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Research article

New approach on controllability of Hilfer fractional derivatives with nondense domain

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Abstract: This work picturizes the results on the controllability of the nondense Hilfer neutral fractional derivative (HNFD). The uniqueness and controllability of HNFD are discussed with Mönch theorem and Banach contraction technique. In addition, a numerical approximation is given to deal with different criteria of our results.

Keywords: controllability; Hilfer derivative; nondense domain; fixed point; Banach space; fractional calculus
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1. Introduction

Fractional calculus consists of fractional differentiation and integration with a described value of undetermined functions at more than one point in the solution domain. This is a more precious tool in technology such as electrical network analysis, population models, economics and engineering, medicine, etc. Nowadays, numerical analysis has been used widely to approximate the solutions of the system. For existing theoretical work, we can refer [3,6,7,11,12,14,15,20,24,25,29,35–38] and monographs [1,9,13,17,22,23,40].

Generalized fractional derivatives that contain Caputo and R-L derivatives as particular cases were introduced by Hilfer [9, 10]. Also, the additional parameter introduced in the above-said model provides more degrees of freedom. The results on Hilfer derivative with initial conditions were discussed by several authors [8, 21, 26, 30–32]. In particular, Furati [5] and Gu [8] discussed the existence, nonexistence, and stability of nonlinear problems with HFD.

Controllability is a qualitative property and being an essential tool in mathematical modeling. It deals with problems on pole assignment, stability, optimal control that the respective system is controllable. Many authors contributed their research work on controllability with dense C_0-semigroup operator [15, 18, 19, 34]. Practically, some problems occur in non-dense domain that are explained in [4, 27, 28, 36].

In the year 2020, Wang [30] exhibited the approximate controllability of HFD. Continuation of this in 2020, Lv and Yang [18] published an article in Hilfer fractional equation deals with approximate controllability. At the same time, Vikram Singh [27] extended the results to non-dense HFD. In 2021, Zhou [39] discussed the results on controllability of fractional evolution system. Also in the year 2020, Kumar [16] discussed the numerical approach to the controllability for fractional differential equations. To the best of our knowledge, we have no discussions exists on the numerical approach on controllability for fractional derivatives in nondense domain.

Here we discussed HNFD of the model as:

$$\mathcal{D}_{0+}^{\alpha,\beta}\Big[\mathfrak{z}(\theta) - \mathfrak{q}\Big(\theta, \mathfrak{z}(\theta), \int_0^\theta e(\theta, t, \mathfrak{z}(t))dt\Big)\Big] = A\mathfrak{z}(\theta) + Bu(\theta) + h\Big(\theta, \mathfrak{z}(\theta), \int_0^\theta f(\theta, t, \mathfrak{z}(t))dt\Big), \quad (1.1)$$

$$\mathcal{I}_{0+}^{(1-\alpha)(1-\beta)}\mathfrak{z}(0) = \mathfrak{z}_0, \quad \theta \in \mathcal{T} = [0, a]. \quad (1.2)$$

Here $\mathfrak{z}(\theta)$ is state variable and $u(\theta)$ be a control variable. $\mathcal{D}_{0+}^{\alpha,\beta}$ refers Hilfer derivative of order α in $(0, 1)$ and type β in $(0, 1]$. Also $A : \mathcal{D} \subset \mathcal{X} \to \mathcal{X}$, the non-densely closed linear operator in the Banach space $\mathcal{X}$, and the domain $\mathcal{D} \in \mathcal{X}$. Here $\mathfrak{q}, h, f, e$ are appropriate functions defined as $\mathfrak{q} : [0, a] \times \mathcal{X} \times \mathcal{X} \to \mathcal{D} \subset \mathcal{X}$, $h : [0, a] \times \mathcal{X} \times \mathcal{X} \to \mathcal{D} \subset \mathcal{X}$, $f : [0, a] \times [0, a] \times \mathcal{X} \to \mathcal{D} \subset \mathcal{X}$ and $e : [0, a] \times [0, a] \times \mathcal{X} \to \mathcal{D} \subset \mathcal{X}$.

For certain sub-classes for the function h, the above problem can be solved explicitly [2]. However, for even slightly less amenable right-hand sides, there are no existing methods for constructing explicit solutions. In such cases it is of interest to study approximations to the solution of the initial value problem (1.1) and (1.2). In the present paper we show that for such initial value problems, we can construct a sequence of successive approximations which converge to the solution in the uniform norm of the space of continuous functions on an interval. This result generalizes the result of [33] to the fractional case of $\alpha \in (0, 1)$.

The results are obtained as:

(*i*) A mild solution of the system (1.1) and (1.2) has been obtained using Laplace transform technique.

(*ii*) We enriched the idea of controllability for the system using the Mönch fixed point theorem.

(*iii*) Hilfer derivative and its particular cases are discussed graphically.

2. Basic facts

Consider a space of continuous functions $C(\mathcal{T},\mathcal{X})$ is defined on $\mathcal{T}$ that satisfies $\|y\| = \sup_{t\in\mathcal{T}}\|y(t)\|$. $C_{1-\vartheta}(\mathcal{T},\mathcal{X}) = \{y : \mathcal{X}\to\mathcal{X} : t^{1-\vartheta}y(t)\in C(\mathcal{T},\mathcal{X})\}$, a Banach space with $\|y\|_{C_{1-\vartheta}} = \sup_{0\le t\le a}|t^{1-\vartheta}y(t)|$, where $\vartheta = \alpha+\beta-\alpha\beta$. Define

$$ {}^{R-L}\mathcal{D}^{p}_{0+}y(t) = \frac{d^n}{dt^n}(y(t)*q_{n-p}(t)), \tag{2.1} $$

$$ {}^{C}\mathcal{D}^{p}_{0+}y(t) = \frac{d^n}{dt^n}y(t)*q_{n-p}(t), \qquad n-1<p<n. \tag{2.2} $$

Equation (2.1) refers Riemann-Liouville (R-L) derivative as well as (2.2) refers Caputo derivative.

Definition 2.1 ([9]). *For $k\in N$, we define HFD for $\alpha\in(k-1,\,k)$,*

$$ \mathcal{D}^{\alpha,\beta}_{0+}h(\theta) = I^{\alpha(k-\beta)}_{0+}\frac{d}{d\theta}I^{(1-\alpha)(k-\beta)}_{0+}h(\theta) = I^{\alpha(k-\beta)}_{0+}\mathcal{D}^{\beta+\alpha n-\beta\alpha}_{0+}h(\theta). $$

Lemma 2.2 ([5]). *If $h\in C^{\vartheta}_{1-\vartheta}[r_1,r_2]$ is such that $\mathcal{D}^{\vartheta}_{0+}h\in C_{1-\vartheta}[r_1,r_2]$ then,*

$$ I^{\vartheta}_{0+}\mathcal{D}^{\vartheta}_{0+}h = I^{\alpha}_{0+}\mathcal{D}^{\alpha,\beta}_{0+} \ \text{ and } \ \mathcal{D}^{\vartheta}_{0+}I^{\vartheta}_{0+}h = \mathcal{D}^{\beta(1-\alpha)}_{0+}h. $$

Lemma 2.3 ([5]). *If $h\in C_{1-\vartheta}[r_1,r_2]$ and $I^{1-\vartheta}_{0+}h\in C^{1}_{\vartheta}[r_1,r_2]$ then,*

$$ I^{\vartheta}_{0+}\mathcal{D}^{\vartheta}_{0+}h(\theta) = h(\theta) - \frac{I^{1-\vartheta}_{0+}h(r_1)}{\Gamma(\vartheta)}(\theta-r_1)^{\vartheta-1}, \quad \text{for all } \ \theta\in(r_1,r_2], $$

where $\alpha\in(0,1), \vartheta\in[0,1)$.

