Mathematics

Research article

Some qualitative properties of solutions to a nonlinear fractional differential equation involving two Φ-Caputo fractional derivatives

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Abstract: The momentous objective of this work is to discuss some qualitative properties of solutions such as the estimate of the solutions, the continuous dependence of the solutions on initial conditions and the existence and uniqueness of extremal solutions to a new class of fractional differential equations involving two fractional derivatives in the sense of Caputo fractional derivative with respect to another function Φ. Firstly, using the generalized Laplace transform method, we give an explicit formula of the solutions to the aforementioned linear problem which can be regarded as a novelty item. Secondly, by the implementation of the Φ-fractional Gronwall inequality, we analyze some properties such as estimates and continuous dependence of the solutions on initial conditions. Thirdly, with the help of features of the Mittag-Leffler functions (MLFs), we build a new comparison principle for the corresponding linear equation. This outcome plays a vital role in the forthcoming analysis of this paper especially when we combine it with the monotone iterative technique alongside facet with the method of upper and lower solutions to get the extremal solutions for the analyzed problem. Lastly, we present some examples to support the validity of our main results.

Keywords: Φ-Caputo fractional derivative; multi-terms; generalized Laplace transforms; extremal solutions; monotone iterative style; upper (lower) solutions
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

For the last few years, the field of fractional calculus has become a powerful tool to support mathematical modeling with several successful results. Moreover, fractional differential equations have been used to model many physical, biological and engineering problems (see [1–4]).

As a result of the continuous develepment in the theory of the fractional calculus, a variety of definitions have appeared in the literature. Some famous definitions are those given by Riemann and Liouville, Caputo, Hadamard, and so on. For instance, see the textbook of Kilbas [5]. A more generalized type of fractional operators that appears in the literature is the fractional derivative of a function by another function. Details and properties of this class of fractional operators can be found in [6–10]. On the other hand, most of the time it is a hard task to search and compute the exact solution of nonlinear FDEs. One possible way to achieve this purpose is to apply the monotone iterative technique [11, 12] alongside facet with the method of upper and lower solutions. In addition, another interesting and fascinating feature of this method is that not only it guarantees the existence of extreme solutions, but it is also an effective method for constructing two explicit monotone iterative sequences that converge to the extremal solutions in a region generated by the upper and lower solutions. The readers can find more details about the utility of this technique as well as its significance in tackling nonlinear FDEs in a series of papers [13–20]. However, to the best of the authors' observation, the aforesaid method is very rarely used for nonlinear FDEs involving two Φ-Caputo fractional derivatives.

Motivated by the above mentioned reasons, in this manuscript, we investigate some qualitative properties of solutions such as the estimate of the solutions, the continuous dependence of the solutions on initial conditions as well as the existence and uniqueness of extremal solutions for the following problem:

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}(\ell) = \mathbb{F}(\ell, \mathfrak{z}(\ell)), & \ell \in \Delta := [a, b], \\ \mathfrak{z}(a) = \mathfrak{z}_a, \end{cases} \tag{1.1}$$

where ${}^c\mathbb{D}_{a^+}^{\mu;\Phi}$ and ${}^c\mathbb{D}_{a^+}^{\kappa;\Phi}$ denote the Φ-Caputo fractional derivatives, with the orders μ and κ respectively such that $0 < \kappa < \mu \leq 1$, $\omega > 0$, $\mathfrak{z}_a \in \mathbb{R}$ and $\mathbb{F} \in C(\Delta \times \mathbb{R}, \mathbb{R})$.

The core contribution of this research work is summarized as follows:

- Unlike previous studies, our suggested model (1.1) includes a more generalized fractional derivative that combines various traditional fractional derivatives.
- By coupling the Φ-Laplace transform method and the properties of Mittag-Leffler functions an explicit formula of the solutions to the aforementioned linear problem is given. At the same time, based on this new formula we build a new comparison result which will be used in the forthcoming analysis.
- Based on the Φ-fractional Gronwall inequality, the monotone iterative technique associated with the method of upper and lower solutions and under some reasonable assumptions on the source function $\mathbb{F}$ of the considered problem, we have analyzed some qualitative properties of solutions such as the estimate of the solutions, the continuous dependence of the solutions on initial conditions as well as the existence and uniqueness of extremal solutions.
- The results of our techniques show that they are very effective and simple to use. Moreover, our findings are generalizations and partial continuation of some results obtained in [16, 17, 21–23].

An outline of the present work is as follows. Section 2 is devoted to some preliminary results that are useful in the sequel. In Section 3, we discuss some qualitative properties of solutions such as the estimate of the solutions, the continuous dependence of the solutions on initial conditions as well as the uniqueness of solutions for the problem (1.1). Section 4 is devoted to studying the existence and uniqueness of extremal solutions for problem (1.1). To prove this, we use the monotone iterative technique together with the technique of upper and lower solutions. To show the applicability and consistency of the method under consideration, some examples are given in Section 5. For all computational work, MATLAB software is used. The paper closes with a brief conclusion and some possible future directions of research.

