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Research article

Robust optimal investment-reinsurance strategies with the preferred reinsurance level of reinsurer

Wanlu Zhang[1] **and Hui Meng**[2,*]

[1] School of Insurance, Central University of Finance and Economics, Beijing, China
[2] China Institute for Actuarial Science, Central University of Finance and Economics, Beijing, China

* **Correspondence:** Email: menghuidragon@126.com.

Abstract: This paper investigates robust equilibrium investment-reinsurance strategy for a mean variance insurer. With a larger market share, a reinsurer has a greater say in negotiating reinsurance contracts and makes the decision to propose the preferred level of reinsurance and charges extra fees as a penalty for losses that deviate from the preferred level of reinsurance. Once the insurer receives a decision from the reinsurer, the insurer weighs its risk-bearing capacity against the cost of reinsurance in order to find the optimal investment-reinsurance strategy under the mean-variance criterion. The insurer who is ambiguity averse to jump risk and diffusion risk obtains a robust optimal investment-reinsurance strategy by dynamic programming principle. Moreover, the reinsurance strategy is no longer excess-of-loss reinsurance or proportional reinsurance. In particular, the insurer may purchase proportional reinsurance for different ranges of loss and the proportion depends on the extra charge rate, which is more consistent with market practice than the standard excess-of-loss reinsurance and proportional reinsurance. The optimal reinsurance policy depends on the degree of ambiguous aversion to jump risk and not on the degree of ambiguous aversion to diffusion risk. Finally, several numerical experiments are presented to illustrate the impacts of key parameters on the value function and the optimal reinsurance contracts.

Keywords: dynamic programming; ambiguity aversion; mean-variance premium principle; mean-variance criterion; preferred reinsurance level; extra charge rate
Mathematics Subject Classification: 62P05, 91G05, 93E20

1. Introduction

Reinsurance and investment are the main ways for the insurers to manage risk. In recent years, optimal reinsurance and investment have become a hot topic in the actuarial field. The research on optimal reinsurance can date back to [2, 7], etc. Many scholars extend these notable works under different objective functions. [10] and [3] obtained optimal investment and reinsurance strategy that maximizes the expected exponential utility of terminal surplus. [23] derived optimal investment and proportional reinsurance policy to minimize the ruin probability. [19] investigated the optimal reinsurance policy that minimizes the discounted probability of exponential parisian ruin. [6] investigated the optimal portfolio select problem under mean-variance criterion. Furthermore, [16] researched equilibrium investment strategy under mean-variance criterion with a random horizon. [26] researched a mean-variance investment-reinsurance problem under the 4/2 stochastic volatility model. [12] studied the optimal dividend and reinsurance strategy to maximize the expected accumulated discounted dividends paid up to ruin under thinning dependence structure.

As an important way to increase their own wealth, insurers and reinsurers typically invest their surpluses in risky assets and risk-free assets. Therefore, the return on risky assets is of key concern. However, it is a disreputable fact that the return on risky assets is difficult to be estimated precisely. Thus, investor should take parameter uncertainty into account. Moreover, most of the literature on optimal reinsurance and investment assumes that the decision-makers knows precisely what the true probability measure is. Is the probability measure we choose necessarily a true representation of the real world? In fact, it is not known which probability measure in reality can describe the real world. For these reasons, it makes sense for investors to consider model ambiguity, or model uncertainty. To ensure decision validity under model uncertainty, one possible option is to adopt a robust method, in which alternative models close to the reference model are introduced. Model uncertainty can date back to [1] that formulates alternative models for the optimal control problem. Distinguished from [1], [25] allowed for different levels of ambiguity in the probability laws of returns for different assets. [20] and [21] derived closed-form of the optimal consumption and portfolio strategies for a robust investor. However, the above mentioned literature only considers the ambiguity of the diffusion risk. In the complete market, [9] declared that ignoring ambiguity of jump risk leads to large utility losses, while ignoring ambiguity of diffusion risk is not as severe. [24] derived robust optimal investment and proportional reinsurance strategies with default risk. [18] investigated the optimal investment and excess of loss reinsurance contracts under model uncertainty. More literature on ambiguity of the diffusion and jump risk can see [15, 28, 29] and references therein. Therefore, in the context of optimal reinsurance and investment, it is essential to consider the ambiguity of diffusion and jump risk (arising from claims).

Most of the existing works focus on optimal reinsurance and investment issues only from the insurer's perspective. However, the insurer and reinsurer jointly enter into reinsurance contracts in reality. As [8] noted: "There are two parties to a reinsurance contract, and an arrangement which is very attractive to one party may be quite unacceptable to the other." [7] researched the optimal policies in reciprocal treaties from the perspective of both parties. [11] investigated optimal premium and reinsurance strategies under the framework of Stackelberg game, in which the reinsurance premium is determined by the reinsurer and the reinsurance strategy is controlled by the insurer. [13] derived the optimal premium and reinsurance contracts in a static optimal insurance model, in which the insurer aims to maximize the expected utility of final surplus and the reinsurer devotes to maximizing the expected profit. However, in practice, the reinsurance contract is often negotiated by both parties. The insurer or reinsurer chooses different objective functions based on their own risk preferences. Since reinsurance accounts for a large share of the insurance market, we assume that the reinsurer imposes the preferred reinsurance level that is acceptable to itself from its own perspective, while it may not be acceptable to the insurer. As mentioned by [14], "A risk-averse re-insurer may impose additional service charge on firms seeking services beyond the target level, other re-insurer may demand additional charges for those seeking services with risk level lower than its preferred level as an aggressive move to gain market shares". In this situation, the problem of optimal reinsurance and investment is very interesting and worth exploring.

In this paper, we research a robust optimal reinsurance and investment problem considering the interests of the insurer and reinsurer. To be specific, the surplus processes of the insurer and reinsurer are modeled by the Cramér–Lundberg model, which can be invested in a risk-free asset and a risky asset. Inspired by [14], we assume that the reinsurance strategy is determined in the following two steps, taking into account the interests of both parties and the market share of the reinsurer. First, the reinsurer can derive the preferred reinsurance level under mean variance criterion as the benchmark. Second, the insurer, as the other party to the reinsurance contract, cedes the loss to the reinsurer and assigns a portion of the premium to the reinsurer. If the ceded loss by the insurer exceeds or falls below the preferred reinsurance level of the reinsurer, the reinsurer will charge additional premiums as a penalty, which depends on the degree of deviation from the reinsurance strategy of both parties. Based on the preferred reinsurance level of the reinsurer, the insurer searches for his own optimal reinsurance strategy with penalty function. The reinsurance countermeasure mechanism in the paper is very different from the Stackelberg reinsurance game approach used in the [11,27], which assumes that the reinsurer is the leader and the insurer is the follower, and the insurer first obtains the optimal policy for any given reinsurance price (safety loading), and then the reinsurer controls the safety loading to obtain the optimal value.

Different from [14], we assume the insurer is ambiguity averse to diffusion and jump risk. Moreover, reinsurance is limited to proportional reinsurance or excess of loss reinsurance, which has been studied in a lot of literature such as [18, 29, 30]. Unlike the above literature, our research adopts a wide reinsurance form, which requires the ceded loss and retained loss to be an increasing function of loss and includes proportional reinsurance and excess of loss reinsurance. Based on the above setting, we formulate a robust optimal reinsurance and investment model and apply the stochastic dynamic programming principle to derive the extended Hamilton–Jacobi-Bellman (HJB) equation, and deduce the explicit optimal strategies for an ambiguity averse insurer. Some numerical analyses are presented to show how the financial parameters affect the reinsurance-investment strategy and the value function.

Comparing with the existing literature, our paper proposes three main innovations. First, we consider the deviation between the preferred reinsurance level of the reinsurer and level of the ceded loss by the insurer. Second, faced with the deviation, the reinsurer charges additional premiums as a penalty. Third, the reinsurance form satisfies incentive compatibility constraint which includes proportional and excess of loss reinsurance.

The remainder of this paper is organized as follows. Section 2 formulates the reinsurer's model and obtains the preferred reinsurance level under mean variance criterion. Section 3 formulates the robust investment-reinsurance problem for an ambiguity averse insurer, and derives the explicit optimal policy using the dynamic programming principle. Section 4 obtains the equilibrium value function of the reinsurer based on reinsurance strategy ultimately determined by the insurer in Section 3. In Section 5, we show numerical analyses to illustrate our results. Finally, Section 6 concludes the paper.

2. Model formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by a compound Poisson process $L(t)$ given in the following and a Brownian motion $W(t)$, where T is a positive finite constant representing the terminal time.

Assume the surplus of the insurer is denoted by the classical Cramér–Lundberg model:

$$X(t) = x_0 + ct - \sum_{i=1}^{N(t)} Z_i,$$

where x_0 is the initial surplus; $c > 0$ is the premium rate; $L(t) := \sum_{i=1}^{N(t)} Z_i$ represents the cumulative claims up to time t; $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$, which is the claim arrival process; $\{Z_i, i \geq 1\}$ are a sequence of positive independent and identically distributed positive random variables independent of $N(t)$ and represent the size of claim. The probability distribution function of Z_i is denoted by $F(z)$. Meanwhile, Z_i has finite mean value $\mathbb{E}[Z_i] = \int_{\mathbb{R}^+} zdF(z) := \tilde{\mu}$ and second moment $\int_{\mathbb{R}^+} z^2 dF(z) := \tilde{\sigma}^2$. We assume the insurance premium is calculated based on the expected value principle, that is, $c = \lambda(1+\eta)\tilde{\mu}$, where $\eta > 0$ is the safe loading of the insurer.

Next, we suppose that the reinsurer can invest surplus in the financial market to manage risk, where the financial market consists of a risk-free asset and a risky asset. The price process of the risk-free asset, $S_0(t)$, follows the following ODE:

$$dS_0(t) = rS_0(t)dt, \tag{2.1}$$

where $r > 0$ is the constant risk-free interest rate. Moreover, the price process of the risky asset, $S(t)$, follows a geometric Brownian motion (GBM):

$$dS(t) = S(t)[\mu dt + \sigma dW(t)], \tag{2.2}$$

in which $\mu > r$ is the appreciation rate, $\sigma > 0$ denotes the volatility of risky asset and $W(t)$ is an $\mathcal{F}_t$-adapted standard Brownian motion.

2.1. *Wealth process of the reinsurer*

To spread risk, we assume the insurer can purchase reinsurance to transfer risk to the reinsurer. Because reinsurance accounts for a large share of the insurance market, reinsurers impose the preferred reinsurance level that is acceptable to themselves from their perspective, while it may not be acceptable to the insurer. Suppose the preferred reinsurance level of the reinsurer is $R_t(z) \in \mathfrak{C}$ for an incoming claim Z_i at time t, where

$$\mathfrak{C} := \{f : [0,\infty] \to [0,\infty] | f(0) = 0,\ 0 \leq f(y) - f(x) \leq y - x, \forall x \leq y\}$$

is called incentive compatibility constraint and can eliminate ex post moral hazard. Let $\pi_1(t,z) := \{(\pi_L(t), R_t(z))\}_{t\in[0,T]}$ denote a strategy of the reinsurer, where $\pi_L(t)$ represents the total amount of money invested in the risky asset at time t. To simplify our presentation, we employ Poisson random measure $N(\cdot,\cdot)$ to denote the compound Poisson process $L(t)$ under the alternative measure $\mathbb{P}$ as follows.

$$L(t) := \sum_{i=1}^{N(t)} Y_i = \int_0^t \int_0^\infty zN(ds, dz).$$

$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is a compensated Poisson random measure, where $\nu(dz)dt = \lambda F(dz)dt$ is a compensator of the random measure $N(dt, dz)$. Therefore, the surplus of the reinsurer with reinsurance and investment, under measure $\mathbb{P}$, is modeled by

$$\begin{aligned} dY^{\pi_1}(t) &= (p(t) + rY^{\pi_1}(t) + (\mu - r)\pi_L(t))\, dt + \sigma\pi_L(t)dW(t) \\ &\quad - \int_{\mathbb{R}^+} R_t(z)N(dt, dz),\ \ Y^{\pi_1}(0) = y_0, \end{aligned} \tag{2.3}$$

where $p(t)$ is the reinsurance premium rate and calculated by mean variance premium principle, that is

$$p(t) = \int_{\mathbb{R}^+} \left[(1 + \theta)R_t(z) + \xi R_t^2(z)\right] \nu(dz).$$

To avoid trivialities, we assume that $c < \lambda[(1+\theta)\tilde{\mu}+\xi\tilde{\sigma}^2]$. That is, the reinsurance premium is not cheap. Otherwise, the insurer can transfer full losses to the reinsurer to achieve arbitrage. In the following, we formally state the definition of admissible strategy (see [17]).

