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Research article

Action-angle variables for the Lie-Poisson Hamiltonian systems associated with the three-wave resonant interaction system

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Abstract: The $\mathfrak{gl}_3(\mathbb{C})$ rational Gaudin model governed by 3×3 Lax matrix is applied to study the three-wave resonant interaction system (TWRI) under a constraint between the potentials and the eigenfunctions. And the TWRI system is decomposed so as to be two finite-dimensional Lie-Poisson Hamiltonian systems. Based on the generating functions of conserved integrals, it is shown that the two finite-dimensional Lie-Poisson Hamiltonian systems are completely integrable in the Liouville sense. The action-angle variables associated with non-hyperelliptic spectral curves are computed by Sklyanin's method of separation of variables, and the Jacobi inversion problems related to the resulting finite-dimensional integrable Lie-Poisson Hamiltonian systems and three-wave resonant interaction system are analyzed.

Keywords: three-wave resonant interaction system; non-hyperelliptic algebraic curve; separated variables; action-angle variables
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1. Introduction

It is well known that Gaudin models describe completely integrable classical and quantum long-range interacting spin chains. Originally it was introduced by M. Gaudin for the simple Lie algebra $\mathfrak{su}(2)$ [1] and later generalized to arbitrary semi-simple Lie algebras [2,3]. This model attracted considerable interest among theoretical and mathematical physicists, playing a distinguished role in the realm of integrable systems. For example, classical Hamiltonian systems associated with Lax matrices of the Gaudin-type in the context of a general group-theoretic approach [4], multi-Hamiltonian formulations (see e.g., [5]) and their integrable discretizations through Bäcklund transformations (see e.g., [6]), separation of variables of the rational Gaudin models (see e.g., [7–10]).

The separation of variables for finite-dimensional integrable systems are important for constructing action-angle variables. Much work on the separation of variables for finite dimensional integrable systems associated with hyperelliptic spectral curves was done(see e.g., [9–16]). It is worth noting that Sklyanin gave an efficient way to construct separated variables for the classical integrable $SL(3)$ magnetic chain and $\mathfrak{sl}(3)$ Gaudin model related to the non-hyperelliptic spectral curves [8]. After this classical work, the general cases [17–19] and a sufficient condition to guarantee the "separation polynomial" of Sklyanin under the corresponding r-matrix were provided in [20]. Action-angle variables of the finite-dimensional integrable systems can be constructed by using separated variables from the Liouville theorem and integration of Hamiltonian equations of motion. They were also derived with the help of the scattering data for the first-order matrix differential operator [21], or by finding real Darboux coordinates for scattering data [22, 23], or by using the algebraic-geometric techniques [23–25]. Action-angle variables of the finite-dimensional integrable systems related to non-hyperelliptic spectral curves can also be obtained in dealing with the stationary equations of soliton hierarchies [24, 25].

Based on the previous research on Gaudin models [8, 10], the motivation of this paper is to use the $\mathfrak{gl}_3(\mathbb{C})$ rational Gaudin model governed by the Lax matrix to study the constraint systems associated with the generalized TWRI system: [26, 27]

$$u_{lk,t} = c_{lk}u_{lk,x} + (c_{ml} - c_{mk})u_{lm}u_{mk}, \quad 1 \leq l, k, m \leq 3, \tag{1.1}$$

where l, k, m are not equal to each other, the real constants $c_{12},\ c_{21},\ c_{13},\ c_{31},\ c_{23},\ c_{32}$ are six real constants and $u_{12},\ u_{21},\ u_{13},\ u_{31},\ u_{23},\ u_{32}$ are complex functions of two real independent variables x and t. Also, as a basic integrable model in mathematical physics, which has important applications in nonlinear optics, plasma physics, acoustics, fluid dynamics, solid-state physics and other fields, many studies have been done on the TWRI system. For example, Zakharov and Manakov [28, 29] and Kaup [30] considered the inverse scattering transformation of the TWRI equations. It is shown that the TWRI equations can be reduced to the generic sixth Painlevé equation [31]. Some explicit solutions of the TWRI system are obtained in [32–39], including soliton solutions, rational solutions, rogue wave solutions and so on. The finite dimensional Hamiltonian system associated with the TWRI system is proved completely integrable in the Liouville sense [26]. Based on the theory of trigonal curves, explicit algebro-geometric solutions of the TWRI system have been achieved in [40] by using the asymptotic properties of the Baker-Akhiezer functions and the meromorphic functions [41, 42].

The outline is as follows. In Section 2, we review some basics about Lie-Poisson structure and coadjoint representaive theory of Lie-algebra $\mathfrak{gl}_3(\mathbb{C})$. In Section 3, Lax matrix V_λ is introduced to study the TWRI system (1.1) in the Lie-Poisson structure, the integrability of this restricted system and the relation between the finite dimensional Hamiltonian system and the TWRI system (1.1) are also discussed. In Section 4, the $9N$ dimensional Poisson manifold $(\mathfrak{gl}_3(\mathbb{C})^*)^N$ is reduced to $6N$ dimensional symplectic manifold by making a restriction on $3N$ common level set of Casimir functions, from which $3N$ pairs of separated variables are constructed on the $6N$ dimensional symplectic manifold. In Section 5, based on the Hamilton-Jacobi theory, the generating function to construct the canonical transformation is obtained from separated variables to action-angle variables in implicit form. Further, the functional independence of conserved integrals is proved in terms of the evolution of angle type variables. In addition, the Jacobi inversion problems for the Lie-Poisson Hamiltonian systems related to the TWRI system (1.1) are established.

2. Preliminary

We start from a general definition of the Lie-Poisson structure [43]. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let also $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ with the natural pairing $\langle A, B\rangle$ between $\mathfrak{g}$ and $\mathfrak{g}^*$. Consider the algebra of smooth function $F \in C^\infty(\mathfrak{g}^*)$, then its gradient $\nabla F \in \mathfrak{g}$ is defined as

$$\langle x, \nabla F(y)\rangle = \lim_{\varepsilon\to 0} \frac{F(y+\varepsilon x) - F(y)}{\varepsilon}, \quad y, x \in \mathfrak{g}^*.$$

For any smooth functions F and $G \in C^\infty(\mathfrak{g}^*)$, Lie-Poisson brakect is defined as

$$\{F, G\}(y) = \langle y, [\nabla F(y), \nabla G(y)]\rangle \tag{2.1}$$

When $\mathfrak{g}$ is the matrix Lie-algebra $\mathfrak{gl}_3(\mathbb{C})$, we take the trace form $\langle A, B\rangle = \mathrm{tr}(AB)$ as the pairing between $\mathfrak{gl}_3(\mathbb{C})$ and $\mathfrak{gl}_3(\mathbb{C})^*$ and fix the basis of $\mathfrak{gl}_3(\mathbb{C})$ as

$$E_{kl} = (\delta_{ml}\delta_{nk})_{mn}, \ 1 \le l, k \le 3,$$

where δ_{ij} are the Kronecker delta functions, which satisfy the commutative relations

$$[E_{mn}, E_{kl}] = \delta_{nk}E_{ml} - \delta_{lm}E_{kn}.$$

Since the trace form $\langle A, B\rangle = \mathrm{tr}(AB)$ is a non-degenerate pairing, we can make an identification $\mathfrak{gl}_3(\mathbb{C}) \cong \mathfrak{gl}_3(\mathbb{C})^*$. In the sense of $\langle E_{mn}, e_{kl}\rangle = \delta_{mk}\delta_{nl}$, one can find that the dual basis of $\{E_{kl},\ 1 \le k, l \le 3\}$ is $\{e_{kl} = E_{lk},\ 1 \le m, n \le 3\}$. Hereafter, we choose $y = \sum_{l,k=1}^{3} y^{kl}e_{kl} \in \mathfrak{gl}_3(\mathbb{C})^*$ and y^{kl} are coordinates on $\mathfrak{gl}_3(\mathbb{C})^*$. In these coordinates, the gradient $\nabla F \in \mathfrak{gl}_3(\mathbb{C})$ is determined by

$$\nabla F = \sum_{k,l=1}^{3} \frac{\partial F}{\partial y^{kl}} E_{kl}.$$

The Lie-Poisson bracket (2.1) become

$$\{F, G\}(y) = \langle y, [\nabla F(y), \nabla G(y)]\rangle = \mathrm{tr}(y[\nabla F(y), \nabla G(y)]) \tag{2.2}$$