Lemma 2.4 ([7]). *Let S and M be subsets of a metric space $(\mathcal{X},d)$ and $\epsilon>0$. Then, S is called an ϵ-net of M in $\mathcal{X}$ if for every $x\in M$ there exists $s\in S$ such that $d(x,s)<\epsilon$. Further, if the set S is finite, then the ϵ-net S of M is called a finite ϵ-net of M, and we say that M has a finite ϵ-net in $\mathcal{X}$.*

Let M_x denotes the collection of all bounded subsets of metric space $(\mathcal{X},d)$ and $\nabla\in M_x$. Then the Hausdorff measure of the set ∇ is defined as $\rho(\nabla) = \inf\{\epsilon>0, \nabla$ has finite $\epsilon-$net in $\mathcal{X}\}$, and meets:

(1) *for each $y\in\mathcal{X}$, $\rho(\{y\}\cup\nabla) = \rho(\nabla)$, where $\nabla\subseteq\mathcal{X}$ is nonempty;*

(2) $\rho(\nabla_1)\le\rho(\nabla_2)$, *for all bounded subsets ∇_1,∇_2 of $\mathcal{X}$ provided $\nabla_1\subseteq\nabla_2$;*

(3) $\rho(\nabla)=0$ *iff ∇ is relatively compact in $\mathcal{X}$;*

(4) $\rho(\nabla_1\cup\nabla_2)\le\max\{\rho(\nabla_1),\rho(\nabla_2)\}$.

(5) *for any $\lambda\in R$, $\rho(\lambda\nabla)\le|\lambda|\rho(\nabla)$;*

(6) $\rho(\nabla_1+\nabla_2)\le\rho(\nabla_1)+\rho(\nabla_2)$, *where $\nabla_1+\nabla_2 = \{y_1+y_2; y_1\in\nabla_1, y_2\in\nabla_2\}$;*

Lemma 2.5. *[8] Let $\mathcal{T}$ be the set $[0,a]$, and for $t\in\mathcal{T}$ with $n\ge1$ the Bochner's sequence $\{z_n\}_{n=1}^{\infty}$ from $\mathcal{T}$ to $\mathcal{X}$ fulfills $|z_n(t)|\le\widetilde{m}(\theta)$, where $\widetilde{m}\in L(\mathcal{T},R^+)$. Also, $G(t)=\rho(\{z_n(t)\}_{n=1}^{\infty})$ in $L(\mathcal{T},R^+)$ satisfies*

$$ \rho\left(\left\{\int_0^{\theta} z_n(s)\,ds : n\ge1\right\}\right) \le 2\int_0^{\theta} G(s)\,ds. $$

Consider $\mathcal{X}_0 = \overline{\mathcal{D}}$ and A_0 be the characteristic element of A in $\mathcal{X}_0$ defined as

$$\mathcal{D}_{A_0} = \{y \in \mathcal{D} : Ay \in \mathcal{X}_0\}, \ A_0 y = Ay.$$

Proposition 2.6. *Let $A_0 \subset A$ generates $\{\Re(\theta)\}_{\theta \geq 0}$ on $\mathcal{X}_0$ with $\mathcal{X}_0 = \overline{\mathcal{D}}$ satisfies $A_0 y = Ay$.*

Consider the following assumptions:

$(H1)$ For every $n \geq 1$ and $\lambda > k$, and for the constants $k \in \mathbb{R}, \mathscr{M}_0$ with $(k, +\infty) \subseteq \hbar(A)$,

$$\|(\lambda I - A)^{-n}\|_{L(\mathcal{X})} \leq \frac{\mathscr{M}_0}{\sup(\lambda - k)^n}.$$

$(H2)$ For some constant $\mathscr{M}_1 > 1$ with $\sup_{\theta \in [0,+\infty]} |\Re(\theta)| < \mathscr{M}_1$.

Also, we define for $\theta \geq 0$,

$$T_\alpha(\theta) = \alpha \int_0^\infty \nu \psi_\chi(\nu) \Re(\theta^\alpha \nu) d\nu, \qquad P_\alpha(\theta) = \theta^{\alpha-1} T_\alpha(\theta), \quad S_{\alpha,\beta}(\theta) = \mathcal{T}_{0+}^{\beta(1-\alpha)} P_\alpha(\theta).$$

For $\nu \in (0, \infty)$,

$$\begin{aligned} \psi_\chi(\nu) &= \frac{1}{\chi} \nu^{(-1-\frac{1}{\chi})} W_\chi(\nu^{-\frac{1}{\chi}}) \geq 0, \\ W_\chi(\nu) &= \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \nu^{-k\chi-1} \frac{\Gamma(k\chi + 1)}{k!} \sin(k\pi\chi). \end{aligned}$$

Lemma 2.7 ([8]). *From $(H2)$,*

(i) *For $\theta > 0$,*

$$|S_{\alpha,\beta}(\theta)| \leq \frac{\mathscr{M}\theta^{\vartheta-1}}{\Gamma(\vartheta)} \quad and \quad |P_\alpha(\theta)| \leq \frac{\mathscr{M}\theta^{\alpha-1}}{\Gamma(\alpha)}.$$

(ii) *For $\mathfrak{z} \in \mathcal{X}_0$, $0 < \theta_1 < \theta_2 \leq a$, $\{S_{\alpha,\beta}(\theta)\}$ and $\{P_\alpha(\theta)\}$ satisfies*

$$|S_{\alpha,\beta}(\theta_1)\mathfrak{z} - S_{\alpha,\beta}(\theta_2)\mathfrak{z}| \to 0 \text{ and } |P_\alpha(\theta_1)\mathfrak{z} - P_\alpha(\theta_2)\mathfrak{z}| \to 0 \text{ as } \theta_2 \to \theta_1.$$

(iii) *$\{T_\alpha(\theta)\}_{\theta \geq 0}$ is uniformly continuous.*

Lemma 2.8 ([5]). *For $\theta \in \mathcal{T}$, the system (1.1) and (1.2) reduces to,*

$$\begin{aligned} \mathfrak{z}(\theta) = & \frac{[\mathfrak{z}_0 - \mathfrak{q}(0, \mathfrak{z}(0), 0)]}{\Gamma(\vartheta)} \theta^{\vartheta-1} + \mathfrak{q}\Big(\theta, \mathfrak{z}(\theta), \int_0^\theta e(\theta, t, \mathfrak{z}(t)) dt\Big) + \frac{1}{\Gamma(\alpha)} \int_0^\theta (\theta - t)^{\alpha-1} \\ & \times [A\mathfrak{z}(t) + h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s)) ds\Big) + Bu(t)] dt. \end{aligned} \tag{2.3}$$

Definition 2.9. *For all $\theta \in \mathcal{X}$ and $h \in \mathcal{X}_0$, integral solution of (1.1) and (1.2) defined as*

$$\mathfrak{z}(\theta) = S_{\alpha,\beta}(\theta)[\mathfrak{z}_0 - \mathfrak{q}(0,\mathfrak{z}(0),0)] + \mathfrak{q}\Big(\theta,\mathfrak{z}(\theta),\int_0^\theta e(\theta,t,\mathfrak{z}(t))dt\Big) + \lim_{\lambda\to\infty}\int_0^\theta P_\alpha(\theta-t)\mathcal{B}_\lambda \times \Big[A\mathfrak{q}\Big(t,\mathfrak{z}(t),\int_0^t e(t,s,\mathfrak{z}(s))ds\Big) + Bu(t) + h\Big(t,\mathfrak{z}(t),\int_0^t f(t,s,\mathfrak{z}(s))ds\Big)\Big]dt, \tag{2.4}$$

where $\mathcal{B}_\lambda = \lambda(\lambda I - A)^{-1}$ such that $\mathcal{B}_\lambda \mathfrak{z} = \mathfrak{z}$ as $\lambda \to \infty$.