Definition 2.1 (Admissible strategy)**.** A pair strategy $\pi_1(t, z) := \{(\pi_L(t), R_t(z))\}_{(t,z)\in[0,T]\times\mathbb{R}^+}$ is said to be admissible if it satisfies the following conditions:

1) $R_t(z) \in \mathfrak{C}$, $\forall (t, z) \in [0, T] \times \mathbb{R}^+$;
2) $\pi_1(t, z)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ predictable and $\mathbb{E}^{\mathbb{P}}\left[\int_t^T \left((\pi_L(s))^2 + (R_s(z))^2\right) ds\right] < \infty$;
3) The stochastic differential equation (2.3) associated with π_1 has a unique strong solution $Y^{\pi_1}(\cdot)$.

Let $\Pi_1 := \Pi_L \times \mathcal{R}$ denote the set of all admissible policies, where Π_L represents the set of all admissible investment policies of the reinsurer, and $\mathcal{R}$ represents the set of all admissible reinsurance policies determined by the reinsurer.

2.2. The preferred reinsurance level of reinsurer

Compared with insurance companies, reinsurance companies have stronger economic strength, which makes reinsurance companies occupy a larger share of the insurance market. Therefore, the reinsurer has more say in the negotiation of the reinsurance contract. The reinsurer gives its own preferred level of reinsurance based on its own risk appetite. In this paper, we consider the optimal investment and reinsurance problem of the reinsurer under mean variance criterion. The problem of the reinsurer is described by

$$\begin{cases} \sup\limits_{\pi_1\in\Pi_1} J_1(t, y, \pi_1) := \sup\limits_{\pi_1(t,z)\in\Pi_1} \left\{\mathbb{E}^{\mathbb{P}}_{t,y}[Y^{\pi_1}(T)] - \dfrac{\gamma_1}{2}\mathrm{Var}^{\mathbb{P}}_{\mathrm{t,y}}[\mathrm{Y}^{\pi_1}(\mathrm{T})]\right\} \\ \text{subject to } Y^{\pi_1}(\cdot) \text{ satisfies (2.3),} \end{cases} \tag{2.4}$$

where $\mathbb{E}^{\mathbb{P}}_{t,y}[\cdot] = \mathbb{E}[\cdot|Y^{\pi_1}(t) = y]$, $\mathrm{Var}^{\mathbb{P}}_{t,y}[\cdot] = \mathrm{Var}[\cdot|Y^{\pi_1}(t) = y]$, $\gamma_1 > 0$ is the risk aversion coefficient of the reinsurer.

However, the problem (2.4) is time inconsistent since the variance term of the objective lacks the iterated expectation property. Therefore, Bellman's optimality principle fails. In order to handle time inconsistent problem, we adopt a non-cooperative game theoretic approach as that in [4, 5], in which we consider one player "t" is at time t, one player "s" is for each time $s > t$ viewed as the future incarnation of the player "t" and look for subgame perfect Nash equilibrium points. We give the definitions of equilibrium strategy and equilibrium value function in the following.

Definition 2.2. For an admissible reinsurance and investment strategy of the reinsurer $\pi_1^*(\cdot,\cdot)$, for $\epsilon > 0$ and any $(t,z) \in [0,T] \times \mathbb{R}^+$, we define the strategy

$$\pi_1^{\epsilon}(s,z) = \begin{cases} \tilde{\pi}_1(z), & s \in [t, t+\epsilon), \\ \pi_1^*(s,z), & s \in [t+\epsilon, T], \end{cases}$$

where $\tilde{\pi}_1(z) \in \Pi_1$. If

$$\lim_{\epsilon \to 0+} \inf \frac{J_1(t,y,\pi_1^*) - J_1(t,y,\pi_1^{\epsilon})}{\epsilon} \geq 0,$$

for all deterministic function $\tilde{\pi}_1(z) \in \Pi_1$, $\pi_1^*(t,z)$ is called an equilibrium strategy of the reinsurer. The equilibrium value function of problem (2.4) is defined by

$$V_1(t,y) = J_1(t,y,\pi_1^*).$$

Before presenting the verification theorem of problem (2.4), we define

$$C^{1,1}([0,T] \times \mathbb{R}^+) = \{\varphi(t,x) | \varphi(t,x) \text{ is continuously differentiable in } t \text{ and } x\}.$$

For $(t,y) \in [0,T] \times \mathbb{R}^+$, $\varphi(t,y) \in C^{1,1}([0,T] \times \mathbb{R}^+)$, we define an operator $\mathcal{L}_1^{\pi_L,R_t}$ on $\varphi(t,y)$ as follows:

$$\begin{aligned} \mathcal{L}_1^{\pi_L,R_t}\varphi(t,y) = &\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\{ry + p(t) + (\mu - r)\pi_L(t)\} + \frac{1}{2}\sigma^2\pi_L^2(t)\frac{\partial^2 \varphi}{\partial y^2} \\ &+ \int_{\mathbb{R}_+} [\varphi(t, y - R_t(z)) - \varphi(t,y)]v(dz), \end{aligned}$$

Theorem 2.1 (Verification theorem of reinsurer's problem)**.** *Suppose there exist $\mathcal{V}_1(t,y)$ and $f_1(t,y) \in C^{1,1}([0,T] \times \mathbb{R}^+)$ satisfying the extended HJB equations for the reinsurer: $\forall (t,y) \in [0,T] \times \mathbb{R}$,*

$$\sup_{\pi_L \in \Pi_L, R_t \in \mathcal{R}} \left\{ \mathcal{L}_1^{\pi_L,R_t}[\mathcal{V}_1(t,y)] - \frac{\gamma_1}{2}\mathcal{L}_1^{\pi_L,R_t}[f_1^2(t,y)] + \gamma_1 f_1(t,y)\mathcal{L}_1^{\pi_L,R_t}[f_1(t,y)] \right\} = 0, \tag{2.5}$$

$$\mathcal{L}_1^{\pi_L^*,R_t^*}[f_1(t,y)] = 0, \tag{2.6}$$

$$\mathcal{V}_1(T,y) = y, f_1(T,y) = y,$$

where

$$(\pi_L^*, R_t^*) = \arg \sup_{\pi_L \in \Pi_L, R_t \in \mathcal{R}} \left\{ \mathcal{L}_1^{\pi_L,R_t}[\mathcal{V}_1(t,y)] - \frac{\gamma_1}{2}\mathcal{L}_1^{\pi_L,R_t}[f_1^2(t,y)] + \gamma_1 f_1(t,y)\mathcal{L}_1^{\pi_L,R_t}[f_1(t,y)] \right\}.$$

Then $V_1(t,y) = \mathcal{V}_1(t,y)$, $\mathbb{E}_{t,y}\left[Y^{\pi_1^*}(T)\right] = f_1(t,y)$ and $\pi_1^* = (\pi_L^*, R_t^*)$ is the equilibrium investment and reinsurance strategy of the reinsurer.

Next, we give the equilibrium investment and reinsurance strategy of the reinsurer under mean variance criterion.

Theorem 2.2. *The optimal equilibrium investment strategy of the reinsurer is*

$$\pi_L^*(t) = \frac{\mu - r}{\gamma_1 e^{r(T-t)}\sigma^2}$$

and the optimal preferred reinsurance level has the following cases:

1) Consider $2\xi - \gamma_1 e^{r(T-t)} > 0$, $R_t^*(z) = R_{1,t}(z) := z$.
2) Consider $2\xi - \gamma_1 e^{r(T-t)} < 0$, $R_t^*(z) = R_{2,t}(z) := z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$.

Proof. In order to solve (2.5) and (2.6), we ansatz

$$V_1(t,y) = e^{r(T-t)}y + B_1(t),\ B_1(T) = 0, \tag{2.7}$$

$$f_1(t,y) = e^{r(T-t)}y + b_1(t),\ b_1(T) = 0. \tag{2.8}$$

Substituting (2.7) and (2.8) into the first extended HJB equation (2.5) gives

$$B_1'(t) + e^{r(T-t)} \sup_{R_t\in\mathcal{R}} \left\{ \int_{\mathbb{R}_+} P_1(t,z,R_t)v(dz) \right\} + e^{r(T-t)} \sup_{\pi_L\in\Pi_L} \left\{ (\mu - r)\pi_L(t) - \frac{\gamma_1}{2} e^{r(T-t)}\sigma^2 \pi_L^2(t) \right\} = 0, \tag{2.9}$$

where

$$P_1(t,z,R_t) = \left(\xi - \frac{\gamma_1}{2} e^{r(T-t)}\right) R_t(z)^2 + \theta R_t(z). \tag{2.10}$$

From (2.9), we can split the optimization problem into the following problems:

$$\sup_{\pi_L\in\Pi_L} \left\{ (\mu - r)\pi_L(t) - \frac{\gamma_1}{2} e^{r(T-t)}\sigma^2\pi_L^2(t) \right\}, \tag{2.11}$$

$$\sup_{R_t\in\mathcal{R}} \left\{ \int_{\mathbb{R}_+} P_1(t,z,R_t)v(dz) \right\}. \tag{2.12}$$

The expression in (2.11) is concave with respect to $\pi_L(t)$. According to the first order condition, we obtain the equilibrium investment strategy of the reinsurer is

$$\pi_L^*(t) = \frac{\mu - r}{\gamma_1 e^{r(T-t)}\sigma^2}.$$

The optimization problem (2.12) is readily solved by identifying the concavity of $P_1(t,z,R_t)$ with respect to R_t. From (2.10), we have $\frac{\partial^2 P_1(t,y,R_t)}{\partial R_t^2} = 2\xi - \gamma_1 e^{r(T-t)}$.

1) When $2\xi - \gamma_1 e^{r(T-t)} > 0$, $P_1(t,y,R_t)$ is strictly convex in R_t. Hence,

$$R_t^*(z) := z;$$

2) When $2\xi - \gamma_1 e^{r(T-t)} < 0$, $P_1(t,y,R_t)$ is strictly concave in R_t. Hence,

$$R_t^*(z) := z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}.$$

According to the expression of $R_t^*(z)$, we can get $R_t^*(z) \in \mathfrak{C}$. In fact, the incentive compatibility constraint requires that $0 \le R_t^*(z) \le z$ and $R_t^*(z)$, $z - R_t^*(z)$ are non-decreasing with z. In the case of $R_t^*(z) = z$, $z - R_t^*(z) = 0$. In the case of $R_t^*(z) = z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$, $z - R_t^*(z) = \left(z - \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}\right)_+ = \max\left\{0, z - \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}\right\}$. Obviously, $z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$ and $\left(z - \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}\right)_+$ are non-decreasing with z. From above two cases, we can easily get that $R_t^*(z)$ satisfies the incentive compatibility constraint. Moreover, it is also easy to verify that $R_t^*(z)$ and $\pi_L^*(t)$ satisfy conditions 2 and 3 of the admissible strategy that is given by Definition 2.1. □

3. Mean-variance problem with model uncertainty and for an AAI

3.1. The wealth process of the insurer

In Section 2, we obtain the preferred reinsurance level of the reinsurer, which may be not the insurer's optimal choice based on the insurer's risk preference and thus may not be accepted by the insurer, as also mentioned in [14].

Next, we assume that the insurer can also invest surplus in a risk-free asset and a risky asset to manage risk, where the price processes of the risk-free asset and the risky asset follow (2.1) and (2.2), respectively. Moreover, the insurer purchases reinsurance contract to spread risk. When the claim Z arrives, the insurer cedes the loss $I(Z)$ to the reinsurer and retains the remaining part of the loss $Z - I(Z)$. $z - I_t(z)$, $I_t(z) \in \mathfrak{C}$ are called retained loss function and ceded loss function that satisfy the incentive compatibility constraint, respectively. Let $\pi_2(t, z) := \{(\pi_F(t), I_t(z))\}_{t\in[0,T]}$ denote a strategy of insurer, where $\pi_F(t)$ represents the total amount of money invested by the insurer in the risky asset at time t.

