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*Research article*

# Going beyond the threshold: Blowup criteria with arbitrary large energy in trapped quantum gases

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**Abstract:** The present paper considers the blowup properties in trapped dipolar quantum gases modelled by the Gross-Pitaevskii equation. More precisely, through analyzing the temporal evolution of $J'(t)$ in the form of uncertain principle [1], we provide some invariant evolution flows. Based on that, we establish the global existence versus blowup dichotomy of solutions above the mass-energy threshold. Meanwhile, we can estimate the behaviour of solutions with arbitrary large energy.

**Keywords:** trapped quantum gases; gross-pitaevskii equation; blowup dynamics; mass-energy threshold; arbitrary large energy
**Mathematics Subject Classification:** 35A01, 35D99

## 1. Introduction

The successful implementation of dilute atomic Bose-Einstein condensates has generated considerable attention in the properties of trapped dipolar quantum gases. In traditional experiments, bosonic quantum gases with isotropic and short-range interactions have taken a dominant role and been commendably described by the scattering length of $s$-wave [2]. However, dipolar interactions, provided with an anisotropic and long-range component, are not negligible for those particles with electric dipole moment or large permanent magnetic. In 2005, the first dipolar BEC of chromium atoms was successfully generated by the combination of magnetic, optical and magneto-optical trapping techniques [3]. In the approximate range of mean field, the dipolar quantum gases at zero temperature have been described, and the nonlinear Schrödinger equation of its macroscopic wave function was derived [4].

In this paper, we study the following Gross-Pitaevskii equation (GPE) for the trapped dipolar quantum gases

$$\begin{cases} i\varphi_t = -\frac{1}{2}\triangle\varphi + \frac{|x|^2}{2}\varphi + \beta_1|\varphi|^2\varphi + \beta_2(K * |\varphi|^2)\varphi, \\ \varphi(0,x) = \varphi_0, \quad t \in \mathbb{R}^+, x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Here $\varphi = \varphi(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{C}$ is a given smooth function, $i$ is the imaginary unit, $\triangle$ is the Laplace operator on $\mathbb{R}^3$ and $\beta_1, \beta_2 \in \mathbb{R}$ satisfy certain constraints. Moreover, we define $*$ by the convolution for $x$. The long-range anisotropic dipolar interaction kernel $K(x)$ is highly singular denoted by

$$K(x) = \frac{1 - 3\cos^2\theta}{|x|^3},$$

where $\theta = \theta(x)$ represents the angle between the point $x \in \mathbb{R}^3$ and the fixed polarization axis $n \in \mathbb{R}^3$, with $|n| = 1$, i.e.

$$\cos\theta = \frac{x \cdot n}{|x|}.$$

These remarkable experimental breakthroughs of dipolar interactions have stimulated various theoretical investigations. When $\beta_2 = 0$, there is no dipolar interaction, and Eq (1.1) describes the BEC with alkali atoms. Many results about the dynamical properties were developed [5]. When $\beta_2 \neq 0$, Eq (1.1) is hard to discuss due to the strong singularity of dipolar interactions. By reducing three-dimensional GPE to one or two dimensions, polarized along an arbitrary polarization angle, the researchers in [6] revealed how the dipolar interactions change the contact interactions of strongly constrained atoms. Moreover, Carles, Markowich and Sparber [7] applied the decomposition of $e^{-ix\cdot\xi}$ into spherical functions to calculate the Fourier transform of kernel $K(x)$ with $n = (0,0,1)$.

Recently, Li [8] extended the known result of the existence of blowup solution for Eq (1.1) in terms of mechanical analogy and a new estimate of the kinetic energy. Moreover, the profile decomposition has been employed to explore the blowup dynamic of Eq (1.1) in [9]. The authors constructed two refined Gagliardo-Nirenberg inequalities at first. Then, they proved that the blowup solution would concentrate at one fixed amount with the help of a compactness lemma. This method has also been applied to the focusing Schrödinger-Hartree equation to investigate blowup dynamic in [10]. With the same method, Zhang [11] obtained the threshold of global existence and blowup for the NLSE without dipolar interaction ($\beta_2 = 0$ in Eq (1.1)) through the Hamilton conservation and the variational characteristic of the classical nonlinear scalar field equation. Then, the limiting profile and the mass concentration property of the blowup solution have been discussed. Zhou [12] has proved that any minimizer of the minization problem blows up at one of the endpoints of the major axis for the variational functional associates with the GPE, if the parameter relate to the attractive interaction strength close to a critical value. Pavlovic [13] considered the solutions of the focusing quintic and cubic GP hierarchies. The authors proved that all solutions at the $L^2$-critical or $L^2$-supercritical level blow up in finite time if the energy per particle is negative in the initial condition. It is worth noting that their results do not admit any factorization of the initial date. The readers can refer to other blowup dynamical properties, the stability and instability of standing waves in [14–21]. As for fractional NLSE, Dai [22] has derived symmetric and anti-symmetric solitons of the fractional second and third order NLSE. Another example can be seen in [23], where the authors used two kinds of fractional dual-function methods to solve the space-time fractional Fokas-Lenells equation. The coupled NLSE that contains partially nonlocal nonlinearity has been investigated in [24]. Ma soliton, Akhmediev breather and rogue wave were derived via projecting expression along with Hirota's bilinear method and Darboux transformation. The fractional bi-function method and fractional mapping equation method can be found in [25]. The readers can refer to [26–28] for other related works.

From the point of mathematics, the collapse in a particular space reflects the limited behaviour of solutions, and the occurrence of finite time blowup is closely connected with standing waves of Eq (1.1), i.e., $\varphi(t,x) = e^{i\omega t}Q(x), \omega > 0$. It is obvious that $Q$ is the unique solution of the following elliptic equation

$$-\frac{1}{2}\triangle Q + \omega Q + \frac{|x|^2}{2}Q + \beta_1|Q|^2Q + \beta_2(K * |Q|^2)Q = 0, \quad Q \in H^1(\mathbb{R}^3), \tag{1.2}$$

which will be the main key throughout this paper. Moreover, Eq (1.1) satisfies the conservation laws of mass and energy, i.e.,

$$M[\varphi(t,x)] := \int |\varphi|^2 dx = M[\varphi_0], \tag{1.3}$$

$$E[\varphi(t,x)] := \frac{1}{2}\int |\nabla\varphi|^2 dx + \frac{1}{2}\int |x|^2|\varphi|^2 dx + \frac{\beta_1}{2}\int |\varphi|^4 dx + \frac{\beta_2}{2}\int (K * |\varphi|^2)|\varphi|^2 dx = E[\varphi_0]. \tag{1.4}$$