Definition 2.10. *The system (1.1) and (1.2) is said to be controllable on $\mathcal{T} = [0, a]$ if for any $\mathfrak{z}_0, \mathfrak{z}_1 \in \mathcal{X}$ and fixed a, there exists a control $u \in U$ such that the corresponding solution $\mathfrak{z}(\cdot)$ of the system (1.1) and (1.2) satisfies $\mathfrak{z}(a) = \mathfrak{z}_1$.*

3. Controllability results

The following hypotheses are introduced for further analysis:

$(H3)$ A function $h : \mathcal{T} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ fulfills:
(i) $\mathfrak{z} \in \mathcal{X}$, $h(\cdot, \mathfrak{z}_1, \mathfrak{z}_2) : \mathcal{X} \to \mathcal{X}$ is strongly measurable and for every $\theta \in \mathcal{T}$, $h(\theta, \cdot, \cdot) : \mathcal{T} \to \mathcal{X}$ is continuous and
(ii) For every $n > 0$, and for $\mu_n \in L^1(0, a)$ such that for each $\theta \in \mathcal{T}$,

$$\sup_{\|\mathfrak{z}\|,|y|\le n} |h(\theta,\mathfrak{z},y| \le \mu_n(\theta).$$

(iii) For $q \in (0,\alpha), m_1 \in L^{\frac{1}{q}}(\mathcal{T}, R^+)$, and for a continuous and nondecreasing function $\mathcal{L}_h : [0,\infty] \to (0,\infty)$,

$$\|h(\theta,\mathfrak{z}_1(\theta),\mathfrak{z}_2(\theta)\| \le m_1(\theta)\mathcal{L}_h(\theta^{1-\vartheta}(\|\mathfrak{z}_1(\theta)\| + \|\mathfrak{z}_2(\theta)\|)).$$

Moreover $\lim_{\nu\to\infty} \frac{\mathcal{L}_h(\nu)}{\nu} = \mathcal{L}_h^*$, for all $(\theta, \mathfrak{z}, y) \in \mathcal{T} \times \mathcal{X} \times \mathcal{X}$ and $m_1^* = \max\{m_1(\theta)\}$.

$(H4)$ For $\bar{\mathcal{L}}_h, \tilde{\mathcal{L}}_h > 0$ and for $\mathfrak{z}, w \in \mathcal{X}$, we have

$$\|h(\theta,\mathfrak{z},\psi_1) - h(\theta,w,\psi_2)\| \le \bar{\mathcal{L}}_h\|\mathfrak{z} - w\| + \tilde{\mathcal{L}}_h\|\psi_1 - \psi_2\|.$$

$(H5)$ For a constant ${l_f}^* > 0$ and bounded sets $D_1, D_2 \subseteq \mathcal{X}$ then

$$\rho\,(h(\theta,D_1,D_2)) \le {l_f}^*\theta^{1-\vartheta}[\rho(D_1) + \rho(D_2)],$$

a.e. $\theta \in \mathcal{T}$.

$(H6)$ $\mathfrak{q}$ satisfies Lipschitz continuous, i.e., for some constants $m_g > 0$, $\mathcal{L}_g \in (0, 1)$ that fulfills

$$\|\mathfrak{q}(\theta,\mathfrak{z}(\theta),y(\theta))\| \le m_g(\theta)\mathcal{L}_g(\theta^{1-\vartheta}(\|\mathfrak{z}(\theta)\| + \|y(\theta)\|)) \quad \text{and}$$
$$\|\mathfrak{q}(\theta,\mathfrak{z}_1(\theta),y_1(\theta)) - \mathfrak{q}(\theta,\mathfrak{z}_2(\theta),y_2(\theta))\| \le \mathcal{L}_g(\|\mathfrak{z}_1 - \mathfrak{z}_2\| + \|y_1 - y_2\|), \quad \forall\ \theta \in \mathcal{T}.$$

$(H7)$ For a constant ${l_p}^* > 0$ and bounded sets $D_1, D_2 \subseteq \mathcal{X}$,

$$\rho\,(\mathfrak{q}(\theta,D_1,D_2)) \le {l_p}^*\theta^{1-\vartheta}(\rho(D_1) + \rho(D_2)),$$

almost everywhere $\theta \in \mathcal{T}$.

(*H*8) We define $W : L^2(\mathcal{T}, U) \to \mathcal{X}$ is by:

$$Wu = \lim_{\lambda \to +\infty} \int_0^a P_\alpha(a-s)\mathcal{B}_\lambda Bu(s)ds,$$

and the inverse operator W^{-1} assumes values in $L^2(\mathcal{T}, U)/\ker W$ also for $\mathcal{M}_b, \mathcal{M}_w \geq 0$, provided $\|B\| \leq \mathcal{M}_b$, $\|W^{-1}\| \leq \mathcal{M}_w$.

(*H*9) For some $l_u{}^* > 0$, such that for any bounded $z \in \mathcal{Y}$, $\xi(u(z,\mu)) \leq l_u{}^* \mathfrak{q}^{1-\mathfrak{c}} \nu(z,\mu)\xi(z(\mu))$, almost everywhere $\mu \in \mathcal{T}$ with $\sup_{\mathfrak{q} \in \mathcal{T}} \int_0^\theta \nu(\theta,\mu)ds = \nu^* < \infty$.

The control term can be defined as,

$$u(\theta, \mathfrak{z}) = W^{-1}\Big[\mathfrak{z}_a - S_{\alpha,\beta}(a)[\mathfrak{z}_0 - \mathfrak{q}(0, \mathfrak{z}(0), 0)] - \mathfrak{q}\Big(a, \mathfrak{z}(a), \int_0^a e(\theta, t, \mathfrak{z}(t))dt\Big)$$
$$- \lim_{\lambda \to \infty} \int_0^a P_\alpha(a-t)\mathcal{B}_\lambda\Big[A\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big) + h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big)\Big]dt\Big](\theta)$$

with the norm

$$\begin{aligned}
&\|u(\theta, \mathfrak{z})\| \\
&\leq \mathscr{M}_w\Bigg[\|\mathfrak{z}_a\| - \frac{\mathscr{M} a^{\vartheta-1}}{\Gamma(\vartheta)}\|[\mathfrak{z}_0 - \mathfrak{q}(0, \mathfrak{z}(0), 0)]\| - \Big\|\mathfrak{q}\Big(a, \mathfrak{z}(a), \int_0^a e(\theta, t, \mathfrak{z}(t))dt\Big)\Big\| \\
&\quad - \frac{\mathscr{M} a^{\alpha-1}}{\Gamma(\alpha)}\Big\|\lim_{\lambda \to \infty} \int_0^a \mathcal{B}_\lambda\Big[A\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big) + h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big)\Big]dt\Big\|\Bigg] \\
&\leq \mathscr{M}_w\Bigg[\|\mathfrak{z}_a\| - \frac{\mathscr{M} a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathscr{M}} - m_g(s)L_p(a^{1-\vartheta}(\|\mathfrak{z}(s)\| + a\|\mathfrak{z}(s)\|)) - \frac{\mathscr{M} a^{\alpha-1}\mathscr{M}_0}{\Gamma(\alpha)} \\
&\quad (\times) \int_0^a [\|A\| m_g(s)L_p(a^{1-\vartheta}(\|\mathfrak{z}(s)\| + a\|\mathfrak{z}(s)\|)) + m_1(s)L_h(a^{1-\vartheta}(\|\mathfrak{z}(s)\| + a\|\mathfrak{z}(s)\|))]ds\Bigg] \\
&\leq \mathscr{M}_w\Big[\|\mathfrak{z}_a\| - \frac{\mathscr{M} a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathscr{M}} - m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) \\
&\quad - \frac{\mathscr{M} a^{\alpha}\mathscr{M}_0}{\Gamma(\alpha)}[\|A\| m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) + (1+a)m_1^*\mathscr{L}_h^*]\Big] \\
&\leq \mathscr{M}_w C_b^*,
\end{aligned}$$

where $C_b^* = \|\mathfrak{z}_a\| - \frac{\mathscr{M} a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathscr{M}} - m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) - \frac{\mathscr{M} a^{\alpha}\mathscr{M}_0}{\Gamma(\alpha)}[\|A\| m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) + (1+a)m_1^*\mathscr{L}_h^*]$ and $\hat{\mathscr{M}} = \|\mathfrak{z}_0 - \mathfrak{q}(0, \mathfrak{z}(0), 0)\|$. Construct a space with the uniform convergence topology as $\mathcal{S} = \{\mathfrak{z} : \mathfrak{z} \in C[\mathcal{T}, \mathcal{X}]\}$.