Based on the above assumptions, the surplus process of the insurer $X^{\pi_2}(t)$ under measure $\mathbb{P}$ is modeled by

$$\begin{aligned} dX^{\pi_2}(t) = &\left(c - c(t) + rX^{\pi_2}(t) + (\mu - r)\pi_F(t) - a\lambda\mathbb{E}[I_t(Z) - R_t^*(Z)]^2\right)dt + \sigma\pi_F(t)dW(t) \\ &- \int_{\mathbb{R}^+} (z - I_t(z))N(dt, dz), \quad X^{\pi_2}(0) = x_0, \end{aligned} \tag{3.1}$$

where $c(t)$ is the reinsurance premium rate; a indicates the extra charge rate for the deviation from the preferred reinsurance level of the reinsurer $R_t^*(Z)$ given in Theorem 2.2; $a\mathbb{E}[I_t(Z) - R_t^*(Z)]^2$ can be viewed as the penalty/the additional reinsurance premium imposed on the ceded loss $I_t(Z)$ deviating from $R_t^*(Z)$. The quadratic term guarantees that larger deviation is punished more severely. We adopt the mean variance premium principle, that is

$$c(t) = \int_{\mathbb{R}^+} \left[(1 + \theta)I_t(z) + \xi I_t^2(z)\right] \nu(dz).$$

The insurer is generally assumed to be ambiguity neutral in traditional reinsurance and investment models. The objective function of the insurer is

$$\begin{cases} \sup\limits_{\pi_2\in\widetilde{\Pi}_2} J_2^{\mathbb{P}}(t, x, \pi_2) := \sup\limits_{\pi_2\in\widetilde{\Pi}_2} \left\{\mathbb{E}_{t,x}^{\mathbb{P}}[X^{\pi_2}(T)] - \dfrac{\gamma_2}{2}\mathrm{Var}_{\mathrm{t,x}}^{\mathbb{P}}[\mathrm{X}^{\pi_2}(\mathrm{T})]\right\} \\ \text{subject to } X^{\pi_2}(\cdot) \text{ satisfies (3.1)}, \end{cases} \tag{3.2}$$

where $\mathbb{E}^{\mathbb{P}}_{t,x}[\cdot] = \mathbb{E}[\cdot|X^{\pi_2}(t) = x]$, $\mathrm{Var}^{\mathbb{P}}_{t,x}[\cdot] = \mathrm{Var}[\cdot|X^{\pi_2}(t) = x]$, $\gamma_2 > 0$ denotes the risk aversion coefficient of the insurer, $\widetilde{\Pi}_2$ is the corresponding admissible set. However, the insurer is generally ambiguity averse in practice and only regard the surplus process under single probability measure $\mathbb{P}$ as the reference model, which is the best description of the real world based on the insurer's current information. In general, the ambiguity averse insurer is doubtful about this reference model. In fact, we do not know what models describe the real world. Faced with model uncertainty, the insurer considers alternative models to guard against worst-case scenarios. Suppose that all alternative models are described by a class of probability measures that are equivalent to $\mathbb{P}$ and absolutely continuous, and denoted as

$$\mathfrak{Q} := \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}.$$

Suppose that a process $\phi(t) := (\phi_1(t), \phi_2(t))$ satisfies the three conditions:

1) $\phi_1(t)$ and $\phi_2(t)$ are $\mathcal{F}_t$-measurable for each $t \in [0, T]$;
2) $\phi_1(t)$, $\phi_2(t) > 0$, for a.s. $(t, w) \in [0, T] \times \Omega$;
3) $\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\int_0^T \frac{1}{2}(\phi_1(t))^2 + \lambda(\phi_2(t)\ln\phi_2(t) - \phi_2(t) + 1)dt\right\}\right] < \infty$.

We denote Φ for the space of all such processes $\phi(t)$ called a density operator.

For each $\phi(t) \in \Phi$, we define a real valued process $\{\Lambda^{\phi}(t), t \in [0, T]\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Lambda^{\phi}(t) := \exp\left\{\int_0^t \phi_1(s)dW(s) - \frac{1}{2}\int_0^t (\phi_1(s))^2 ds + \int_0^t \int_0^{\infty} \ln\phi_2(s)N(ds, dz) + \lambda\int_0^t (1 - \phi_2(s))ds\right\}. \tag{3.3}$$

According to the assumption of $\phi(t)$, we get that $\Lambda^{\phi}(t)$ is a $\mathbb{P}$-martingale with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For each $\phi(t) \in \Phi$, a new probability measure $\mathbb{Q}$ absolutely continuous with $\mathbb{P}$ on $\mathcal{F}_t$ is defined by setting

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \Lambda^{\phi}(t).$$

By Girsanov's Theorem, under the alternative measure $\mathbb{Q}$, $W^{\mathbb{Q}}(t)$ is a standard Brownian motion, where

$$dW^{\mathbb{Q}}(t) = dW(t) + \phi_1(t)dt,$$

and $N(t)$ becomes the Poisson process $N^{\mathbb{Q}}(t)$ with jump intensity $\lambda^{\mathbb{Q}}(t) = \lambda\phi_2(t)$ under the alternative measure $\mathbb{Q}$. $N^{\mathbb{Q}}(dt, dz)$ is Poisson random measure under the alternative measure $\mathbb{Q}$. $\widetilde{N}^{\mathbb{Q}}(dt, dz) = N^{\mathbb{Q}}(dt, dz) - \nu^{\mathbb{Q}}(dz)dt$ is a compensated Poisson random measure, where $\nu^{\mathbb{Q}}(dz)dt = \lambda\phi_2(t)F(dz)dt$ is a compensator of the random measure $N^{\mathbb{Q}}(\cdot, \cdot)$. Furthermore, the dynamics of the surplus process $X^{\pi_2}(t)$ under $\mathbb{Q}$ is

$$\begin{aligned} dX^{\pi_2}(t) = &\left(c - c(t) + rX^{\pi_2}(t) + ((\mu - r) - \phi_1(t)\sigma)\,\pi_F(t) - a\lambda\mathbb{E}[I_t(Z) - R_t^*(Z)]^2\right)dt \\ &+ \sigma\pi_F(t)dW^{\mathbb{Q}}(t) - \int_{\mathbb{R}^+}(z - I_t(z))N^{\mathbb{Q}}(dt, dz), \quad X^{\pi_2}(0) = x_0, \end{aligned} \tag{3.4}$$

We give the definition of insurer's admissible strategy (see [28]).

Definition 3.1 (Admissible strategy). A pair strategy $\pi_2(t,z) := \{(\pi_F(t), I_t(z))\}_{t\in[0,T]}$ is said to be admissible if it satisfies the following conditions:

1) $I_t(z) \in \mathfrak{C}$, $\forall (t,z) \in [0,T]\times\mathbb{R}^+$;
2) $\pi_2(t,z)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ predictable and $\mathbb{E}^{\mathbb{Q}^*}_{t,x}\left[\int_t^T (\pi_F(s))^2 + (I_s(z))^2 ds\right] < \infty$, where $\mathbb{Q}^*$ is the chosen probability measure to describe the worst-case scenario;
3) The stochastic differential equation (3.4) associated with π_2 has a unique strong solution, $X^{\pi_2}(\cdot)$.

Let $\Pi_2 := \Pi_F \times \mathcal{I}$ denote the set of all admissible policies, where Π_F represents the set of all admissible investment policies of the insurer, and $\mathcal{I}$ represents the set of all admissible reinsurance policies determined by the insurer.

3.2. *The robust reinsurance and investment problem of the insurer*

We assume that the ambiguity averse insurer tries to design a robust optimal reinsurance and investment contract, which is the best option in some worst cases. Inspired by [9, 20], we formulate a robust control problem to modify problem (3.2) as follows:

$$V(t,x) = \sup_{\pi_2\in\Pi_2} J_2(t,x,\pi_2) := \sup_{\pi_2\in\Pi_2}\inf_{\phi\in\Phi}\left\{\mathbb{E}^{\mathbb{Q}}_{t,x}[X^{\pi_2}(T)] - \frac{\gamma_2}{2}\mathrm{Var}^{\mathbb{Q}}_{\mathrm{t,x}}[\mathrm{X}^{\pi_2}(\mathrm{T})] + \mathbb{E}^{\mathbb{Q}}_{\mathrm{t,x}}\left[\int_{\mathrm{t}}^{\mathrm{T}} \Gamma(\mathrm{s}, \mathrm{X}^{\pi_2}(\mathrm{s}), \phi(\mathrm{s}))\mathrm{ds}\right]\right\}, \tag{3.5}$$

where

$$\Gamma(s, X^{\pi_2}(s), \phi(s)) = \frac{(\phi_1(s))^2}{2\Psi_1(s, X^{\pi_2}(s))} + \frac{\lambda\,(\phi_2(s)\ln\phi_2(s) - \phi_2(s) + 1)}{\Psi_2(s, X^{\pi_2}(s))}. \tag{3.6}$$

The third term of (3.5) denotes the penalty for deviation from the reference model, which relies on the relative entropy caused by the diffusion and jump risks. Similar to [9], The growth of relative entropy from t to $t + dt$ is equal to

$$\left[\frac{1}{2}(\phi_1(t))^2 + \lambda(\phi_2(t)\ln\phi_2(t) - \phi_2(t) + 1)\right]dt. \tag{3.7}$$

In (3.6), relative entropy is scaled by $\Psi_1(t, X^{\pi_2}(s))$ and $\Psi_1(t, X^{\pi_2}(s))$, representing preference parameters for ambiguity aversion regarding diffusion risk and jump risk. The smaller $\Psi(t, \hat{X}^I_t)$ is, the larger the penalty for deviation from the reference model is, and the more confidence in the reference model the insurer has, vice versa. In this paper, we assume $\Psi_1(s, X^{\pi_2}(s))$ and $\Psi_2(s, X^{\pi_2}(s))$ are fixed and state independent functions (see [20, 22, 28]). We take

$$\Psi_1(s, X^{\pi_2}(s)) = \beta_1,\ \Psi_2(s, X^{\pi_2}(s)) = \beta_2, \tag{3.8}$$

where $\beta_1 \geq 0$ and $\beta_2 \geq 0$. After substituting (3.8) into (3.5), the value function $V_2(t,x)$ then becomes

$$\begin{aligned} V_2(t,x) = \sup_{\pi_2\in\Pi_2}\inf_{\phi\in\Phi}\Bigg\{&\mathbb{E}^{\mathbb{Q}}_{t,x}[X^{\pi_2}(T)] - \frac{\gamma_2}{2}\mathrm{Var}^{\mathbb{Q}}_{\mathrm{t,x}}[\mathrm{X}^{\pi_2}(\mathrm{T})] + \mathbb{E}^{\mathbb{Q}}_{\mathrm{t,x}}\Bigg[\int_{\mathrm{t}}^{\mathrm{T}}\Bigg(\frac{(\phi_1(\mathrm{s}))^2}{2\beta_1} \\ &+ \frac{\lambda\,(\phi_2(s)\ln\phi_2(s) - \phi_2(s) + 1)}{\beta_2}\Bigg)ds\Bigg]\Bigg\}. \end{aligned} \tag{3.9}$$

Generally, the parameters β_1 and β_2 measure the strength of the preference for robustness. When $\beta_1 = \beta_2 = 0$, robust problem (3.9) becomes mean variance utility maximization. The larger the parameters β_1 and β_2 are, the more robust the insurer is, and the more robust insurer has less faith in the reference model. In order to obtain time consistent investment and reinsurance strategy, we give the definitions of the equilibrium strategy and the equilibrium value function as [4].