A crucial question for Eq (1.1) is to find the sharp threshold conditions. Now, we recall some helpful results of blowup dynamics for the nonlinear Schrödinger equation (NLSE): $i\partial_t u + \triangle u + |u|^{p-1}u = 0, u(0,x) = u_0, (t,x) \in \mathbb{R}\times\mathbb{R}^N, p > 1 + \frac{4}{N}$. One useful scaling quantity is called *mass-energy* $M[u]^{\frac{1-S_c}{S_c}}E[u]$ ($S_c = \frac{N}{2} - \frac{2}{p-1}$), which can be normalized as

$$\mathcal{ME} = \frac{M[u]^{\frac{1-S_c}{S_c}}E[u]}{M[W]^{\frac{1-S_c}{S_c}}E[W]}, \quad 0 < S_c \leq 1,$$

where $W$ is the unique $H^1$ radial solution of

$$\Delta W - (1 - S_c)W + |W|^{p-1}W = 0.$$

When $0 < S_c < 1$, we regard $\mathcal{ME} = 1$ as the *mass-energy threshold*. First of all, let us start with recalling some well-known fact at the *mass-energy threshold*, i.e., $\mathcal{ME} = 1$. Duyckaerts and Roudenko [29] begin with exhibiting two radial solutions $Q^+$ and $Q^-$, with initial data $Q_0^{\pm}$ satisfies $Q_0^{\pm} \in \bigcap_{s\in R} H^s(R^3)$, such that $Q^{\pm}$ exponentially approach the ground state solution $Q$ in the positive time direction and $Q^+$ blows up in finite time as well as $Q^-$ scatters in the negative time direction. Then, all the solutions can be characterized as: (i) If $\|\nabla u_0\|_2\|u_0\| < \|\nabla Q\|_2\|Q\|$, then either $u$ scatters or $u = Q^-$ up to the symmetries, (ii) If $\|\nabla u_0\|_2\|u_0\| = \|\nabla Q\|_2\|Q\|$, then $u = e^{it}Q$ up to the symmetries, (iii) If $\|\nabla u_0\|_2\|u_0\| > \|\nabla Q\|_2\|Q\|$ and $u_0$ is radial or of finite variance, then either the interval of existence of $u$ is of finite length or $u = Q^+$ up to the symmetries. As for the cases under the threshold, i.e., $\mathcal{ME} < 1$. The focusing mass-critical NLSE($S_c = 0$) was first studied by Weinstein [30], who showed a sharp splitting takes place: (i) If $M[u] < M[Q]$, then the solution exists globally, (ii) if $M[u] \geq M[Q]$, then the solution blows up in finite time, where $Q$ is the solution of $-Q + \Delta Q + |Q|^{\frac{4}{d}}$, $Q = Q(r)$, $r = |x|$, $x \in R^d$. The focusing energy-critical NLSE($S_c = 1, N = 3,4,5$) with $\dot{H}^1_{rad}$ initial data was investigated by Kenig and Merle [31]. They showed that there exists a sharp threshold, which split the behaviours of solutions into two cases under a priori condition $E[u_0] < E[W]$: (i) If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then the solution exists globally and scatters, (ii) if $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ and $u_0 \in L^2$, then the solution may blows up in finite time. Here, $W(x) = W(x,t) = 1/(1 + \frac{|x|}{N(N-2)})^{\frac{N-2}{2}}$ in $H^1(R^N)$ and solves $\Delta W + |W|^{\frac{4}{N-2}} W = 0$. Briefly speaking, the global behaviour of solutions ($0 < \mathcal{ME} \leq 1$) is wholly investigated, which can be summarized as

*Let $u_0 \in H^1(\mathbb{R}^3)$, $0 < S_c < 1$, $u(t,x)$ be the solution of $i\partial_t u + \triangle u + |u|^{p-1}u = 0$ and the corresponding initial datum $u_0$ satisfy*

$$0 < \mathcal{ME} < 1.$$

***Part 1*** *If $M[u_0]^{1-S_c}(\int |u_0|^{p+1}dx)^{S_c} < M[W]^{1-S_c}(\int |W|^{p+1}dx)^{S_c}$, then the solution $u(t,x)$ exists globally.*
***Part 2*** *If $M[u_0]^{1-S_c}(\int |u_0|^{p+1}dx)^{S_c} > M[Q]^{1-S_c}(\int |W|^{p+1}dx)^{S_c}$, either the solution $u(t,x)$ blows up in finite time or there exists a sequence $t_n \to +\infty$ such that $lim_{n\to+\infty}\|\nabla u(t_n)\|_{L^2} = +\infty$.*

When $S_c = 1$ or $p = \frac{4}{N-2} + 1, N \geq 3$, we regard $\mathcal{ME} = 1$ as the *energy threshold* and the dynamical behavior of solutions is described as

*Let $u_0 \in \dot{H}^1(\mathbb{R}^N)$, $S_c = 1$, $u(t,x)$ be the solution of $i\partial_t u + \triangle u + |u|^{p-1}u = 0$ and the corresponding initial datum $u_0$ satisfy*

$$0 < \mathcal{ME} < 1.$$

***Part 1*** *If $\int |u_0|^{p+1}dx < \int |W|^{p+1}dx$ and $u$ is radial with $N = 3, 4$, then the solution $u(t,x)$ exists globally.*
***Part 2*** *If $\int |u_0|^{p+1}dx > \int |W|^{p+1}dx$ and $u_0$ is radial with $u_0 \in L^2(\mathbb{R}^N)$, then the solution $u(t,x)$ blows up in finite time.*

**Remark 1.** *Both the above cases used a concentration compactness argument, proposed by Kenig and Merle [31] in the energy critical case.*

**Remark 2.** *Motivated by "critical phenomena" in physics, Nakanish and Schlag [32] gave a complete picture of the dynamical properties for the focusing nonlinear Klein-Gordon equation, as initial datum energy slightly larger than that of ground state (still denoted by W):*

$$\mathcal{H}^\varepsilon := \{|x|u \in L^2(\mathbb{R}^3)|E[u] < E[W] + \varepsilon^2\}.$$

*Then, they extended this approach to the focusing cubic NLSE, which is slightly above the* mass-energy threshold *($\mathcal{ME} < 1 + \varepsilon$), and above condition turns to*

$$\bar{\mathcal{H}}^\varepsilon := \{|x|u \in L^2(\mathbb{R}^3)|M[u]E[u] < M[W](E[W] + \varepsilon^2)\}.$$