Theorem 3.1. *If (H1)–(H6) hold, then (1.1) and (1.2) has a unique solution such that*

$$\frac{\mathscr{M} a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathscr{M}} + m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) + \frac{\mathscr{M} a^{\alpha}\mathscr{M}_0}{\Gamma(\alpha)}$$
$$\times [\|A\| m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) + \mathscr{M}_b\mathscr{M}_w C_b^* + (1+a)m_1^*\mathscr{L}_h^*] < \zeta^*, \tag{3.1}$$

and

$$(1+a)\mathscr{L}_g + \frac{\mathscr{M} a^{\alpha} \mathscr{M}_0}{\Gamma(\alpha)}\Big[(1+a)\|A\|\mathscr{L}_g + \mathscr{M}_b \mathscr{M}_w (1+a)$$
$$\times \Big[\mathscr{L}_g + \|A\|\mathscr{L}_g + (\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\Big] + (1+a)(\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\Big] < 1. \tag{3.2}$$

Proof. Consider $\mathcal{B}_\iota(0,\mathcal{S}) = \{\mathfrak{z} \in \mathcal{X}, \|\mathfrak{z}\| \le \zeta\}$. Hence $\mathcal{B}_\iota(0,\mathcal{S}) \subset C[\mathcal{T},\mathcal{X}]$ is convex, bounded, and closed. Define $\Gamma_\eta : \mathcal{B}_\iota(0,\mathcal{S}) \to \mathcal{B}_\iota(0,\mathcal{S})$ for $\eta > 0$, as

$$\Gamma_\eta(\mathfrak{z}(\theta)) = S_{\alpha,\beta}(\theta)[\mathfrak{z}_0 - \mathfrak{q}(0,\mathfrak{z}(0),0)] + \mathfrak{q}\bigg(\theta,\mathfrak{z}(\theta),\int_0^\theta e(\theta,t,\mathfrak{z}(t))dt\bigg) + \lim_{\lambda\to\infty}\int_0^\theta P_\alpha(\theta-t)$$
$$\times\mathcal{B}_\lambda\bigg[A\mathfrak{q}\bigg(t,\mathfrak{z}(t),\int_0^t e(t,s,\mathfrak{z}(s))ds\bigg) + Bu(t) + h\bigg(t,\mathfrak{z}(t),\int_0^t f(t,s,\mathfrak{z}(s))ds\bigg)\bigg]dt.$$

Step 1: For $\mathfrak{z} \in \mathcal{B}_\iota(0,\mathcal{S})$,

$$\begin{aligned}
&\|\Gamma_\eta(\mathfrak{z}(\theta))\| \\
&\le \bigg\|S_{\alpha,\beta}(\theta)[\mathfrak{z}_0 - \mathfrak{q}(0,\mathfrak{z}(0),0)] + \mathfrak{q}\Big(\theta,\mathfrak{z}(\theta),\int_0^\theta e(\theta,t,\mathfrak{z}(t))dt\Big) + \lim_{\lambda\to\infty}\int_0^\theta P_\alpha(\theta-t) \\
&\times \mathcal{B}_\lambda\Big[A\mathfrak{q}\Big(t,\mathfrak{z}(t),\int_0^t e(t,s,\mathfrak{z}(s))ds\Big) + Bu(t) + h\Big(t,\mathfrak{z}(t),\int_0^t f(t,s,\mathfrak{z}(s))ds\Big)\Big]dt\bigg\| \\
&\le \frac{\mathscr{M} a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathscr{M}} + m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) + \frac{\mathscr{M} a^{\alpha}\mathscr{M}_0}{\Gamma(\alpha)} \\
&\times [\|A\|m_g(s)L_p(a^{1-\vartheta}(1+a)\|\mathfrak{z}(s)\|) + \mathscr{M}_b\mathscr{M}_w C_b^* + (1+a)m_1^*\mathscr{L}_h^*] \\
&\le \zeta^*.
\end{aligned}$$

$\therefore$ Γ_η maps $\mathcal{B}_\iota(0,\mathcal{S})$ into $\mathcal{B}_\iota(0,\mathcal{S})$.
Step 2: For any $w, \mathfrak{z} \in \mathcal{B}_\iota(0,\mathcal{S})$,

$$\begin{aligned}
&\|\Gamma_\eta(\mathfrak{z}(\theta)) - \Gamma_\eta(w(\theta))\| \\
&\le (1+a)\mathscr{L}_g\|\mathfrak{z} - w\| + \frac{\mathscr{M} a^{\alpha}\mathscr{M}_0}{\Gamma(\alpha)}\bigg[(1+a)\|A\|\mathscr{L}_g + \mathscr{M}_b\mathscr{M}_w(1+a) \\
&\times \Big[\mathscr{L}_g + \|A\|\mathscr{L}_g + (\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\Big] + (1+a)(\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\bigg]\|\mathfrak{z} - w\| \\
&\le (1+a)\mathscr{L}_g + \frac{\mathscr{M} a^{\alpha}\mathscr{M}_0}{\Gamma(\alpha)}\Big[(1+a)\|A\|\mathscr{L}_g + \mathscr{M}_b\mathscr{M}_w(1+a) \\
&\times \Big[\mathscr{L}_g + \|A\|\mathscr{L}_g + (\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\Big] + (1+a)(\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\Big]\|\mathfrak{z} - w\| \\
&\le \mu^*\|\mathfrak{z} - w\|,
\end{aligned}$$

and $\mu^* = (1+a)\mathscr{L}_g + \frac{\mathscr{M} a^{\alpha}\mathscr{M}_0}{\Gamma(\alpha)}\Big[(1+a)\|A\|\mathscr{L}_g + \mathscr{M}_b\mathscr{M}_w(1+a)\Big[\mathscr{L}_g + \|A\|\mathscr{L}_g + (\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\Big] + (1+a)(\bar{\mathscr{L}}_h + \tilde{\mathscr{L}}_h)\Big]$.
Hence Γ_η is contraction. Using Banach contraction principle, Γ_η has a unique solution on $C[\mathcal{T},\mathcal{X}]$. □

Lemma 3.2. *Let (H1)–(H9) hold, then* $\Gamma_\eta : \mathfrak{z} \in \mathcal{B}_\iota(0,\mathcal{S})$ *is equicontinuous.*

