \mathbf{s}^{\wp}}{\wp} \right)^{\vartheta-1} \Lambda(\mathbf{s}, \Omega(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}.
\tag{4.3}
$$

It is worth mentioning that the model (4.1), which is analogous to the problem (1.2), has elucidations only, if and only if the map Θ contains f_p.

Theorem 4.1. *Surmise that there be a continuous mapping $\Lambda : [0, Q] \mapsto \mathcal{R}$ such that there exists a constant $\mathbb{L}_\Lambda > 0$, then*

$$
\left| \Lambda(\varrho, \Omega_1(\varrho)) - \Lambda(\varrho, \Omega_2(\varrho)) \right| \leq \mathbb{L}_\Lambda \left| \Omega_1(\varrho) - \Omega_2(\varrho) \right|, \quad \Omega_1, \Omega_1 \in \mathbb{G}, \text{ and } \forall \varrho \in [0, Q].
\tag{4.4}
$$

If

$$
\mathbb{L}_\Lambda Q^{\wp\vartheta} < \wp^{\vartheta} \Gamma(\vartheta + 1),
\tag{4.5}
$$

then the model (4.1) has a fixed point on $[0, Q]$. Finally, the model (1.2) has a unique solution on $[0, Q]$.

Proof. Now, we convert the problem (4.1) into f_p problem, $\Omega = \Theta\Omega$, where Θ is illustrated in (4.3). Implementing the Banach contraction principle, we illustrate that Θ has a unique f_p. To do this, suppose $\sup_{\varrho \in [0, Q]} |\Lambda(\varrho, 0)| = \Upsilon_1 < \infty$. Choosing $\mathbb{G}_{r_1} = \{\Omega \in \mathbb{G} : \|\Omega\| \leq r_1\}$ having

$$
r_1 \geq \frac{\|\Omega_0\| \wp^{\vartheta} \Gamma(\vartheta + 1) + \Upsilon_1 Q^{\wp\vartheta}}{\wp^{\vartheta} \Gamma(\vartheta + 1) - \mathbb{L}_\Lambda Q^{\wp\vartheta}}.
\tag{4.6}
$$

It is noting that $\mathbb{G}_{r_1}$ is a bounded, closed and convex subset of $\mathbb{G}$. Also, proving that $\Upsilon \mathbb{G}_{r_1} \subset \mathbb{G}_{r_1}$. For any $\Omega \in \mathbb{G}_r$, we have

$$\left\|(\Theta\Omega)(\varrho)\right\| \quad \leq \left\|\Omega_0\right\| + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \left|\Lambda(\mathbf{s}, \Omega(\mathbf{s}))\right| \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}$$

$$\leq \left\|\Omega_0\right\| + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \left[\left|\Lambda(\mathbf{s}, \Omega(\mathbf{s})) - \Lambda(\mathbf{s}, 0)\right| + \left|\Lambda(\mathbf{s}, 0)\right|\right] \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}$$

$$\leq \left\|\Omega_0\right\| + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} (\mathbb{L}_\Lambda r_1 + \Upsilon_1) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}$$

$$\leq \left\|\Omega_0\right\| + \frac{Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta+1)} (\mathbb{L}_\Lambda r_1 + \Upsilon_1) \leq r_1,$$

which implies that $\Upsilon \mathbb{G}_{r_1} \subset \mathbb{G}_{r_1}$.

Furthermore we prove that $\Upsilon : \mathbb{G}_{r_1} \mapsto \mathbb{G}$ is a contraction mapping. For any $\Omega_1, \Omega_2 \in \mathbb{G}$ and every $\varrho \in [0, Q]$, we have

$$\left|(\Theta\Omega_1)(\varrho) - (\Theta\Omega_1)(\varrho)\right| \quad \leq \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \left[\left|\Lambda(\mathbf{s}, \Omega_1(\mathbf{s})) - \Lambda(\mathbf{s}, \Omega_2(\mathbf{s}))\right|\right] \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}$$

$$\leq \frac{\mathbb{L}_\Lambda}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \left|\Omega_1(\mathbf{s}) - \Omega_2(\mathbf{s})\right| \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}$$

$$\leq \frac{\mathbb{L}_\Lambda Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta+1)} \left\|\Omega_1 - \Omega_2\right\|. \tag{4.7}$$

It follows that

$$\left\|\Theta\Omega_1 - \Theta\Omega_2\right\| \leq \left[\frac{\mathbb{L}_\Lambda Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta+1)}\right] \left\|\Omega_1 - \Omega_2\right\|.$$

As $\mathbb{L}_\Lambda Q^{\wp\vartheta}/\wp^\vartheta \Gamma(\vartheta+1) < 1$, then operator Υ is contraction mapping. Utilizing the fact of Lemma 2.5, the operator Υ has a f_p. As a consequence, the problem (4.1) has a fixed solution on $[0, Q]$. Therefore, according to above analysis, we conclude that the model (1.2) has a unique solution on $[0, Q]$. This completes the proof. $\square$

Our next result based on the Leray-Schauder nonlinear alternative (Lemma 2.6) is demonstrated as a new existence theorem.

Theorem 4.2. *Suppose that:*
(A_1) $\exists$ *a mapping* $\tilde{q} \in \mathbb{C}([0, Q, \mathcal{R}^+])$ *and a decreasing mapping* $\mathbf{H} : [0, +\infty) \mapsto [0, +\infty)$ *satisfying sub-homogeneous assumption (i.e.,* $\mathbf{H}(\theta\Omega) \leq \theta\mathbf{H}(\Omega)$, $\forall \theta \geq 1$ *and* $\Omega \in \mathcal{R}$) *such that*

$$\left|\Lambda(\varrho, \Omega(\varrho))\right| \leq \tilde{q}(\varrho)\mathbf{H}(\Omega(\varrho)), \ \forall \ (\varrho, \Omega) \in [0, Q] \times \mathcal{R}, \tag{4.8}$$

where $\tilde{q}_0 = \sup_{\Omega \in [0,Q]} \{\tilde{q}(\varrho)\}$.
(A_2) *There exists a constant* $\Upsilon_2 > 0$ *such that*

$$\frac{\Upsilon_2 \wp^\vartheta \Gamma(\vartheta+1)}{\|\Omega_0\| \wp^\vartheta \Gamma(\vartheta+1) + \tilde{q}_0 \mathbf{H}(\Upsilon_2) Q^{\wp\vartheta}} > 1. \tag{4.9}$$

Then, the Eq (4.1) *that is analogous to the system* (1.2) *one or more solution on* $[0, Q]$.

Proof. By means of the map Θ proposed in (4.3). Initially, we prove that the operator Θ maps bounded set into bounded sets in $\mathbb{G}$. For a positive constant $r_2 > 0$, assume that $\mathbb{G}_{r_2} = \{\Omega \in \mathbb{G} : \|\Omega\| \leq r_2\}$ be a bounded ball in $\mathbb{G}$. Under the hypothesis ($\mathbf{A}_1$), for $\varrho \in [0, Q]$, we have

$$
\begin{aligned}
\left|(\Theta\Omega)(\varrho)\right| \quad &\leq \left\|\Omega_0\right\| + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \Lambda(\mathbf{s}, \Omega(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \\[2mm]
&\leq \left\|\Omega_0\right\| + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \left|\Lambda(\mathbf{s}, \Omega(\mathbf{s}))\right| \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \\[2mm]
&\leq \left\|\Omega_0\right\| + \frac{\tilde{q}_0 \mathbf{H}(\|\Omega\|) Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta + 1)},
\end{aligned}
$$

which yields

$$
\left\|(\Theta\Omega)\right\| \leq \left\|\Omega_0\right\| + \frac{\tilde{q}_0 \mathbf{H}(r_2) Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta + 1)}.
$$

Furthermore, we illustrate that the operator Θ maps bounded sets into equi-continuous sets of $\mathbb{G}$. Surmise that $l_1, l_2 \in [0, Q]$ having $l_1 < l_2$ and $\Omega \in \mathbb{G}_{r_2}$. Then we find

$$
\begin{aligned}
&\left|(\Theta\Omega)(l_2) - (\Theta\Omega)(l_1)\right| \\[2mm]
&\leq \left|\frac{1}{\Gamma(\vartheta)} \int_0^{l_2} \left(\frac{l_2^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \Lambda(\mathbf{s}, \Omega(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} - \frac{1}{\Gamma(\vartheta)} \int_0^{l_1} \left(\frac{l_1^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1} \Lambda(\mathbf{s}, \Omega(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}\right| \\[2mm]
&\leq \frac{\tilde{q}_0 \mathbf{H}(\|\Omega\|)}{\wp^\vartheta \Gamma(\vartheta + 1)}\left(\left|l_2^{\wp\vartheta} - l_1^{\wp\vartheta}\right| + 2\left|l_2^\wp - l_1^\wp\right|^\vartheta\right). \qquad\qquad (4.10)
\end{aligned}
$$

It is clear that $\Omega \in \mathbb{G}_{r_2}$, the right hand side of the inequality (4.10) approaches to zero as $l_2 \mapsto l_1$. Thus, in view of the Arzelá-Ascoli theorem , $\theta : \mathbb{G} \mapsto \mathbb{G}$ is completely continuous.