2. Preliminaries

In the current section, we state some basic concepts of fractional calculus related to our work.

Let $\Delta = [a, b]$, $0 \le a < b < \infty$ be a finite interval and $\Phi \colon \Delta \to \mathbb{R}$ be an increasing differentiable function such that $\Phi'(\ell) \ne 0$ for all $\ell \in \Delta$.

Definition 2.1. *[5, 6] The RL fractional integral of order $\mu > 0$ for an integrable function $\mathfrak{z} \colon \Delta \to \mathbb{R}$ with respect to Φ is described by*

$$\mathbb{I}_{a^+}^{\mu;\Phi} \mathfrak{z}(\ell) = \int_a^\ell \frac{\Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}}{\Gamma(\mu)} \mathfrak{z}(\rho) \mathrm{d}\rho,$$

where $\Gamma(\mu) = \int_0^{+\infty} \ell^{\mu-1} e^{-\ell} \mathrm{d}\ell$, $\mu > 0$ is called the Gamma function.

Definition 2.2. *[6] Let $\Phi, \mathfrak{z} \in C^n(\Delta, \mathbb{R})$. The Caputo fractional derivative of $\mathfrak{z}$ of order $n - 1 < \mu < n$ with respect to Φ is defined by*

$$^c\mathbb{D}_{a^+}^{\mu;\Phi} \mathfrak{z}(\ell) = \mathbb{I}_{a^+}^{n-\mu;\Phi} \mathfrak{z}_\Phi^{[n]}(\ell),$$

where $n = [\mu] + 1$ for $\mu \notin \mathbb{N}$, $n = \mu$ for $\mu \in \mathbb{N}$, and

$$\mathfrak{z}_\Phi^{[n]}(\ell) = \left(\frac{\frac{\mathrm{d}}{\mathrm{d}\ell}}{\Phi'(\ell)} \right)^n \mathfrak{z}(\ell).$$

From the above definition, it is clear that

$$^c\mathbb{D}_{a^+}^{\mu;\Phi} \mathfrak{z}(\ell) = \begin{cases} \displaystyle\int_a^\ell \frac{\Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{n-\mu-1}}{\Gamma(n - \mu)} \mathfrak{z}_\Phi^{[n]}(\rho) \mathrm{d}\rho, & \mu \notin \mathbb{N}, \\ \mathfrak{z}_\Phi^{[n]}(\ell), & \mu \in \mathbb{N}. \end{cases}$$

Some basic properties of the Φ-fractional operators are listed in the following lemma.

Lemma 2.1. *[6] Let $\mu, \kappa > 0$ and $\mathfrak{z} \in C(\Delta, \mathbb{R})$. Then for each $\ell \in \Delta$,*

(i) $^c\mathbb{D}_{a^+}^{\mu;\Phi} \mathbb{I}_{a^+}^{\mu;\Phi} \mathfrak{z}(\ell) = \mathfrak{z}(\ell)$,

(ii) $\mathbb{I}_{a^+}^{\mu;\Phi} {}^c\mathbb{D}_{a^+}^{\mu;\Phi} \mathfrak{z}(\ell) = \mathfrak{z}(\ell) - \mathfrak{z}(a)$, for $0 < \mu \le 1$,

(iii) $\mathbb{I}_{a^+}^{\mu;\Phi} (\Phi(\ell) - \Phi(a))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa+\mu)} (\Phi(\ell) - \Phi(a))^{\kappa+\mu-1}$,

(iv) ${}^{c}\mathbb{D}_{a^+}^{\mu;\Phi}(\Phi(\ell) - \Phi(a))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa-\mu)}(\Phi(\ell) - \Phi(a))^{\kappa-\mu-1}$,

(v) ${}^{c}\mathbb{D}_{a^+}^{\mu;\Phi}(\Phi(\ell) - \Phi(a))^{k} = 0$, *for all* $k \in \{0, \ldots, n-1\}$, $n \in \mathbb{N}$.

Definition 2.3. *[24] For* $p, q > 0$ *and* $\varpi \in \mathbb{R}$, *the Mittag-Leffler functions (MLFs) of one and two parameters are respectively given by*

$$\begin{aligned}
\mathbb{E}_{p}(\varpi) &= \sum_{k=0}^{\infty} \frac{\varpi^{k}}{\Gamma(pk+1)}, \\
\mathbb{E}_{p,q}(\varpi) &= \sum_{k=0}^{\infty} \frac{\varpi^{k}}{\Gamma(pk+q)}.
\end{aligned} \tag{2.1}$$

Clearly, $\mathbb{E}_{p,1}(\varpi) = \mathbb{E}_{p}(\varpi)$.