Proof. It is evident that $S_{\alpha,\beta}(\theta)$ is strongly continuous on $\mathcal{T}$ by Lemma 2.7.
For $\mathfrak{z} \in \mathcal{B}_\iota(0, \mathcal{S}), \theta_1, \theta_2 \in \mathcal{T}$ and $\epsilon > 0$ such that $0 \le \epsilon < \theta_1 < \theta_2 \le a$ and there exists a $\delta > 0$ provided if $0 < |\theta_2 - \theta_1| < \delta$, then

$$
\begin{aligned}
&\|(\Gamma_\eta \mathfrak{z})(\theta_2) - (\Gamma_\eta \mathfrak{z})(\theta_1)\| \\
&\le \Big\| \Big[\mathfrak{q}\Big(\theta_2, \mathfrak{z}(\theta_2), \int_0^{\theta_2} e(\theta_2, t, \mathfrak{z}(t))dt\Big) - \mathfrak{q}\Big(\theta_1, \mathfrak{z}(\theta_1), \int_0^{\theta_1} e(\theta_1, t, \mathfrak{z}(t))dt\Big)\Big] \\
&+ \lim_{\lambda\to\infty} \theta_2^{\vartheta-1} \int_0^{\theta_2} (\theta_2 - t)^{\alpha-1} T_\alpha(\theta_2 - t) B_\lambda A \Big[\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big)\Big]dt \\
&- \theta_1^{\vartheta-1} \int_0^{\theta_1} (\theta_1 - t)^{\alpha-1} T_\alpha(\theta_1 - t) B_\lambda A \Big[\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big)\Big]dt \Big\| \\
&+ \Big\| \lim_{\lambda\to\infty} \theta_2^{\vartheta-1} \int_0^{\theta_2} (\theta_2 - t)^{\alpha-1} T_\alpha(\theta_2 - t) B_\lambda \Big[h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big)\Big]dt \\
&- \theta_1^{\vartheta-1} \int_0^{\theta_1} (\theta_1 - t)^{\alpha-1} T_\alpha(\theta_1 - t) B_\lambda \Big[h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big)\Big]dt \Big\| \\
&+ \Big\| \lim_{\lambda\to\infty} \theta_2^{\vartheta-1} \int_0^{\theta_2} (\theta_2 - t)^{\alpha-1} T_\alpha(\theta_2 - t) B_\lambda [Bu(t)]dt \\
&- \theta_1^{\vartheta-1} \int_0^{\theta_1} (\theta_1 - t)^{\alpha-1} T_\alpha(\theta_1 - t) B_\lambda [Bu(t)]dt \Big\| \\
&\le (1 + a)\mathscr{L}_g \|\theta_2 - \theta_1\| + \Big\| \lim_{\lambda\to\infty} \theta_2^{\vartheta-1} \int_{\theta_1}^{\theta_2} (\theta_2 - s)^{\alpha-1} T_\alpha(\theta_2 - s) \\
&\times B_\lambda \Big[A\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big) + h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big) + Bu(t)\Big]ds \Big\| \\
&+ \Big\| \lim_{\lambda\to\infty} \int_0^{\theta_1} \Big[\theta_2^{\vartheta-1}(\theta_2 - t)^{\alpha-1} - \theta_1^{\vartheta-1}(\theta_1 - t)^{\alpha-1}\Big] T_\alpha(\theta_2 - t) \\
&\times B_\lambda \Big[A\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big) + h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big) + Bu(t)\Big]dt \Big\| \\
&+ \Big\| \lim_{\lambda\to\infty} \theta_1^{\vartheta-1} \int_0^{\theta_1-\epsilon} (\theta_1 - s)^{\alpha-1} \Big[T_\alpha(\theta_2 - t) - T_\alpha(\theta_1 - t)\Big] \\
&\times B_\lambda \Big[A\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big) + h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big) + Bu(t)\Big]dt \Big\| \\
&+ \Big\| \lim_{\lambda\to\infty} \theta_1^{\vartheta-1} \int_{\theta_1-\epsilon}^{\theta_1} (\theta_1 - t)^{\alpha-1} \Big[T_\alpha(\theta_2 - t) - T_\alpha(\theta_1 - t)\Big] \\
&\times B_\lambda \Big[A\mathfrak{q}\Big(t, \mathfrak{z}(t), \int_0^t e(t, s, \mathfrak{z}(s))ds\Big) + h\Big(t, \mathfrak{z}(t), \int_0^t f(t, s, \mathfrak{z}(s))ds\Big) + Bu(t)\Big]ds \Big\|.
\end{aligned}
$$

By Lebesgue theorem, we conclude for ϵ sufficiently small, $\|(\Gamma_\eta \mathfrak{z})(\theta_2) - (\Gamma_\eta \mathfrak{z})(\theta_1)\| \to 0$ *as* $\theta_2 \to \theta_1$. Hence Γ_η is equicontinuous. □

Lemma 3.3. *Let (H1)–(H9) hold, then* $\Gamma_\eta : y \in \mathcal{B}_\iota(0, \mathcal{S})$ *is continuous as well as*

$$
k_b^* \Big[\Big[\frac{\mathscr{M} a^\alpha \mathscr{M}_0 (1 + a)}{\Gamma(\alpha)} [\|A\|\mathscr{L}_g + m_1^* \mathscr{L}_h^*] + (1 + a)\mathscr{L}_g \Big] \Big[1 + \frac{\mathscr{M} a^\alpha \mathscr{M}_0 \mathscr{M}_b \mathscr{M}_w}{\Gamma(\alpha)} \Big] \Big] < 1. \tag{3.3}
$$

Proof. **Step 1:** $\Gamma_\eta(\mathcal{B}_\iota(0,\mathcal{S}))\subset\mathcal{B}_\iota(0,\mathcal{S})$.
Assume contradictory, for each $\iota>0$, $\mathfrak{z}^\iota\in\mathcal{B}_\iota(0,\mathcal{S})$, $\theta^\iota\in\mathcal{T}$ yields $\iota<\|(\Gamma_\eta\mathfrak{z}^\iota)(\theta^\iota)\|_C$.
Consider $0<t<\theta^\iota$ such that $\lim_{\iota\to\infty}\frac{\iota^*}{\iota}=k_b^*$, $\|\mathfrak{z}^\iota\|_\mathcal{X}\le\iota^*$, we get

$$\begin{aligned}\iota&<\|(\Gamma_\eta\mathfrak{z}^\iota)(\theta^\iota)\|_C\\&<\frac{\mathscr{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathscr{M}}+(1+a)\mathscr{L}_g\iota^*+\frac{(1+a)\|A\|\mathscr{M}a^\alpha\mathscr{M}_0\mathscr{L}_g\iota^*}{\Gamma(\alpha)}+\frac{(1+a)\mathscr{M}a^\alpha\mathscr{M}_0\iota^*m_1^*\mathscr{L}_h^*}{\Gamma(\alpha)}\\&+\frac{\mathscr{M}a^\alpha\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}\Big[\|\mathfrak{z}_a\|+\frac{\mathscr{M}a^{\vartheta-1}}{\Gamma(\vartheta)}\hat{\mathscr{M}}+(1+a)\mathscr{L}_g\iota^*\\&+\frac{\mathscr{M}a^\alpha\mathscr{M}_0\iota^*}{\Gamma(\alpha)}[(1+a)\|A\|\mathscr{L}_g+(1+a)m_1^*\mathscr{L}_h^*]\end{aligned}$$