As a result, we illustrate that the boundedness of the collection of findings to $\Omega = \kappa\theta\Omega$ for $\kappa \in (0, 1)$. Now, assume that there be a solution Ω. So, for $\varrho \in [0, Q]$, and subsequent technique analogous to the previous case, we attain

$$
|\Omega(\varrho)| = |\kappa(\theta\Omega)(\varrho)| \leq \|\Omega_0\| + \frac{\tilde{q}_0 \mathbf{H}(\|\Omega\|) Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta + 1)}. \qquad\qquad (4.11)
$$

Implementing the norm of the aforesaid inequality, for $\Omega \in [0, Q]$, it follows that

$$
\frac{\wp^\vartheta \Gamma(\vartheta + 1)\|\Omega\|}{\|\Omega_0\|\wp^\vartheta \Gamma(\vartheta + 1) + \tilde{q}_0 \mathbf{H}(\|\Omega\|)Q^{\wp\vartheta}} \leq 1. \qquad\qquad (4.12)
$$

Using the fact of ($\mathbf{A}_2$), there exists a constant $\Upsilon_2 > 0$ such that $\mathbb{K} := \{\Omega \in \mathbb{G} : \|\Omega\Upsilon_2|\}$. Observe that, the mapping $\Theta : \bar{\mathbb{K}} \mapsto \mathbb{C}$ is continuous as well as completely continuous. So, the appropriate selection of $\bar{\mathbb{K}}$, no as such $\Omega \in \mathbb{K}$ exist $\Omega = \kappa\Theta\Omega$ for $\kappa \in (0, 1)$. Thus, by Leray-Schauder nonlinear alternative (Lemma 2.6), it is concluded that the map Θ has a f_p as $\Omega \in \bar{\mathbb{K}}$ that proves that the system (1.2) has a unique solution on $[0, Q]$. This completes the proof. $\qquad\qquad\square$

5. Stability analysis

Here, we shall establish certain adequate assumptions in this article for the system (1.2) to fulfill the requirements of multiple types of stability. The accompanying definitions are required prior to stating Ulam stability theorem.

Surmise that there be a positive real number $\epsilon > 0$ and a continuous mapping $\Phi_\Lambda : [0, Q] \mapsto \mathcal{R}^+$. We have the subsequent variants

$$\left| {}^{C\wp}\mathbf{D}^\vartheta_{0^+}\mathbf{y}(\varrho) - \Lambda(\varrho, \mathbf{y}(\varrho)) \right| \leq \epsilon, \quad \forall \varrho \in [0, Q], \tag{5.1}$$

$$\left| {}^{C\wp}\mathbf{D}^\vartheta_{0^+}\mathbf{y}(\varrho) - \Lambda(\varrho, \mathbf{y}(\varrho)) \right| \leq \epsilon \Phi_\Lambda(\varrho), \quad \forall \varrho \in [0, Q], \tag{5.2}$$

$$\left| {}^{C\wp}\mathbf{D}^\vartheta_{0^+}\mathbf{y}(\varrho) - \Lambda(\varrho, \mathbf{y}(\varrho)) \right| \leq \Phi_\Lambda(\varrho), \quad \forall \varrho \in [0, Q], \tag{5.3}$$

where $\epsilon = \max(\epsilon_J)^\varrho$ for $J = 1, ..., 5$.

Definition 5.1. ([36]) We say that the Eq (4.1) is Ulam-Hyers stable if there exists a fixed $\mathbb{C}_\Lambda$ such that for every $\epsilon > 0$ and for every elucidation $\mathbf{y} \in \mathbb{G}$ of the variant (5.1) $\exists$ a elucidation $\Omega \in \mathbb{G}$ of the Eq (4.1) having

$$\left| \mathbf{y}(\varrho) - \Omega(\varrho) \right| \leq \mathbb{C}_\Lambda \epsilon, \quad \varrho \in [0, Q], \tag{5.4}$$

where $\mathbb{C}_\Lambda = \max(\mathbb{C}_\Lambda)^\varrho$ for $J = 1, .., 5$.

Definition 5.2. ([36]) We say that the Eq (4.1) is generalized Ulam-Hyers stable if $\exists$ a mapping $\Phi_\Lambda \in \mathbb{C}(\mathcal{R}^+, \mathcal{R}^+)$ having $\Phi_\Lambda = 0$ such that for every elucidation $\mathbf{y} \in \mathbb{G}$ of variant (5.2) $\exists$ a elucidation $\Omega \in \mathbb{G}$ of the Eq (4.1) with

$$\left| \mathbf{y}(\varrho) - \Omega(\varrho) \right| \leq \Phi_\Lambda(\epsilon), \quad \varrho \in [0, Q], \tag{5.5}$$

where $\Phi_\Lambda = \max(\Phi_{\Lambda_J})^\varrho$ for $J = 1, .., 5$.

Definition 5.3. ([36]) We say that the Eq (4.1) is Ulam-Hyers-Rassias stable in respective to $\Phi_\Lambda \in \mathbb{C}([0, Q], \mathcal{R}^+)$ if $\exists$ a real constant Υ_{Φ_Λ} such that for every $\epsilon > 0$ and for every elucidation $\mathbf{y} \in \mathbb{G}$ of variant (5.2) $\exists$ a elucidation $\Omega \in \mathbb{G}$ of the Eq (4.1) with

$$\left| \mathbf{y}(\varrho) - \Omega(\varrho) \right| \leq \Upsilon_{\Phi_\Lambda} \in \Phi_\Lambda(\varrho), \quad \varrho \in [0, Q]. \tag{5.6}$$

Definition 5.4. ([36]) We say that the (4.1) is generalized Ulam-Hyers-Rassias stable in respective Φ_Λ if $\exists$ a real number $\Upsilon_{\Phi_\Lambda} > 0$ such that for every elucidation $\mathbf{y} \in \mathbb{G}$ of variant (5.3), $\exists$ a elucidation $\Omega \in \mathbb{G}$ of the Eq (4.1) with

$$\left| \mathbf{y}(\varrho) - \Omega(\varrho) \right| \leq \Upsilon_{\Phi_\Lambda} \Phi_\Lambda(\varrho), \quad \varrho \in [0, Q]. \tag{5.7}$$

Remark 1. Clearly, the aforesaid variants leads to the following conclusion that:
(i) Inequality (5.4) $\implies$ Inequality (5.5);
(ii) Inequality (5.6) $\implies$ Inequality (5.7);
(iii) Inequality (5.6) for $\Phi_\Lambda(.) = 1 \implies$ Inequality (5.4).

Remark 2. A mapping $\mathbf{y} \in \mathbb{G}$ is a elucidation of the variant (5.1) if and only if $\exists$ a mapping $\omega \in \mathbb{G}$ (be influenced by $\mathbf{y}$) such that the subsequent assumptions hold:

(a) $|\omega(\varrho)| \leq \epsilon, \quad \omega = \max(\omega_J)^\varrho, \quad \forall \; \varrho \in [0, Q],$

(b) $^{C\wp}\mathbb{D}_{0^+}^\vartheta \mathbf{y}(\varrho) = \Lambda(\varrho, \mathbf{y}(\varrho)) + \omega(\varrho), \quad \forall \varrho \in [0, Q].$

Remark 3. A mapping $\mathbf{y} \in \mathbb{G}$ is a elucidation of the variant (5.2) if and only if $\exists$ a mapping $v \in \mathbb{G}$ (be influenced by $\mathbf{y}$) such that the subsequent assumptions hold:

(a) $|v(\varrho)| \leq \epsilon \Phi_\Lambda(\varrho), \quad v = \max(\omega_J)^\varrho, \quad \forall \; \varrho \in [0, Q],$

(b) $^{C\wp}\mathbb{D}_{0^+}^\vartheta \mathbf{y}(\varrho) = \Lambda(\varrho, \mathbf{y}(\varrho)) + v(\varrho), \quad \forall \varrho \in [0, Q].$

We illustrate a vital purpose that can be employed to prove Ulam-Hyers and generalized Ulam-Hyers stability.

Lemma 5.5. *For $0 < \vartheta \leq 1$ and $\wp > 0$. Let there be a solution $\mathbf{y} \in \mathbb{G}$ of the variant (5.1), then $\mathbf{y}$ is a elucidation of the subsequent variant*

$$\left| \mathbf{y}(\varrho) - \mathbf{y}_0 - \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta - 1} \Lambda(\mathbf{s}, \mathbf{y}(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \right| \leq \frac{Q^{\vartheta\wp}}{\wp^\vartheta \Gamma(\vartheta + 1)\epsilon}. \tag{5.8}$$

Proof. Assume that there be a solution $\mathbf{y}$ of the variant (5.1). Using the fact of Remark 2-(ii), we find

$$\begin{cases} ^{C\wp}\mathbb{D}_{0^+}^\vartheta \mathbf{y}(\varrho) = \Lambda(\varrho, \mathbf{y}(\varrho)) + \omega(\varrho), \quad \forall \varrho \in [0, Q], \\ \qquad\qquad \mathbf{y}(0) = \mathbf{y}_0 \geq 0. \end{cases} \tag{5.9}$$

Therefore, the elucidation of the Eq (5.9) can be expressed as

$$\mathbf{y}(\varrho) = \mathbf{y}_0 + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \Lambda(\mathbf{s}, \mathbf{y}(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \omega(\mathbf{s}) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}.$$

Employing Remark 2-(i), we have

$$\left| \mathbf{y}(\varrho) - \mathbf{y}_0 - \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \Lambda(\mathbf{s}, \mathbf{y}(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \right| \leq \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} |\omega(\mathbf{s})| \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}}$$

$$\leq \frac{Q^{\vartheta\wp}}{\wp^\vartheta \Gamma(\vartheta + 1)\epsilon}. \tag{5.10}$$

Hence, the variant (5.8) is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In our next result, we addressed the Ulam-Hyers stability and generalized Ulam-Hyers stability results.