Definition 3.2. For an admissible reinsurance and investment strategy of the insurer $\pi_2^*(\cdot,\cdot)$, for $\epsilon > 0$ and any $(t,z) \in [0,T] \times \mathbb{R}^+$, we define the strategy

$$\pi_2^\epsilon(s,z) = \begin{cases} \tilde{\pi}_2(z), & s \in [t, t+\epsilon), \\ \pi_2^*(s,z), & s \in [t+\epsilon, T], \end{cases}$$

where $\tilde{\pi}_2(z) \in \Pi_2$. If

$$\liminf_{\epsilon \to 0+} \frac{J_2(t,x,\pi_2^*) - J_2(t,x,\pi_2^\epsilon)}{\epsilon} \geq 0,$$

for all deterministic function $\tilde{\pi}_2(z) \in \Pi_2$, $\pi_2^*(t,z)$ is called an equilibrium strategy of the reinsurer. The equilibrium value function of problem (3.9) is defined by

$$V_2(t,x) = J_2(t,x,\pi_2^*).$$

Before presenting the verification theorem of problem (3.9), we define the infinitesimal operator $\mathcal{L}_2^{\phi,\pi_F,I_t}$ on $\varphi(t,y)$: for $(t,y) \in [0,T] \times \mathbb{R}^+$, $\varphi(t,y) \in C^{1,1}([0,T] \times \mathbb{R}^+)$,

$$\begin{aligned} \mathcal{L}_2^{\phi,\pi_F,I_t}\varphi(t,x) =& \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\left\{c - c(t) + rx + ((\mu - r) - \phi_1(t)\sigma)\pi_F(t) - a\int_{\mathbb{R}_+}(I_t(z) - R_t^*(z))^2 v(dz)\right\} \\ &+ \frac{1}{2}\sigma^2\pi_F^2(t)\frac{\partial^2\varphi}{\partial x^2} + \phi_2(t)\int_{\mathbb{R}_+}[\varphi(t, x - (z - I_t(z))) - \varphi(t,x)]v(dz). \end{aligned}$$

Theorem 3.1 (Verification theorem)**.** *Suppose there exist $\mathcal{V}_2(t,x)$ and $f_2(t,x) \in C^{1,1}([0,T] \times \mathbb{R}^+)$ satisfying the extended HJB equations for the reinsurer: $\forall (t,x) \in [0,T] \times \mathbb{R}$,*

$$\begin{aligned} \sup_{\pi_F \in \Pi_F, I_t \in \mathcal{I}} \inf_{\phi \in \Phi} \Big\{ & \mathcal{L}_2^{\phi,\pi_F,I_t}[\mathcal{V}_2(t,x)] - \frac{\gamma_2}{2}\mathcal{L}_2^{\phi,\pi_F,I_t}[f_2^2(t,x)] + \gamma_2 f_2(t,x)\mathcal{L}_2^{\phi,\pi_F,I_t}[f_2(t,x)] \\ & + \frac{(\phi_1(t))^2}{2\beta_1} + \frac{\lambda(\phi_2(t)\ln\phi_2(t) - \phi_2(t) + 1)}{\beta_2} \Big\} = 0, \end{aligned} \tag{3.10}$$

$$\mathcal{L}_2^{\phi^*,\pi_F^*,I_t^*}[f_2(t,x)] = 0, \mathcal{V}_2(T,x) = x, f_2(T,x) = x, \tag{3.11}$$

where

$$\begin{aligned} (\phi^*, \pi_F^*, I_t^*) = \arg \sup_{\pi_F \in \Pi_F, I_t \in \mathcal{I}} \inf_{\phi \in \Phi} \Big\{ & \mathcal{L}_2^{\phi,\pi_F,I_t}[\mathcal{V}_2(t,y)] - \frac{\gamma_1}{2}\mathcal{L}_2^{\phi,\pi_F,I_t}[f_2^2(t,y)] + \gamma_2 f_2(t,y)\mathcal{L}_2^{\phi,\pi_F,I_t}[f_2(t,y)] \\ & + \frac{(\phi_1(t))^2}{2\beta_1} + \frac{\lambda(\phi_2(t)\ln\phi_2(t) - \phi_2(t) + 1)}{\beta_2} \Big\}. \end{aligned}$$

Then $V_2(t,x) = \mathcal{V}_2(t,x)$, $\mathbb{E}_{t,x}^{\mathbb{Q}^*}\left[X^{\pi_2^*}(T)\right] = f_2(t,x)$, and $\pi_2^* = (\pi_F^*, I_t^*)$ is the equilibrium investment and reinsurance strategy of the insurer.

Before presenting the main results, we define some functions as follows:

$$a_1(w) := \frac{\gamma_2 e^{r(T-t)}w + 2a}{\gamma_2 e^{r(T-t)}w + 2a + 2\xi}, \quad a_2(w) := \frac{\gamma_2 e^{r(T-t)}w}{\gamma_2 e^{r(T-t)}w + 2a + 2\xi}, \tag{3.12}$$

$$c_1(w) := \frac{1+\theta-w}{\gamma_2 e^{r(T-t)}w + 2a + 2\xi}, \quad c_2(w) := \frac{2a\theta - (1+\theta-w)(\gamma_1 e^{r(T-t)} - 2\xi)}{(\gamma_2 e^{r(T-t)}w + 2a + 2\xi)(\gamma_1 e^{r(T-t)} - 2\xi)}, \tag{3.13}$$

$$I_{t,1}(z;w) := \begin{cases} 0, & 0 \le z < \dfrac{1+\theta-w}{\gamma_2 e^{r(T-t)}w + 2a}, \\ a_1(w)z - c_1(w), & z \ge \dfrac{1+\theta-w}{\gamma_2 e^{r(T-t)}w + 2a}. \end{cases} \tag{3.14}$$

$$I_{t,2}(z;w) := \begin{cases} z, & 0 \le z < \dfrac{w-1-\theta}{2\xi}, \\ a_1(w)z - c_1(w), & z \ge \dfrac{w-1-\theta}{2\xi}. \end{cases} \tag{3.15}$$

$$I_{t,3}(z;w) := \begin{cases} 0, & 0 \le z < \dfrac{(1+\theta-w)(\gamma_1 e^{r(T-t)} - 2\xi) - 2a\theta}{\gamma_2 e^{r(T-t)}w(\gamma_1 e^{r(T-t)} - 2\xi)}, \\ a_2(w)z + c_2(w), & z \ge \dfrac{(1+\theta-w)(\gamma_1 e^{r(T-t)} - 2\xi) - 2a\theta}{\gamma_2 e^{r(T-t)}w(\gamma_1 e^{r(T-t)} - 2\xi)}. \end{cases} \tag{3.16}$$

$$I_{t,4}(z;w) := \begin{cases} 0, & 0 \le z < \dfrac{1+\theta-w}{\gamma_2 e^{r(T-t)}w + 2a}, \\ a_1(w)z - c_1(w), & \dfrac{1+\theta-w}{\gamma_2 e^{r(T-t)}w + 2a} \le z < \dfrac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}, \\ a_2(w)z + c_2(w), & z \ge \dfrac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi} \end{cases} \tag{3.17}$$

$$I_{t,5}(z;w) := \begin{cases} z, & 0 \le z < \dfrac{w-1-\theta}{2\xi}, \\ a_1(w)z - c_1(w), & \dfrac{w-1-\theta}{2\xi} \le z < \dfrac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}, \\ a_2(w)z + c_2(w), & z \ge \dfrac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}. \end{cases} \tag{3.18}$$

$$I_{t,6}(z;w) := \begin{cases} z, & 0 \le z < \dfrac{(1+\theta-w)(2\xi - \gamma_1 e^{r(T-t)}) + 2a\theta}{2(e^{r(T-t)}\gamma_1 - 2\xi)(a+\xi)}, \\ a_2(w)z + c_2(w), & z \ge \dfrac{(1+\theta-w)(2\xi - \gamma_1 e^{r(T-t)}) + 2a\theta}{2(e^{r(T-t)}\gamma_1 - 2\xi)(a+\xi)}. \end{cases} \tag{3.19}$$

In the following theorems, we compute the robust optimal strategy and the value function of the insurer.

Theorem 3.2. *For $I_t(z) \in \mathfrak{C}$, we define h by*

$$h(I) = \mathbb{E}[z - I_t(z)] + \frac{\gamma_2}{2} e^{r(T-t)} \mathbb{E}[(z - I_t(z))^2].$$

Then the density operator is given by $\phi^* = (\phi_1^*(t), \phi_2^*(t)) = \left(\frac{\beta_1(\mu-r)}{\sigma(\beta_1+\gamma_2)}, \exp\left\{\beta_2 e^{r(T-t)} h(I_t^*(z))\right\}\right)$*, and the optimal investment strategy is* $\pi_F^*(t) = \frac{\mu-r}{e^{r(T-t)}\sigma^2(\beta_1+\gamma_2)}$*, the optimal reinsurance strategy* $I_t^*(z)$ *is given in several cases:*

1) When $2\xi - \gamma_1 e^{r(T-t)} > 0$,

$$I_t^*(z) = \begin{cases} I_{t,1}(z;\phi_2^*(t)), & 0 < \phi_2^*(t) < 1+\theta, \\ I_{t,2}(z;\phi_2^*(t)), & \phi_2^*(t) > 1+\theta. \end{cases}$$

2) When $2\xi - \gamma_1 e^{r(T-t)} < 0$,

$$I_t^*(z) = \begin{cases} I_{t,3}(z;\phi_2^*(t)), & 0 < \phi_2^*(t) < \max\left\{0, \dfrac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta\gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\}, \\ I_{t,4}(z;\phi_2^*(t)), & \max\left\{0, \dfrac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta\gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\} < \phi_2^*(t) < 1+\theta, \\ I_{t,5}(z;\phi_2^*(t)), & 1+\theta < \phi_2^*(t) < \dfrac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi}, \\ I_{t,6}(z;\phi_2^*(t)), & \phi_2^*(t) > \dfrac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi}, \end{cases}$$

where $I_{t,1}, I_{t,2}, I_{t,3}, I_{t,4}, I_{t,5}, I_{t,6}$ *are given by* (3.14)–(3.19).

Proof. For the insurer's value function, we ansatz

$$V_2(t,x) = e^{r(T-t)}x + B_2(t),\ B_2(T) = 0, \tag{3.20}$$
$$f_2(t,x) = e^{r(T-t)}x + b_2(t),\ b_2(T) = 0. \tag{3.21}$$

Substituting (3.20) and (3.21) into the extended HJB equation (3.10), we derive

$$\begin{aligned} &B_2'(t) + ce^{r(T-t)} + \sup_{\pi_F\in\Pi_F, I_t\in I}\ \inf_{\phi\in\Phi}\Bigg\{e^{r(T-t)}\Big((\mu-r)\pi_F(t) - \phi_1(t)\sigma\pi_F(t) - \frac{\gamma_2}{2}e^{r(T-t)}\sigma^2\pi_F^2(t) \\ &-\lambda\mathbb{E}\Big[(1+\theta)I_t(z) + \phi_2(t)(z - I_t(z)) + \xi I_t^2(z) + \frac{\gamma_2}{2}e^{r(T-t)}\phi_2(t)(I_t(z)-z)^2 + a(I_t(z) - R_t^*(z))^2\Big]\Big) \\ &+ \frac{(\phi_1(t))^2}{2\beta_1} + \frac{\lambda\left(\phi_2(t)\ln\phi_2(t) - \phi_2(t) + 1\right)}{\beta_2}\Bigg\} = 0. \end{aligned} \tag{3.22}$$

We first solve the minimization optimization problem. Since the expression in the curly braces in (3.22) is convex with respect to ϕ_1, ϕ_2 and the first order condition gives

$$\begin{aligned} &\frac{\phi_1(t)}{\beta_1} - e^{r(T-t)}\sigma\pi_F(t) = 0, \\ &\frac{1}{\beta_2}ln\phi_2(t) - e^{r(T-t)}\left(\mathbb{E}[z - I_t(z)] + \frac{\gamma_2}{2}e^{r(T-t)}\mathbb{E}[(z - I_t(z))^2]\right) = 0, \end{aligned}$$

we can get the worst-case density generator

$$\begin{cases} \phi_1^*(t,\pi_F) = \beta_1 e^{r(T-t)}\sigma\pi_F(t), \\ \phi_2^*(t,I) = \exp\left\{\beta_2 e^{r(T-t)}h(I)\right\}. \end{cases} \tag{3.23}$$