Divide by ι and take $\iota\to\infty$,

$$\begin{aligned}1&<k_b^*\Big[\frac{\mathscr{M}a^\alpha\mathscr{M}_0(1+a)}{\Gamma(\alpha)}[\|A\|\mathscr{L}_g+m_1^*\mathscr{L}_h^*]+(1+a)\mathscr{L}_g+\frac{\mathscr{M}a^\alpha\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}\\&\times(1+a)\mathscr{L}_g+\frac{\mathscr{M}a^\alpha\mathscr{M}_0(1+a)}{\Gamma(\alpha)}[\|A\|\mathscr{L}_g+m_1^*\mathscr{L}_h^*]\Big]\\&<k_b^*\Big[\frac{\mathscr{M}a^\alpha\mathscr{M}_0(1+a)}{\Gamma(\alpha)}[\|A\|\mathscr{L}_g+m_1^*\mathscr{L}_h^*]\Big[1+\frac{\mathscr{M}a^\alpha\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}\Big]\\&+(1+a)\mathscr{L}_g\Big[1+\frac{\mathscr{M}a^\alpha\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}\Big]\Big]\\&<k_b^*\Big[\Big[\frac{\mathscr{M}a^\alpha\mathscr{M}_0(1+a)}{\Gamma(\alpha)}[\|A|\mathscr{L}_g+m_1^*\mathscr{L}_h^*]+(1+a)\mathscr{L}_g\Big]\Big[1+\frac{\mathscr{M}a^\alpha\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}\Big]\Big],\end{aligned}$$

leads the contradiction to (3.3). Then for $\iota>0$, $\Gamma_\eta(\mathcal{B}_\iota(0,\mathcal{S}))\subset\mathcal{B}_\iota(0,\mathcal{S})$.
Step 2: Consider $\mathfrak{z}_k,\mathfrak{z}$ in $\mathcal{B}_\iota(0,\mathcal{S})$, for all $\theta\in\mathcal{T}, k=1,2,...$ such that $\lim_{k\to\infty}\|\mathfrak{z}_k-\mathfrak{z}\|\to0$ and $\lim_{k\to\infty}\mathfrak{z}_k(\theta)\to\mathfrak{z}(\theta)$. Therefore

$$\lim_{k\to\infty}\Big\|h\Big(\theta,\mathfrak{z}_k(\theta),\int_0^t f(\theta,s,\mathfrak{z}_k(s))ds\Big)-h\Big(\theta,\mathfrak{z}(\theta),\int_0^t f(\theta,s,\mathfrak{z}(s))ds\Big)\Big\|\to0.$$

By $(H1)$,

$$\begin{aligned}&(\theta-s)^{\alpha-1}\Big\|h\Big(\theta,\mathfrak{z}_k(\theta),\int_0^t f(\theta,s,\mathfrak{z}_k(s))ds\Big)-h\Big(\theta,\mathfrak{z}(\theta),\int_0^t f(\theta,s,\mathfrak{z}(s))ds\Big)\Big\|\\&\le(\theta-s)^{\alpha-1}m_1(s)(1+a)\mathscr{L}_h\Big[\|\mathfrak{z}_k-\mathfrak{z}\|\Big]\ \text{ a.e }\ s\in(0,\theta).\end{aligned}$$

For $\theta\in[0,a]$, $s\in(0,\theta)$ $(\theta-s)^{\alpha-1}m_1(s)(1+a)\mathscr{L}_h\Big[\|\mathfrak{z}_k-\mathfrak{z}\|\Big]$ is integrable. Also

$$\int_0^\theta(\theta-s)^{\alpha-1}\Big\|h\Big(t,\mathfrak{z}_k(t),\int_0^t f(t,s,\mathfrak{z}_k(s))ds\Big)-h\Big(t,\mathfrak{z}(t),\int_0^t f(t,s,\mathfrak{z}(s))ds\Big)\Big\|dt\to0\ \text{ as }\ k\to\infty.\quad(3.4)$$

Hence

$$
\begin{aligned}
&\|(\Gamma_\eta \mathfrak{z}_k)(\theta) - (\Gamma_\eta \mathfrak{z})(\theta)\| \\
&\le a^{\vartheta-1}\Big\| \lim_{\lambda\to\infty}\int_0^\theta (\theta-t)^{\alpha-1} B_\lambda T_\alpha(\theta-t) \\
&\times \Big[A\Big[\mathfrak{q}\Big(t,\mathfrak{z}_k(t),\int_0^t e(t,s,\mathfrak{z}_k(s))ds\Big) - \mathfrak{q}\Big(t,\mathfrak{z}(t),\int_0^t e(t,s,\mathfrak{z}(s))ds\Big)\Big] \\
&+ \Big[h\Big(t,\mathfrak{z}_k(t),\int_0^t f(t,s,\mathfrak{z}_k(s))ds\Big) - h\Big(t,\mathfrak{z}(t),\int_0^t f(t,s,\mathfrak{z}(s))ds\Big)\Big] + B[u_{\mathfrak{z}_k} - u_{\mathfrak{z}}]\Big]dt\Big\| \\
&\le \frac{\mathscr{M} a^{\vartheta-1}\mathscr{M}_0}{\Gamma(\alpha)}\Big\|\int_0^\theta (\theta-t)^{\alpha-1}\Big[A\Big[\mathfrak{q}\Big(t,\mathfrak{z}_k(t),\int_0^t e(t,s,\mathfrak{z}_k(s))ds\Big) - \mathfrak{q}\Big(t,\mathfrak{z}_k(t),\int_0^t f(t,s,\mathfrak{z}(s))ds\Big)\Big] \\
&+ \Big[h\Big(t,\mathfrak{z}_k(t),\int_0^t f(t,s,\mathfrak{z}_k(s))ds\Big) - h\Big(t,\mathfrak{z}(t),\int_0^t f(t,s,\mathfrak{z}(s))ds\Big)\Big] + B[u_{\mathfrak{z}_k} - u_{\mathfrak{z}}]\Big]ds\Big\|. \qquad (3.5)
\end{aligned}
$$

By (3.4) and (3.5)

$$\|(\Gamma_\eta \mathfrak{z}_k)(\theta) - (\Gamma_\eta \mathfrak{z})(\theta)\| \to 0 \text{ as } n \to \infty.$$

Therefore Γ_η is continuous on $\mathcal{B}_\iota(0,\mathcal{S})$.

□

Theorem 3.4. *Let (H1)–(H9) hold, then (1.1) and (1.2) is controllable on $\mathcal{T}$ such that*

$$
\begin{aligned}
&{l_p}^*\theta^{1-\vartheta}(1+a)\rho(\mathbf{S}) + \frac{2\mathscr{M}\mathscr{M}_0}{\Gamma(\alpha)}(1+a)({l_p}^* + {l_f}^*)\theta^{1-\vartheta} \\
&\times\Big[1 + \frac{2\mathscr{M}\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}{l_u}^*\nu^*\Big]\int_0^a (a-t)^{\alpha-1}\rho(\mathbf{S})ds < r. \qquad (3.6)
\end{aligned}
$$

Proof. Construct $\mathbf{S}$ of $\mathcal{B}_\iota(0,\mathcal{S})$ be a countable subset and $\mathbf{S} \subset \overline{\mathrm{co}}(\{0\}\cup\Gamma_\eta(\mathbf{S}))$, and we show that $\rho(\mathbf{S}) = 0$. Consider $\mathbf{S} = \{\mathfrak{z}_n\}_{n=1}^\infty$. $\Gamma_\eta\{\mathfrak{z}_n\}_{n=1}^\infty$ is equicontinuous on $\mathcal{T}$ by Lemma 3.2. Also $\mathbf{S} \subset \overline{\mathrm{co}}(\{0\}\cup\Gamma_\eta(\mathbf{S}))$ is also equicontinuous on $\mathcal{T}$.