As we can see, the mechanism of global existence and blowup for the focusing NLSE have already been considered and fully studied below *mass-energy threshold*, or at the *mass-energy threshold*. However, the case above the *mass-energy threshold* is mostly open. The purpose of our paper is to derive sharp criteria for global existence and blowup of Cauchy problem (1.1), above the *mass-energy threshold*, which are not necessarily "$\varepsilon^2$-close" to it. Our main result is demonstrated in Theorem 1. Briefly speaking, under the same manners of Duyckaerts and Roudenko [33], we study the global existence and blowup for Eq (1.1) under the condition of

$$\mathcal{ME} := \frac{M[\varphi_0]E[\varphi_0]}{M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2dx}{2}\right)} \geq 1. \tag{1.5}$$

As a result, we have derived the criterion for the global existence and blowup of solution for Eq (1.1), above the *mass-energy threshold* and such criterion is sharp (Theorem 1). Based on Theorem 1, we are able to predict the dynamical behavior of certain solution that possesses arbitrary large energy. Moreover, we show the relation of two methods, one is the technique used in Theorem 1, the other is given by establishing cross-constrained invariant sets in [34]. The structure of this paper is as follows: In Section 2, we give some valuable preliminaries which will be used in the following work. Section 3 constructs the invariant evolution flows generated by the Cauchy problem (1.1). In Section 4, Theorem 1 and Corollary 1 have been proposed and proved for the existence of blowup solutions with arbitrary initial energy. As for the proof, we give a new calculation of $J'(t)$ in the form of an uncertain principle. Then, we show by contradiction that Theorem 1 stands. Furthermore, Corollary 1 implies we can deduce the behaviour of solutions with arbitrary large energy. Concerning the complementary case of (1.5), Ma and Wang established it in [34], i.e.,

*Let $\varphi_0 \in H^1(\mathbb{R}^3), |x|\varphi_0 \in L^2(\mathbb{R}^3)$ and $\varphi(t,x)$ be the solution of the Cauchy problem (1.1) corresponding to the initial datum $\varphi_0$ satisfying*

$$M[\varphi_0]E[\varphi_0] < M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2dx}{2}\right).$$

***Part 1** (Blowup) If*

$$M[\varphi_0]\left(-\beta_1\int|\varphi_0|^4dx - \beta_2\int(K*|\varphi_0|^2)|\varphi_0|^2dx\right) > M[Q]\left(-\beta_1\int|Q|^4dx - \beta_2\int(K*|Q|^2)|Q|^2dx\right),$$

*then the solution $\varphi(t,x)$ blows up in finite time.*

***Part 2** (Global existence) If*

$$M[\varphi_0]\left(-\beta_1\int|\varphi_0|^4dx - \beta_2\int(K*|\varphi_0|^2)|\varphi_0|^2dx\right) < M[Q]\left(-\beta_1\int|Q|^4dx - \beta_2\int(K*|Q|^2)|Q|^2dx\right),$$

*then the solution $\varphi(t,x)$ exists globally.*

**Remark 3.** *Actually, the sharp criterion by Ma and Wang [34], obtained by establishing cross-constrained invariant sets, is equivalent to our method and we will show it in Section 2.*

## 2. Preliminaries

We provide some useful preliminaries in this section. Throughout the paper, we denote $L^p(\mathbb{R}^3)$, $H^1(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3}\cdot dx$ as $L^p$, $H^1$ and $\int\cdot dx$, respectively. In general, we take $n=(0,0,1)$, then $K(x)$ can be expressed as

$$K(x) = \frac{x_1^2+x_2^2-2x_3^2}{|x|^5}.$$

Define the Fourier transform in $\mathbb{R}^3$ as

$$\hat{u}(\xi) := \int u(x)e^{-ix\cdot\xi}dx,\ u\in\mathcal{S}(\mathbb{R}^3),$$

where $\mathcal{S}$ is the Schwartz space. The Fourier transform of $K(x)$ is taken by

$$\hat{K}(\xi) = \frac{4\pi}{3}\left(\frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2}\right).$$

The calculation procedure of $\hat{K}(\xi)$ has been studied carefully in [7].

Moreover, we denote the energy space as

$$\Sigma := \left\{u \in H^1 | \int |x|^2|u|^2 dx < +\infty\right\}$$

with the corresponding norm $\|u\|_\Sigma^2 = \int (|u|^2 + |\nabla u|^2 + |x|^2|u|^2)dx$.

Next, we recall the local well-posedness of Eq (1.1), which is the main key of Carles, Markowich and Sparber in [7].

**Lemma 1.** *( [7]) Let $\varphi_0 \in \Sigma$, $\beta_1, \beta_2 \in \mathbb{R}$. Then there exists a unique solution $\varphi(t,x)$ of the Cauchy problem (1.1) on the maximal time interval $[0, T^*)$ such that*

$$\varphi \in \left\{\varphi \in C([0,T^*];\Sigma);\ \varphi, \nabla\varphi, x\varphi \in C([0,T^*];L^2) \cap L^{\frac{8}{3}}([0,T^*];L^4)\right\}$$

*and either $T^* = +\infty$(global existence), or else $0 < T^* < +\infty$ and $\lim_{t\to T^*} \|\nabla\varphi(t,x)\|_{L^2} = +\infty$ (blowup).*

**Lemma 2.** *( [21, 35]) Let $\varphi_0 \in \Sigma$ and $\varphi(t,x)$ be a solution of the Cauchy problem (1.1) on $[0,T^*)$. We set up a variable*

$$J(t) := \int |x|^2|\varphi|^2 dx.$$

*Then the following identities stand:*

$$J'(t) = 2\mathrm{Im}\int x\overline{\varphi} \cdot \nabla\varphi dx, \tag{2.1}$$

$$J''(t) = 2\int |\nabla\varphi|^2 dx - 2\int |x|^2|\varphi|^2 dx + 3\beta_1 \int |\varphi|^4 dx + 3\beta_2 \int (K * |\varphi|^2)|\varphi|^2 dx. \tag{2.2}$$

Combining Lemmas 1 and 2, a straightforward computation shows that

$$-\beta_1 \int |\varphi|^4 dx - \beta_2 \int (K * |\varphi|^2)|\varphi|^2 dx = 4E[\varphi_0] - J''(t) - 4\int |x|^2|\varphi|^2 dx, \tag{2.3}$$

and

$$\int |\nabla\varphi|^2 dx = 6E[\varphi_0] - J''(t) - 5\int |x|^2|\varphi|^2 dx. \tag{2.4}$$

Next, we recall the refined Gagliardo-Nirenberg inequality constructed by Antonelli and Sparber in [35].