Theorem 5.6. *Suppose for every $\Omega \in \mathbb{G}$ and there be a continuous mapping $\Lambda : [0, Q] \times \mathcal{R} \mapsto \mathcal{R}$. If (4.4) and (4.5) are fulfilled, then the Eq (4.1) which is analogous to the system (1.2) is Ulam-Hyers and, finally, generalized Ulam-Hyers stable on $[0, Q]$.*

Proof. Consider $\epsilon > 0$ and $\mathbf{y} \in \mathbb{G}$ assumed to be a elucidation of the variant (5.1). Suppose $\Omega \in \mathbb{G}$ be the fixed solution of the Eq (4.1),

$$\begin{cases} {}^{C_\wp}\mathbf{D}_{0^+}^\vartheta \Omega(\varrho) = \Lambda(\varrho, \Omega(\varrho)), & \varrho \in [0, Q], \\ \Omega(0) = \Omega_0. \end{cases}$$

Utilizing the fact of (4.1), Lemma 5.5 and by means of triangular inequality, we attain

$$
\begin{aligned}
\left| \mathbf{y}(\varrho) - \Omega(\varrho) \right| \quad &\leq \left| \mathbf{y}(\varrho) - \Omega_0 - \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \Lambda(\mathbf{s}, \Omega(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \right| \\
&\leq \left| \mathbf{y}(\varrho) - \mathbf{y}_0 - \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \Lambda(\mathbf{s}, z(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \right. \\
&\quad \left. + \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \left| \Lambda(\mathbf{s}, z(\mathbf{s})) - \Lambda(\mathbf{s}, \Omega(\mathbf{s})) \right| \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \right| \\
&\leq \left| \mathbf{y}(\varrho) - \mathbf{y}_0 - \frac{1}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \Lambda(\mathbf{s}, z(\mathbf{s})) \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \right| \\
&\quad + \frac{\mathbb{L}_\Lambda}{\Gamma(\vartheta)} \int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp} \right)^{\vartheta-1} \left| \mathbf{y}(\varrho) - \Omega(\varrho) \right| \frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \\
&\leq \frac{Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta+1)} \epsilon + \frac{\mathbb{L}_\Lambda Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta+1)} \left| \mathbf{y}(\varrho) - \Omega(\varrho) \right|.
\end{aligned}
\tag{5.11}
$$

In view of Definition 5.1, we have

$$\mathbb{C}_\Lambda = \frac{Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta+1)} \Big/ \left(1 - \frac{\mathbb{L}_\Lambda Q^{\wp\vartheta}}{\wp^\vartheta \Gamma(\vartheta+1)} \right). \tag{5.12}$$

Thus, the system (1.2) is Ulam-Hyers stable. Now, by employing $\Phi_\Lambda(\epsilon) = \mathbb{C}_\Lambda \epsilon$ such that $\Phi_\Lambda(0) = 0$ provides that the system (1.2) is generalized Ulam-Hyers stable. This completes the proof. $\square$

To prove our next result, we consider the following hypothesis:
(**A**$_3$) There exists an increasing mapping $\Phi_\Lambda \in \mathbb{G}$ and $\exists \lambda_{\Phi_\Lambda} > 0$, such that for fixed $\varrho \in [0, Q]$, the subsequent formulation holds:

$${}^\wp \mathcal{I}_{0^+}^\vartheta \Phi_\Lambda(\varrho) \leq \lambda_{\Phi_\Lambda} \Phi_\Lambda(\varrho). \tag{5.13}$$

Further, we demonstrate a significant result that will be considered in our coming findings of the Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability consequences.

Lemma 5.7. *For $0 < \vartheta \leq 1$ and $\wp > 0$ and there be a solution $\mathbf{y} \in \mathbb{G}$ of the variant (5.2), then $\mathbf{y}$ is a elucidation of the subsequent variant*

$$\left| \mathbf{y}(\varrho) - \mathbf{y}_0 - {}^\wp \mathcal{I}_{0^+}^\vartheta \Lambda(\varrho, \mathbf{y}(\varrho)) \right| \leq \epsilon \lambda_{\Phi_\Lambda} \Phi_\Lambda(\varrho). \tag{5.14}$$

Proof. Suppose there be a solution **y** of the variant (5.2). In view of Remark 3-(ii), we find

$$\begin{cases} {}^{C\wp}\mathbf{D}_{0^+}^{\vartheta}\mathbf{y}(\varrho) = \Lambda(\varrho, \mathbf{y}(\varrho)) + \nu(\varrho), & \varrho \in [0, Q], \\ \mathbf{y}(0) = \mathbf{y}_0. \end{cases} \tag{5.15}$$

Therefore, the solution of the problem (5.15) can be expressed as

$$\mathbf{y}(\varrho) = \mathbf{y}_0 + {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}\Lambda(\varrho, \mathbf{y}(\varrho)) + {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}\nu(\varrho).$$

Employing Remark 3-(i), we have

$$\left|\mathbf{y}(\varrho) - \mathbf{y}_0 - {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}\Lambda(\varrho, \mathbf{y}(\varrho))\right| \le {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}|\nu(\varrho)| \le \epsilon\, {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}\Phi_\Lambda(\varrho) \le \epsilon\lambda_{\Phi_\Lambda}\Phi_\Lambda(\varrho).$$

Thus, the variant (5.14) is acquired. $\square$

As a final outcome, we are able to establish Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability.

Theorem 5.8. *Suppose that there be a continuous mapping* $\Lambda : [0, Q] \times \mathcal{R} \mapsto \mathcal{R}$ *for each* $\Omega \in \mathcal{R}$. *If* (4.4), *(A$_2$) and* (4.5) *are fulfilled, then the Eq* (4.1) *which is similar to the system* (1.2) *is Ulam-Hyers-Rassias and, consequently, generalized Ulam-Hyers-Rassias stable on* $[0, Q]$.

Proof. Surmise that $\epsilon > 0$ and there be a elucidation $\mathbf{y} \in \mathbb{G}$ of the variant (5.3). Also, assume that $\Omega \in \mathbb{G}$ be a fixed solution of (4.1). Employing (4.2) and Lemma 5.7, we have

$$\begin{aligned} \left|\mathbf{y}(\varrho) - \Omega(\varrho)\right| &\le \left|\mathbf{y}(\varrho) - \Omega_0 - {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}\Lambda(\varrho, \Omega(\varrho))\right| \\ &\le \left|\mathbf{y}(\varrho) - \mathbf{y}_0 - {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}\Lambda(\varrho, z(\varrho))\right| + \frac{1}{\Gamma(\vartheta)}\int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1}\left|\Lambda(\mathbf{s}, z(\mathbf{s})) - \Lambda(\mathbf{s}, \Omega(\mathbf{s}))\right|\frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \\ &\le \left|\mathbf{y}(\varrho) - \mathbf{y}_0 - {}^{\wp}\mathcal{I}_{0^+}^{\vartheta}\Lambda(\varrho, z(\varrho))\right| + \frac{\mathbb{L}_\Lambda}{\Gamma(\vartheta)}\int_0^\varrho \left(\frac{\varrho^\wp - \mathbf{s}^\wp}{\wp}\right)^{\vartheta-1}\left|z(\mathbf{s}) - \Omega(\mathbf{s})\right|\frac{d\mathbf{s}}{\mathbf{s}^{1-\wp}} \\ &\le \epsilon\lambda_{\Phi_\Lambda}\Phi_\Lambda(\varrho) + \frac{\mathbb{L}_\Lambda Q^{\wp\vartheta}}{\wp^\vartheta\Gamma(\vartheta+1)}\left|\mathbf{y}(\varrho) - \Omega(\varrho)\right|, \end{aligned} \tag{5.16}$$

which implies that $\left|\mathbf{y}(\varrho) - \Omega(\varrho)\right| \le \epsilon\lambda_{\Phi_\Lambda}\Phi_\Lambda(\varrho)\big/\left(1 - \frac{\mathbb{L}_\Lambda Q^{\wp\vartheta}}{\wp^\vartheta\Gamma(\vartheta+1)}\right)$. By choosing

$$\Upsilon_{\Phi_\Lambda} = \epsilon\lambda_{\Phi_\Lambda}\Big/\left(1 - \frac{\mathbb{L}_\Lambda Q^{\wp\vartheta}}{\wp^\vartheta\Gamma(\vartheta+1)}\right). \tag{5.17}$$

Therefore, we attain the subsequent variant

$$\left|\mathbf{y}(\varrho) - \Omega(\varrho)\right| \le \Upsilon_{\Phi_\Lambda}\epsilon\Phi_\Lambda(\varrho). \tag{5.18}$$

Consequently, the model (1.2) is Ulam-Hyers-Rassias stable. Also, taking $\epsilon = 1$, in (5.18), with $\Phi_\Lambda(0) = 0$, then the framework (1.2) is generalized Ulam-Hyers-Rassias stable. This completes the proof. $\square$

6. Configuration of generic algorithm for the TB model

In order to establish the series formulation of the proposed problem, we employ the modified Laplace transform on both sides of (1.2). We develop the subsequent formulation as follows:

$$
\begin{cases}
\mathbf{s}^{\vartheta}\mathcal{L}_{\wp}[\mathbf{M}(\varrho)] - \mathbf{s}^{\vartheta-1}\mathbf{M}(0) = \mathcal{L}_{\wp}[\theta\zeta - (\eta + \mu)\mathbf{M}(\varrho)], \\
\mathbf{s}^{\vartheta}\mathcal{L}_{\wp}[\mathbf{N}(\varrho)] - \mathbf{s}^{\vartheta-1}\mathbf{N}(0) = \mathcal{L}_{\wp}[(1-\theta)\zeta + \eta\mathbf{M}(\varrho) - \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - \zeta\mathbf{N}(\varrho)], \\
\mathbf{s}^{\vartheta}\mathcal{L}_{\wp}[\mathbf{O}(\varrho)] - \mathbf{s}^{\vartheta-1}\mathbf{O}(0) = \mathcal{L}_{\wp}[\beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - (\varpi + \varphi + \mu)\mathbf{O}(\varrho)], \\
\mathbf{s}^{\vartheta}\mathcal{L}_{\wp}[\mathbf{P}(\varrho)] - \mathbf{s}^{\vartheta-1}\mathbf{P}(0) = \mathcal{L}_{\wp}[\varphi\mathbf{O}(\varrho) - (\phi + \mu + \psi)\mathbf{P}(\varrho)], \\
\mathbf{s}^{\vartheta}\mathcal{L}_{\wp}[\mathbf{Q}(\varrho)] - \mathbf{s}^{\vartheta-1}\mathbf{Q}(0) = \mathcal{L}_{\wp}[\varpi\mathbf{O}(\varrho) + \phi\mathbf{P}(\varrho) - \mu\mathbf{Q}(\varrho)].
\end{cases}
$$