Plugging (3.23) into (3.22), we have

$$
\begin{aligned}
&-\lambda \inf_{I_t\in\mathcal{I}}\left\{e^{r(T-t)}\Big((1+\theta)\,\mathbb{E}[I_t(z)]+\xi\mathbb{E}[I_t^2(z)]+a\mathbb{E}[(I_t(z)-R_t^*(z))^2]\Big)+\frac{1}{\beta_2}\Big[\exp\{\beta_2 e^{r(T-t)}h(I)\}-1\Big]\right\}\\
&+B_2'(t)+ce^{r(T-t)}+e^{r(T-t)}\sup_{\pi_F\in\Pi_F}\left\{(\mu-r)\pi_F(t)-\frac{1}{2}(\beta_1+\gamma_2)e^{r(T-t)}\sigma^2\pi_F^2(t)\right\}=0.
\end{aligned}
\tag{3.24}
$$

Because $(\mu-r)\pi_F(t)-\frac{1}{2}(\beta_1+\gamma_2)e^{r(T-t)}\sigma^2\pi_F^2(t)$ is concave with respect to $\pi_F(t)$, and the first order condition gives

$$
\mu-r-e^{r(T-t)}\sigma^2\,(\beta_1+\gamma_2)\,\pi_F(t)=0,
$$

the optimal investment strategy is given by

$$
\pi_F^*(t)=\frac{\mu-r}{e^{r(T-t)}\sigma^2(\beta_1+\gamma_2)}. \tag{3.25}
$$

Combining (3.23) with (3.25), we can get

$$
\phi_1^*(t)=\phi_1^*(t,\pi_F^*)=\frac{\beta_1(\mu-r)}{\sigma(\beta_1+\gamma_2)}.
$$

Substituting (3.25) into (3.24), we obtain

$$
\begin{aligned}
&B_2'(t)+ce^{r(T-t)}+\frac{(\mu-r)^2}{2\sigma^2(\beta_1+\gamma_2)}-\lambda\inf_{I\in\mathcal{I}}\left\{e^{r(T-t)}\Big((1+\theta)\,\mathbb{E}[I_t(z)]+\xi\mathbb{E}[I_t^2(z)]\right.\\
&\left.+a\mathbb{E}[(I_t(z)-R_t^*(z))^2]\Big)+\frac{1}{\beta_2}\Big[\exp\{\beta_2 e^{r(T-t)}h(I)\}-1\Big]\right\}=0.
\end{aligned}
\tag{3.26}
$$

Next, we need to solve the following optimization problem:

$$
\inf_{I\in\mathcal{I}} P(I) \tag{3.27}
$$

where

$$
P(I)=e^{r(T-t)}\Big((1+\theta)\,\mathbb{E}[I_t(z)]+\xi\mathbb{E}[I_t^2(z)]+a\mathbb{E}[(I_t(z)-R_t^*(z))^2]\Big)+\frac{1}{\beta_2}\Big[\exp\{\beta_2 e^{r(T-t)}h(I)\}-1\Big]. \tag{3.28}
$$

For the problem (3.27), we solve it in two steps. First, fix $h(I)$; then the problem is equivalent to minimizing

$$
\inf_{I\in\mathcal{I}}\Big\{(1+\theta)\,\mathbb{E}[I_t(z)]+\xi\mathbb{E}[I_t^2(z)]+a\mathbb{E}[(I_t(z)-R_t^*(z))^2]\Big\}
$$

subject to $h(I)=h$. Then, we denote the Lagrangian $\mathfrak{L}$ by

$$
\begin{aligned}
\mathfrak{L}(I,\varrho)=&(1+\theta)\,\mathbb{E}[I_t(z)]+\xi\mathbb{E}[I_t^2(z)]+a\mathbb{E}[(I_t(z)-R_t^*(z))^2\\
&+\varrho\Big(\mathbb{E}[z-I_t(z)]+\frac{\gamma_2}{2}e^{r(T-t)}\mathbb{E}[(z-I_t(z))^2]-h\Big),
\end{aligned}
\tag{3.29}
$$

where $\varrho > 0$ is a Lagrange multiplier. (3.29) can be rewritten as

$$\mathfrak{L}(I, \varrho) = \int_0^\infty \Gamma(t, I, z) dF_Z(z) - \varrho h,$$

where

$$\Gamma(t, I, z) = \varrho z + (1 + \theta - \varrho) I_t(z) + \xi I_t^2(z) + a(I_t(z) - R_t^*(z))^2 + \frac{\gamma_2}{2} e^{r(T-t)} \varrho (z - I_t(z))^2.$$

We can easily obtain that $\Gamma(t, I, z)$ is a convex function of I and the first order condition becomes

$$\frac{\partial \Gamma(t, I, z)}{\partial I} = 1 + \theta - \varrho + (2\xi + 2a + \gamma_2 e^{r(T-t)} \varrho) I_t(z) - 2aR_t^*(z) - \gamma_2 e^{r(T-t)} \varrho z = 0.$$

Then, the optimal reinsurance strategy of the insurer is

$$I_t^*(z, \varrho) = \max\left\{0, \min\left\{z, \frac{\gamma_2 e^{r(T-t)} \varrho z + 2aR_t^*(z) - (1 + \theta - \varrho)}{\gamma_2 e^{r(T-t)} \varrho + 2a + 2\xi}\right\}\right\}, \tag{3.30}$$

where $R_t^*(z)$ is given as Theorem 2.2. Specifically, we discuss the specific form of $I_t^*(z; \varrho)$ in the following two cases.

1) When $2\xi - \gamma_1 e^{r(T-t)} \geq 0$, we know $R_t^*(z) = z$ from Theorem 2.2. Therefore, according to (3.30), we get

$$I_t^*(z; \varrho) = \max\{0, \min\{z, a_1(\varrho) z - c_1(\varrho)\}\},$$

that is,

$$I_t^*(z; \varrho) := \begin{cases} I_{t,1}(z; \varrho), & 0 < \varrho \leq 1 + \theta, \\ I_{t,2}(z; \varrho), & \varrho > 1 + \theta. \end{cases} \tag{3.31}$$

2) When $2\xi - \gamma_1 e^{r(T-t)} < 0$, we have $R_t^*(z) = z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$ from Theorem 2.2. Hence, based on (3.30), we deduce

$$I_t^*(z; \varrho) := \max\{0, \min\{z, [a_1(\varrho) z - c_1(\varrho)] \wedge [a_2(\varrho) z + c_2(\varrho)]\}\},$$

that is,

$$I_t^*(z; \varrho) := \begin{cases} I_{t,3}(z; \varrho), & 0 < \varrho \leq \max\left\{0, \dfrac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta \gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\}, \\ I_{t,4}(z; \varrho), & \max\left\{0, \dfrac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta \gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\} < \varrho \leq 1 + \theta, \\ I_{t,5}(z; \varrho), & 1 + \theta < \varrho \leq \dfrac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi}, \\ I_{t,6}(z; \varrho), & \varrho > \dfrac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi}. \end{cases} \tag{3.32}$$

Clearly, we can conclude that the optimal strategy depends on ϱ from (3.31) and (3.32).

Second, our goal is to seek the optimal Lagrange multiplier $\varrho^* > 0$ that solves the following optimization problem:

$$\inf_{\varrho(t)>0} L(\varrho), \tag{3.33}$$

where

$$L(\varrho) = e^{r(T-t)}\left((1+\theta)\, h_1(\varrho) + \xi h_2(\varrho) + a h_3(\varrho)\right) + \frac{1}{\beta_2}\left[\exp\left\{\beta_2 e^{r(T-t)}\left(h_4(\varrho) + \frac{\gamma_2}{2} e^{r(T-t)} h_5(\varrho)\right)\right\} - 1\right],$$

in which

$$\begin{aligned}
h_1(\varrho) &= \mathbb{E}[I_t^*(z;\varrho)] = \int_0^\infty I_t^*(z;\varrho) dF_Z(z),\\
h_2(\varrho) &= \mathbb{E}[I_t^{*2}(z;\varrho)] = \int_0^\infty I_t^{*2}(z;\varrho) dF_Z(z),\\
h_3(\varrho) &= \mathbb{E}[(I_t^*(z;\varrho) - R_t^*(z))^2] = \int_0^\infty (I_t^*(z;\varrho) - R_t^*(z))^2 dF_Z(z),\\
h_4(\varrho) &= \mathbb{E}[z - I_t^*(z;\varrho)] = \int_0^\infty (z - I_t^*(z;\varrho)) dF_Z(z),\\
h_5(\varrho) &= \mathbb{E}[(z - I_t^*(z;\varrho))^2] = \int_0^\infty (z - I_t^*(z;\varrho))^2 dF_Z(z).
\end{aligned}$$

We differentiate $L(\varrho)$ with respect to ϱ, then obtain L_ϱ in two cases:

1) When $2\xi - \gamma_1 e^{r(T-t)} > 0$, based on (3.31), we can determine that

$$\begin{aligned}
L_\varrho = & e^{r(T-t)} \int_{w(\varrho)}^\infty \frac{(2a + 2\xi + e^{r(T-t)}\gamma_2(1+\theta+2\xi z))^2}{(2a + e^{r(T-t)}\gamma_2\varrho + 2\xi)^3} dF_Z(z)\\
& \times\left[\varrho - \exp\left\{\beta_2 e^{r(T-t)}\left(h_4(\varrho) + \frac{\gamma_2}{2} e^{r(T-t)} h_5(\varrho)\right)\right\}\right],
\end{aligned} \tag{3.34}$$

where

$$w(\varrho) = \begin{cases} \dfrac{1+\theta-\varrho}{\gamma_2 e^{r(T-t)}\varrho + 2a}, & 0 < \varrho < 1+\theta,\\[2mm] \dfrac{\varrho - 1 - \theta}{2\xi}, & \varrho \geq 1+\theta. \end{cases}$$

2) When $2\xi - \gamma_1 e^{r(T-t)} < 0$, based on (3.32), we can determine that

$$L_\varrho = \begin{cases}
W_1(p_1), & 0 < \varrho < \max\left\{0, \dfrac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta\gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\},\\[3mm]
W_2(q_1), & \max\left\{0, \dfrac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta\gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\} < \varrho < 1+\theta,\\[3mm]
W_2(q_2), & 1+\theta < \varrho < \dfrac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi},\\[3mm]
W_1(p_2), & \varrho > \dfrac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi},
\end{cases} \tag{3.35}$$

in which

$$q_1 = \frac{1+\theta-\varrho}{\gamma_2 e^{r(T-t)}\varrho + 2a}, \quad p_1 = \frac{(1+\theta-\varrho)(\gamma_1 e^{r(T-t)} - 2\xi) - 2a\theta}{\gamma_2 e^{r(T-t)}\varrho(\gamma_1 e^{r(T-t)} - 2\xi)},$$

$$q_2 = \frac{\varrho - 1 - \theta}{2\xi}, \quad p_2 = \frac{(1+\theta-\varrho)(2\xi - \gamma_1 e^{r(T-t)}) + 2a\theta}{2(e^{r(T-t)}\gamma_1 - 2\xi)(a+\xi)},$$

$$\begin{aligned} W_1(p) = e^{r(T-t)} & \frac{\left[\varrho - \exp\left\{\beta_2 e^{r(T-t)}\left(h_4(\varrho) + \frac{\gamma_2}{2} e^{r(T-t)} h_5(\varrho)\right)\right\}\right]}{(e^{r(T-t)}\gamma_1 - 2\xi)^2(2a + e^{r(T-t)}\gamma_2\varrho + 2\xi)^3} \\ & \times \int_p^\infty \left[\left(e^{r(T-t)}\gamma_1 - 2\xi\right)\left(2\xi + e^{r(T-t)}\gamma_2(1+\theta+2\xi z)\right)\right. \\ & \left. + 2a\left(e^{2r(T-t)}\gamma_1\gamma_2 z - 2\xi + e^{r(T-t)}\left(\gamma_1 - \gamma_2(\theta + 2\xi z)\right)\right)\right]^2 dF_Z(z), \end{aligned}$$

$$\begin{aligned} W_2(q) = & e^{r(T-t)} \frac{\left[\varrho - \exp\left\{\beta_2 e^{r(T-t)}\left(h_4(\varrho) + \frac{\gamma_2}{2} e^{r(T-t)} h_5(\varrho)\right)\right\}\right]}{(2a + e^{r(T-t)}\gamma_2\varrho + 2\xi)^3} \\ & \times \left(\int_q^{\frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}} \left(2a + 2\xi + e^{r(T-t)}\gamma_2(1+\theta+2\xi z)\right)^2 dF_Z(z) \right. \\ & + \frac{1}{(e^{r(T-t)}\gamma_1 - 2\xi)^2} \int_{\frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}}^\infty \left[\left(e^{r(T-t)}\gamma_1 - 2\xi\right)\left(2\xi + e^{r(T-t)}\gamma_2(1+\theta+2\xi z)\right)\right. \\ & \left.\left. + 2a\left(e^{2r(T-t)}\gamma_1\gamma_2 z - 2\xi + e^{r(T-t)}\left(\gamma_1 - \gamma_2(\theta + 2\xi z)\right)\right)\right]^2 dF_Z(z) \right). \end{aligned}$$