Specially, the brackets between the coordinates are

$$\{y^{lk}, y^{mn}\} = \langle y, [E_{kl}, E_{nm}]\rangle = \delta_{ln}y^{mk} - \delta_{mk}y^{ln}, \ 1 \le l, k, m, n \le 3. \tag{2.3}$$

Using the cyclicity of the trace, the Hamiltonian vector field associated with (2.5) for a smooth function F is represented as

$$X_F(y) = [\nabla F(y), y].$$

For any $g \in GL_3(\mathbb{C})$ and $X \in \mathfrak{gl}_3(\mathbb{C})$, the adjoint action of $g \in GL_3(\mathbb{C})$ on its Lie algebra $\mathfrak{gl}_3(\mathbb{C})$ is

$$\mathrm{Ad}_g X = gXg^{-1}.$$

Then for any $\xi \in \mathfrak{gl}_3(\mathbb{C})^*$, the coadjoint action Ad_g^* of $GL_3(\mathbb{C})$ in the dual space $\mathfrak{gl}_3(\mathbb{C})^*$ is defined as

$$\langle \mathrm{Ad}_g^* y, X\langle = \langle y, \mathrm{Ad}_{g^{-1}} X\rangle = \langle y, g^{-1}Xg\rangle = \mathrm{tr}(yg^{-1}Xg) = \mathrm{tr}(gyg^{-1}X) = \langle gyg^{-1}, X\rangle.$$

It follows that $\mathrm{Ad}_g^* y = gyg^{-1} = \mathrm{Ad}_g y$. The coadjoint orbit passing through any $y \in \mathfrak{gl}_3(\mathbb{C})^*$ is

$$O_y = \{\mathrm{Ad}_g^* y | g \in GL_3(\mathbb{C})\} = \{y = gyg^{-1} | g \in GL_3(\mathbb{C})\}$$

According to the representative theory of Lie group, we know that the coadjoint orbits of $GL_3(\mathbb{C})$ are the symplectic leaves of the Lie-Poisson structure on $\mathfrak{gl}_3(\mathbb{C})^*$.

A smooth function h on $\mathfrak{gl}_3(\mathbb{C})^*$ is a Casimir if an only if it is invariant under the coadjoint action

$$h(\mathrm{Ad}_g^* y) = h(y) \quad \forall g \in GL_3(\mathbb{C}),\ y \in \mathfrak{gl}_3(\mathbb{C})^*.$$

Choose $\tilde{h}_k(y) = \mathrm{tr}(y^k)$, where $y \in \mathfrak{gl}_3(\mathbb{C})^*,\ k = 1, 2, 3, \cdots$, we have

$$\tilde{h}_k(Ad_g^* y) = \mathrm{tr}((Ad_g^* y)^k) = \mathrm{tr}((gyg^{-1})^k) = \mathrm{tr}(gy^k g^{-1}) = \mathrm{tr}(y^k) = \tilde{h}_k(y).$$

Thus, we choose three Casimir functions as $\mathrm{tr}(y)$, $\frac{1}{2}\mathrm{tr}(y^2)$ and $\frac{1}{3}\mathrm{tr}(y^3)$.

To study the Lie-Poisson Hamiltonian systems, we will use the direct product of N copies of $\mathfrak{gl}_3(\mathbb{C})^*$ which is denoted by $(\mathfrak{gl}_3(\mathbb{C})^*)^N$. The standard Lie-Poisson structure on $(\mathfrak{gl}_3(\mathbb{C})^*)^N$ is given by

$$\{F, G\} = \sum_{j=1}^{N} \left\langle y_j, \left[\nabla_j F, \nabla_j G\right]\right\rangle,\ \nabla_j F = \sum_{k,l=1}^{3} \frac{\partial F}{\partial y_j^{kl}} E_{lk} \tag{2.4}$$

where $y_j \in \mathfrak{gl}_3(\mathbb{C})^*,\ j = 1, 2, \cdots, N$.

The Hamiltonian vector field associated with Lie-Poisson bracket (2.4) for a smooth function F is

$$X_{jF}(y_j) = \left[\nabla_j F(y_j), y_j\right], \quad j = 1, \ldots, N \tag{2.5}$$

and the $3N$ Casimir functions are $\mathrm{tr}(y_j)$, $\frac{1}{2}\mathrm{tr}(y_j^2)$, $\frac{1}{3}\mathrm{tr}(y_j^3)$, $j = 1, \ldots, N$.

3. The Lie-Poisson Hamiltonian systems

In this section, we shall use the $\mathfrak{gl}_3(\mathbb{C})$ rational Gaudin model governed by the Lax matrix

$$V_\lambda = (V_{lk}(\lambda))_{3\times 3} = \beta + \sum_{j=1}^{N} \frac{y_j}{\lambda - \lambda_j}, \quad \beta = \mathrm{diag}(\beta_1, \beta_2, \beta_3) \tag{3.1}$$

to study the TWRI system (β_l are different constants).

To obtain the Lie-Poisson Hamiltonian systems from the Lax matrix (3.1), we choose

$$\mathcal{F}_1(\lambda) = \mathrm{tr}V_\lambda,\ \mathcal{F}_2(\lambda) = \frac{1}{2}\mathrm{tr}(V_\lambda^2),\ \mathcal{F}_3(\lambda) = \frac{1}{3}\mathrm{tr}(V_\lambda^3) \tag{3.2}$$

as the Hamiltonians. Let $t_{k\lambda}$ be the variables of $\mathcal{F}_k(\lambda)$, according to Hamiltonian vector field (2.5), the Lie-Poisson Hamiltonian systems for $\mathcal{F}_k(\lambda), k = 1, 2, 3$ are

$$y_{j,t_{k\lambda}} = [\nabla_j \mathcal{F}_k(\lambda), y_j] = \frac{1}{\lambda - \lambda_j}[V_\lambda^{k-1}, y_j],\ j = 1, \ldots, N. \tag{3.3}$$

Proposition 3.1. *The Lax matrix V_τ satisfies the Lax equations along the $\mathcal{F}_k(\lambda)$-flows:*

$$\frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}}V_\tau = \left[\frac{1}{\lambda-\tau}V_\lambda^{k-1}, V_\tau\right],\ k=2,3$$

with λ, τ are two different constant spectral parameters.

Proof. By making use of (3.3), one infers

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}}V_\tau &= \sum_{j=}^{N}\frac{1}{\tau-\lambda_j}y_{j,t_{k\lambda}}\\ &= \frac{1}{\lambda-\tau}\left[V_\lambda^{k-1}, \sum_{j=1}^{N}\frac{y_j}{\tau-\lambda_j}\right] - \frac{1}{\lambda-\tau}\left[V_\lambda^{k-1}, \sum_{j=1}^{N}\frac{y_j}{\lambda-\lambda_j}\right]\\ &= \frac{1}{\lambda-\tau}\left[V_\lambda^{k-1}, V_\tau-\beta\right] - \frac{1}{\lambda-\tau}\left[V_\lambda^{k-1}, V_\lambda-\beta\right] = \left[\frac{1}{\lambda-\tau}V_\lambda^{k-1}, V_\tau\right].\end{aligned}$$

□

Based on Proposition 3.1, for any λ, τ, a direct calculation shows that

$$\{\mathcal{F}_l(\tau), \mathcal{F}_k(\lambda)\} = \frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}}\mathcal{F}_l(\tau) = \frac{1}{l}\mathrm{tr}\Big(\frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}}V_\tau^l\Big) = \frac{1}{l}\mathrm{tr}\left(\left[\frac{1}{\lambda-\tau}V_\lambda^{k-1}, V_\tau^l\right]\right) = 0,\ k,l=2,3,$$