Employing the initial conditions and suitable arrangements yields

$$
\begin{cases}
\mathcal{L}_{\wp}[\mathbf{M}(\varrho)] = \frac{\mathbf{M}_0}{s} + \frac{1}{\mathbf{s}^{\vartheta}}\mathcal{L}_{\wp}[\theta\zeta - (\eta + \mu)\mathbf{M}(\varrho)], \\
\mathcal{L}_{\wp}[\mathbf{N}(\varrho)] = \frac{\mathbf{N}_0}{s} + \frac{1}{\mathbf{s}^{\vartheta}}\mathcal{L}_{\wp}[(1-\theta)\zeta + \eta\mathbf{M}(\varrho) - \beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - \zeta\mathbf{N}(\varrho)], \\
\mathcal{L}_{\wp}[\mathbf{O}(\varrho)] = \frac{\mathbf{O}_0}{s} + \frac{1}{\mathbf{s}^{\vartheta}}\mathcal{L}_{\wp}[\beta\mathbf{N}(\varrho)\mathbf{P}(\varrho) - (\varpi + \varphi + \mu)\mathbf{O}(\varrho)], \\
\mathcal{L}_{\wp}[\mathbf{P}(\varrho)] = \frac{\mathbf{P}_0}{s} + \frac{1}{\mathbf{s}^{\vartheta}}\mathcal{L}_{\wp}[\varphi\mathbf{O}(\varrho) - (\phi + \mu + \psi)\mathbf{P}(\varrho)], \\
\mathcal{L}_{\wp}[\mathbf{Q}(\varrho)] = \frac{\mathbf{Q}_0}{s} + \frac{1}{\mathbf{s}^{\vartheta}}\mathcal{L}_{\wp}[\varpi\mathbf{O}(\varrho) + \phi\mathbf{P}(\varrho) - \mu\mathbf{Q}(\varrho)].
\end{cases}
$$

Let us surmise that the solution we calculate in an infinite series formulation is as follows:

$$
\mathbf{M}(\varrho) = \sum_{\mathbf{n}=0}^{+\infty}\mathbf{M_n}(\varrho), \quad \mathbf{N}(\varrho) = \sum_{\mathbf{n}=0}^{+\infty}\mathbf{N_n}(\varrho), \quad \mathbf{O}(\varrho) = \sum_{\mathbf{n}=0}^{+\infty}\mathbf{O_n}(\varrho), \quad \mathbf{P}(\varrho) = \sum_{\mathbf{n}=0}^{+\infty}\mathbf{P_n}(\varrho),
$$

$$
\mathbf{Q}(\varrho) = \sum_{\mathbf{n}=0}^{+\infty}\mathbf{Q_n}(\varrho) \tag{6.1}
$$

and the non-linearity factor **NP** can be decomposed by the Adomian polynomial as follows:

$$
\mathbf{N}(\varrho)\mathbf{P}(\varrho) = \sum_{\mathbf{n}=0}^{+\infty}\mathcal{J}_n(\varrho), \tag{6.2}
$$

where

$$
\mathcal{J}_{\mathbf{n}} = \frac{1}{\Gamma(n+1)}\frac{d^{\mathbf{n}}}{d\lambda^{\mathbf{n}}}\Big[\Big(\sum_{\kappa=0}^{\mathbf{n}}\lambda^{\kappa}\mathbf{N}_{\kappa}\Big)\Big(\sum_{\kappa=0}^{\mathbf{n}}\lambda^{\kappa}\mathbf{P}_{\kappa}\Big)\Big]_{\lambda=0}.
$$

Some first few Adomian polynomials are expressed as

$$
\mathcal{J}_{\mathbf{n}} = \begin{cases}
\mathbf{N}_0(\varrho)\mathbf{P}_0(\varrho), \quad \mathbf{n} = 0, \\
\mathbf{N}_0(\varrho)\mathbf{P}_1(\varrho) + \mathbf{N}_1(\varrho)\mathbf{P}_0(\varrho), \quad \mathbf{n} = 1, \\
\mathbf{N}_0(\varrho)\mathbf{P}_2(\varrho) + \mathbf{N}_1(\varrho)\mathbf{P}_1(\varrho) + \mathbf{N}_2(\varrho)\mathbf{P}_1(\varrho), \quad \mathbf{n} = 2, \\
\mathbf{N}_0(\varrho)\mathbf{P}_3(\varrho) + \mathbf{N}_1(\varrho)\mathbf{P}_2(\varrho) + \mathbf{N}_2(\varrho)\mathbf{P}_1(\varrho) + \mathbf{N}_3(\varrho)\mathbf{P}_0(\varrho), \quad \mathbf{n} = 3, \\
\vdots \\
\mathbf{N}_0(\varrho)\mathbf{P}_n(\varrho) + \mathbf{N}_1(\varrho)\mathbf{P}_{(n-1)}(\varrho) + ... + \mathbf{N}_{(n-1)}(\varrho)\mathbf{P}_1(\varrho) + \mathbf{N}_{(n)}(\varrho)\mathbf{P}_0(\varrho), \quad \mathbf{n} = n.
\end{cases} \tag{6.3}
$$

By the virtue of (6.1)–(6.3), we have

$$
\begin{cases}
\mathcal{L}_\wp[\sum\limits_{\kappa=0}^{+\infty} \mathbf{M}_\kappa(\varrho)] = \frac{\mathbf{M}_0}{s} + \frac{1}{s^\vartheta}\mathcal{L}_\wp[\theta\zeta - (\eta+\mu)\sum\limits_{\kappa=0}^{+\infty}\mathbf{M}_\kappa(\varrho)], \\
\mathcal{L}_\wp[\sum\limits_{\kappa=0}^{+\infty} \mathbf{N}_\kappa(\varrho)] = \frac{\mathbf{N}_0}{s} + \frac{1}{s^\vartheta}\mathcal{L}_\wp[(1-\theta)\zeta + \eta\sum\limits_{\kappa=0}^{+\infty}\mathbf{M}_\kappa(\varrho) - \beta\sum\limits_{\kappa=0}^{+\infty}\mathcal{J}_\kappa(\varrho) - \zeta\sum\limits_{\kappa=0}^{+\infty}\mathbf{N}_\kappa(\varrho)], \\
\mathcal{L}_\wp[\sum\limits_{\kappa=0}^{+\infty} \mathbf{O}_\kappa(\varrho)] = \frac{\mathbf{O}_0}{s} + \frac{1}{s^\vartheta}\mathcal{L}_\wp[\beta\sum\limits_{\kappa=0}^{+\infty}\mathcal{J}_\kappa - (\varpi+\varphi+\mu)\sum\limits_{\kappa=0}^{+\infty}\mathbf{O}_\kappa(\varrho)], \\
\mathcal{L}_\wp[\sum\limits_{\kappa=0}^{+\infty} \mathbf{P}_\kappa(\varrho)] = \frac{\mathbf{P}_0}{s} + \frac{1}{s^\vartheta}\mathcal{L}_\wp[\varphi\sum\limits_{\kappa=0}^{+\infty}\mathbf{O}_\kappa(\varrho) - (\phi+\mu+\psi)\sum\limits_{\kappa=0}^{+\infty}\mathbf{P}_\kappa(\varrho)], \\
\mathcal{L}_\wp[\sum\limits_{\kappa=0}^{+\infty} \mathbf{Q}_\kappa(\varrho)] = \frac{\mathbf{Q}_0}{s} + \frac{1}{s^\vartheta}\mathcal{L}_\wp[\varpi\sum\limits_{\kappa=0}^{+\infty}\mathbf{O}_\kappa(\varrho) + \phi\sum\limits_{\kappa=0}^{+\infty}\mathbf{P}_\kappa(\varrho) - \mu\sum\limits_{\kappa=0}^{+\infty}\mathbf{Q}_\kappa(\varrho)].
\end{cases}
$$

Now, equating terms on both sides and after employing the $\wp$-Laplace inverse transform, we get

Step I. For **n** = 0, we have

$$
\begin{cases}
\mathbf{M}_0(\varrho) = \mathbf{M}_0 + \frac{\theta\zeta}{\Gamma(\vartheta+1)}\left(\frac{\varrho^\wp}{\wp}\right)^\vartheta, \\
\mathbf{N}_0(\varrho) = \mathbf{N}_0 + (1-\theta)\zeta\frac{1}{\Gamma(\vartheta+1)}\left(\frac{\varrho^\wp}{\wp}\right)^\vartheta, \\
\mathbf{O}_0(\varrho) = \mathbf{O}_0, \\
\mathbf{P}_0(\varrho) = \mathbf{P}_0, \\
\mathbf{Q}_0(\varrho) = \mathbf{Q}_0.
\end{cases}
$$

Step II. For **n** = 1, we have

$$
\begin{cases}
\mathbf{M}_1(\varrho) = \frac{1}{s^\vartheta}\mathcal{L}_\wp[-(\eta+\mu)\mathbf{M}_0(\varrho)], \\
\mathbf{N}_1(\varrho) = \frac{1}{s^\vartheta}\mathcal{L}_\wp[\eta\mathbf{M}_0(\varrho) - \beta\mathcal{J}_0(\varrho) - \zeta\mathbf{N}_0(\varrho)], \\
\mathbf{O}_1(\varrho) = \frac{1}{s^\vartheta}\mathcal{L}_\wp[\beta\mathcal{J}_0 - (\varpi+\varphi+\mu)\mathbf{O}_0(\varrho)], \\
\mathbf{P}_1(\varrho) = \frac{1}{s^\vartheta}\mathcal{L}_\wp[\varphi\mathbf{O}_0(\varrho) - (\phi+\mu+\psi)\mathbf{P}_0(\varrho)], \\
\mathbf{Q}_1(\varrho) = \frac{1}{s^\vartheta}\mathcal{L}_\wp[\varpi\mathbf{O}_0(\varrho) + \phi\mathbf{P}_0(\varrho) - \mu\mathbf{Q}_0(\varrho)].
\end{cases}
$$