**Lemma 3.** *( [21, 35]) Let $\beta_1, \beta_2 \in \mathbb{R}$ such that the following condition stands:*

$$\beta_1 < \begin{cases} \frac{4\pi}{3}\beta_2, & \text{if } \beta_2 > 0; \\ -\frac{8\pi}{3}\beta_2, & \text{if } \beta_2 < 0. \end{cases} \tag{2.5}$$

*Then, for any $f \in H^1$, there exists a positive constant $C_Q$ such that*

$$-\beta_1 \int |f|^4 dx - \beta_2 \int (K * |f|^2)|f|^2 dx \leq C_Q \left( \int |f|^2 dx \right)^{\frac{1}{2}} \left( \int |\nabla f|^2 dx \right)^{\frac{3}{2}}, \tag{2.6}$$

*where the optimal constant $C_Q = C_*$ is given by*

$$C_* = \frac{-\beta_1 \int |Q|^4 dx - \beta_2 \int (K * |Q|^2)|Q|^2 dx}{(\int |Q|^2 dx)^{\frac{1}{2}} (\int |\nabla Q|^2 dx)^{\frac{3}{2}}} \tag{2.7}$$

*with $Q$ being the solution of the following nonlinear elliptic equation*

$$-\frac{1}{2}\triangle Q + \omega Q + \beta_1 |Q|^2 Q + \beta_2 (K * |Q|^2) Q = 0, \quad Q \in H^1(\mathbb{R}^3). \tag{2.8}$$

Multiplying (2.8) by $x \cdot \nabla Q$ and $Q$ respectively, we derive the following identities:

$$-\beta_1 \int |Q|^4 dx - \beta_2 \int (K * |Q|^2)|Q|^2 dx = \frac{2}{3} \int |\nabla Q|^2 dx, \tag{2.9}$$

$$\int |Q|^2 dx = \frac{1}{6\omega} \int |\nabla Q|^2 dx. \tag{2.10}$$

Thus, we can rewrite $C_Q$ as

$$C_Q = \frac{2}{3^{\frac{3}{2}}} \frac{(2E[Q] - \int |x|^2 |Q|^2 dx)^{-\frac{1}{2}}}{(M[Q])^{\frac{1}{2}}}. \tag{2.11}$$

**Remark 4.** *It may be an mistake that causes the constraint condition of $\beta_1$ and $\beta_2$ to be wrong in [35]. Here, we put forward the correct one as shown by (2.5), which leads to*

$$\beta_1 \int |f|^4 dx + \beta_2 \int (K * |f|^2)|f|^2 dx = (2\pi)^{-3} \int (\beta_1 + \beta_2 \hat{K}) \widehat{|f|^2}^2 d\xi < 0, \quad f \in H^1$$

*with the assistance of Parseval formula.*

**Proposition 1.** *[33] Let $f \in \Sigma$. Then*

$$\left( \mathrm{Im} \int x\overline{f} \cdot \nabla f dx \right)^2 \leq \int |x|^2 |f|^2 dx \left[ \int |\nabla f|^2 dx - \frac{(-\beta_1 \int |f|^4 dx - \beta_2 \int (K * |f|^2)|f|^2 dx)^{\frac{2}{3}}}{(C_Q)^{\frac{2}{3}} (\int |f|^2 dx)^{\frac{1}{3}}} \right]. \tag{2.12}$$

*Proof.* This proof keeps consistent with that in [33]. We prove it to preserve integrity. It is obvious to check that $e^{i\delta|x|^2} f \in \Sigma$ under the condition of $f \in \Sigma$, where $\delta \in \mathbb{R}$. Applying (2.6) to $e^{i\delta|x|^2} f$, we derive that

$$\begin{aligned} \int \left| \nabla e^{i\delta|x|^2} f \right|^2 dx &= \int |\nabla f|^2 dx + 4\delta \mathrm{Im} \int x\overline{f} \cdot \nabla f dx + 4\delta^2 \int |x|^2 |f|^2 dx. \\ &\geq \frac{(-\beta_1 \int |f|^4 dx - \beta_2 \int (K * |f|^2)|f|^2 dx)^{\frac{2}{3}}}{(C_Q)^{\frac{2}{3}} (\int |f|^2 dx)^{\frac{1}{3}}}. \end{aligned}$$

For each $\delta \in \mathbb{R}$, we have

$$(C_Q)^{\frac{2}{3}}\left(\int |f|^2 dx\right)^{\frac{1}{3}}\left(\int |\nabla f|^2 dx + 4\delta \mathrm{Im}\int x\overline{f}\cdot\nabla f dx + 4\delta^2\int |x|^2|f|^2 dx\right)$$
$$-\left(-\beta_1\int |f|^4 dx - \beta_2\int (K*|f|^2)|f|^2 dx\right)^{\frac{2}{3}} \geq 0,$$

where the left of above inequality is a quadratic polynomial of $\delta$. The discriminant of this polynomial is non-positive, which directly deduce the result of Proposition 1. □

Combining Lemma 2 with Proposition 1, we obtain

$$(J'(t))^2 \leq 4J(t)\left(\int |\nabla\varphi|^2 dx - \frac{(-\beta_1\int |\varphi|^4 dx - \beta_2\int (K*|\varphi|^2)|\varphi|^2 dx)^{\frac{2}{3}}}{(C_Q)^{\frac{2}{3}}(\int |\varphi|^2 dx)^{\frac{1}{3}}}\right).$$

Setting $V(t) = \sqrt{J(t)}$, we have

$$V'(t) = \frac{J'(t)}{2\sqrt{J(t)}}.$$

Consequently, we infer that

$$\begin{aligned}(V'(t))^2 &\leq \int |\nabla\varphi|^2 dx - \frac{(-\beta_1\int |\varphi|^4 dx - \beta_2\int (K*|\varphi|^2)|\varphi|^2 dx)^{\frac{2}{3}}}{(C_Q)^{\frac{2}{3}}(\int |\varphi|^2 dx)^{\frac{1}{3}}}\\ &= 6E[\varphi_0] - J''(t) - 5\int |x|^2|\varphi|^2 dx - \frac{(4E[\varphi_0] - J''(t) - 4\int |x|^2|\varphi|^2 dx)^{\frac{2}{3}}}{(C_Q)^{\frac{2}{3}}(M(\varphi_0))^{\frac{1}{3}}}\\ &< 6E[\varphi_0] - J''(t) - 4\int |x|^2|\varphi|^2 dx - \frac{(4E[\varphi_0] - J''(t) - 4\int |x|^2|\varphi|^2 dx)^{\frac{2}{3}}}{(C_Q)^{\frac{2}{3}}(M(\varphi_0))^{\frac{1}{3}}}\\ &= F\left(J''(t) + 4\int |x|^2|\varphi|^2 dx\right),\end{aligned} \tag{2.13}$$

where

$$F(S) := 6E[\varphi_0] - S - \frac{1}{(C_Q)^{\frac{2}{3}}(M[\varphi_0])^{\frac{1}{3}}}(4E[\varphi_0] - S)^{\frac{2}{3}}, \quad S \in (-\infty, 4E[\varphi_0]].$$