Therefore, $\mathcal{F}_k(\lambda), k=1,2,3$ can be regarded as the generating function of integrals of the Hamiltonian systems generated from it. Using (3.1), we arrive at

$$\begin{aligned}\mathcal{F}_1(\lambda) &= \mathrm{tr}\beta + \sum_{j=1}^{N}\frac{h_{1j}}{\lambda-\lambda_j} := \mathrm{tr}\beta + \sum_{l=0}^{\infty}\frac{F_{1,l}}{\lambda^{l+1}},\\ \mathcal{F}_2(\lambda) &= \frac{1}{2}\mathrm{tr}(\beta^2) + \sum_{j=1}^{N}\frac{E_{1,j}}{\lambda-\lambda_j} + \sum_{j=1}^{N}\frac{h_{2j}}{(\lambda-\lambda_j)^2}\\ &:= \frac{1}{2}\mathrm{tr}(\beta^2) + \sum_{l=0}^{\infty}\frac{F_{2,l}}{\lambda^{l+1}},\end{aligned}\tag{3.4}$$

where

$$h_{1j} = \mathrm{tr}(y_j),\ F_{1,l} = \sum_{j=1}^{N}\lambda_j^l h_{1j},\ E_{1,j} = \mathrm{tr}(\beta y_j) + \sum_{\substack{k=1\\k\neq j}}^{N}\frac{\mathrm{tr}(y_j y_k)}{\lambda_j-\lambda_k},$$

$$h_{2j} = \frac{1}{2}\mathrm{tr}(y_j^2),\ F_{2,l} = \sum_{j=1}^{N}\lambda_j^l E_{1,j} + l\sum_{j=1}^{N}\lambda_j^{l-1}h_{2j},\ l=0,1,\ldots.$$

Similarly, we have

$$\mathcal{F}_3(\lambda) = \frac{1}{3}\mathrm{tr}(V_\lambda^3) = \frac{1}{3}\mathrm{tr}(\beta^3) + \sum_{j=1}^N \frac{E_{2,j}}{\lambda-\lambda_j} + \sum_{j=1}^N \frac{E_{3,j}}{(\lambda-\lambda_j)^2} + \sum_{j=1}^N \frac{h_{3j}}{(\lambda-\lambda_j)^3}$$
$$:= \frac{1}{3}\mathrm{tr}(\beta^3) + \sum_{l=0}^\infty \frac{F_{3,l}}{\lambda^{l+1}}, \tag{3.5}$$

where

$$E_{2,j} = \mathrm{tr}(\beta^2 y_j) + \sum_{\substack{k=1\\k\neq j}}^N \frac{1}{\lambda_j-\lambda_k}\left[\mathrm{tr}(\beta y_j y_k + \beta y_k y_j) + \sum_{\substack{i=1\\i\neq k,j}}^N \frac{\mathrm{tr}(y_j y_k y_i + y_j y_i y_k)}{3(\lambda_k-\lambda_i)}\right.$$
$$\left. + \sum_{\substack{i=1\\i\neq k,j}}^N \frac{\mathrm{tr}(y_j y_k y_i + y_j y_i y_k)}{3(\lambda_j-\lambda_i)}\right] + \sum_{\substack{k=1\\k\neq j}}^N \frac{\mathrm{tr}(y_k^2 y_j - y_j^2 y_k)}{(\lambda_j-\lambda_k)^2}$$

$$E_{3,j} = \mathrm{tr}(\beta y_j^2) + \sum_{\substack{k=1\\k\neq j}}^N \frac{\mathrm{tr}(y_j^2 y_k)}{\lambda_j-\lambda_k},\ h_{3j} = \frac{1}{3}\mathrm{tr}(y_j^3),$$

$$F_{3,l} = \sum_{j=1}^N \lambda_j^l E_{2,j} + l\sum_{j=1}^N \lambda_j^{l-1}E_{3,j} + \frac{1}{2}l(l-1)\sum_{j=1}^N \lambda_j^{l-2}h_{3j},\ l = 0,1,\dots.$$

which implies $\{E_{lj}, E_{km}\} = 0,\ l,k = 1,2,3,\ j,m = 1,\dots,N$ and the following fact.

Corollary 3.1. *Functions $F_{1,l},\ F_{2,l},\ F_{3,l}$ $(l \geq 1)$ are in involution in pairs with respect to the Lie-Poisson bracket* (2.4).

Now, we consider two Lie-Poisson Hamiltonian systems generated by the Hamiltonians

$$\begin{aligned} H =& \gamma_0 F_{1,1} + \gamma_1\Big(\mathrm{tr}\beta F_{1,1} - F_{2,1} + \frac{1}{2}(F_{1,0})^2\Big) + \gamma_3 P(\beta_1)P(\beta_2) \\ &+ \gamma_2\left[F_{3,1} + \frac{1}{2}[(\mathrm{tr}\beta)^2 - \mathrm{tr}(\beta^2)]F_{1,1} - \mathrm{tr}\beta F_{2,1} + \frac{1}{2}\mathrm{tr}\beta(F_{1,0})^2 - F_{1,0}F_{2,0}\right] \\ &+ \gamma_4 P(\beta_1)P(\beta_3) + \gamma_5 P(\beta_2)P(\beta_3) \end{aligned} \tag{3.6}$$

$$= \alpha_1\sum_{j=1}^N \lambda_j y_j^{11} + \alpha_2\sum_{j=1}^N \lambda_j y_j^{22} + \alpha_3\sum_{j=1}^N \lambda_j y_j^{33} + c_{12}^{-1}\sum_{j=1}^N y_j^{12}\sum_{j=1}^N y_j^{21} + c_{13}^{-1}\sum_{j=1}^N y_j^{13}\sum_{j=1}^N y_j^{31} + c_{23}^{-1}\sum_{j=1}^N y_j^{23}\sum_{j=1}^N y_j^{32}$$

where

$$\gamma_0 = \frac{\alpha_1(\beta_2-\beta_3)\beta_1^2 + \alpha_2(\beta_3-\beta_1)\beta_2^2 + \alpha_3(\beta_1-\beta_2)\beta_3^2}{(\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_1-\beta_3)},$$

$$\gamma_1 = \frac{\alpha_1(\beta_3-\beta_2)\beta_1+\alpha_2(\beta_1-\beta_3)\beta_2+\alpha_3(\beta_2-\beta_1)\beta_3}{(\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_1-\beta_3)},$$
$$\gamma_2 = \frac{\alpha_1(\beta_2-\beta_3)+\alpha_2(\beta_3-\beta_1)+\alpha_3(\beta_1-\beta_2)}{(\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_1-\beta_3)},$$
$$\gamma_3 = \frac{\alpha_2-\alpha_1}{(\beta_1-\beta_2)^3(\beta_2-\beta_3)(\beta_1-\beta_3)},\ \gamma_4 = \frac{\alpha_1-\alpha_3}{(\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_1-\beta_3)^2},$$
$$\gamma_5 = \frac{\alpha_3-\alpha_2}{(\beta_1-\beta_2)(\beta_2-\beta_3)^2(\beta_1-\beta_3)},$$
$$P(\beta_j) = \beta_j^2F_{1,0}-\beta_j(\mathrm{tr}\beta F_{1,0}-F_{2,0})+F_{3,0}+\frac{1}{3}[(\mathrm{tr}\beta)^2-\mathrm{tr}(\beta^2)]F_{1,0}-\mathrm{tr}\beta F_{2,0}$$

and

$$\begin{aligned} H_1 &= F_{2,1}-\frac{(\beta_2-\beta_3)^2P^2(\beta_1)-(\beta_1-\beta_3)^2P^2(\beta_2)+(\beta_1-\beta_2)^2P^2(\beta_3)}{2(\beta_1-\beta_2)^2(\beta_1-\beta_3)^2(\beta_2-\beta_3)^2} \\ &= \beta_1\sum_{j=1}^N\lambda_jy_j^{11}+\beta_2\sum_{j=1}^N\lambda_jy_j^{22}+\beta_3\sum_{j=1}^N\lambda_jy_j^{33}+\sum_{j=1}^Ny_j^{12}\sum_{j=1}^Ny_j^{21}+\sum_{j=1}^Ny_j^{13}\sum_{j=1}^Ny_j^{31}+\sum_{j=1}^Ny_j^{23}\sum_{j=1}^Ny_j^{32}. \end{aligned} \tag{3.7}$$

respectively, where $\alpha_1,\ \alpha_2,\ \alpha_3$ are different constants and $c_{lk}=\frac{\beta_l-\beta_k}{\alpha_l-\alpha_k},\ 1\le l,k\le 3,\ l\ne k$.

The equations of motion for H and H_1 are

$$y_{j,x}=\left[\nabla_jH,y_j\right],\quad j=1,\ldots,N \tag{3.8}$$

and

$$y_{j,t}=[\nabla_jH_1,y_j],\quad j=1,\ldots,N \tag{3.9}$$

which are exactly the Lie-Poisson Hamiltonian systems associated with TWRI system (1.1).