Again, employing the inverse $\wp$-Laplace transform on both sides of the above system, we have

$$
\begin{cases}
\mathbf{M}_1(\varrho) = -(\eta+\mu)\mathbf{M}_0\frac{1}{\Gamma(\vartheta+1)}\left(\frac{\varrho^\wp}{\wp}\right)^\vartheta - (\eta+\mu)\frac{\theta\zeta}{\Gamma(2\vartheta+1)}\left(\frac{\varrho^\wp}{\wp}\right)^{2\vartheta}, \\
\mathbf{N}_1(\varrho) = (\eta\mathbf{M}_0 - \beta\mathbf{P}_0\mathbf{N}_0 - \zeta\mathbf{N}_0)\frac{\left(\frac{\varrho^\wp}{\wp}\right)^\vartheta}{\Gamma(\vartheta+1)} + (\eta\theta\zeta - \beta\zeta(1-\theta)\mathbf{P}_0 - \zeta^2(1-\theta))\frac{\left(\frac{\varrho^\wp}{\wp}\right)^{2\vartheta}}{\Gamma(2\vartheta+1)}, \\
\mathbf{O}_1(\varrho) = \left(\beta\mathbf{P}_0\mathbf{N}_0 - (\varpi+\varphi+\mu)\mathbf{O}_0\right)\frac{\left(\frac{\varrho^\wp}{\wp}\right)^\vartheta}{\Gamma(\vartheta+1)} + \beta\mathbf{P}_0(1-\theta)\zeta\frac{\left(\frac{\varrho^\wp}{\wp}\right)^{2\vartheta}}{\Gamma(2\vartheta+1)}, \\
\mathbf{P}_1(\varrho) = (\varphi\mathbf{O}_0 - (\phi+\mu+\psi)\mathbf{P}_0)\frac{\left(\frac{\varrho^\wp}{\wp}\right)^\vartheta}{\Gamma(\vartheta+1)}, \\
\mathbf{Q}_1(\varrho) = (\varpi\mathbf{O}_0 + \phi\mathbf{P}_0 - \mu\mathbf{Q}_0)\frac{\left(\frac{\varrho^\wp}{\wp}\right)^\vartheta}{\Gamma(\vartheta+1)}.
\end{cases}
$$

Step III. For $\mathbf{n} = 2$, we have

$$\begin{cases} \mathbf{M}_2(\varrho) = \frac{1}{\mathbf{s}^\vartheta}\mathcal{L}_\wp\big[-(\eta+\mu)\mathbf{M}_1(\varrho)\big], \\ \mathbf{N}_2(\varrho) = \frac{1}{\mathbf{s}^\vartheta}\mathcal{L}_\wp\big[\eta\mathbf{M}_1(\varrho) - \beta\mathcal{J}_1(\varrho) - \zeta\mathbf{N}_1(\varrho)\big], \\ \mathbf{O}_2(\varrho) = \frac{1}{\mathbf{s}^\vartheta}\mathcal{L}_\wp\big[\beta\mathcal{J}_1 - (\varpi+\varphi+\mu)\mathbf{O}_1(\varrho)\big], \\ \mathbf{P}_2(\varrho) = \frac{1}{\mathbf{s}^\vartheta}\mathcal{L}_\wp\big[\varphi\mathbf{O}_1(\varrho) - (\phi+\mu+\psi)\mathbf{P}_1(\varrho)\big], \\ \mathbf{Q}_2(\varrho) = \frac{1}{\mathbf{s}^\vartheta}\mathcal{L}_\wp\big[\varpi\mathbf{O}_1(\varrho) + \phi\mathbf{P}_1(\varrho) - \mu\mathbf{Q}_1(\varrho)\big]. \end{cases}$$

Further, employing the inverse $\wp$-Laplace transform on both sides of the above system, we have

$$\begin{cases} \mathbf{M}_2(\varrho) = (\eta+\mu)^2\mathbf{M}_0\frac{1}{\Gamma(2\vartheta+1)}\big(\frac{\varrho^\wp}{\wp}\big)^{2\vartheta} + (\eta+\mu)^2\frac{\theta\zeta}{\Gamma(3\vartheta+1)}\big(\frac{\varrho^\wp}{\wp}\big)^{3\vartheta}, \\ \mathbf{N}_2(\varrho) = \big(\eta\mathbf{M}_0\mathbf{P}_0 - \beta\mathbf{P}_0^2\mathbf{N}_0 - \zeta\mathbf{N}_0\mathbf{P}_0 + \mathbf{N}_0(\varphi\mathbf{O}_0 - (\phi+\mu+\psi)\mathbf{P}_0) - \eta\mathbf{M}_0(\eta+\mu) \\ \quad -\zeta(\eta\mathbf{M}_0 - \beta\mathbf{P}_0\mathbf{N}_0 - \zeta\mathbf{N}_0)\big)\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{2\vartheta}}{\Gamma(2\vartheta+1)} + \big(\zeta(1-\theta)(\varphi\mathbf{O}_0 - (\phi+\mu+\psi)\mathbf{P}_0) \\ \quad -\eta\theta\zeta(\mu+\eta) - \zeta^2\eta\theta - \beta\zeta^4(1-\theta)^2\mathbf{P}_0\big)\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{3\vartheta}}{\Gamma(3\vartheta+1)} + \mathbf{P}_0\big(\eta\theta\zeta - \beta\zeta(1-\theta)\mathbf{P}_0 - \zeta^2(1-\theta)\big)\frac{\Gamma(2\vartheta+1)\big(\frac{\varrho^\wp}{\wp}\big)^{3\vartheta}}{\Gamma^2(\vartheta+1)\Gamma(3\vartheta+1)}, \\ \mathbf{O}_2(\varrho) = \big(\mathbf{N}_0(\beta\varphi\mathbf{O}_0 - \beta\mathbf{P}_0(\phi+\mu+\psi)) + \beta(\eta\mathbf{P}_0\mathbf{M}_0 - \beta\mathbf{P}_0^2\mathbf{N}_0 - \zeta\mathbf{P}_0\mathbf{N}_0) - \beta\mathbf{P}_0\mathbf{N}_0(\varpi+\varphi+\mu) \\ \quad -\mathbf{O}_0(\varpi+\varphi+\mu)^2\big)\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{2\vartheta}}{\Gamma(2\vartheta+1)} + \zeta(1-\theta)\big(\beta\varphi\mathbf{O}_0 - \beta\mathbf{P}_0\big)\frac{\Gamma(2\vartheta+1)\big(\frac{\varrho^\wp}{\wp}\big)^{3\vartheta}}{\Gamma^2(\vartheta+1)\Gamma(3\vartheta+1)} \\ \quad +\beta\mathbf{P}_0\big(\eta\theta\zeta - \beta\zeta(1-\theta)\mathbf{P}_0 - \zeta^2(1-\theta) - \mathbf{P}_0\zeta(1-\theta)(\varpi+\varphi+\mu)\big)\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{3\vartheta}}{\Gamma(3\vartheta+1)}, \\ \mathbf{P}_2(\varrho) = \big(\varphi\beta\mathbf{P}_0\mathbf{N}_0 - \varphi\mathbf{O}_0(\varpi+\varphi+\mu) - (\phi+\mu+\psi)(\varphi\mathbf{O}_0 - \mathbf{P}_0(\mu+\phi+\psi))\big)\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{2\vartheta}}{\Gamma(2\vartheta+1)} + \varphi\beta(1-\theta)\mathbf{P}_0\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{2\vartheta}}{\Gamma(2\vartheta+1)}, \\ \mathbf{Q}_2(\varrho) = \big(\varpi\beta\mathbf{P}_0\mathbf{N}_0 - \varpi(\varpi+\varphi+\mu)\mathbf{O}_0 + \phi(\varphi\mathbf{O}_0 - \mathbf{P}_0(\phi+\mu+\psi)) - \mu(\varpi\mathbf{O}_0 + \phi\mathbf{P}_0 - \mu\mathbf{Q}_0)\big)\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{2\vartheta}}{\Gamma(2\vartheta+1)} \\ \quad +\varpi\beta\zeta(1-\theta)\mathbf{P}_0\frac{\big(\frac{\varrho^\wp}{\wp}\big)^{3\vartheta}}{\Gamma(3\vartheta+1)}. \end{cases}$$

Continuing in the same way, we can obtained the recursive terms $\mathbf{n} \geq 3$. Thus, we attain the desired series solution as follows:

$$\begin{cases} \mathbf{M}(\varrho) = \mathbf{M}_0(\varrho) + \mathbf{M}_1(\varrho) + \mathbf{M}_2(\varrho) + \ldots, \\ \mathbf{N}(\varrho) = \mathbf{N}_0(\varrho) + \mathbf{N}_1(\varrho) + \mathbf{N}_2(\varrho) + \ldots, \\ \mathbf{O}(\varrho) = \mathbf{O}_0(\varrho) + \mathbf{O}_1(\varrho) + \mathbf{O}_2(\varrho) + \ldots, \\ \mathbf{P}(\varrho) = \mathbf{P}_0(\varrho) + \mathbf{P}_1(\varrho) + \mathbf{P}_2(\varrho) + \ldots, \\ \mathbf{Q}(\varrho) = \mathbf{Q}_0(\varrho) + \mathbf{Q}_1(\varrho) + \mathbf{Q}_2(\varrho) + \ldots. \end{cases} \tag{6.4}$$