$$
\begin{aligned}
&\rho\Big(u(\theta,\{\mathfrak{z}_n\}_{n=1}^\infty)\Big) \\
&\le \rho\Big\{W^{-1}\Big[\mathfrak{z}_a - S_{\alpha,\beta}(a)[\mathfrak{z}_0 - \mathfrak{q}(0,\mathfrak{z}(0),0)] - \mathfrak{q}\Big(a,\mathfrak{z}(a),\int_0^a e(\theta,t,\{\mathfrak{z}_n(s)\}_{n=1}^\infty)dt\Big) \\
&- \lim_{\lambda\to\infty}\int_0^a P_\alpha(a-t)\mathcal{B}_\lambda\Big[A\mathfrak{q}\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^\infty,\int_0^t e(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty)ds\Big) \\
&+ h\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^\infty,\int_0^t f(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty)ds\Big)\Big]dt\Big]\Big\} \\
&\le {l_u}^*\nu(\theta)\frac{2\mathscr{M}\mathscr{M}_0\mathscr{M}_w}{\Gamma(\alpha)}\int_0^a (a-t)^{\alpha-1}\rho\Big(A\mathfrak{q}\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^\infty,\int_0^t e(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty)ds\Big) \\
&+ h\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^\infty,\int_0^t f(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^\infty)ds\Big)\Big)dt \\
&\le {l_u}^*\nu(\theta)\frac{2\mathscr{M}\mathscr{M}_0\mathscr{M}_w}{\Gamma(\alpha)}\int_0^a (a-t)^{\alpha-1}(1+a)({l_p}^* + {l_f}^*)s^{1-\vartheta}\rho\Big(\{\mathfrak{z}_n(s)\}_{n=1}^\infty\Big)ds.
\end{aligned}
$$

Moreover, by Lemma 2.6,

$$
\begin{aligned}
&\rho\Big(\Gamma_\eta(\{\mathfrak{z}_n(\theta)\}_{n=1}^{\infty})\Big)\\
&=\rho\bigg\{S_{\alpha,\beta}(\theta)[\mathfrak{z}_0-\mathfrak{q}(0,\mathfrak{z}(0),0)]+\mathfrak{q}\Big(\theta,\{\mathfrak{z}_n(\theta)\}_{n=1}^{\infty},\int_0^t e(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^{\infty})ds\Big)\\
&+\lim_{\lambda\to\infty}\int_0^\theta(\theta-t)^{\alpha-1}T_\alpha(\theta-t)\mathcal{B}_\lambda\Big[A\mathfrak{q}\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^{\infty},\int_0^t e(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^{\infty})ds\Big)\\
&+Bu(t)+h\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^{\infty},\int_0^t f(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^{\infty})ds\Big)\Big]dt\bigg\}\\
&\le\rho\bigg(\mathfrak{q}\Big(\theta,\{\mathfrak{z}_n(\theta)\}_{n=1}^{\infty},\int_0^t e(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^{\infty})ds\Big)+\lim_{\lambda\to\infty}\int_0^\theta(\theta-t)^{\alpha-1}T_\alpha(\theta-t)\mathcal{B}_\lambda\\
&\times\Big[A\mathfrak{q}\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^{\infty},\int_0^t e(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^{\infty})ds\Big)+h\Big(t,\{\mathfrak{z}_n(t)\}_{n=1}^{\infty},\int_0^t f(t,s,\{\mathfrak{z}_n(s)\}_{n=1}^{\infty})ds\Big)\Big]dt\\
&+\lim_{\lambda\to\infty}\int_0^\theta(\theta-t)^{\alpha-1}T_\alpha(\theta-t)\mathcal{B}_\lambda\Big[Bu(t,\{\mathfrak{z}_n(t)\}_{n=1}^{\infty})\Big]dt\bigg)\\
&\le {l_p}^*\theta^{1-\vartheta}(1+a)\rho(\{\mathfrak{z}_n(\theta)\}_{n=1}^{\infty})+\frac{2\mathscr{M}\mathscr{M}_0}{\Gamma(\alpha)}\int_0^a(a-t)^{\alpha-1}(1+a)({l_p}^*+{l_f}^*)t^{1-\vartheta}\rho\Big(\{\mathfrak{z}_n(t)\}_{n=1}^{\infty}\Big)dt\\
&+\frac{2\mathscr{M}\mathscr{M}_0\mathscr{M}_b}{\Gamma(\alpha)}\int_0^a(a-t)^{\alpha-1}\rho\Big(u(t,\{\mathfrak{z}_n(t)\}_{n=1}^{\infty})\Big)dt\\
&\le {l_p}^*\theta^{1-\vartheta}(1+a)\rho(\{\mathfrak{z}_n(\theta)\}_{n=1}^{\infty})+\frac{2\mathscr{M}\mathscr{M}_0}{\Gamma(\alpha)}\int_0^a(a-t)^{\alpha-1}(1+a)({l_p}^*+{l_f}^*)t^{1-\vartheta}\rho\Big(\{\mathfrak{z}_n(t)\}_{n=1}^{\infty}\Big)dt\\
&+\frac{2\mathscr{M}\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}\int_0^a(a-t)^{\alpha-1}\Big[{l_u}^*\nu(t)\frac{2\mathscr{M}\mathscr{M}_0}{\Gamma(\alpha)}\\
&\times\int_0^a(a-s)^{\alpha-1}(1+a)({l_p}^*+{l_f}^*)s^{1-\vartheta}\rho\Big(\{\mathfrak{z}_n(s)\}_{n=1}^{\infty}\Big)ds\Big]dt\\
&\le {l_p}^*\theta^{1-\vartheta}(1+a)\rho(\mathbf{S})+\frac{2\mathscr{M}\mathscr{M}_0}{\Gamma(\alpha)}(1+a)({l_p}^*+{l_f}^*)\theta^{1-\vartheta}\\
&\times\left[1+\frac{2\mathscr{M}\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}{l_u}^*\nu^*\right]\int_0^a(a-t)^{\alpha-1}\rho(\mathbf{S})ds.
\end{aligned}
$$

By Mönch theorem,

$$
\begin{aligned}
&\rho(\mathbf{S})\le\overline{\mathrm{co}}(\{0\}\cup\Gamma_\eta(\mathbf{S}))=\rho(\Gamma_\eta(\mathbf{S}))\\
&\le {l_p}^*\theta^{1-\vartheta}(1+a)\rho(\mathbf{S})+\frac{2\mathscr{M}\mathscr{M}_0}{\Gamma(\alpha)}(1+a)({l_p}^*+{l_f}^*)\theta^{1-\vartheta}\\
&\times\left[1+\frac{2\mathscr{M}\mathscr{M}_0\mathscr{M}_b\mathscr{M}_w}{\Gamma(\alpha)}{l_u}^*\nu^*\right]\int_0^a(a-t)^{\alpha-1}\rho(\mathbf{S})ds.
\end{aligned}
$$

Grownwall's inequality leads to the conclusion $\rho(\mathbf{S})=0$. Evidently (1.1) and (1.2) has a fixed point satisfies $\mathfrak{z}(a)=\mathfrak{z}_a$. □

4. Numerical analysis

In this section, we have used Python3 language for numerical estimation and Mathematica-ver12.2 for graphical representation.

Consider the problem

$$\mathcal{D}_{0+}^{\alpha,\beta}\Big[\mathfrak{z}(t) - \frac{1}{8}\int_0^t e^{2s-t}\mathfrak{z}(t)dt\Big] = A\mathfrak{z}(t) + \frac{t^2}{12}\int_0^t s^2\cos\Big(\frac{\mathfrak{z}(s)}{s}\Big)ds + e^{-t}\sin t, \tag{4.1}$$

$$I_{0+}^{(1-\alpha)(1-\beta)}\mathfrak{z}(0) = 1, \qquad t \in \mathcal{T} = [0,1]. \tag{4.2}$$

Let $D = \{\mathfrak{z} \in C^2([0,1],R) : \mathfrak{z}(0) = \mathfrak{z}(1) = 0\}$, $A\mathfrak{z} = \mathfrak{z}''$. Here $\mathcal{H} : C([0,1],R)$ equipped with the uniform topology and $A : D \subset \mathcal{H} \to \mathcal{H}$.