It is obvious that $F(S)$ is decreasing on $(-\infty, S^*)$, increasing on $(S^*, 4E[\varphi_0]]$, where $S^*$ is written as

$$S^* = 4E[\varphi_0] - \left(\frac{2}{3}\right)^3 \frac{1}{(C_Q)^2 M[\varphi_0]}. \tag{2.14}$$

and

$$F(S^*) = \frac{S^*}{2}. \tag{2.15}$$

Substituting (2.11) into (2.14), we obtain

$$S^* = 4E[\varphi_0] - \left(\frac{2}{3}\right)^3 \frac{1}{(C_Q)^2 M[\varphi_0]} = 4E[\varphi_0] - \frac{2M[Q](2E[Q] - \int |x|^2|Q|^2 dx)}{M[\varphi_0]}.$$

As a consequence, the following identity holds:

$$M[\varphi_0]\left(2E[\varphi_0]-\frac{S^*}{2}\right)=M[Q]\left(2E[Q]-\int|x|^2|Q|^2dx\right). \tag{2.16}$$

At the end of this section, we will show the equivalence relation of these two techniques mentioned in Introduction. In [34], the researchers considered the following functionals and invariant sets on the space $\Sigma$:

$$\begin{aligned}
S[\varphi]&=\frac{\omega}{2}\int|\varphi|^2dx+\frac{1}{4}\int|\nabla\varphi|^2dx+\frac{1}{4}\int|x|^2|\varphi|^2dx+\frac{\beta_1}{4}\int|\varphi|^4dx+\frac{\beta_2}{4}\int(K*|\varphi|^2)|\varphi|^2dx\\
&=\frac{\omega}{2}M[\varphi_0]+\frac{1}{2}E[\varphi_0],\\
R[\varphi]&=\frac{1}{2}\int|\nabla\varphi|^2dx+\frac{1}{2}\int|x|^2|\varphi|^2dx+\frac{3}{4}\beta_1\int|\varphi|^4dx+\frac{3}{4}\beta_2\int(K*|\varphi|^2)|\varphi|^2dx\\
&=E[\varphi_0]+\frac{\beta_1}{4}\int|\varphi|^4dx+\frac{\beta_2}{4}\int(K*|\varphi|^2)|\varphi|^2dx,\\
K^+&=\{\varphi\in\Sigma:S[\varphi]<m,R[\varphi]>0\},\\
K^-&=\{\varphi\in\Sigma:S[\varphi]<m,R[\varphi]<0\}.
\end{aligned}$$

**Remark 5.** *In [34], it failed to calculate the exact value of the upper bounded of mass and energy for Eq (1.1). Delightedly, Huang and Zhang [21] derived that value represented by m equaled to* $\frac{1}{6}\|\nabla Q\|_2^2$.

**Proposition 2.** *(Equivalence of two descriptions) Let* $\varphi$ *be in* $\Sigma$*. Then*

$(a)\varphi\in K^+(K^-)\Leftrightarrow$

$$(b)\begin{cases}M[\varphi]E[\varphi]<M[Q]\left(E[Q]-\dfrac{\int|x|^2|Q|^2dx}{2}\right)\\ M[\varphi]\left(-\beta_1\int|\varphi|^4dx-\beta_2\int(K*|\varphi|^2)|\varphi|^2dx\right)<(>)M[Q]\left(-\beta_1\int|Q|^4dx-\beta_2\int(K*|Q|^2)|Q|^2dx\right).\end{cases}$$

*Proof of Proposition 2*: We only show the case $\varphi\in K^+$, and similar argument is applicable to $\varphi\in K^-$. Assume $\varphi\in K^+$, according to the Young's inequality, we derive

$$\frac{1}{6}\|\nabla Q\|_2^2>\frac{\omega}{2}M[\varphi]+\frac{1}{2}E[\varphi]\geq(\omega M[\varphi])^{\frac{1}{2}}(E[\varphi])^{\frac{1}{2}}=(\omega M[\varphi]E[\varphi])^{\frac{1}{2}}.$$

Recalling the identities (2.9) and (2.10), we know

$$\frac{1}{6}\|\nabla Q\|_2^2=\left(\omega M[Q]\left(E[Q]-\frac{\int|x|^2|Q|^2dx}{2}\right)\right)^{\frac{1}{2}},$$

Thus, the first inequality in (b) is derived.

Also by Young' inequality, we have

$$\begin{aligned}\frac{1}{6}\|\nabla Q\|_2^2 &= \frac{1}{2}\left(\omega M[Q]\left(-\beta_1\int|Q|^4dx-\beta_2\int(K*|Q|^2)|Q|^2dx\right)\right)^{\frac{1}{2}}\\ &> \frac{\omega}{2}M[\varphi]+\frac{1}{2}E[\varphi]\\ &> \frac{\omega}{2}M[\varphi]-\frac{\beta_1}{8}\int|\varphi|^4dx-\frac{\beta_2}{8}\int(K*|\varphi|^2)|\varphi|^2dx\\ &\geq (\omega M[\varphi])^{\frac{1}{2}}\left(-\frac{\beta_1}{4}\int|\varphi|^4dx-\frac{\beta_2}{4}\int(K*|\varphi|^2)|\varphi|^2dx\right)^{\frac{1}{2}},\end{aligned}$$

Thus, the second inequality in (b) is derived.

Now, we turn to deduce (b) ⇒ (a). Notice that (b) is maintained under the scaling

$$\varphi_\nu = \nu\varphi(\nu^2 t, \nu x),$$

and that $M[\varphi_\nu] = \nu^{-1}M[\varphi]$, $E[\varphi_\nu] = \nu E[\varphi]$. By Young' inequality,

$$\frac{\omega}{2}M[\varphi_\nu]+\frac{1}{2}E[\varphi_\nu] \geq (\omega M[\varphi_\nu]E[\varphi_\nu])^{\frac{1}{2}},$$

where the inequality holds if and only if

$$\omega M[\varphi_{\nu_0}] = E[\varphi_{\nu_0}] \;\Leftrightarrow\; \varphi_{\nu_0} = \left(\frac{\omega M[\varphi]}{E[\varphi]}\right)^{\frac{1}{2}}.$$

Substituting $\nu_0$ into the equality, we derive

$$S[\varphi] = S[\varphi_{\nu_0}] = \frac{\omega}{2}M[\varphi_{\nu_0}]+\frac{1}{2}E[\varphi_{\nu_0}] = (\omega M[\varphi]E[\varphi])^{\frac{1}{2}} < \left(\omega M[Q]\left(E[Q]-\frac{\int|x|^2|Q|^2dx}{2}\right)\right)^{\frac{1}{2}} = m.$$

$R[\varphi] > 0$ is directly held by using the refined Gagliardo-Nirenberg inequality, which completes the proof of the Proposition 2.