From (3.34) and (3.35), we discover that L_ϱ is proportional to $b(\varrho)$, where

$$b(\varrho) := \varrho - \exp\left\{\beta_2 e^{r(T-t)}\left(h_4(\varrho) + \frac{\gamma_2}{2} e^{r(T-t)} h_5(\varrho)\right)\right\}.$$

It is easy to conclude that $I_t^*(z;\varrho)$ increases with respect to ϱ, which implies that $h_4(\varrho)$ and $h_5(\varrho)$ decrease with ϱ. Then, $b(\varrho)$ increases from

$$b(1) = 1 - \exp\left\{\beta_2 e^{r(T-t)}\left(h_4(1) + \frac{\gamma_2}{2} e^{r(T-t)} h_5(1)\right)\right\} < 0$$

to ∞ as ϱ increases from 1 to ∞. Therefore, there exists a unique value $\varrho^*(t)$ that solves $b(\varrho) = 0$, and minimizes (3.33). Thus, the optimal reinsurance strategy $I^*(t)$ is given by $I_t^*(z;\varrho^*)$. Moreover, according to (3.23), we obtain

$$\phi_2^*(t) = \phi_2^*(t, I_t^*(z;\varrho^*)) = \exp\left\{\beta_2 e^{r(T-t)} h(I_t^*(z;\varrho^*))\right\} = \varrho^*.$$

In summary, we obtain the result as described in the theorem. According to the expression of $I_t^*(z)$, we can get $I_t^*(z) \in \mathfrak{C}$. In fact, the incentive compatibility constraint is equal to

$$\mathfrak{C} = \{f : [0,\infty] \to [0,\infty] | f(0) = 0,\ 0 \le f'(y) \le 1,\ a.e.\}.$$

We note that it is quite obvious that $I_t^*(0) = 0$. Moreover, from (3.12), we can find $0 < a_1(\varrho^*),\ a_2(\varrho^*) < 1$. Thus, we get that $0 < I_t^{*\,\prime}(z) < 1$. From above, we deduce $I_t^*(z)$ satisfies the incentive compatibility constraint. Moreover, it is also easy to verify that $I_t^*(z)$ and $\pi_F^*(t)$ satisfy conditions 2 and 3 of the admissible strategy that is given by Definition 3.1. □

Remark 3.1. *The optimal investment strategy depends only on the degree of ambiguous aversion β_1 to diffusion risk. While, the optimal reinsurance policy depends only on the degree of ambiguous aversion β_2 to jump risk.*

Next, we present the second key result about the value function of the insurer. For notional convenience, we define several sets below.

$$
\begin{aligned}
P_1 &:= \left\{t : 2\xi - \gamma_2 e^{r(T-t)} > 0, 0 < \phi_2^*(t) < 1 + \theta\right\},\\
P_2 &:= \left\{t : 2\xi - \gamma_2 e^{r(T-t)} > 0, \phi_2^*(t) > 1 + \theta\right\},\\
P_3 &:= \left\{t : 2\xi - \gamma_2 e^{r(T-t)} < 0, 0 < \phi_2^*(t) < \max\left\{0, \frac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta\gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\}\right\},\\
P_4 &:= \left\{t : 2\xi - \gamma_2 e^{r(T-t)} < 0, \max\left\{0, \frac{(1+\theta)(\gamma_1 e^{r(T-t)} - 2\xi) - 2\theta a}{\theta\gamma_2 e^{r(T-t)} + (\gamma_1 e^{r(T-t)} - 2\xi)}\right\} < \phi_2^*(t) < 1 + \theta\right\},\\
P_5 &:= \left\{t : 2\xi - \gamma_2 e^{r(T-t)} < 0, 1 + \theta < \phi_2^*(t) < \frac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi}\right\},\\
P_6 &:= \left\{t : 2\xi - \gamma_2 e^{r(T-t)} < 0, \phi_2^*(t) > \frac{(1+\theta)\gamma_1 e^{r(T-t)} - 2\xi}{\gamma_1 e^{r(T-t)} - 2\xi}\right\}.
\end{aligned}
$$

Theorem 3.3. *The equilibrium value function of the insurer is given by*

$$
V_2(t, x) = e^{r(T-t)}x + \frac{c(e^{r(T-t)} - 1)}{r} + \frac{(\mu - r)^2(T - t)}{2\sigma^2(\beta_1 + \gamma_2)} - \sum_{i=1}^{6}\int_t^T C_i(s)\mathbb{I}_{P_i}(s)ds, \tag{3.36}
$$

and the expectation of the insurer's terminal surplus is

$$
f_2(t, x) = e^{r(T-t)}x + \frac{c(e^{r(T-t)} - 1)}{r} + \frac{(\mu - r)^2\gamma_2(T - t)}{\sigma^2(\beta_1 + \gamma_2)^2} - \sum_{i=1}^{6}\int_t^T D_i(s)\mathbb{I}_{P_i}(s)ds, \tag{3.37}
$$

where $C_i(s)$, $D_i(s)$, $i = 1, 2, \cdots, 6$ are given in (3.38) *and* (3.39).

Proof. Substituting $I_t^*(z)$, which are given in Theorem 3.2, into (3.26), together with terminal conditions $B_2(T) = 0$, we obtain

$$
B_2(t) = \frac{c(e^{r(T-t)} - 1)}{r} + \frac{(\mu - r)^2(T - t)}{2\sigma^2(\beta_1 + \gamma_2)} - \sum_{i=1}^{6}\int_t^T C_i(s)\mathbb{I}_{P_i}(s)ds,
$$

where

$$
\begin{aligned}
C_i(s) =& e^{r(T-s)}\left(\int_0^\infty \left[(1+\theta)I_{s,i}(z;\phi_2^*) + \xi I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - z)^2\right]\nu(dz)\right)\\
&+ \frac{1}{\beta_2}\left[\exp\left\{\beta_2 e^{r(T-s)}\int_0^\infty \left[z - I_{s,i}(z;\phi_2^*) + \frac{\gamma_2}{2}e^{r(T-t)}(z - I_{s,i}(z;\phi_2^*))^2\right]\nu(dz)\right\} - 1\right], i = 1,2;\\
C_i(s) =& e^{r(T-s)}\left(\int_0^\infty \left[(1+\theta)I_{s,i}(z;\phi_2^*) + \xi I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - R_{2,t}(z))^2\right]\nu(dz)\right)\\
&+ \frac{1}{\beta_2}\left[\exp\left\{\beta_2 e^{r(T-s)}\int_0^\infty \left[z - I_{s,i}(z;\phi_2^*) + \frac{\gamma_2}{2}e^{r(T-t)}(z - I_{s,i}(z;\phi_2^*))^2\right]\nu(dz)\right\} - 1\right], i = 3,4,5,6,
\end{aligned}
\tag{3.38}
$$

in which $R_{2,t}(z) = z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$. From (3.20), we obtain $V_2(t,x)$ as shown in (3.36).

Next, we present the expectation of the insurer's terminal surplus. Substituting (3.21) and $\pi_F^*(t)$, $I_t^*(z)$, $\phi^*(t)$, $R_t^*(z)$, which are given in Theorem 2.2 and 3.2, into (3.11), yields

$$
\begin{aligned}
&b_2'(t) + ce^{r(T-t)} + \frac{(\mu - r)^2\gamma_2}{\sigma^2(\beta_1+\gamma_2)^2} - \lambda e^{r(T-t)}\Big\{(1+\theta)\mathbb{E}[I_t^*(Z)] + \xi\mathbb{E}[(I_t^*(Z))^2] + a\mathbb{E}[(I_t^*(Z) - R_t^*(Z))^2]\\
&+ \exp\{\beta_2 e^{r(T-t)} h(I_t^*(z))\}\mathbb{E}[Z - I_t^*(Z)]\Big\} = 0.
\end{aligned}
$$

Together with terminal condition $b_2(T) = 0$, we obtain

$$
b_2(t) = \frac{c(e^{r(T-t)} - 1)}{r} + \frac{(\mu - r)^2\gamma_2(T-t)}{\sigma^2(\beta_1+\gamma_2)^2} - \sum_{i=1}^{6}\int_t^T D_i(s)\mathbb{I}_{P_i}(s)ds,
$$

where

$$
\begin{aligned}
D_i(s) =& e^{r(T-s)}\left(\int_0^\infty \left[(1+\theta)I_{s,i}(z;\phi_2^*) + \xi I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - z)^2 + (z - I_{s,i}(z;\phi_2^*))\right.\right.\\
&\left.\left.\times \exp\left\{\beta_2 e^{r(T-s)}\int_0^\infty z - I_{s,i}(z;\phi_2^*) + \frac{\gamma_2}{2}e^{r(T-s)}(z - I_{s,i}(z;\phi_2^*))^2\nu(dz)\right\}\right]\nu(dz)\right), i = 1,2;\\
D_i(s) =& e^{r(T-s)}\left(\int_0^\infty \left[(1+\theta)I_{s,i}(z;\phi_2^*) + \xi I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - R_{2,t}(z))^2 + (z - I_{s,i}(z;\phi_2^*))\right.\right.\\
&\left.\left.\times \exp\left\{\beta_2 e^{r(T-s)}\int_0^\infty \left(z - I_{s,i}(z;\phi_2^*) + \frac{\gamma_2}{2}e^{r(T-s)}(z - I_{s,i}(z;\phi_2^*))^2\right)\nu(dz)\right\}\right]\nu(dz)\right), i = 3,4,5,6,
\end{aligned}
\tag{3.39}
$$

in which, $R_{2,t}(z) = z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$. Finally, from (3.21), we have (3.37). □

4. The optimal strategy and the value function of the reinsurer

In this section, we investigate the optimal investment strategy under mean variance criterion when the reinsurance strategy adopts $I_t^*(z)$ which satisfies the insurer's risk preference and is eventually implemented. Assume $\pi_3(t)$ is the money amount invested in a risky asset and the reminding $Y^{\pi_3}(t)$ −

$\pi_3(t)$ is invested in a risk-free asset, in which $Y^{\pi_3}(t)$ is the surplus of the reinsurer associated with the strategy π_3. With the presence of the reinsurance and investment, the surplus of the reinsurer is modeled by

$$
\begin{aligned}
dY^{\pi_3}(t) = &\Big(c^*(t) + rY^{\pi_3}(t) + (\mu - r)\pi_3(t) + a\lambda\mathbb{E}[I_t^*(z) - R_t^*(z)]^2\Big)dt + \sigma\pi_3(t)dW(t) \\
&- \int_{\mathbb{R}^+} I_t^*(z)N(dt, dz), \ Y^{\pi_3}(0) = y_0,
\end{aligned}
\tag{4.1}
$$

where $c^*(t)$ is the reinsurance premium rate and is denoted as

$$
c^*(t) = \int_{\mathbb{R}^+} \Big((1+\theta)I_t^*(z) + \xi I_t^{*2}(z)\Big)\nu(dz).
$$

In the following, we state the admissible strategy.

Definition 4.1 (Admissible strategy)**.** A strategy $\{\pi_3(t)\}_{t\in[0,T]}$ is said to be admissible if it satisfies the following conditions:

1) $\pi_3(t)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ predictable and $\mathbb{E}[\int_t^T (\pi_3(s))^2 ds] < \infty$;
2) The stochastic differential equation (4.1) associated with π_3 has a unique strong solution, $Y^{\pi_3}(\cdot)$.