A successive approximation of (4.1) and (4.2) is

$$\mathfrak{z}_n(t) = \frac{t^{\theta-1}}{\Gamma(\theta)} + \frac{1}{8}\int_0^t e^{2s-t}\mathfrak{z}_{n-1}(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\frac{t^2}{12}\int_0^s \tau^2\cos\Big(\frac{\mathfrak{z}_{n-1}(\tau)}{\tau}\Big)d\tau + e^{-t}\sin t\Big]ds,$$

and n varies from 1 to 6 with initial approximation $\mathfrak{z}_0 = 1$. Also by the boundedness of h and q,

$$\begin{aligned}
|\mathfrak{z}_n(t)| \le & \Big|\frac{t^{\theta-1}}{\Gamma(\theta)} + \frac{1}{8}\int_0^t e^{2s-t}\mathfrak{z}_{n-1}(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
& (\times)\Big[\frac{t^2}{12}\int_0^s \tau^2\cos\Big(\frac{\mathfrak{z}_{n-1}(\tau)}{\tau}\Big)d\tau + |e^{-t}\sin t\Big]ds\Big|, \\
& = \frac{t^{\theta-1}}{\Gamma(\theta)} + \frac{1}{8}\int_0^t \Big|e^{2s-t}\mathfrak{z}_{n-1}(t)\Big|dt + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
& (\times)\Big|\Big[\frac{t^2}{12}\int_0^s \tau^2\cos\Big(\frac{\mathfrak{z}_{n-1}(\tau)}{\tau}\Big)\Big|d\tau + \Big|e^{-t}\sin t\Big]\Big|ds \\
& = k_i,\ \ k > 0,
\end{aligned}$$

implies the sequence $\mathfrak{z}_{n-1}(t)$ are well defined and uniformly bounded on [0, 1]. Now, $\limsup_{n\to\infty}\{\mathfrak{z}_n(t) - \mathfrak{z}_{n-1}(t)\} = 0$. This problems is approximated as (1.1) and (1.2) which satisfies (H1)–(H9). Then by Arzela- Ascoli's theorem and the closeness of the derivative operator, we conclude the sequence $\{\mathfrak{z}_n(t)\}$ converges uniformly to the continuous functions $\mathfrak{z}$ on [0, 1] when $n \to \infty$.

We have analyzed the controllability of the system (4.1) and (4.2) via numerical approximation.

Figures 1–6 represents the solutions in Hilfer, R-L and Caputo's derivatives for the different values of $\alpha = 0.1, 0.6$ and $\beta = 0, 0.3, 1$. This figures gives the comparison among the differential operators. Figures 1 and 2 shows the numerical approximation for $x(t)$ for the values $\alpha = 0.1, 0.6$; $\beta = 0, 0.3, 1$. Figures 3–6 shows the numerical approximation for $u(t)$ for the values $\alpha = 0.1, 0.6$; $\beta = 0, 0.3, 1$.

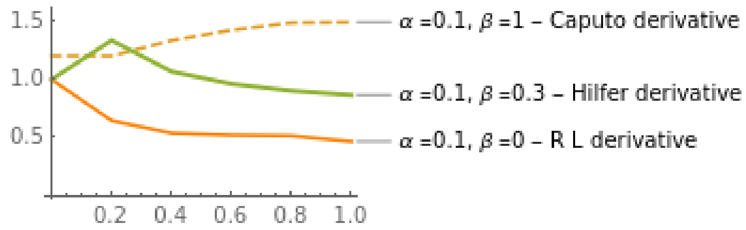

Figure 1. Numerical approximation for $x(t)$ $\alpha = 0.1, \beta = 0, 0.3, 1$.

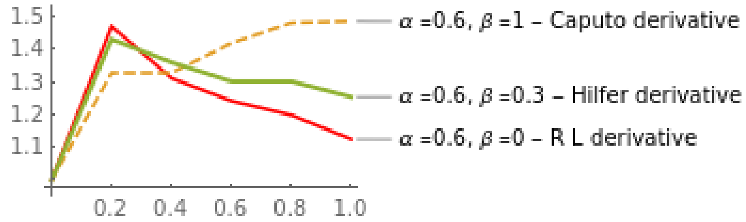

Figure 2. Numerical approximation for $x(t)$ $\alpha = 0.6, \beta = 0, 0.3, 1$.

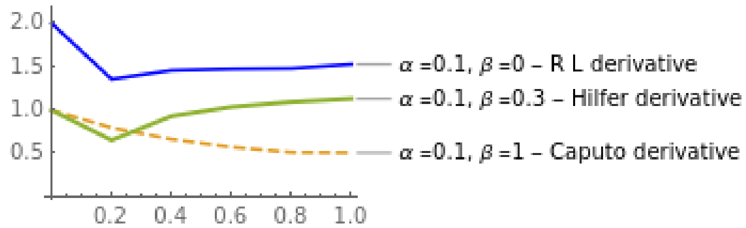

Figure 3. Numerical approximation for $u(t)$ $\alpha = 0.1, \beta = 0, 0.3, 1$.

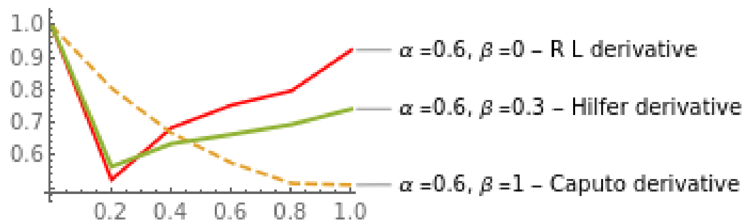

Figure 4. Numerical approximation for $u(t)$ $\alpha = 0.6, \beta = 0, 0.3, 1$.

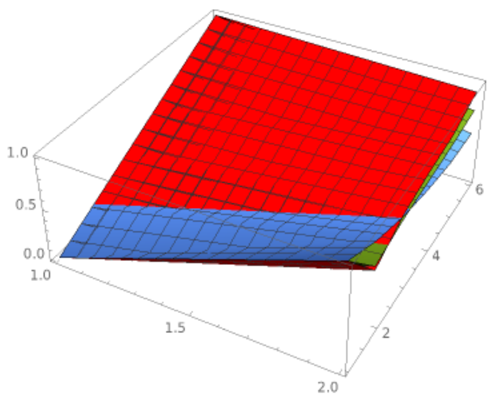

Figure 5. Numerical approximation for $u(t)$.

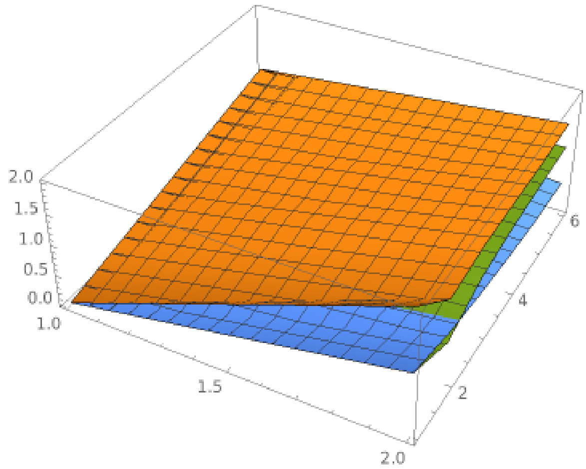

Figure 6. Numerical approximation for $u(t)$.

5. Conclusions

This article we discuss the results on controllability of nondensely defined HNFD via fundamentals of fractional calculus and fixed point technique. Initially, we discussed the controllability results using suitable concepts. We discussed some numerical approaches for different criteria of parameters. Further, we can extend the theory with different fixed point approaches.

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