In fact, the Lie-Poisson Hamiltonian systems (3.8) and (3.9) are generated respectively by the N copies of adjoint representations of the spectral problems

$$\varphi_x=U\varphi,\ \varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix},U=\lambda A+U_0=\lambda\begin{pmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{pmatrix}+\begin{pmatrix}0&u_{12}&u_{13}\\u_{21}&0&u_{23}\\u_{31}&u_{32}&0\end{pmatrix} \tag{3.10}$$

and

$$\varphi_t=W\varphi,\ W=\begin{pmatrix}\beta_1\lambda&c_{12}u_{12}&c_{13}u_{13}\\c_{21}u_{21}&\beta_2\lambda&c_{23}u_{23}\\c_{31}u_{31}&c_{32}u_{32}&\beta_3\lambda\end{pmatrix} \tag{3.11}$$

under the Bargmann constraint

$$u_{lk}=c_{lk}^{-1}\sum_{j=1}^Ny_j^{lk},\quad (1\le l,k\le 3,l\ne k). \tag{3.12}$$

Proposition 3.2. *The Lie-Poisson Hamiltonian systems* (3.8) *and* (3.9) *admit the Lax representations*

$$\frac{\mathrm{d}}{\mathrm{d}x}V_\lambda=[U,V_\lambda]$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\lambda = [W, V_\lambda],$$

respectively, where

$$U = \begin{pmatrix} \alpha_1\lambda & u_{12} & u_{13} \\ u_{21} & \alpha_2\lambda & u_{23} \\ u_{31} & u_{32} & \alpha_3\lambda \end{pmatrix}, \; W = \begin{pmatrix} \beta_1\lambda & c_{12}u_{12} & c_{13}u_{13} \\ c_{21}u_{21} & \beta_2\lambda & c_{23}u_{23} \\ c_{31}u_{31} & c_{32}u_{32} & \beta_3\lambda \end{pmatrix}$$

with

$$U_0 = \begin{pmatrix} 0 & u_{12} & u_{13} \\ u_{21} & 0 & u_{23} \\ u_{31} & u_{32} & 0 \end{pmatrix} = f(y_1, \cdots, y_N) = \begin{pmatrix} 0 & c_{12}^{-1}\sum_{j=1}^N y_j^{12} & c_{13}^{-1}\sum_{j=1}^N y_j^{13} \\ c_{21}^{-1}\sum_{j=1}^N y_j^{21} & 0 & c_{23}^{-1}\sum_{j=1}^N y_j^{23} \\ c_{31}^{-1}\sum_{j=1}^N y_j^{31} & c_{32}^{-1}\sum_{j=1}^N y_j^{32} & 0 \end{pmatrix}.$$

It follows that the integrals of motion of the Lie-Poisson Hamiltonian systems (3.8) and (3.9) are provided by $\mathcal{F}_k(\lambda), k = 1, 2, 3$.

Now we know that there are $3N$ Casimir functions of the Lie-Poisson structure (2.4): $\mathrm{tr}(y_j)$, $\frac{1}{2}\mathrm{tr}(y_j^2)$, $\frac{1}{3}\mathrm{tr}(y_j^3)$, $j = 1, \ldots, N$ and $3N$ involutive first integrals $E_{1,j}$, $E_{2,j}$, $E_{3,j}$, $j = 1, \cdots, N$. Hence, they are integrable.

The first two typical members of TWRI vector fields $\{X_m\}$ are [26]

$$X_0 = \begin{pmatrix} 0 & (\beta_1-\beta_2)u_{12} & (\beta_1-\beta_3)u_{13} \\ (\beta_2-\beta_1)u_{21} & 0 & (\beta_2-\beta_3)u_{23} \\ (\beta_3-\beta_1)u_{31} & (\beta_3-\beta_2)u_{32} & 0 \end{pmatrix}, \tag{3.13}$$

$$X_1 = \begin{pmatrix} 0 & c_{12}u_{12x} + (c_{13}-c_{23})u_{13}u_{32} & c_{13}u_{13x} + (c_{12}-c_{23})u_{12}u_{23} \\ c_{21}u_{21x} + (c_{23}-c_{13})u_{23}u_{31} & 0 & c_{23}u_{23x} + (c_{12}-c_{13})u_{21}u_{13} \\ c_{31}u_{31x} + (c_{23}-c_{12})u_{21}u_{32} & c_{32}u_{32x} + (c_{13}-c_{12})u_{12}u_{31} & 0 \end{pmatrix}. \tag{3.14}$$

The zero-curvature equation $U_t = W_x - [U, W]$ leads to the TWRI system (1.1)

$$(U_0)_t = X_1.$$

From Corollary 3.1, it is not difficult to verify the involutivity $\{H, H_1\} = 0$, which implies the commutativity of Hamiltonian vector fields. The importance of Hamiltonians H, H_1 are that the differential f_* maps $[\nabla_j H, y_j]$, $[\nabla_j H_1, y_j]$ exactly into the TWRI vector fields X_0, X_1

$$f_*([\nabla_1 H, y_1], \cdots, [\nabla_N H, y_N]) = X_0, \qquad f_*([\nabla_1 H_1, y_1], \cdots, [\nabla_N H_1, y_N]) = X_1,$$

respectively. Thus solutions of TWRI system (1.1) can be obtained by solving two compatible Lie-Poisson Hamiltonian systems with ordinary differential equations:

Proposition 3.3. *Let y_j be a compatible solution of the Lie-Poisson Hamiltonian systems* (3.8) *and* (3.9). *Then*

$$u_{lk} = c_{lk}^{-1}\sum_{j=1}^N y_j^{lk}, \; 1 \le l, k \le 3, l \ne k$$

solves TWRI system (1.1).

4. Separation of variables

In this section, we will give the separated canonical coordinates on the common level set of the Casimir functions to deal with the Lie-Poisson Hamiltonian systems.

Remark 1. Let $(C_{1j},\ C_{2j},\ C_{3j})$ be regular values of the map defined by the Casimirs: $y_j \rightarrow (\mathrm{tr}(y_j), \frac{1}{2}\mathrm{tr}(y_j^2), \frac{1}{3}\mathrm{tr}(y_j^3))$. Then restricted on the common level set of Casimir functions

$$\{y_1, \cdots, y_j, \cdots, y_N | \mathrm{tr}(y_j) = C_{1j},\ \frac{1}{2}\mathrm{tr}(y_j^2) = C_{2j},\ \frac{1}{3}\mathrm{tr}(y_j^3) = C_{3j},\ j = 1, \ldots, N\}, \tag{4.1}$$

the $9N$ dimensional Poisson manifold $(\mathfrak{gl}_3(\mathbb{C})^*)^N$ is naturally reduced to a $6N$ dimensional symplectic manifold, by which $3N$ pairs of canonical variables can be introduced.

In the following, we give the first $3N-2$ pairs of separated variables μ_i, ν_i by Sklyanin's method [8].

In fact, the characteristic polynomial of Lax matrix V_λ for the TWRI system (1.1) is a constant independent of variables x and t with the expansion

$$\det(zI - V_\lambda) = z^3 - \mathcal{F}_1(\lambda)z^2 + (\frac{1}{2}\mathcal{F}_1^2(\lambda) - \mathcal{F}_2(\lambda))z - \Big(\mathcal{F}_3(\lambda) - \mathcal{F}_2(\lambda)\mathcal{F}_1(\lambda) + \frac{1}{6}\mathcal{F}_1^3(\lambda)\Big) = 0, \tag{4.2}$$

which defines a non-hyperelliptic algebraic curve by introducing variable $\zeta = a(\lambda)z$:

$$\zeta^3 - a(\lambda)\mathcal{F}_1(\lambda)\zeta^2 + a^2(\lambda)(\frac{1}{2}\mathcal{F}_1^2(\lambda) - \mathcal{F}_2(\lambda))\zeta - a^3(\lambda)(\mathcal{F}_3(\lambda) - \mathcal{F}_2(\lambda)\mathcal{F}_1(\lambda) + \frac{1}{6}\mathcal{F}_1^3(\lambda)) = 0$$

with

$$a(\lambda) = \prod_{j=1}^{N}(\lambda - \lambda_j).$$

Follow the method in [8], the canonical separated variables μ_i $(i = 1, \ldots, 3N-2)$ are defined by zeros of some polynomial $B(\lambda)$ and the corresponding conjugate coordinates ν_i $(i = 1, \ldots, 3N-2)$ related to μ_i by the equation

$$\nu_i^3 - \mathcal{F}_1(\mu_i)\nu_k^2 + (\frac{1}{2}\mathcal{F}_1^2(\mu_i) - \mathcal{F}_2(\mu_i))\nu_i - (\mathcal{F}_3(\mu_i) - \mathcal{F}_2(\mu_i)\mathcal{F}_1(\mu_i) + \frac{1}{6}\mathcal{F}_1^3(\mu_i)) = 0, \tag{4.3}$$