Let Π_3 denote the set of all admissible investment policies.
In this section, the problem of the reinsurer is described by

$$
\begin{cases}
\sup\limits_{\pi_3\in\Pi_3} J_3(t, y, \pi_3) := \sup\limits_{\pi_3\in\Pi_3} \Big\{\mathbb{E}_{t,y}^{\mathbb{P}}[Y^{\pi_3}(T)] - \dfrac{\gamma_1}{2} Var_{t,y}^{\mathbb{P}}[Y^{\pi_3}(T)]\Big\} \\
\text{subject to } Y^{\pi_3}(\cdot) \text{ satisfies (4.1)},
\end{cases}
\tag{4.2}
$$

where $\mathbb{E}_{t,y}^{\mathbb{P}}[\cdot] = \mathbb{E}[\cdot|Y^{\pi_3}(t) = y]$, $\mathrm{Var}_{t,y}^{\mathbb{P}}[\cdot] = \mathrm{Var}[\cdot|Y^{\pi_3}(t) = y]$, $\gamma_1 > 0$ is the risk aversion coefficient of the reinsurer.

Similar to Section 2, the problem (4.2) is time inconsistent and solved by a non-cooperative game theoretic approach. Therefore, we give the definitions of equilibrium strategy and equilibrium value function in the following.

Definition 4.2. For an admissible investment strategy of the reinsurer $\pi_3^*(\cdot)$, for $\epsilon > 0$ and any $t \in [0, T]$ and fixed number $\tilde{\pi}_3 \in \mathbb{R}^+$, we define the strategy

$$
\pi_3^\epsilon(s) = \begin{cases} \tilde{\pi}_3, & s \in [t, t+\epsilon), \\ \pi_3^*(s), & s \in [t+\epsilon, T]. \end{cases}
$$

If

$$
\liminf_{\epsilon\to 0+} \frac{J_3(t, y, \pi_3^*) - J_3(t, y, \pi_3^\epsilon)}{\epsilon} \geq 0,
$$

for all deterministic function $\tilde{\pi}_3 \in \mathbb{R}^+$, $\pi_3^*(t)$ is called an equilibrium strategy of the investment. The equilibrium value function of problem (4.2) is defined by

$$
V_3(t, y) = J_3(t, y, \pi_3^*).
$$

Before presenting the Verification theorem of the preoblem (4.2), we defined a variational operator: for $\forall(t, y) \in [0, T] \times \mathbb{R}^+$, $\varphi(t, y) \in C^{1,1}([0, T] \times \mathbb{R}^+)$,

$$\begin{aligned}\mathcal{L}_3^{\pi_3}\varphi(t, y) =& \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\left\{ry + c^*(t) + (\mu - r)\pi_3(t) + a\int_{\mathbb{R}_+}(I_t^*(z) - R_t^*(z))^2 v(dz)\right\}\\ &+ \frac{1}{2}\sigma^2\pi_3^2(t)\frac{\partial^2 \varphi}{\partial y^2} + \int_{\mathbb{R}_+}[\varphi(t, y - I_t(z)) - \varphi(t, y)]\, v(dz).\end{aligned}$$

Theorem 4.1 (Verification theorem of the preoblem (4.2))**.** *Suppose there exist $\mathcal{V}_3(t, y)$ and $f_3(t, y) \in C^{1,1}([0, T] \times \mathbb{R}^+)$ satisfying the extended HJB equations for the reinsurer: $\forall(t, y) \in [0, T] \times \mathbb{R}$,*

$$\sup_{\pi_3\in\Pi_3}\left\{\mathcal{L}_3^{\pi_3}[\mathcal{V}_3(t, y)] - \frac{\gamma_1}{2}\mathcal{L}_3^{\pi_3}[f_3^2(t, y)] + \gamma_1 f_3(t, y)\mathcal{L}_3^{\pi_3}[f_3(t, y)]\right\} = 0, \tag{4.3}$$

$$\mathcal{L}_3^{\pi_3^*}[f_3(t, y)] = 0, \tag{4.4}$$

$$\mathcal{V}_3(T, y) = y, f_3(T, y) = y,$$

where

$$\pi_3^* = \arg\sup_{\pi_3\in\Pi_3}\left\{\mathcal{L}_3^{\pi_3}[\mathcal{V}_3(t, y)] - \frac{\gamma_1}{2}\mathcal{L}_3^{\pi_3}[f_3^2(t, y)] + \gamma_1 f_3(t, y)\mathcal{L}_3^{\pi_3}[f_3(t, y)]\right\}.$$

Then $V_3(t, y) = \mathcal{V}_3(t, y)$, $\mathbb{E}_{t,y}\left[Y^{\pi_3^*}(T)\right] = f_3(t, y)$ and π_3^* is the equilibrium investment of the reinsurer.

Theorem 4.2. *The optimal investment strategy of* (4.2) *is*

$$\pi_3^*(t) = \frac{\mu - r}{\gamma_1 e^{r(T-t)}\sigma^2}.$$

The equilibrium value function of the reinsurer is given by

$$V_3(t, y) = e^{r(T-t)}y + \frac{(\mu - r)^2(T - t)}{2\sigma^2\gamma_1} + \sum_{i=1}^{6}\int_t^T E_i(s)\mathbb{I}_{P_i}(s)ds, \tag{4.5}$$

and the expectation of the insurer's terminal surplus is

$$f_3(t, y) = e^{r(T-t)}y + \frac{(\mu - r)^2(T - t)}{\sigma^2\gamma_1} + \sum_{i=1}^{6}\int_t^T F_i(s)\mathbb{I}_{P_i}(s)ds, \tag{4.6}$$

where $E_i(s)$, $F_i(s)$, $i = 1, 2, \cdots, 6$ are given in (4.10) *and* (4.11).

Proof. For the reinsurer's value function, we ansatz

$$V_3(t, y) = e^{r(T-t)}y + B_3(t),\ B_3(T) = 0, \tag{4.7}$$

$$f_3(t, y) = e^{r(T-t)}y + b_3(t),\ b_3(T) = 0. \tag{4.8}$$

Substituting (4.7) and (4.8) into the extended HJB equation (4.3) gives

$$\begin{aligned}&B_3'(t) + e^{r(T-t)}\int_{\mathbb{R}_+}\left[\theta I_t^*(z) + \left(\xi - \frac{\gamma_1}{2}e^{r(T-t)}\right)I_t^{*2}(z) + a\left(I_t^*(z) - R_t^*(z)\right)^2\right]v(dz)\\ &+ e^{r(T-t)}\sup_{\pi_3}\left\{(\mu - r)\pi_3(t) - \frac{\gamma_1}{2}e^{r(T-t)}\sigma^2\pi_3^2(t)\right\} = 0.\end{aligned} \tag{4.9}$$

The first order condition gives

$$(\mu - r) - \gamma_1 e^{r(T-t)}\sigma^2\pi_3(t) = 0,$$

we can get

$$\pi_3^*(t) = \frac{\mu - r}{\gamma_1 e^{r(T-t)}\sigma^2}.$$

Because $(\mu-r)\pi_3(t) - \frac{\gamma_1}{2}e^{r(T-t)}\sigma^2\pi_3^2(t)$ is strictly convex in $\pi_3(t)$, $\pi_3^*(t)$ is the optimal investment strategy of the reinsurer.

Substituting $R_t^*(z)$, $I_t^*(z)$, which are given in Theorem 2.2 and 3.2, and $\pi_3^*(t)$ into (4.9), together with terminal condition $B_3(T) = 0$, we obtain

$$B_3(t) = \frac{(\mu - r)^2(T-t)}{2\sigma^2\gamma_1} + \sum_{i=1}^{6}\int_t^T E_i(s)\mathbb{I}_{P_i}(s)ds,$$

where

$$\begin{aligned} E_i(s) =& e^{r(T-s)}\int_0^\infty \left[\theta I_{s,i}(z;\phi_2^*) + \left(\xi - \frac{\gamma_1}{2}e^{r(T-s)}\right)I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - z)^2\right]\nu(dz), i = 1,2;\\ E_i(s) =& e^{r(T-s)}\int_0^\infty \left[\theta I_{s,i}(z;\phi_2^*) + + \left(\xi - \frac{\gamma_1}{2}e^{r(T-s)}\right)I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - R_{2,s}(z))^2\right]\nu(dz), i = 3,4,5,6, \end{aligned} \tag{4.10}$$

in which $R_{2,s}(z) = z \wedge \frac{\theta}{\gamma_1 e^{r(T-s)} - 2\xi}$. From (4.7), we obtain $V_3(t,y)$ as shown in (4.5).

Next, we present the expectation of the reinsurer's terminal surplus. Inserting $R_t^*(z)$, $I_t^*(z)$, which are given in Theorem 2.2 and 3.2, and $\pi_3^*(t)$, (4.8) into (4.4), yields

$$b_3'(t) + \frac{(\mu - r)^2}{\gamma_1\sigma^2} + e^{r(T-t)}\int_0^\infty \left[\theta I_t^*(z) + \xi(I_t^*(z))^2 + a(I_t^*(z) - R_t^*(z))^2\right]\nu(dz) = 0.$$

Together with terminal condition $b_3(T) = 0$, we obtain

$$b_3(t) = \frac{(\mu - r)^2(T-t)}{\gamma_1\sigma^2} + \sum_{i=1}^{6}\int_t^T F_i(s)\mathbb{I}_{P_i}(s)ds,$$

where

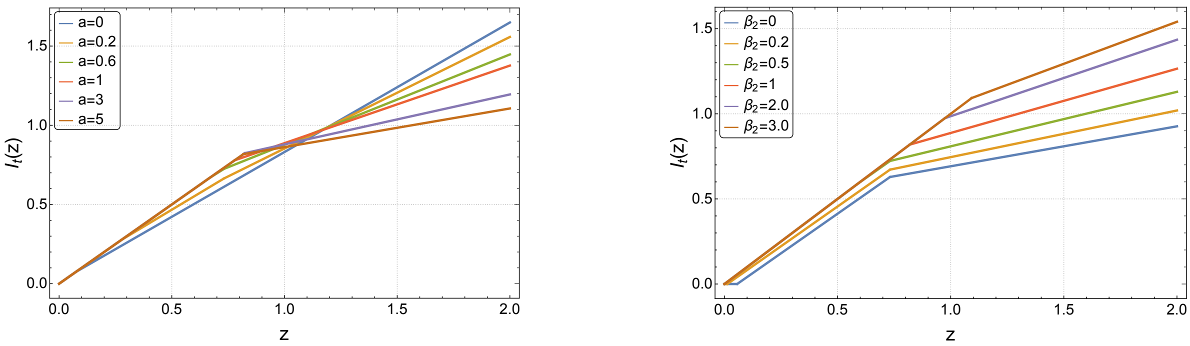
$$\begin{aligned} F_i(s) =& e^{r(T-s)}\int_0^\infty \left[\theta I_{s,i}(z;\phi_2^*) + \xi I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - z)^2\right]\nu(dz), i = 1,2;\\ F_i(s) =& e^{r(T-s)}\int_0^\infty \left[\theta I_{s,i}(z;\phi_2^*) + \xi I_{s,i}^2(z;\phi_2^*) + a(I_{s,i}(z;\phi_2^*) - R_{2,s}(z))^2\right]\nu(dz), i = 3,4,5,6, \end{aligned} \tag{4.11}$$

in which, $R_{2,s}(z) = z \wedge \frac{\theta}{\gamma_1 e^{r(T-s)} - 2\xi}$. Finally, from (4.8), we have (4.6). □

5. Numerical analysis

In this section, we analyze the impact of some parameters, such as the extra charge rate a, parameters of the ambiguous aversion β_1, β_2 and the risk aversion coefficient γ_1, γ_2, on the optimal reinsurance strategy as well as the value functions of the insurer and the reinsurer by several numerical experiments. Unless otherwise specified, the basic parameters in the following analysis are as follows: $x = 10, y = 100, \sigma = 0.2, \mu = 0.06, \beta_1 = 0.6, \lambda = 1, \xi = 0.2, \theta = 0.3, \gamma_1 = 0.6, \gamma_2 = 1, r = 0.03, t = 0, T = 10, \rho = 1, \beta_2 = 1, \eta = 0.2, a = 2$. And we presume that the loss obeys an exponential distribution with a parameter of 1, i.e., $F(z) = 1 - e^{-z}$.