## 3. Invariant evolution flow

In this section, we provide two invariant evolution flows generated by the Cauchy problem (1.1). As a matter of convenience, we define

$$G_+ := \left\{\varphi\in\Sigma | J''(t)+4\int|x|^2|\varphi|^2dx < S^*\right\},$$

$$G_- := \left\{\varphi\in\Sigma | J''(t)+4\int|x|^2|\varphi|^2dx > S^*\right\},$$

where $S^*$ is given by (2.14). We have the following propositions.

**Proposition 3.** *Let* $V'(0) \leq -\sqrt{F(S^*)}$*. Then* $G_+$ *is invariant evolution flows generated by the Cauchy problem (1.1). More specifically, if* $\varphi_0 \in G_+$*, then for all* $t \in [0, T^*)$ *the solution* $\varphi(t, x)$ *corresponding to the initial datum* $\varphi_0$ *still satisfies* $\varphi(t, x) \in G_+$*.*

*Proof.* Supposing $\varphi_0 \in G_+$ and $\varphi(t, x)$ is the unique solution of the Eq (1.1) corresponding to the initial datum $\varphi_0$. According to the definition of $V(t)$, we have

$$V''(t) = \frac{1}{V(t)}\left(\frac{J''(t)}{2} - (V'(t))^2\right). \tag{3.1}$$

Consequently, the assumption $V'(0) \leq -\sqrt{F(S^*)}$ yields that

$$V''(0) = \frac{1}{V(0)}\left(\frac{J''(0)}{2} - (V'(0))^2\right) \leq \frac{J''(0) - S^*}{2V(0)} < -\frac{2\int |x|^2|\varphi_0|^2 dx}{V(0)} < 0. \tag{3.2}$$

Next, we will show by contradiction that

$$\forall t \in [0, T^*), \quad V''(t) < 0. \tag{3.3}$$

Assume that (3.3) does not hold, then there exists a time $t_0 \in (0, T^*)$ such that $V''(t_0) \geq 0 (t_0 \neq 0$ for $V''(0) < 0)$. By continuity, we can find a time $t_a \in (0, t_0)$ such that

$$\forall t \in [0, t_a), \quad V''(t) < 0, \quad V''(t_a) = 0. \tag{3.4}$$

By $V'(0) \leq -\sqrt{F(S^*)}$, we have

$$\forall t \in (0, t_a], \quad V'(t) < V'(0) \leq -\sqrt{F(S^*)}. \tag{3.5}$$

Hence, $(V'(t))^2 \geq F(S^*)$, which connected with (2.13), reveals

$$\forall t \in (0, t_a], \quad F\left(J''(t) + 4\int |x|^2|\varphi|^2 dx\right) > F(S^*). \tag{3.6}$$

As a consequence, $J''(t) + 4\int |x|^2|\varphi|^2 dx \neq S^*$ for all $t \in [0, t_a]$. Due to $\varphi_0 \in G_+$ and by continuity,

$$\forall t \in [0, t_a], \quad J''(t) + 4\int |x|^2|\varphi|^2 dx < S^*. \tag{3.7}$$

Thus, we derive that

$$V''(t_a) = \frac{1}{V(t_a)}\left(\frac{J''(t_a)}{2} - (V'(t_a))^2\right) < -\frac{2\int |x|^2|\varphi(t_a)|^2 dx}{V(t_a)} < 0. \tag{3.8}$$

Hence, (3.3) holds and indicates that

$$\forall t \in [0, T^*), \quad F\left(J''(t) + 4\int |x|^2|\varphi|^2 dx\right) > F(S^*). \tag{3.9}$$

According to the continuity and monotony of $F(S)$, we obtain

$$\forall t \in [0, T^*), \quad J''(t) + 4\int |x|^2|\varphi|^2 dx < S^*, \tag{3.10}$$

which completes the proof of Proposition 3. □

**Proposition 4.** *Let $\varphi_0 \in G_-$ and $\varphi(t,x)$ be the solution of the Cauchy problem (1.1) corresponding to the initial datum $\varphi_0$. Suppose that there exist a time $t_0 \geq 0$ and a small enough parameter $\varepsilon > 0$ such that*

$$V'(t_0) \geq \sqrt{F(S^*)} + 2\varepsilon, \tag{3.11}$$

*then we have*

$$\forall t \in [t_0, T^*), \quad V'(t) > \sqrt{F(S^*)} + \varepsilon. \tag{3.12}$$

*Proof.* We will prove it by contradiction. Suppose that (3.12) dose not stand, and set

$$t_b = \inf\left\{t \in [t_0, T^*) : V'(t) \leq \sqrt{F(S^*)} + \varepsilon\right\}. \tag{3.13}$$

It is obvious (3.11) and (3.13) imply $t_b > t_0$. By continuity,

$$V'(t_b) = \sqrt{F(S^*)} + \varepsilon \tag{3.14}$$

and

$$\forall t \in [t_0, t_b], \quad V'(t) \geq \sqrt{F(S^*)} + \varepsilon. \tag{3.15}$$

Combining (2.13) and (3.15), we have

$$\forall t \in [t_0, t_b], \quad \left(\sqrt{F(S^*)} + \varepsilon\right)^2 \leq (V'(t))^2 \leq F\left(J''(t) + 4\int |x|^2|\varphi|^2 dx\right). \tag{3.16}$$

As a consequence,

$$\forall t \in [t_0, t_b], \quad F(S^*) < F\left(J''(t) + 4\int |x|^2|\varphi|^2 dx\right).$$

Thus, we have $S^* \neq J''(t) + 4\int |x|^2|\varphi|^2 dx$. In virtue of $\varphi_0 \in G_-$ and the continuity of $F(S)$,

$$\forall t \in [t_0, t_b], \quad J''(t) + 4\int |x|^2|\varphi|^2 dx > S^*.$$

Next, we show that there exists a positive constant $M$ satisfying

$$\forall t \in [t_0, t_b], \quad J''(t) + 4\int |x|^2|\varphi|^2 dx \geq S^* + \frac{\sqrt{\varepsilon}}{M}. \tag{3.17}$$

As a matter of fact, by the Taylor's expansion of $F(S)$ at $S = S^*$, there exist $\delta > 0$ and $\lambda > 0$ such that

$$|S - S^*| \leq \delta \quad \Rightarrow \quad F(S) \leq F(S^*) + \lambda(S - S^*)^2. \tag{3.18}$$

If $J''(t) + 4\int |x|^2|\varphi|^2 dx \geq S^* + \delta$, then (3.17) holds as long as $M$ is large enough. If $S^* < J''(t) + 4\int |x|^2|\varphi|^2 dx \leq S^* + \delta$, then by (3.16) and (3.18), we obtain

$$\left(\sqrt{F(S^*)} + \varepsilon\right)^2 \leq (V'(t))^2 \leq F\left(J''(t) + 4\int |x|^2|\varphi|^2 dx\right) \leq F(S^*) + \lambda(V'(t) - S^*)^2,$$

thus we derive (3.17) with $M = \sqrt{\frac{\lambda}{2}}(F(S^*))^{-\frac{1}{4}}$.