Theorem 6.1. *Suppose there be a Banach space and $\mathcal{T} : \chi \mapsto \chi$ be a contractive nonlinear operator such that $\forall\ \mathcal{U}, \tilde{\mathcal{U}} \in \chi$, $\big\|\mathcal{T}(\mathcal{U}) - \mathcal{T}(\tilde{\mathcal{U}})\big\|_\chi \leq \mathcal{K}\big\|\mathcal{U} - \tilde{\mathcal{U}}\big\|_\chi$, $\mathcal{K} \in (0,1)$. Applying the Banach contraction principle, $\mathcal{T}$ has a unique fixed point $\mathcal{U}$ such that $\mathcal{T}\mathcal{U} = \mathcal{U}$, where $\mathcal{U} = (u, v, w)$. Employing $\wp$-LADM, the series presented in (6.4) can be expressed as*

$$\mathcal{U}_\mathbf{n} = \mathcal{T}\mathcal{U}_{\mathbf{n}-1}, \quad \mathcal{U}_{\mathbf{n}-1} = \sum_{J=1}^{\mathbf{n}-1}\mathcal{U}_J, \quad J = 1, 2, 3, \ldots, \tag{6.5}$$

and suppose that $\mathcal{U}_0 = \mathcal{U}_0 \in \mathbb{G}_{r_1}(\mathcal{U})$, where $\mathbb{G}_{r_1}(\mathcal{U}) = \mathbf{u} \in \chi : \|\mathbf{u} - \mathcal{U}\|_\chi < r_1$, then we have

(i) $\mathcal{U}_{n1} \in \mathbb{G}_{r_1}(\mathcal{U})$;

(ii) $\lim\limits_{\mathbf{n} \mapsto +\infty} \mathcal{U}_\mathbf{n} = \mathcal{U}$.

Proof. The proof of the following theorem can be developed in an analogous manner as in [37]. □

7. Numerical experiments and explanation

In what follows, we provide simulation solutions as well as representations for the estimation algorithm of the model under investigation in this section of the article. As a starting point, we use the approximate values from Table 1. In view of these variables, we estimate the following series solution:

Table 1. Table of parameterized settings considered in the problem (1.2) for simulations.

Population/parameters	Explanation
$\mathbb{N} = 500$	Overall Population
$\mathbf{M}_0 = 90$	Immunized Population
$\mathbf{N}_0 = 400$	Susceptible Population
$\mathbf{O}_0 = 100$	latently infected Population
$\mathbf{P}_0 = 50$	Infected Population
$\mathbf{Q}_0 = 50$	Recovered Population
$\zeta = 1$	Consistent procurement
$\theta = 65/1000$	Proportion immunized at birth
$\eta = 256/10000$	proportion of weaning off the inoculation
$\mu = 21/1000$	Natural death rate
$\beta = 9091/100000$	**TB** contraction proportion
$\varpi = 342/10000$	Efficacious treatment of infectious proportion
$\varphi = 124/1000$	Proportion of latent **TB** into highly contagious **TB**
$\phi = 16709/1000000$	Satisfactorily healed of contagious **TB** patients
$\psi = 30/1000$	Death consequence of **TB** transmission.

$$\begin{cases} \mathbf{M}(\varrho) = 90 - 4.194\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{\vartheta}}{\Gamma(\vartheta+1)} + 0.192411\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{2\vartheta}}{\Gamma(2\vartheta+1)} - 0.008966\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{3\vartheta}}{\Gamma(3\vartheta+1)}, \\ \mathbf{N}(\varrho) = 400 - 1823.36\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{\vartheta}}{\Gamma(\vartheta+1)} + 8408.2894\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{2\vartheta}}{\Gamma(2\vartheta+1)} - 39200.2682\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{3\vartheta}}{\Gamma(3\vartheta+1)} - 355.6365\frac{\Gamma(2\vartheta+1)\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{3\vartheta}}{\Gamma^2(\vartheta+1)\Gamma(3\vartheta+1)}, \\ \mathbf{O}(\varrho) = 100 + 18211.44\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{\vartheta}}{\Gamma(\vartheta+1)} - 8488.5579\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{2\vartheta}}{\Gamma(2\vartheta+1)} + 39597.60959\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{3\vartheta}}{\Gamma(3\vartheta+1)} + 355.6365\frac{\Gamma(2\vartheta+1)\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{3\vartheta}}{\Gamma^2(\vartheta+1)\Gamma(3\vartheta+1)}, \\ \mathbf{P}(\varrho) = 50 - 2.14545\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{\vartheta}}{\Gamma(\vartheta+1)} + 22.6071\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{2\vartheta}}{\Gamma(2\vartheta+1)} - 106.7888\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{3\vartheta}}{\Gamma(3\vartheta+1)}, \\ \mathbf{Q}(\varrho) = 10 + 4.0454\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{\vartheta}}{\Gamma(\vartheta+1)} + 61.8304\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{2\vartheta}}{\Gamma(2\vartheta+1)} + 289.93398\frac{\left(\frac{\varrho^{\varphi}}{\varphi}\right)^{3\vartheta}}{\Gamma(3\vartheta+1)}. \end{cases} \quad (7.1)$$

Now we display the result up to four components in Figures 1–5, which are associated with various fractional orders, which can be seen in (7.1) in Figures 1–5. The vaccinated community diminishes with distinct fractional orders at various proportions, as seen in Figure 1(a). Similarly, as demonstrated in Figure 2(a), the susceptible community is expanding. As illustrated in Figures 3(a) and 4(a), both infected and innately affected communities are proliferating. As a result of the susceptible community

growing sick or insidiously afflicted. If a suitable treatment is adopted, the proportion of people who have been cured will rise, as illustrated in Figure 5(a). The mechanism of demographic rise or decline is generally quickest at smaller fractional orders and then flips, and the higher the fractional order, the more rapid the procedure of growing population or reduction in the appropriate compartment.On the other hand, fractional order derivatives can describe behaviour extremely broadly. As time progresses, the recoverable community progressively grows and tends to a dynamic equilibrium, as noticed in Figure 5(a).

Figures 1(b)–5(b), the approximate solutions for vaccinated $\mathbf{M}(\varrho)$, susceptible $\mathbf{N}(\varrho)$, innately affected $\mathbf{O}(\varrho)$, infected $\mathbf{P}(\varrho)$ and recovered $\mathbf{R}(\varrho)$ communities, derived with the $\wp$-LADM exhibit a remarkable degree of precision when contrasted to the LADM solution computed by Ahmad et al. [38]. As a result, we may conclude that the $\wp$-LADM approach is an adequate and trustworthy mathematical approach for solving linear and nonlinear differential equation systems in demographic dynamics. The visual depictions explicitly indicate that $\wp$-LADM produces excellent outcomes once a given amount of space has passed. This is a highly handy strategy that will surely find usage in a variety of situations. So, the generalized Caputo fractional derivative operator has an added benefit over the Caputo-Fabrizio in that it does not require the determination of intricate integrals, and it has an edge over the $\wp$-LADM in that it has a reasonable aim when utilizing predictive control width. As a result, it delivers an approximate solution that is effective.