Figure 1(a) shows the optimal cession loss increases with the extra charge rate a when the loss z is relatively small. However, when the loss z is relatively large, the cession loss $I_t^*(z)$ decreases as a increases. In fact, to reduce the cost of service, cession loss of the insurer should be closer to the reinsurer's preferred level of reinsurance. Due to $2\xi - \gamma_1 e^{r(T-t)} < 0$, the preferred reinsurance level of the reinsurer is $z \wedge \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$. When $z < \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$, the reinsurer prefers full reinsurance. As a increases, the insurer will pay more for losses that deviate from full reinsurance. Thus, to reduce costs, the insurer's reinsurance strategy will be more inclined to full reinsurance. Similarly, when losses $z > \frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$, the reinsurer prefers to take $\frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$. To reduce the penalty for deviating from the reinsurer's preferred reinsurance level, the insurer's reinsurance strategy should be more inclined to $\frac{\theta}{\gamma_1 e^{r(T-t)} - 2\xi}$. Figure 1(b) shows the cession losses of insurer $I_t^*(z)$ increases with β_2, which is the ambiguity aversion parameter for jump risk. As β_2 increases, the insurer becomes more uncertain about the intensity of claim arrival and thus will reduce the retained losses to mitigate this uncertainty. Therefore, more losses are ceded to the reinsurer.

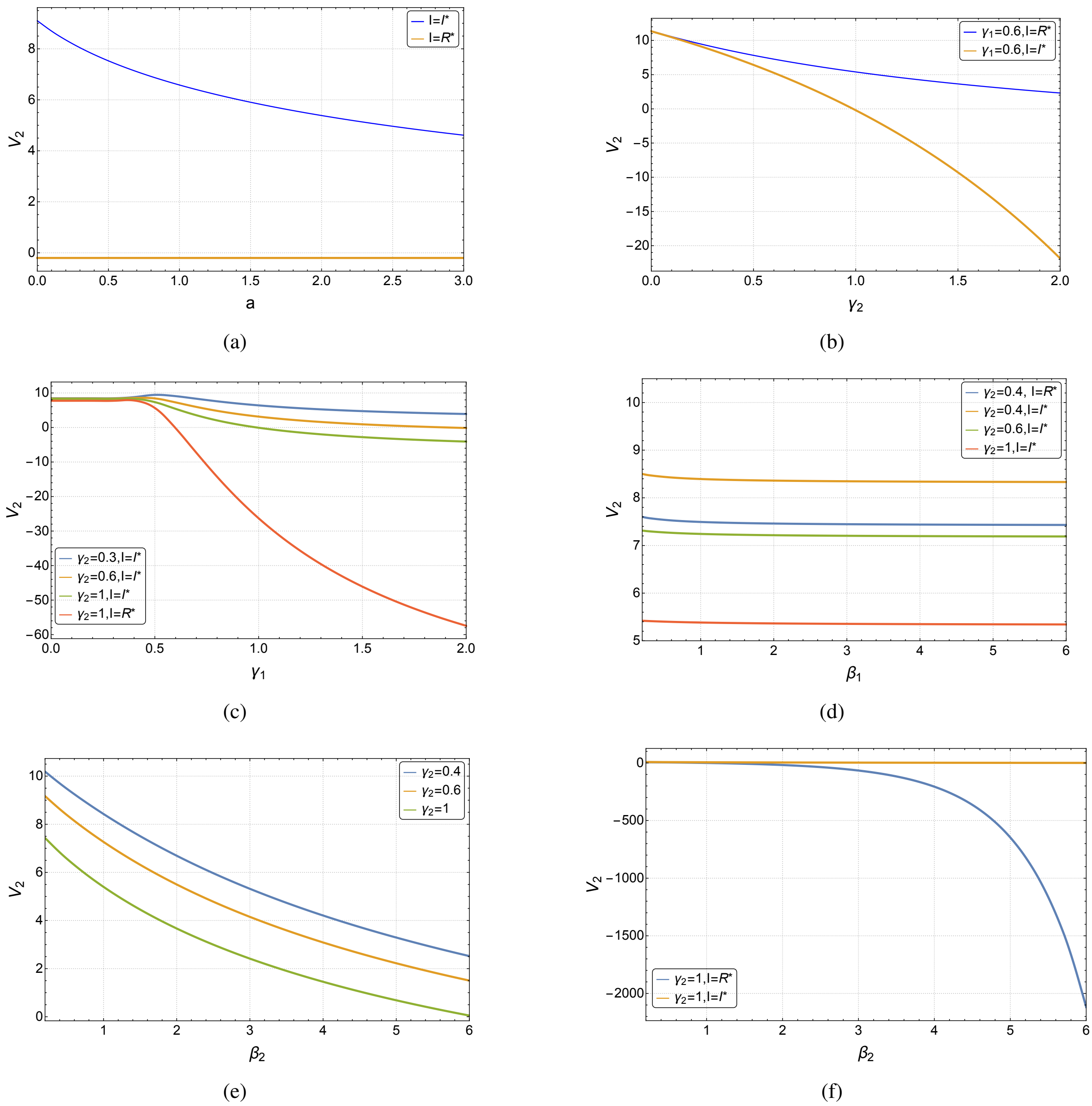


(a) (b)

Figure 1. The effects of a, β_2 on the optimal reinsurance strategy of the insurer.

Figure 2 illustrates the effect of parameters a, γ_1, γ_2, β_1 and β_2 on the value function $V_2(t, x)$. From Figure 2(a), we conclude that as the extra charge rate a increases, $V_2(t, x)$ decreases. The higher the extra charge rate is, the more reinsurance premiums the insurer pays for ceded losses and the less wealth the insurer invests in the financial markets. Thus surplus of the insurer decreases. Figure 2(b) shows that as the risk aversion coefficient γ_2 increases, the value function $V_2(t, x)$ has a decreasing trend, which is in line with the form of the objective function (3.9). Moreover, Figure 2(c) shows that as the risk aversion γ_1 of the reinsurer increases, the insurer's value function $V_2(t, x)$ first remains constant, then increases and finally decreases. From Figure 2(d), we find that the value function $V_2(t, x)$

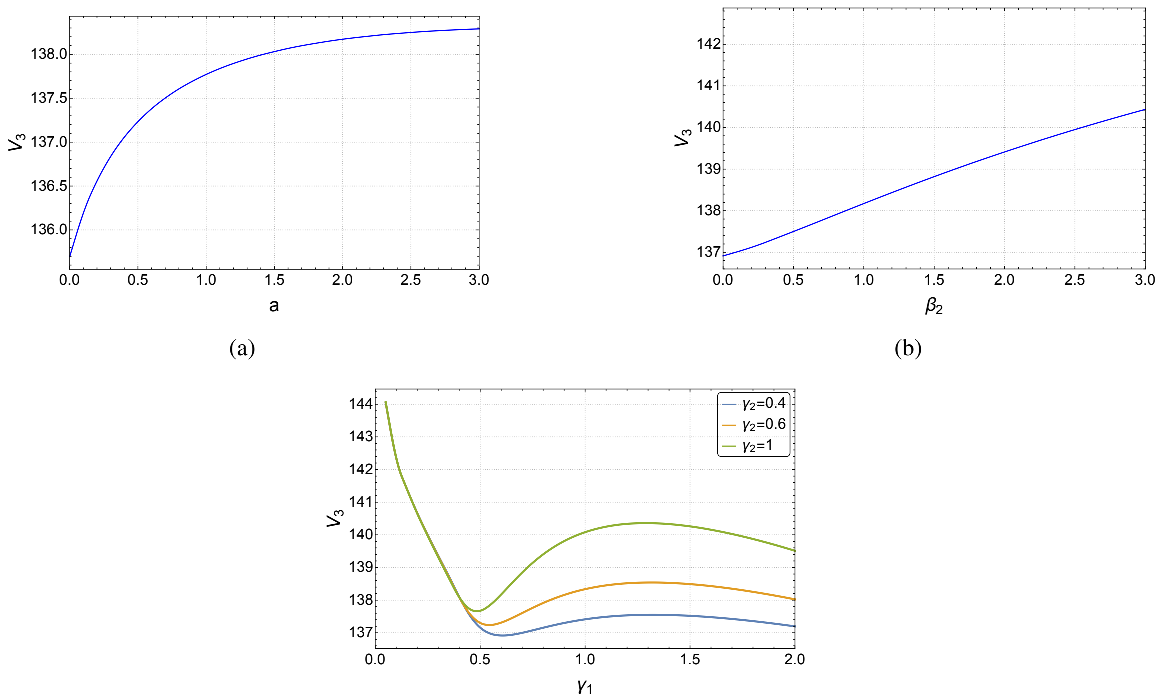
tends to decrease as the ambiguity aversion to diffusion risk β_1 increases, but this downward trend is not significant. Figure 2(e) presents that the value function $V_2(t, x)$ decreases as the ambiguity aversion to jump risk β_2 increases. Combining Figure 2(d) with Figure 2(e), we can find that the ambiguity aversion to jump risk has a greater impact on the value function $V_2(t, x)$ than the ambiguity aversion to diffusion risk. Moreover, in theory, if the insurer adopts the preferred reinsurance level $R_t^*(z)$ of reinsurer, the objective function of the insurer is smaller than the value function $V_2(t, x)$ with $I_t^*(z)$. Figures 2(a)–2(d) and 2(f) support this conclusion. In fact, from Figures 2(a)–2(d) and 2(f), we find that the curve of the value function with $I = R^*$ is always below the curve of the value function with $I = I^*$.



(a) (b) (c) (d) (e) (f)

Figure 2. The effects of a, γ_1, γ_2, β_1 and β_2 on the value function of the insurer.

Figure 3 presents the effect of the extra charge rate a and the ambiguity aversion to jump risk β_2 on the value function $V_3(t, y)$. As the extra charge rate a increases, $V_3(t, y)$ tends to rise as is shown in Figure 3(a). In fact, it is quite obvious that the increase in the extra charge rate leads to an increase in the surplus of the reinsurer, which in turn causes an increase in the value function. Figure 3(b) shows the increase of β_2 will raise the reinsurer's value function $V_3(t, y)$. As Figure 1(b) demonstrates, as ambiguity aversion to jump risk increases, the insurer cedes more losses to the reinsurer, which may exceed the reinsurer's preferred level of reinsurance. As a result, the insurer pays additional costs to the reinsurer, which in turn increases the reinsurer's surplus. Figure 3(c) shows that the reinsurer's value function $V_3(t, y)$ first decreases, then increases and finally decreases as its own risk aversion γ_1 increases.



(a) (b)

(c)

Figure 3. The effects of a, β_2, γ_1 on the value function of the reinsurer.

6. Conclusions

In this paper, we investigate the optimal reinsurance investment strategies of the insurer and the reinsurer under mean-variance criterion, where the reinsurer proposes a preferred level of reinsurance and charges an extra fee as a penalty for losses that deviate from the reinsurer's preferred level of reinsurance. Since the reinsurer has a larger market share, the reinsurer has a greater say in negotiating the reinsurance contracts. Specifically, the reinsurer first proposes its preferred level of reinsurance to the insurer.Secondly, when the insurer receives information about the reinsurer's decision, the insurer trades off its own risk bearing capacity against the reinsurance premium to find the optimal reinsurance-investment strategy under mean-variance criterion. In addition to being risk aversion, we suppose that the insurer is ambiguity averse to jump risk and diffusion risk and obtain a robust optimal reinsurance-investment strategy that is no longer excess of loss reinsurance or proportional reinsurance. We discover that the optimal reinsurance policy relies on the degree of ambiguous aversion β_2 to jump risk and not on ambiguous aversion β_1 to diffusion risk. In particular, the insurer may purchase proportional reinsurance in different ranges of losses and the percentage purchased depends on the extra charge rate a, which is more in line with market practice than excess-of-loss reinsurance and proportional reinsurance.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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