However, by (3.1) and (3.14) we deduce

$$\begin{aligned} V''(t_b) &= \frac{1}{V(t_b)}\left(\frac{J''(t_b)}{2} - (V'(t_b))^2\right) \\ &\geq \frac{1}{V(t_b)}\left(\frac{S^*}{2} + \frac{\sqrt{\varepsilon}}{2M} - (\sqrt{F(S^*)} + \varepsilon)^2\right) \\ &\geq \frac{1}{V(t_b)}\left(\frac{\sqrt{\varepsilon}}{2M} - 2\varepsilon\sqrt{F(S^*)} - \varepsilon^2\right) \\ &> 0, \end{aligned}$$

if $\varepsilon$ is small enough, which contradicts to (3.14) and (3.15). Thus we complete the proof of Proposition 4. □

## 4. Blowup criteria

In this section, we construct the blowup versus global existence dichotomy for the Cauchy problem (1.1), which demonstrated by invariant evolution flows and propositions derived in Section 3. Moreover, we can deduce the behaviour of solutions with arbitrary large energy.

**Theorem 1.** *Let $\varphi_0 \in \Sigma$ and $\varphi(t,x)$ be the solution of the Cauchy problem (1.1) corresponding to the initial datum $\varphi_0$. Assume*

$$M[\varphi_0]E[\varphi_0] \geq M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2dx}{2}\right), \tag{4.1}$$

$$\frac{M[\varphi_0]E[\varphi_0]}{M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2dx}{2}\right)}\left(1 - \frac{(J'(0))^2}{8E[\varphi_0]J(0)}\right) \leq 1. \tag{4.2}$$

***Part 1*** *(Blowup) If*

$$M[\varphi_0]\left(-\beta_1\int|\varphi_0|^4dx - \beta_2\int(K*|\varphi_0|^2)|\varphi_0|^2dx\right) > M[Q]\left(-\beta_1\int|Q|^4dx - \beta_2\int(K*|Q|^2)|Q|^2dx\right) \tag{4.3}$$

*and*

$$J'(0) \leq 0, \tag{4.4}$$

*then the solution $\varphi(t,x)$ blows up in finite time, $T^* < +\infty$.*
***Part 2*** *(Global existence) If*

$$M[\varphi_0]\left(-\beta_1\int|\varphi_0|^4dx - \beta_2\int(K*|\varphi_0|^2)|\varphi_0|^2dx\right) < M[Q]\left(-\beta_1\int|Q|^4dx - \beta_2\int(K*|Q|^2)|Q|^2dx\right) \tag{4.5}$$

*and*

$$J'(0) \geq 0, \tag{4.6}$$

*then the solution $\varphi(t,x)$ exists globally. Moreover,*

$$\limsup_{t\to T^*} M[\varphi]\left(-\beta_1\int|\varphi|^4dx - \beta_2\int(K*|\varphi|^2)|\varphi|^2dx\right) < M[Q]\left(-\beta_1\int|Q|^4dx - \beta_2\int(K*|Q|^2)|Q|^2dx\right). \tag{4.7}$$

*Proof.* In view of (2.16), it is obvious to check that (4.1) is equivalent to

$$S^* \geq 0 \tag{4.8}$$

and (4.2) is equivalent to

$$(V'(0))^2 \geq \frac{S^*}{2} = F(S^*). \tag{4.9}$$

Part 1. Notice that (4.4) means exactly $V'(0) \leq 0$, combining with (4.9), we have

$$V'(0) \leq -\sqrt{F(S^*)}. \tag{4.10}$$

In view of (2.3) and (2.9), the assumption (4.3) is equivalent to

$$M[\varphi_0]\left(4E[\varphi_0] - J''(0) - 4\int |x|^2|\varphi_0|^2 dx\right) > M[Q]\left(4E[Q] - 2\int |x|^2|\varphi|^2 dx\right),$$

that is, by (2.16),

$$J''(0) + 4\int |x|^2|\varphi_0|^2 dx < S^*. \tag{4.11}$$

From the proof of the Proposition 3, we derive that

$$\forall t \in [0, T^*), \quad V''(t) < 0. \tag{4.12}$$

Assuming that $T^* = +\infty$. It follows that for all $t \geq 0$, there exists a constant $m$ such that

$$V''(t) \leq m < 0.$$

Thus, we derive

$$\begin{aligned} V(t) &= V(0) + V'(0)t + \int_0^t V''(\tau)(t-\tau)d\tau \\ &\leq V(0) + V'(0)t + \frac{m}{2}t^2. \end{aligned}$$

As a consequence, we have $\lim_{t\to+\infty} V(t) < 0$ which contradicts to the fact that $V(t)$ is positive. then the solution $\varphi(t, x)$ blows up in finite time, $T^* < +\infty$.

Part 2. A short calculation revealed that these assumptions in Part 2 could be replaced by the following inequalities

$$V'(0) \geq 0, \tag{4.13}$$

$$J''(0) + 4\int |x|^2|\varphi_0|^2 dx > S^*. \tag{4.14}$$

Notice that there exists $t_0 \geq 0$ such that

$$V'(t_0) > \sqrt{F(S^*)}. \tag{4.15}$$

In fact, according to (4.9) and (4.13), we have $V'(0) \geq \sqrt{F(S^*)}$. If the inequality holds strictly, then we let $t_0 = 0$. If not, then by (3.1) and (4.14), we have $V''(0) > 0$ and (4.15) follows for small $t_0 > 0$.