Lemma 2.2. *[24, 25] Let* $p \in (0, 1)$, $q > p$ *be arbitrary and* $\varpi \in \mathbb{R}$. *The functions* $\mathbb{E}_{p}$, $\mathbb{E}_{p,p}$ *and* $\mathbb{E}_{p,q}$ *are nonnegative and have the following properties:*

(i) $\mathbb{E}_{p}(\varpi) \leq 1$, $\mathbb{E}_{p,q}(\varpi) \leq \frac{1}{\Gamma(q)}$, *for any* $\varpi < 0$,

(ii) $\mathbb{E}_{p,q}(\varpi) = \varpi\mathbb{E}_{p,p+q}(\varpi) + \frac{1}{\Gamma(q)}$, *for* $p, q > 0$, $\varpi \in \mathbb{R}$.

Definition 2.4. *[8] A function* $u : [a, \infty) \to \mathbb{R}$ *is said to be of* $\Phi(\ell)$*-exponential order if there exist non-negative constants* M, c, b *such that*

$$|u(\ell)| \leq Me^{c(\Phi(\ell) - \Phi(a))},$$

for $\ell \geq b$.

Definition 2.5. *[8] Let* $\mathfrak{z}, \Phi : [a, \infty) \to \mathbb{R}$ *be real valued functions such that* $\Phi(\ell)$ *is continuous and* $\Phi'(\ell) > 0$ *on* $[a, \infty)$. *The generalized Laplace transform of* $\mathfrak{z}$ *is denoted by*

$$\mathbb{L}_{\Phi}\{\mathfrak{z}(\ell)\} = \int_{a}^{\infty} e^{-\lambda(\Phi(\ell) - \Phi(a))}\mathfrak{z}(\ell)\Phi'(\ell)\,\mathrm{d}\ell, \tag{2.2}$$

for all $\lambda > 0$ *provided that the integral in* (2.2) *exists.*

Definition 2.6. *[8] Let* u *and* v *be two piecewise continuous functions on* $[a, b]$ *and of* $\Phi(\ell)$*-exponential order. The generalized convolution of* u *and* v *is defined by*

$$(u *_{\Phi} v)(\ell) = \int_{a}^{\ell} \Phi'(\rho)u(\rho)v(\Phi^{-1}(\Phi(\ell) + \Phi(a) - \Phi(\rho)))\,d\rho.$$

Lemma 2.3. *[8] Let* u *and* v *be two piecewise continuous functions on* $[a, b]$ *and of* $\Phi(\ell)$*-exponential order. Then*

$$\mathbb{L}_{\Phi}\{u *_{\Phi} v\} = \mathbb{L}_{\Phi}\{u\}\mathbb{L}_{\Phi}\{v\}.$$

In the following lemma, we present the generalized Laplace transforms of some elementary functions as well as the generalized Laplace transforms of the generalized fractional integrals and derivatives.

Lemma 2.4. *[8] The following properties are satisfied:*

(i) $\mathbb{L}_\Phi\{1\} = \frac{1}{\lambda}$, *where* $\lambda > 0$,

(ii) $\mathbb{L}_\Phi\{(\Phi(\ell) - \Phi(a))^{r-1}\} = \frac{\Gamma(r)}{\lambda^r}$, *where* r *where* $\lambda > 0$,

(iii) $\mathbb{L}_\Phi\{\mathbb{E}_p(\pm\omega(\Phi(\ell) - \Phi(a))^p)\} = \frac{\lambda^{p-1}}{\lambda^p \mp \omega}$, *for* p > 0 *and* $\left|\frac{\omega}{\lambda^p}\right| < 1$,

(iv) $\mathbb{L}_\Phi\{(\Phi(\ell) - \Phi(a))^{q-1}\mathbb{E}_{p,q}(\pm\omega(\Phi(\ell) - \Phi(a))^p)\} = \frac{\lambda^{p-q}}{\lambda^p \mp \omega}$, *where* p > 0 *and* $\left|\frac{\omega}{\lambda^p}\right| < 1$,

(v) $\mathbb{L}_\Phi\{\mathbb{I}_{a^+}^{\mu;\Phi}\mathfrak{z}(\ell)\} = \frac{\mathbb{L}_\Phi\{\mathfrak{z}(\ell)\}}{\lambda^\mu}$, *for* $\mu, \lambda > 0$,

(vi) $\mathbb{L}_\Phi\{{}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}(\ell)\} = \lambda^\mu \mathbb{L}_\Phi\{\mathfrak{z}(\ell)\} - \lambda^{\mu-1}\mathfrak{z}(a)$, *for* $0 < \mu \leq 1$ *and* $\lambda > 0$.