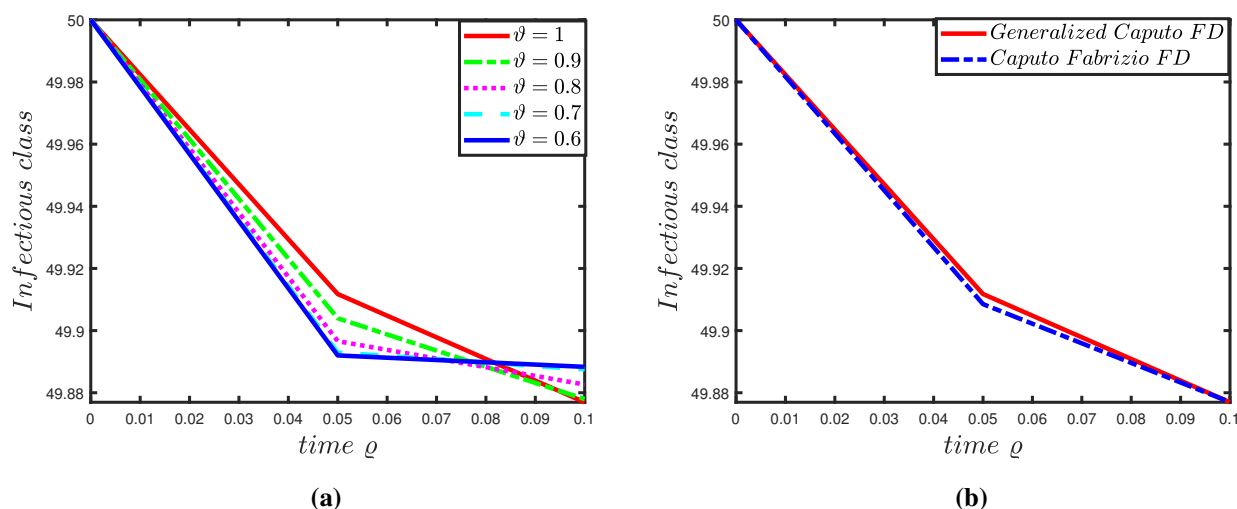**(a)**   **(b)**
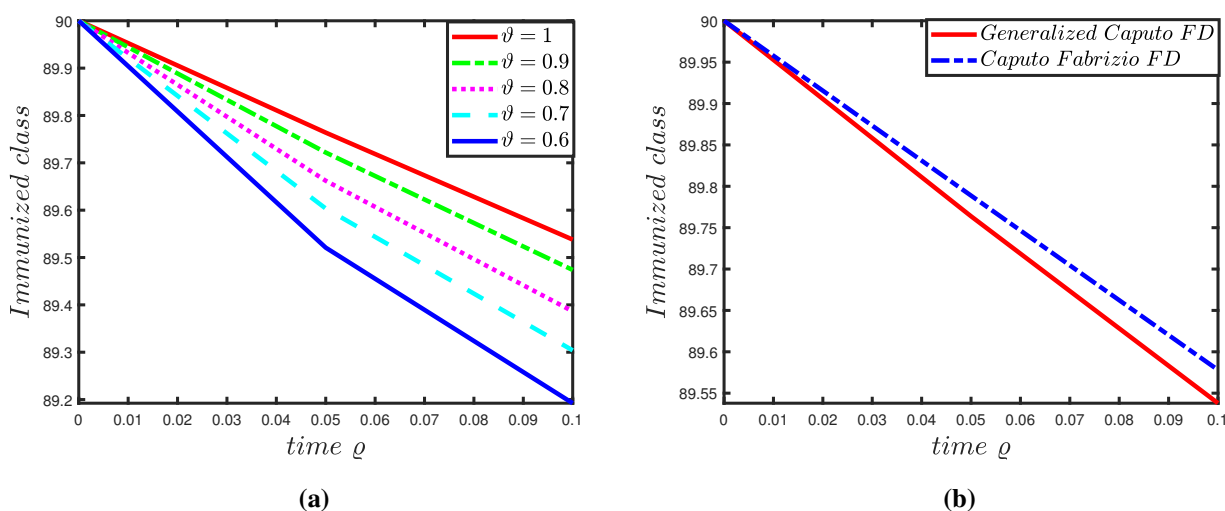
Figure 1. (a) Simulation of the approximate results of the Immunized class $\mathbf{M}(\varrho)$ for various fractional order $\vartheta = 1$, 0.9, 0.8, 0.7, 0.6. (b) Comparison analysis of the generalized Caputo fractional derivative and the Caputo Fabrizio fractional derivative operator of the Immunized class $\mathbf{M}(\varrho)$ when $\vartheta = 1$ and $\wp = 0.67$.
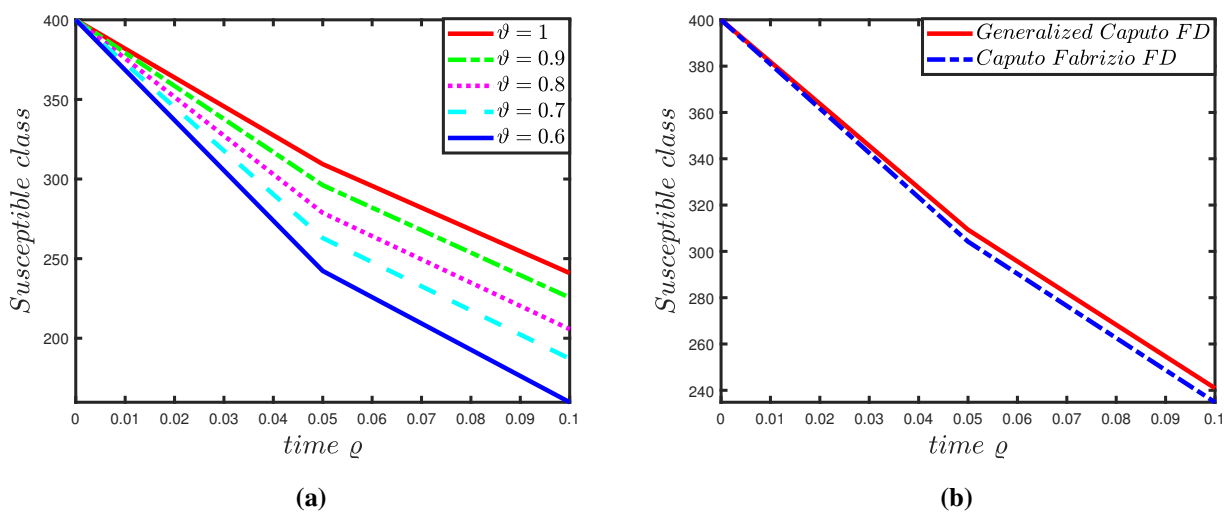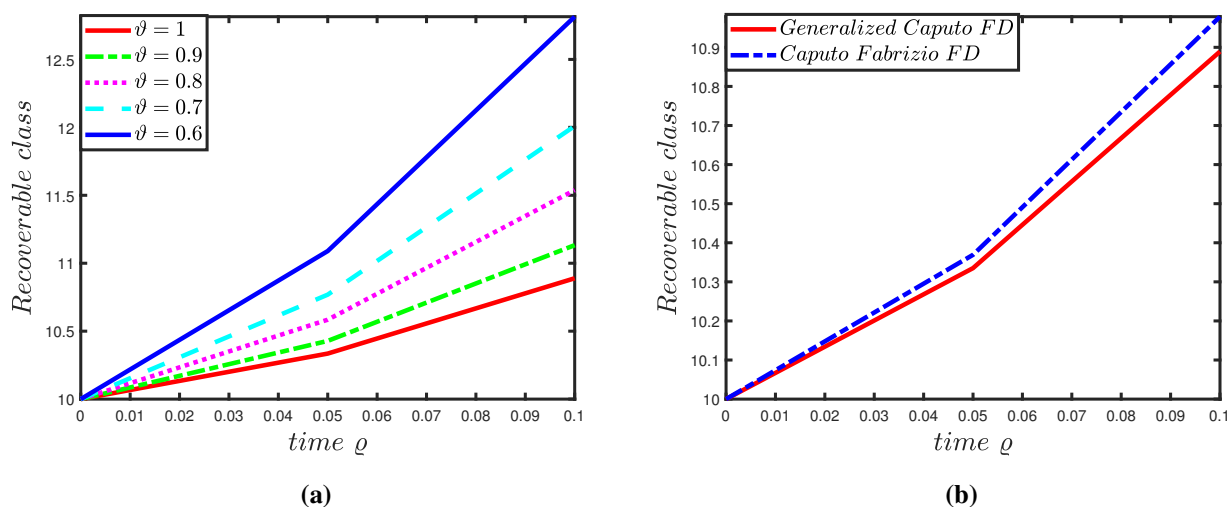
Figure 2. (a) Simulation of the approximate results of the Susceptible class $\mathbf{N}(\varrho)$ for various fractional order $\vartheta = 1,\ 0.9,\ 0.8,\ 0.7,\ 0.6$. (b) Comparison analysis of the generalized Caputo fractional derivative and the Caputo Fabrizio fractional derivative operator of the Susceptible class $\mathbf{N}(\varrho)$ when $\vartheta = 1$ and $\wp = 0.67$.
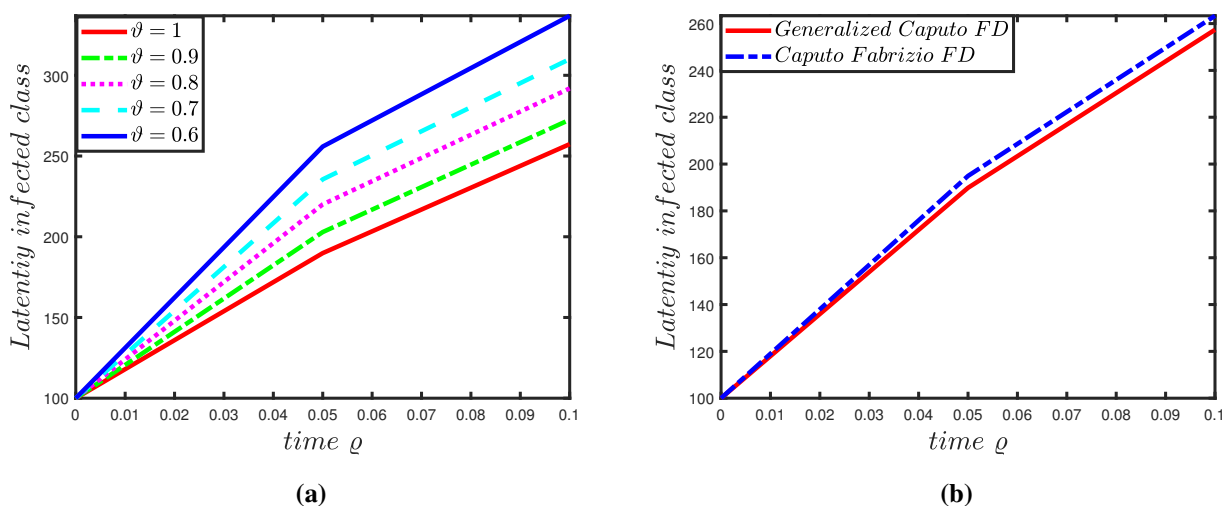
Figure 3. (a) Simulation of the approximate results of the latently infected class $\mathbf{O}(\varrho)$ for various fractional order $\vartheta = 1,\ 0.9,\ 0.8,\ 0.7,\ 0.6$. (b) Comparison analysis of the generalized Caputo fractional derivative and the Caputo Fabrizio fractional derivative operator of the latently infected class $\mathbf{O}(\varrho)$ when $\vartheta = 1$ and $\wp = 0.67$.

Figure 4. (a) Simulation of the approximate results of the Infectious class $\mathbf{P}(\varrho)$ for various fractional order $\vartheta = 1$, 0.9, 0.8, 0.7, 0.6. (b) Comparison analysis of the generalized Caputo fractional derivative and the Caputo Fabrizio fractional derivative operator of the Infectious class $\mathbf{P}(\varrho)$ when $\vartheta = 1$ and $\wp = 0.67$.

Figure 5. (a) Simulation of the approximate results of the Recoverable class $\mathbf{Q}(\varrho)$ for various fractional order $\vartheta = 1$, 0.9, 0.8, 0.7, 0.6. (b) Comparison analysis of the generalized Caputo fractional derivative and the Caputo Fabrizio fractional derivative operator of the Recoverable class $\mathbf{Q}(\varrho)$ when $\vartheta = 1$ and $\wp = 0.67$.