It follows from (4.2) that ν_i are eigenvalues of the matrix V_{μ_i}. Therefore, there must exists a similarity transformation $V_{\mu_i} \rightarrow \widetilde{V}_{\mu_i} = PV_{\mu_i}P^{-1}$ for each i such that the matrix $\widetilde{V}_{\mu_i}$ is block-triangular

$$\widetilde{V}_{21}(\mu_i) = \widetilde{V}_{31}(\mu_i) = 0 \tag{4.4}$$

and ν_i is the eigenvalue of V_{μ_i} splitted from the upper block,

$$\nu_i = \widetilde{V}_{11}(\mu_i). \tag{4.5}$$

Thus, the problem is reduced to determining the matrix P and polynomial $B(\lambda)$. Let P be of the form:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the matrix

$$\begin{aligned}\widetilde{V}_\lambda &= PV_\lambda P^{-1}\\ &= \begin{pmatrix} V_{11}(\lambda) - pV_{12}(\lambda) & V_{12}(\lambda) & V_{13}(\lambda) \\ V_{21}(\lambda) + pV_{11}(\lambda) - p(pV_{12}(\lambda) + V_{22}(\lambda)) & V_{22}(\lambda) + pV_{12}(\lambda) & V_{23} + pV_{13}(\lambda) \\ V_{31}(\lambda) - pV_{32}(\lambda) & V_{32}(\lambda) & V_{33}(\lambda) \end{pmatrix}\end{aligned}$$

depends on two parameters λ and p. Hence, we can consider the condition (4.4) as the set of two algebraic equations

$$\begin{cases} \widetilde{V}_{21}(\lambda) = V_{21}(\lambda) + pV_{11}(\lambda) - p(pV_{12}(\lambda) + V_{22}(\lambda)) = 0, \\ \widetilde{V}_{31}(\lambda) = V_{31}(\lambda) - pV_{32}(\lambda) = 0 \end{cases} \tag{4.6}$$

for two variable λ and p. Eliminating p from (4.6), one obtains the polynomial equation for λ:

$$B(\lambda) = V_{32}(\lambda)V_{31}(\lambda)[V_{11}(\lambda) - V_{22}(\lambda)] + V_{32}^2(\lambda)V_{21}(\lambda) - V_{31}^2(\lambda)V_{12}(\lambda) = 0. \tag{4.7}$$

Based on (3.1) and (4.7), the polynomial $B(\lambda)$ of degree $3N-2$ can be denote as

$$B(\lambda) := (\beta_1 - \beta_2)\sum_{j=1}^{N} y_j^{31} \sum_{j=1}^{N} y_j^{32} \frac{n(\lambda)}{a^3(\lambda)}, \tag{4.8}$$

where

$$n(\lambda) = \prod_{i=1}^{3N-2}(\lambda - \mu_i),\ a(\lambda) = \prod_{j=1}^{N}(\lambda - \lambda_j) := \sum_{j=0}^{N} a_j \lambda^{N-j},\ (a_0 = 1). \tag{4.9}$$

Expressing p from $\widetilde{V}_{31}(\lambda) = 0$ as $p = -V_{31}(\lambda)/V_{32}(\lambda)$ and substituting it into Eq (4.5) of ν_i, we arrive at

$$\nu_i = \widetilde{V}_{11}(\mu_i) = V_{11}(\mu_i) - \frac{V_{12}(\mu_i)V_{31}(\mu_i)}{V_{32}(\mu_i)}, i = 1, \dots, 3N-2, \tag{4.10}$$

which produces $3N-2$ pairs of variables μ_i, ν_i. It is easy to see from (2.3) and the above expressions that

$$\{V_{lk}(\tau), V_{mn}(\lambda)\} = \frac{1}{\lambda - \tau}[(V_{mk}(\tau) - V_{mk}(\lambda))\delta_{ln} - (V_{ln}(\tau) - V_{ln}(\lambda))\delta_{mk}],\ 1 \le l, k, m, n \le 3 \tag{4.11}$$

with $\lambda,\ \tau$ are two different constant spectral parameters. We have from (4.11) that

$$A(\lambda) = V_{11}(\lambda) - \frac{V_{12}(\lambda)V_{31}(\lambda)}{V_{32}(\lambda)}, \tag{4.12}$$

which, together with $B(\lambda)$ defined by (4.7), $B(\lambda)$ and $A(\tau)$ satisfy

$$\begin{cases} \{A(\tau), A(\lambda)\} = 0, \\ \{B(\tau), B(\lambda)\} = 0, \\ \{A(\tau), B(\lambda)\} = \dfrac{1}{\lambda - \tau}\left(\dfrac{V_{32}^2(\lambda)}{V_{32}^2(\tau)}B(\tau) - B(\lambda)\right). \end{cases} \tag{4.13}$$

Proposition 4.1. *The canonical separated variables μ_j and ν_j constructed from $B(\lambda)$ in* (4.8) *and $A(\mu_j)$ in* (4.10) *satisfy the relations*

$$\{\mu_i,\mu_j\}=0,\ \{\nu_i,\nu_j\}=0,\ \{\nu_i,\mu_j\}=\delta_{ij}1\leq i,j\leq 3N-2.$$

Proof. The commutativity of B in (4.13) implies the commutativity of μ_j (zeros of $B(\lambda)$). The Poisson brackets of ν_j can be calculated by using the definition of μ_j. From $B(\mu_j)=0$ for $j=1,\ldots,3N-2$, it follows that

$$0=\{F,B(\mu_j)\}=\{F,B(\lambda)\}|_{\lambda=\mu_j}+B'(\mu_j)\{F,\mu_j\}$$

that is

$$\{F,\mu_j\}=-\frac{\{F,B(\lambda)\}|_{\lambda=\mu_j}}{B'(\mu_j)}, \tag{4.14}$$

for any function F. In the same way we have

$$\{\nu_i,F\}=\{A(\mu_i),F\}=\{A(\tau),F\}|_{\tau=\mu_i}+A'(\mu_i)\{\mu_i,F\}.$$

Now we turn to prove $\{\nu_i,\mu_j\}=\delta_{ij}$. Noting that

$$\{\nu_i,\mu_j\}=\{A(\tau),\mu_j\}|_{\tau=\mu_i}+A'(\mu_i)\{\mu_i,\mu_j\}=\{A(\tau),\mu_j\}|_{\tau=\mu_i},$$

we obtain by using (4.14) and the third equation in (4.13) that

$$\{\nu_i,\mu_j\}=-\frac{\{A(\tau),B(\lambda)\}|_{\lambda=\mu_j}^{\tau=\mu_i}}{B'(\mu_j)}=\frac{1}{\mu_j-\mu_i}\left(\frac{V_{32}^2(\mu_j)}{V_{32}^2(\mu_i)}B(\mu_i)-B(\mu_j)\right)\frac{1}{B'(\mu_j)}, \tag{4.15}$$

which vanishes for $\mu_i\neq\mu_j$ due to $B(\mu_i)=B(\mu_j)=0$ and is evaluated via L'Hôpital rule for $\mu_i=\mu_j$ to produce the proclaimed result. Similarly, the commutativity of ν_i $(1\leq i\leq 3N-2)$ can be shown by the first equation of (4.13). □

Apart from the $3N-2$ pairs of separated variables above, we should add 2 pairs of conjugate variables to prove the canonical structure on the common level set of Casimir functions (4.1).

The last 2 pairs of conjugate variables can be defined as follows by direct calculation.