Similar methods to Proposition 4 can prove that the inequality (3.17) holds for all $t \in [t_0, T^*)$. Hence, using (2.3), (2.9) and the characterization (2.16) of $S^*$, we deduce

$$\begin{aligned} M[\varphi]\left(-\beta_1 \int |\varphi|^4 dx - \beta_2 \int (K * |\varphi|^2)|\varphi|^2 dx\right) &= M[\varphi]\left(4E[\varphi] - J''(t) - 4\int |x|^2|\varphi|^2 dx\right) \\ &\le M[\varphi]\left(4E[\varphi] - S^* - \frac{\sqrt{\varepsilon}}{M}\right) \\ &< M[\varphi](4E[\varphi] - S^*) \\ &= 2M[Q]\left(2E[Q] - \int |x|^2|Q|^2 dx\right) \\ &= M[Q]\left(-\beta_1 \int |Q|^4 dx - \beta_2 \int (K * |Q|^2)|Q|^2 dx\right), \end{aligned}$$

which implies (4.5). This completes the proof of Theorem 1. □

**Remark 6.** *We claim that the assumption (4.3) in Theorem 1 can be replaced by* $M[\varphi_0]\|\nabla\varphi_0\|_2^2 < M[Q]\|\nabla Q\|_2^2$ *under the condition of* $M[\varphi_0]E[\varphi_0] \le M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2 dx}{2}\right)$.

As a result of Theorem 1, we are able to predict the dynamical behavior of certain solutions that are composed by multiplying a finite variance solution by $e^{i\mu|x|^2} (\mu \in \mathbb{R})$.

**Corollary 1.** *Let* $\mu \in \mathbb{R} \setminus \{0\}$, $u_0 \in \Sigma$ *with finite variance such that* $M[u_0]E[u_0] \le M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2 dx}{2}\right)$ *and* $\varphi$ *be the solution of Eq (1.1) with the initial data*

$$\varphi_0 = e^{i\mu|x|^2} u_0.$$

*If* $u_0$ *satisfies the assumption (4.3) for all* $\mu < 0$*, then* $\varphi(x,t)$ *blows up in finite time. If* $u_0$ *satisfies the assumption (4.5) for all* $\mu > 0$*, then* $\varphi(x,t)$ *exists globally and (4.7) holds.*

*Proof.* We assume

$$M[\varphi_0]E[\varphi_0] \ge M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2 dx}{2}\right). \tag{4.16}$$

Some direct calculation shows that

$$M[\varphi_0] = M[u_0], \tag{4.17}$$

$$E[\varphi_0] = E[u_0] + 4\mu^2 \int |x|^2|u_0|^2 dx + 4\mu \mathrm{Im} \int x \cdot \nabla u_0 \bar{u}_0 dx, \tag{4.18}$$

and

$$\mathrm{Im} \int x \cdot \nabla\varphi_0 \bar{\varphi}_0 dx = \mathrm{Im} \int x \cdot \nabla u_0 \bar{u}_0 dx + 2\mu \int |x|^2|u_0|^2 dx. \tag{4.19}$$

Moreover, in connection with (4.18) and (4.19), we deduce that

$$E[\varphi_0] - \frac{(\mathrm{Im} \int x \cdot \nabla\varphi_0 \bar{\varphi}_0 dx)^2}{\int |x|^2|\varphi_0|^2 dx} = E[u_0] - \frac{(\mathrm{Im} \int x \cdot \nabla u_0 \bar{\varphi}_0 dx)^2}{\int |x|^2|u_0|^2 dx}. \tag{4.20}$$

We will only deal with the case when

$$\mu > 0, \quad M[u_0]\left(-\beta_1 \int |u_0|^4 dx - \beta_2 \int (K * |u_0|^2)|u_0|^2 dx\right) < M[Q]\left(-\beta_1 \int |Q|^4 dx - \beta_2 \int (K * |Q|^2)|Q|^2 dx\right),$$

the proof of the other case is similar. Obviously,

$$\begin{aligned} M[\varphi_0]\left(-\beta_1 \int |\varphi_0|^4 dx - \beta_2 \int (K * |\varphi_0|^2)|\varphi_0|^2 dx\right) &= M[u_0]\left(-\beta_1 \int |u_0|^4 dx - \beta_2 \int (K * |u_0|^2)|u_0|^2 dx\right) \\ &< M[Q]\left(-\beta_1 \int |Q|^4 dx - \beta_2 \int (K * |Q|^2)|Q|^2 dx\right), \end{aligned}$$

which implies that (4.5) is satisfied. Due to $\mu > 0$, we know by (4.18) that (4.16) reveals that $\mu \geq \mu_0^+$, where $\mu_0^+$ is the unique positive solution of

$$M[u_0]\left(E[u_0] + 4\mu_0^+ \int |x|^2|u_0|^2 dx + 4(\mu_0^+)^2 \mathrm{Im} \int x \cdot \nabla u_0 \bar{u}_0 dx\right) = M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2 dx}{2}\right).$$

Due to $M[u_0]E[u_0] \leq M[Q]\left(E[Q] - \frac{\int |x|^2|Q|^2 dx}{2}\right)$, the above inequality shows

$$\int |x|^2|u_0|^2 dx + \mu_0^+ \mathrm{Im} \int x \cdot \nabla u_0 \bar{u}_0 dx \geq 0.$$

By the aid of $\mu \geq \mu_0^+$, we derive

$$\mathrm{Im} \int x \cdot \nabla \varphi_0 \bar{\varphi}_0 dx = \mathrm{Im} \int x \cdot \nabla u_0 \bar{u}_0 dx + 2\mu \int |x|^2|u_0|^2 dx \geq \mu \int |x|^2|u_0|^2 dx > 0.$$

As a consequence, the condition (4.6) in Theorem 1 stands. This completes the proof of corollary 1. □

**Remark 7.** *The above Corollary reveals that we can deduce the dynamical behaviour of certain solutions with arbitrary large energy. In fact, if* $u_0$ *satisfies these assumptions in Corollary 1 and* $\mu > 0$ *is sufficiently large, then* $E[\varphi_0] \to +\infty$ *as* $\mu \to \pm\infty$.

## 5. Conclusions

Physically, the significance of the following questions is obvious. Under what circumstances will the solutions of GPE in trapped quantum gases become unstable, turn to blow up? Moreover, under what conditions will these solutions exist globally? Regarding the sharp threshold for the existence of blowup solutions, most of them are illustrated by establishing cross-constrained invariant sets for bounded $M[\varphi_0]$ and $E[\varphi_0]$. In this paper, estimating the temporal evolution of $V'(t) = \frac{J'(t)}{2\sqrt{J(t)}}$ by refined Gagliardo-Nirenberg inequality, we establish the relationship between initial mass-energy and that of the ground state. Then, some invariant evolution flows generated by the Cauchy problem (1.1) are constructed according to the continuity of derivable functions $V(t)$ and $F(S)$. Based on these analyses and discussion, we consider the global existence versus blowup dichotomy of solutions above the *mass-energy threshold*, which can be extended to the dynamical behaviour of certain solutions with arbitrary large energy. Furthermore, it is a natural and critical issue to prove that the global solutions scatter, we intend to study this question in the future.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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