8. Conclusions

In this article, the generalized Caputo fractional derivative is employed to examine the mathematical formulation of a tuberculosis model using preventative medication. With the assistance of the Larey-Schauder dynamic substitute and Banach's fixed point hypothesis, we demonstrated that

the response of the analyzed paradigm (1.2) is developed by the generalized Caputo fractional derivative. Furthermore, Ulam stability, such as Ulam-Hyers stable, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stable, and generalized Ulam-Hyers-Rassias stable, were adopted to assess the method's reliability. Then, using the Matlab program, we implemented the $\wp$-LADM approach to provide estimated values for various fractional-order problems, and we discovered that the findings of the descriptive and predictive models (1.2) approaches to the classical ones when $\vartheta \mapsto 1$. The comparison analysis shows that the projected scheme is in close agreement with the existing one. One can be successfully extended to several forms of fractional derivatives with analogous methodologies in numerous real-world applications for future development. We expect that this effort will serve as a viable replacement for numerous scientific projects.

Acknowledgements

Conflict of interest

The authors declare that they have no competing interests.

References

1. Tackling the dual burden of TB and diabetes for patients and their families, World Health Organization, 2019. Available from: `https://www.who.int/news/item/14-11-2019-dept-newstackling-the-dual-burden -of-tb-and-diabetes-for-patients-and-their-families`.

2. S. E. Geerlings, A. I. M. Hoepelman, Immune dysfunction in patients with diabetes mellitus (DM), *FEMS Immunol. Med. Mic.*, **26** (1999), 259–265. https://doi.org/10.1111/j.1574-695X.1999.tb01397.x

3. D. Morse, D. R. Brothwell, P. J. Ucko, Tuberculosis in ancient Egypt, *Am. Rev. Respir. Dis.*, **90** (1964), 524–541.

4. J. P. Aparicio, A. F. Capurro, C. Castillo-Chavez, Transmission and dynamics of tuberculosis on generalized households, *J. Theor. Biol.*, **206** (2000), 327–341. https://doi.org/10.1006/jtbi.2000.2129

5. K. Floyd, P. Glaziou, A. Zumla, M. Raviglione, The global tuberculosis epidemic and progress in care, prevention, and research: An overview in year 3 of the end TB era, *Lancet Respir. Med.*, **6** (2018), 299–314. https://doi.org/10.1016/S2213-2600(18)30057-2

6. C. Dye, Global epidemiology of tuberculosis, *Lancet*, **367** (2006), 938–940. https://doi.org/10.1016/S0140-6736(06)68384-0

7. G. A. Colditz, T. F. Brewer, C. S. Berkey, M. E. Wilson, E. Burdick, H. V. Fineberg, et al., Efficacy of BCG vaccine in the prevention of tuberculosis: Meta-analysis of the published literature, *JAMA*, **271** (1994), 698–702. https://doi.org/10.1001/jama.1994.03510330076038

8. O. A. Arqub, A. El-Ajou, Solution of the fractional epidemic model by homotopy analysis method, *J. King Saud Univ. Sci.*, **25** (2013), 73–81. https://doi.org/10.1016/j.jksus.2012.01.003

9. M. Rafei, D. D. Ganji, H. Daniali, Solution of the epidemic model by homotopy perturbation method, *Appl. Math. Comput.*, **187** (2007), 1056–1062. https://doi.org/10.1016/j.amc.2006.09.019

10. S. Zhao, Z. Xu, Y. Lu, A mathematical model of hepatitis B virus transmission and its application for vaccination strategy in China, *Int. J. Epidemiol.*, **29** (2000), 744–752. https://doi.org/10.1093/ije/29.4.744

11. F. Haq, K. Shah, A. Khan, M. Shahzad, G. Rahman, Numerical solution of fractional order epidemic model of a vector born disease by laplace adomian decomposition method, *Punjab Univ. J. Math.*, **49** (2017), 13–22.

12. I. Ullah, S. Ahmad, Q. Al-Mdallal, Z. A. Khan, H. Khan, A. Khan, Stability analysis of a dynamical model of tuberculosis with incomplete treatment, *Adv. Differ. Equ.*, **2020** (2020), 499. https://doi.org/10.1186/s13662-020-02950-0

13. A. I. Enagi, M. O. Ibrahim, N. I. Akinwande, M. Bawa, A. Wachin, A mathematical model of tuberculosis control incorporating vaccination, latency and infectious treatments (case study of Nigeria), *Int. J. Math. Comput. Sci.*, **12** (2017), 97–106.

14. F. Wang, M. N. Khan, I. Ahmad, H. Ahmad, H. Abu-Zinadah, Y. M. Chu, Numerical solution of traveling waves in chemical kinetics: Timefractional fishers equations, *Fractals*, **30** (2022), 22400051.

15. S. Rashid, E. I. Abouelmagd, A. Khalid, F. B. Farooq, Y. M. Chu, Some recent developments on dynamical *h*-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels, *Fractals*, **30** (2022), 2240110.

16. F. Jin, Z. S. Qian, Y. M. Chu, M. ur Rahman, On nonlinear evolution model for drinking behavior under Caputo-Fabrizio derivative, *J. Appl. Anal. Comput.*, 2022. https://doi.org/10.11948/20210357

17. Z. Y. He, A. Abbes, H. Jahanshahi, N. D. Alotaibi, Y. Wang, Fractionalorder discrete-time SIR epidemic model with vaccination: Chaos and complexity, *Mathematics*, **10** (2022), 165. https://doi.org/10.3390/math10020165

18. S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, Y. M. Chu, On multistep methods for singular fractional q-integro-differential equations, *Open Math.*, **19** (2021), 1378–1405. https://doi.org/10.1515/math-2021-0093

19. S. Rashid, S. Sultana, Y. Karaca, A. Khalid, Y. M. Chu, Some further extensions considering discrete proportional fractional operators, *Fractals*, **30** (2022), 2240026.

20. Y. M. Chu, U. Nazir, M. Sohail, M. M. Selim, J. R. Lee, Enhancement in thermal energy and solute particles using hybrid nanoparticles by engaging activation energy and chemical reaction over a parabolic surface via finite element approach, *Fractal Fract.*, **5** (2021), 119. https://doi.org/10.3390/fractalfract5030119

21. K. Karthikeyan, P. Karthikeyan, H. M. Baskonus, K. Venkatachalam, Y. M. Chu, Almost sectorial operators on ψ-Hilfer derivative fractional impulsive integro-differential equations, *Math. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7954

22. M. A. Iqbal, Y. Wang, M. M. Miah, M. S. Osman, Study on Date-Jimbo-Kashiwara-Miwa equation with conformable derivative dependent on time parameter to find the exact dynamic wave solutions, *Fractal Fract.*, **6** (2022), 4. https://doi.org/10.3390/fractalfract6010004

23. T. H. Zhao, M. I. Khan, Y. M. Chu, Artificial neural networking (ANN) analysis for heat and entropy generation in flow of non-Newtonian fluid between two rotating disks, *Math. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7310

24. Y. M. Chu, B. M. Shankaralingappa, B. J. Gireesha, F. Alzahrani, M. I. Khan, S. U. Khan, Combined impact of Cattaneo-Christov double diffusion and radiative heat flux on bio-convective flow of Maxwell liquid configured by a stretched nano-material surface, *Appl. Math. Comput.*, **419** (2021), 126883. https://doi.org/10.1016/j.amc.2021.126883

25. M. Nazeer, F. Hussain, M. I. Khan, A. ur-Rehman, E. R. ElZahar, Y. M. Chu, et al., Theoretical study of MHD electro-osmotically flow of third-grade fluid in micro channel, *Appl. Math. Comput.*, **420** (2021), 126868. https://doi.org/10.1016/j.amc.2021.126868

26. T. H. Zhao, O. Castillo, H. Jahanshahi, A. Yusuf, M. O. Alassafi, F. E. Alsaadi, et al., A fuzzy-based strategy to suppress the novel coronavirus (2019-NCOV) massive outbreak, *Appl. Comput. Math.*, **20** (2021), 160–176.

27. U. N. Katugampola, New approach to generalized fractional integral, *Appl. Math. Comput.*, **218** (2011), 860–865. https://doi.org/10.1016/j.amc.2011.03.062

28. I. Podlubny, *Fractional differential equations: Mathematics in science and engineering*, Academic Press, New York 1999.

29. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland, Amsterdam, 2006.

30. F. Jarad, T. Abdeljawad, A modified Laplace transform for certain generalized fractional operators, *Res. Nonlinear Anal.*, **1** (2018), 88–98.

31. K. Deimling, *Nonlinear functional analysis*, New York: Springer-Verlag, 1985.

32. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2003.

33. Z. M. Odibat, N. T. Shawagfeh, Generalized Taylor's formula, *Appl. Math. Comput.*, **186** (2007), 286–293. https://doi.org/10.1016/j.amc.2006.07.102

34. X. Q. Zhao, The theory of basic reproduction ratios, In: *Dynamical systems in population biology*, Springer, Cham, 2017. https://doi.org/10.1007/978-3-319-56433-3_11

35. E. Ahmed, A. M. A. El-Sayed, H. A. A. El-Saka, On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rossler, Chua and Chen systems, *Phys. Lett. A*, **358** (2006), 1–4. https://doi.org/10.1016/j.physleta.2006.04.087

36. I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.*, **26** (2010), 103–107.

37. S. Rashid, K. T. Kubra, H. Jafari, S. U. Lehre, A semi-analytical approach for fractional order Boussinesq equation in a gradient unconfined aquifers, *Math. Meth. Appl. Sci.*, **45** (2022), 1033–1062. https://doi.org/10.1002/mma.7833 DOI: 10.1002/mma.7833

38. S. Ahmad, R. Ullah, D. Baleanu, Mathematical analysis of tuberculosis control model using nonsingular kernel type Caputo derivative, *Adv. Diff. Equ.*, **2021** (2021), 26. https://doi.org/10.1186/s13662-020-03191-x