Proposition 4.2. *Assume that the 2 pairs of additional canonical separated variables are defined by*

$$\begin{aligned}\mu_{3N-1}&=\ln\sum_{j=1}^{N}y_j^{31}, \quad \nu_{3N-1}=\frac{(\beta_2+\beta_3)F_{2,0}-F_{3,0}}{(\beta_2-\beta_1)(\beta_3-\beta_1)},\\ \mu_{3N}&=\ln\sum_{j=1}^{N}y_j^{32}, \quad \nu_{3N}=\frac{G_0-(\beta_1+\beta_3)F_{2,0}}{(\beta_2-\beta_1)(\beta_3-\beta_2)}.\end{aligned} \tag{4.16}$$

Then on the common level set of Casimir functions (4.1)*, we have*

$$\{\mu_i,\mu_j\}=0,\ \{\nu_i,\nu_j\}=0,\ \{\mu_i,\nu_j\}=\delta_{ij},\ i,j=1,\ldots,3N. \tag{4.17}$$

It is shown that $\mu_j,\ \nu_j,\ j=1,\cdots,N$ are $3N$ pairs of conjugate variables.

5. Action-angle variables and Jacobi inversion problems

In this section, the action-angle variables will be introduced by resorting to Hamilton-Jacobi theory. As a by-product, the functional independence of conserved integrals for the Liouville integrability of Lie-Poisson Hamiltonian systems (3.8) and (3.9) will be proved. Further, the Jacobi inversion problems for systems (3.8), (3.9) and TWRI Eq (1.1) will be built by using the canonical transformation from the separated variables to the action-angle variables

Let

$$\frac{1}{2}\mathrm{tr}\beta^2+\sum_{j=1}^{N}\frac{E_{1,j}}{\lambda-\lambda_j}:=\frac{b_2(\lambda)}{a(\lambda)}:=\frac{1}{2}\mathrm{tr}\beta^2+\sum_{l=0}^{\infty}\frac{f_l}{\lambda^{l+1}},$$
$$\frac{1}{3}\mathrm{tr}\beta^3+\sum_{j=1}^{N}\frac{E_{2,j}}{\lambda-\lambda_j}+\sum_{j=1}^{N}\frac{E_{3,j}}{(\lambda-\lambda_j)^2}:=\frac{b_3(\lambda)}{a^2(\lambda)}:=\frac{1}{3}\mathrm{tr}\beta^3+\sum_{l=0}^{\infty}\frac{g_l}{\lambda^{l+1}},$$

where

$$\begin{aligned}b_2(\lambda)&=\frac{1}{2}\mathrm{tr}\beta^2\lambda^N+I_0\lambda^{N-1}+I_1\lambda^{N-2}+\cdots+I_{N-3}\lambda^2+I_{N-2}\lambda+I_{N-1},\\ b_3(\lambda)&=\frac{1}{3}\mathrm{tr}\beta^3\lambda^{2N}+I_{\bar{0}}\lambda^{2N-1}+I_N\lambda^{2N-2}+\cdots+I_{3N-3}\lambda+I_{3N-2},\end{aligned}\tag{5.1}$$

from which we can rewrite the generating functions $\mathcal{F}_1(\lambda)$, $\mathcal{F}_2(\lambda)$, $\mathcal{F}_3(\lambda)$ as

$$\begin{aligned}\mathcal{F}_1(\lambda)&=\mathrm{tr}\beta+\sum_{j=1}^{N}\frac{C_{1j}}{\lambda-\lambda_j}:=\frac{R_1(\lambda)}{a(\lambda)},\\ \mathcal{F}_2(\lambda)&=\frac{b_2(\lambda)}{a(\lambda)}+\sum_{j=1}^{N}\frac{C_{2j}}{(\lambda-\lambda_j)^2}=\frac{1}{2}\mathrm{tr}\beta^2+\sum_{l=0}^{\infty}\frac{f_l}{\lambda^{l+1}}+\sum_{j=1}^{N}\frac{C_{2j}}{(\lambda-\lambda_j)^2}:=\frac{R_2(\lambda)}{a^2(\lambda)},\\ \mathcal{F}_3(\lambda)&=\frac{b_3(\lambda)}{a^2(\lambda)}+\sum_{j=1}^{N}\frac{C_{3j}}{(\lambda-\lambda_j)^3}=\frac{1}{3}\mathrm{tr}\beta^3+\sum_{l=0}^{\infty}\frac{g_l}{\lambda^{l+1}}+\sum_{j=1}^{N}\frac{C_{3j}}{(\lambda-\lambda_j)^3}\end{aligned}\tag{5.2}$$

with

$$R_1(\lambda)=a(\lambda)\Big(\mathrm{tr}\beta+\sum_{j=1}^{N}\frac{C_{1j}}{\lambda-\lambda_j}\Big),R_2(\lambda)=a(\lambda)b_2(\lambda)+a^2(\lambda)\sum_{j=1}^{N}\frac{C_{2j}}{(\lambda-\lambda_j)^2},$$

The comparison of the coefficients of $\lambda^l,\ l=0,\ldots,N-1$ in equation

$$b_2(\lambda)=a(\lambda)\Big(\frac{1}{2}\mathrm{tr}\beta^2+\sum_{l=0}^{\infty}\frac{f_l}{\lambda^{l+1}}\Big)$$

and the comparison of the coefficients of $\lambda^l,\ l=0,1,\ldots,2N-1$ in equation

$$b_3(\lambda)=a^2(\lambda)\Big(\frac{1}{3}\mathrm{tr}\beta^3+\sum_{l=0}^{\infty}\frac{g_l}{\lambda^{l+1}}\Big),$$

respectively, yield

$$\begin{aligned}
I_0 &= \frac{1}{2}a_1\mathrm{tr}\beta^2 + f_0 = \frac{1}{2}a_1\mathrm{tr}\beta^2 + (\beta_3-\beta_1)\nu_{3N-1} + (\beta_3-\beta_2)\nu_{3N},\\
I_{\widetilde{0}} &= \frac{2}{3}a_1\mathrm{tr}\beta^3 + g_0 = \frac{2}{3}a_1\mathrm{tr}\beta^3 + (\beta_3^2-\beta_1^2)\nu_{3N-1} + (\beta_3^2-\beta_2^2)\nu_{3N},\\
I_j &= \sum_{i=0}^{j} a_i f_{j-i} + \frac{1}{2}a_{j+1}\mathrm{tr}\beta^2,\ j=1,\dots,N-1,\\
I_{N+k} &= \sum_{l=0}^{k+1}\Big(\sum_{\substack{i,j\geq 0\\ i+j=l}} a_i a_j\Big) g_{k+1-l} + \frac{1}{3}\sum_{\substack{i,j\geq 0\\ i+j=k+2}} a_i a_j \mathrm{tr}\beta^3,\ k=0,\dots,2N-2.
\end{aligned}$$

Let

$$\nu_i = \frac{\partial S}{\partial \mu_i},\ i=1,\dots,3N-2.$$

We obtain from (4.3) that the completely separated Hamilton-Jacobi equations:

$$\begin{aligned}
&\left(\frac{\partial S}{\partial \mu_i}\right)^3 - \frac{R_1(\mu_i)}{a(\mu_i)}\left(\frac{\partial S}{\partial \mu_i}\right)^2 + \left(\frac{R_1^2(\mu_i)}{2a^2(\mu_i)} - \frac{b_2(\mu_i)}{a(\mu_i)} - \sum_{j=1}^{N}\frac{C_{2j}}{(\mu_i-\lambda_j)^2}\right)\frac{\partial S}{\partial \mu_i}\\
&-\left(\frac{b_3(\mu_i)}{a^2(\mu_i)} + \sum_{j=1}^{N}\frac{C_{3j}}{(\mu_i-\lambda_j)^3} - \Big(\frac{b_2(\mu_i)}{a(\mu_i)} + \sum_{j=1}^{N}\frac{C_{2j}}{(\mu_i-\lambda_j)^2}\Big)\frac{R_1(\mu_i)}{a(\mu_i)} + \frac{R_1^3(\mu_i)}{6a^3(\mu_i)}\right) = 0,
\end{aligned}$$

with $i=1,\dots,3N-2$, from which we get an implicit complete integral of Hamilton-Jacobi equations for the generating functions $\mathcal{F}_2(\lambda)$ and $\mathcal{F}_3(\lambda)$:

$$S = \sum_{j=1}^{3N-2} S_j(\mu_j) = S(\mu_1,\cdots,\mu_{3N-2}; I_1,\cdots,I_{3N-2}) = \sum_{j=1}^{3N-2}\int_0^{\mu_j} z\,\mathrm{d}\lambda, \tag{5.3}$$

where z satisfies (4.2).

Now let us consider a canonical transformation from the variables $\mu_i, \nu_i, (i=1,\dots,3N-2)$ to variables ϕ_i and $I_i, (i=1,\dots,3N-2)$, generated by the generating function S:

$$\sum_{i=1}^{3N-2}\nu_i \mathrm{d}\mu_i + \sum_{i=1}^{3N-2}\phi_i \mathrm{d}I_i = \mathrm{d}S,$$

that satisfies

$$\nu_i = \frac{\partial S}{\partial \mu_i},\ \phi_i = \frac{\partial S}{\partial I_i},\ i=1,\cdots,3N-2. \tag{5.4}$$

Resorting to Eqs (5.3), (5.4), (4.2) and (5.2), we arrive at

$$\phi_i = \frac{\partial S}{\partial I_i} = \sum_{j=1}^{3N-2}\int_0^{\mu_j}\frac{\partial z}{\partial I_i}\,\mathrm{d}\lambda = \begin{cases}\displaystyle\sum_{j=1}^{3N-2}\int_0^{\mu_j}\frac{(a(\lambda)z - R_1(\lambda))\lambda^{N-i-1}}{R(\lambda)}\,\mathrm{d}\lambda,\ i=1,\dots,N-1,\\ \displaystyle\sum_{j=1}^{3N-2}\int_0^{\mu_j}\frac{\lambda^{3N-i-2}}{R(\lambda)}\,\mathrm{d}\lambda,\ i=N,\dots,3N-2,\end{cases} \tag{5.5}$$

where $R(\lambda) = 3a^2(\lambda)z^2 - 2a(\lambda)R_1(\lambda)z + \frac{1}{2}R_1^2(\lambda) - R_2(\lambda)$. From Eqs (5.1) and (5.2), the generating functions of integrals can be rewritten as

$$\begin{aligned}\mathcal{F}_2(\lambda) &= \sum_{j=1}^{N} \frac{C_{2j}}{(\lambda-\lambda_j)^2} + \frac{\frac{1}{2}\mathrm{tr}\beta^2\lambda^N + I_0\lambda^{N-1} + I_1\lambda^{N-2} + \cdots + I_{N-1}}{a(\lambda)} \\ &:= K_2(I_1, \cdots, I_{N-1}, \lambda),\end{aligned}$$

$$\begin{aligned}\mathcal{F}_3(\lambda) &= \sum_{j=1}^{N} \frac{C_{3j}}{(\lambda-\lambda_j)^3} + \frac{\frac{1}{3}\mathrm{tr}\beta^3\lambda^{2N} + I_{\tilde{0}}\lambda^{2N-1} + I_N\lambda^{2N-2} + \cdots + I_{3N-2}}{a^2(\lambda)} \\ &:= K_3(I_N, \ldots, I_{3N-2}, \lambda).\end{aligned}$$

Functions $I_1, \cdots, I_{3N-2}$ and $\phi_1, \ldots, \phi_{3N-2}$ are variables of action type and the corresponding variables of angles, respectively. In the following, we will use these action-angle variables to discuss the equations of motion for the Lie-Poisson Hamiltonian systems generated by the Lax matrix (3.1). The Hamiltonian canonical equations for the generating functions $\mathcal{F}_2(\lambda)$ and $\mathcal{F}_3(\lambda)$ in terms of action-angle variables I_j and ϕ_j, $j = 1, \ldots, 3N-2$, are as follows

$$\phi_{j,t_{2\lambda}} = \begin{cases} \dfrac{\partial K_2(\lambda)}{\partial I_j} = \dfrac{\lambda^{N-j-1}}{a(\lambda)}, & 1 \le j \le N-1 \\ \dfrac{\partial K_2(\lambda)}{\partial I_j} = 0, & N \le j \le 3N-2 \end{cases}, \tag{5.6}$$

$$I_{j,t_{2\lambda}} = -\frac{\partial K_2(\lambda)}{\partial \phi_j} = 0, \ 1 \le j \le 3N-2, \tag{5.7}$$

$$\phi_{j,t_{3\lambda}} = \begin{cases} \dfrac{\partial K_3(\lambda)}{\partial I_j} = 0, & 1 \le j \le N-1 \\ \dfrac{\partial K_3(\lambda)}{\partial I_j} = \dfrac{\lambda^{3N-j-2}}{a^2(\lambda)}, & N \le j \le 3N-2 \end{cases}, \tag{5.8}$$

$$I_{j,t_{3\lambda}} = -\frac{\partial K_3(\lambda)}{\partial \phi_j} = 0, \ 1 \le j \le 3N-2. \tag{5.9}$$

Let $t_{2,l}$ and $t_{3,l}$ represent the variables of $F_{2,l}$-flow and $F_{3,l}$-flow, respectively. According to the definition of the Lie-Poisson bracket, one infers

$$\left.\begin{aligned} I_{j,t_{2\lambda}} &= \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}}\{I_j, F_{2,l}\} = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}} \frac{\mathrm{d}I_j}{\mathrm{d}t_{2,l}} \\ I_{j,t_{3\lambda}} &= \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}}\{I_j, F_{3,l}\} = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}} \frac{\mathrm{d}I_j}{\mathrm{d}t_{3,l}} \\ \phi_{j,t_{2\lambda}} &= \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}}\{\phi_j, F_{2,l}\} = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}} \frac{\mathrm{d}\phi_j}{\mathrm{d}t_{2,l}} \\ \phi_{j,t_{3\lambda}} &= \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}}\{\phi_j, F_{3,l}\} = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l+1}} \frac{\mathrm{d}\phi_j}{\mathrm{d}t_{3,l}} \end{aligned}\right., \quad j = 1, \ldots, 3N-2. \tag{5.10}$$

Proposition 5.1.

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t_{2,1}},\cdots,\frac{\mathrm{d}\phi}{\mathrm{d}t_{2,N-1}},\frac{\mathrm{d}\phi}{\mathrm{d}t_{3,1}},\cdots,\frac{\mathrm{d}\phi}{\mathrm{d}t_{3,2N-1}}\right)=\begin{pmatrix}Q_{11} & 0\\ 0 & Q_{22}\end{pmatrix},\tag{5.11}$$

with

$$Q_{11}=\begin{pmatrix}1 & A_1 & A_2 & \cdots & A_{N-2}\\ & 1 & A_1 & \cdots & A_{N-3}\\ & & \ddots & \ddots & \vdots\\ & & & 1 & A_1\\ & & & & 1\end{pmatrix},\quad Q_{22}=\begin{pmatrix}1 & B_1 & B_2 & \cdots & B_{2N-2}\\ & 1 & B_1 & \cdots & B_{2N-3}\\ & & \ddots & \ddots & \vdots\\ & & & 1 & B_1\\ & & & & 1\end{pmatrix}$$

where A_k's are the coefficients in the expansion

$$\frac{\lambda^N}{a(\lambda)}=\sum_{k=0}^{\infty}\frac{A_k}{\lambda^k},$$

which could be represented through the power sums of λ_j, that is

$$A_0=1,\ A_1=s_1,\ A_2=\frac{1}{2}(s_2+s_1^2)$$

with the recursive formula

$$A_k=\frac{1}{k}\left(s_k+\sum_{\substack{m,\ n\geq 1\\ m+n=k}}s_mA_n\right),\quad s_k=\sum_{j=1}^{N}\lambda_j^k,$$

and B_r's are obtained by comparing the coefficients of λ^r, $r=0,1,\ldots,$ in equation

$$\frac{\lambda^{2N}}{a^2(\lambda)}=\left(\sum_{k=0}^{\infty}\frac{A_k}{\lambda^k}\right)^2=\sum_{r=0}^{\infty}\frac{B_r}{\lambda^r},$$

where $B_0=A_0^2=1,\ B_1=2A_1,\cdots,B_r=\sum\limits_{\substack{m,n\geq 0\\ m+n=r}}A_mA_n$ with the supplementary definition $A_{-k}=B_{-k}=0$, $k=1,2,\ldots.$

Proof. Using (5.6), (5.8) and (5.10), it is easy to see that

$$\begin{aligned}
&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\{I_j,F_{2,l}\}=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\{I_j,F_{3,l}\}=0,\ j=1,\ldots,3N-2,\\
&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\{\phi_j,F_{2,l}\}=\frac{\lambda^{N-j-1}}{a(\lambda)}=\sum_{k=0}^{\infty}\frac{A_k}{\lambda^{k+j+1}},\ j=1,\ldots,N-1,\\
&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\{\phi_j,F_{3,l}\}=0,\ j=1,\ldots,N-1,\\
&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\{\phi_j,F_{2,l}\}=0,\ j=N,\ldots,3N-2,\\
&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\{\phi_j,F_{3,l}\}=\frac{\lambda^{3N-j-2}}{a^2(\lambda)}=\sum_{k=0}^{\infty}\frac{B_k}{\lambda^{k+j+2-N}},\ j=N\ldots,3N-2.
\end{aligned}\tag{5.12}$$

By comparing the coefficients of λ^{-l-1} in (5.12), we deduce the Poisson brackets

$$\begin{aligned}
&\{I_j, F_{2,l}\} = 0,\ \{I_j, F_{3,l}\} = 0,\ j = 1, \dots, 3N-2,\\
&\{\phi_j, F_{2,l}\} = 0,\ \{\phi_j, F_{3,l}\} = 0,\ j = 1, \dots, 3N-2,\\
&\{\phi_j, F_{2,l}\} = A_{l-j},\ \{\phi_j, F_{3,l}\} = 0,\ j = 1, \dots, N-1,\\
&\{\phi_j, F_{2,l}\} = 0,\ \{\phi_j, F_{3,l}\} = B_{l+N-j-1},\ j = N, \dots, 3N-2.
\end{aligned} \tag{5.13}$$

Thus, the non-degenerate matrix takes on the form (5.11). □

Proposition 5.2. $F_{2,1}, \cdots, F_{2,N-1},\ F_{3,1}, \cdots, F_{3,2N-1}$ *given in* (3.4) *and* (3.5) *are functionally independent.*

Proof. We need only prove the linear independence of the gradients: $\nabla F_{2,1}, \cdots, \nabla F_{2,N-1}$, $\nabla F_{3,1}, \cdots \nabla F_{3,2N-1}$. Suppose

$$\sum_{k=1}^{N-1} c_k\, \nabla F_{2,k} + \sum_{m=1}^{2N-1} c_{N+m-1}\, \nabla F_{3,m} = 0.$$

It is easy to calculate that

$$\begin{aligned}
0 &= \sum_{k=1}^{N-1} c_k \{\phi_j, F_{2,k}\} + \sum_{m=1}^{2N-1} c_{N+m-1} \{\phi_j, F_{3,m}\}\\
&= \sum_{k=1}^{N-1} c_k \frac{\mathrm{d}\phi_j}{\mathrm{d}t_{2,k}} + \sum_{m=1}^{2N-1} c_{N+m-1} \frac{\mathrm{d}\phi_j}{\mathrm{d}t_{3,m}},
\end{aligned}$$

which implies that $c_1 = c_2 = \cdots = c_{3N-2} = 0$ because the coefficient determinant is equal to 1 by (5.11). □

Remark 2. From Corollary 3.1 and the above Proposition, it is proved that the Lie-Poisson Hmailtonian systems (3.8) and (3.9) with the Hamiltonians (3.6) and (3.7) are complete integrable in the Liouville sense because their $3N-2$ integrals $F_{2,1}, \cdots, F_{2,N-1}, F_{3,1}, \cdots, F_{3,2N-1}$ are involutive in pairs and functionally independent.

For given values of the $3N$ Casimir functions in (4.1), $F_{0,l} = \sum_{j=1}^{N} \lambda_j^l C_{1j}$ are constants, which means that $\{\phi_j, F_{0,l}\} = 0$. Based on (5.13) and (3.6), the solution of the Lie-Poisson Hmailtonian system (3.8) in terms of action-angle variables ϕ_j and I_j is

$$I_j(x) = I_j(0),\ \phi_j(x) = \begin{cases} \phi_j(0) - (\gamma_2 \mathrm{tr}\beta + \gamma_1) A_{1-j} x, & j = 1, \dots, N-1,\\ \phi_j(0) + \gamma_2 B_{N-j} x, & j = N, \dots, 3N-2. \end{cases} \tag{5.14}$$

Thus, combining (5.5) with (5.14) give rise to the Jacobi inversion problem

$$\begin{cases} \phi_j(0) - (\gamma_2 \mathrm{tr}\beta + \gamma_1) A_{1-j} x = \displaystyle\sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{(a(\lambda)z - R_1(\lambda))\lambda^{N-j-1}}{R(\lambda)}\, \mathrm{d}\lambda,\ j = 1, \dots, N-1,\\ \phi_j(0) + \gamma_2 B_{N-j} x = \displaystyle\sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{\lambda^{3N-j-2}}{R(\lambda)}\, \mathrm{d}\lambda,\ j = N, \dots, 3N-2. \end{cases}$$

For the Lie-Poisson Hamiltonian system (3.9) with respect to Lie-Poisson bracket (5.13), we obtain by using (3.7) that

$$I_j(t) = I_j(0),\ \phi_j(t) = \begin{cases} \phi_j(0) + A_{1-j}t, & j = 1, \ldots, N-1, \\ \phi_j(0), & j = N, \ldots, 3N-2. \end{cases} \tag{5.15}$$

With the help of (5.5) and (5.15), we deduce the Jacobi inversion problem

$$\begin{cases} \phi_j(0) + A_{1-j}t &= \sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{(a(\lambda)z - R_1(\lambda))\lambda^{N-j-1}}{R(\lambda)}\,\mathrm{d}\lambda,\ j = 1, \ldots, N-1, \\ \phi_j(0) &= \sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{\lambda^{3N-j-2}}{R(\lambda)}\,\mathrm{d}\lambda,\ j = N, \ldots, 3N-2. \end{cases}$$

The compatible solution of Lie-Poisson Hamiltonian systems (3.8) and (3.9) in terms of action-angle variables ϕ_j and I_j is

$$\begin{aligned} & I_j(x,t) = I_j(0,0), \\ & \phi_j(x,t) = \begin{cases} \phi_j(0,0) - (\gamma_2 \mathrm{tr}\beta + \gamma_1)A_{1-j}x + A_{1-j}t, & j = 1, \ldots, N-1, \\ \phi_j(0,0) + \gamma_2 B_{N-j}x, & j = N, \ldots, 3N-2. \end{cases} \end{aligned} \tag{5.16}$$

Thus, by making use of (5.5) and (5.16), we arrive at the Jacobi inversion problem for the TWRI Eq (1.1)

$$\begin{cases} \phi_j(0,0) - (\gamma_2 \mathrm{tr}\beta + \gamma_1)A_{1-j}x + A_{1-j}t = \sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{(a(\lambda)z - R_1(\lambda))\lambda^{N-j-1}}{R(\lambda)}\,\mathrm{d}\lambda,\ j = 1, \ldots, N-1, \\ \phi_j(0,0) + \gamma_2 B_{N-j}x = \sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{\lambda^{3N-j-2}}{R(\lambda)}\,\mathrm{d}\lambda,\ j = N, \ldots, 3N-2. \end{cases}$$

6. Conclusions

In this paper, two finite dimensional Lie-Poisson Hamiltonian systems associated with a 3×3 spectral problem related to three-wave resonant interaction system are presented with the help of nonlinearization method. In the framework of Lie-Poisson structure, it is easier to prove the integrability for these finite-dimensional Lie-Poisson Hamiltonian systems in Liouville sense. $3N$ pairs of separation of variables for these integrable systems with non-hyperelliptic spectral curves are constructed and $3N-2$ pairs of them are proposed by using Sklyanin's method. In addition, apart from the variables $\mu_k,\ \nu_k (k = 1, \cdots, 3N-2)$, we add two pairs of conjugate variables. $3N-2$ pairs of action-angle variables are introduced with the help of Hamilton-Jacobi theory. The Jacobi inversion problems for the these Lie-Poisson Hamiltonian systems and three-wave resonant interaction system are discussed. Furthermore, based upon the Jacobi inversion problems, we may use the algebro-geometric method to get the multi-variable sigma-function solutions, which will be left to a future publication. The methods in this paper can be applied to other systems of soliton hierarchies with 3×3 matrix spectral problems.

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Conflict of interest

The authors declare that they have no competing interests.

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