Lemma 2.5. *For a given* $\mathbb{H} \in C(\Delta, \mathbb{R})$, $0 < \kappa < \mu \leq 1$ *and* $\omega > 0$, *the linear fractional initial value problem*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}(\ell) = \mathbb{H}(\ell), & \ell \in \Delta := [a, b], \\ \mathfrak{z}(a) = \mathfrak{z}_a, \end{cases} \tag{2.3}$$

has a unique solution given explicitly by

$$\mathfrak{z}(\ell) = \mathfrak{z}_a + \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{H}(\rho)d\rho. \tag{2.4}$$

Proof. Applying the generalized Laplace transform to both sides of Eq (2.3) and then using Lemma 2.4, one gets

$$\lambda^\mu \mathbb{L}_\Phi\{\mathfrak{z}(\ell)\} - \lambda^{\mu-1}\mathfrak{z}(a) + \omega\lambda^\kappa \mathbb{L}_\Phi\{\mathfrak{z}(\ell)\} - \omega\lambda^{\kappa-1}\mathfrak{z}(a) = \mathbb{L}_\Phi\{\mathbb{H}(\ell)\}.$$

So,

$$\begin{aligned} \mathbb{L}_\Phi\{\mathfrak{z}(\ell)\} &= \omega\frac{\lambda^{-1}}{\lambda^{\mu-\kappa} + \omega}\mathfrak{z}_a + \frac{\lambda^{\mu-\kappa-1}}{\lambda^{\mu-\kappa} + \omega}\mathfrak{z}_a + \frac{\lambda^{-\kappa}}{\lambda^{\mu-\kappa} + \omega}\mathbb{L}_\Phi\{\mathbb{H}(\ell)\} \\ &= \omega\mathbb{L}_\Phi\{(\Phi(\ell) - \Phi(a))^{\mu-\kappa}\mathbb{E}_{\mu-\kappa,\mu-\kappa+1}(-\omega(\Phi(\ell) - \Phi(a))^{\mu-\kappa})\}\mathfrak{z}_a \\ &\quad + \mathbb{L}_\Phi\{\mathbb{E}_{\mu-\kappa}(-\omega(\Phi(\ell) - \Phi(a))^{\mu-\kappa})\}\mathfrak{z}_a \\ &\quad + \mathbb{L}_\Phi\{(\Phi(\ell) - \Phi(a))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(a))^{\mu-\kappa})\}\mathbb{L}_\Phi\{\mathbb{H}(\ell)\}. \end{aligned}$$

Taking the inverse generalized Laplace transform of both sides of the last expression, we get

$$\begin{aligned} \mathfrak{z}(\ell) &= \Big[\mathbb{E}_{\mu-\kappa}(-\omega(\Phi(\ell) - \Phi(a))^{\mu-\kappa}) \\ &\quad + \omega(\Phi(\ell) - \Phi(a))^{\mu-\kappa}\mathbb{E}_{\mu-\kappa,\mu-\kappa+1}(-\omega(\Phi(\ell) - \Phi(a))^{\mu-\kappa})\Big]\mathfrak{z}_a \\ &\quad + \mathbb{H}(\ell) *_\Phi (\Phi(\ell) - \Phi(a))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(a))^{\mu-\kappa}) \\ &= \mathfrak{z}_a + \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{H}(\rho)d\rho. \end{aligned}$$

$\square$

Lemma 2.6. *(Comparison result) Let $\kappa, \mu \in (0, 1]$ such that $\kappa < \mu$ and $\omega > 0$. If $\gamma \in C(\Delta, \mathbb{R})$ satisfying*

$$^{c}\mathbb{D}_{a^{+}}^{\mu;\Phi}\gamma(\ell), \, ^{c}\mathbb{D}_{a^{+}}^{\kappa;\Phi}\gamma(\ell) \in C(\Delta, \mathbb{R})$$

and

$$\begin{cases} ^{c}\mathbb{D}_{a^{+}}^{\mu;\Phi}\gamma(\ell) + \omega\,^{c}\mathbb{D}_{a^{+}}^{\kappa;\Phi}\gamma(\ell) \geq 0, & \ell \in (a, b], \\ \gamma(a) \geq 0, \end{cases}$$

then $\gamma(\ell) \geq 0$ for all $\ell \in \Delta$.

Proof. Let

$$\mathbb{H}(\ell) = {}^{c}\mathbb{D}_{a^{+}}^{\mu;\Phi}\gamma(\ell) + \omega\,^{c}\mathbb{D}_{a^{+}}^{\kappa;\Phi}\gamma(\ell) \geq 0,$$

$\gamma(a) = \mathfrak{z}_{a} \geq 0$ in Lemma 2.5. Then it follows by Eq (2.4) and Lemma 2.2 that the conclusion of Lemma 2.6 holds. $\square$

The following lemma is a generalization of Gronwall's inequality.

Lemma 2.7. *[26] Let Δ be the domain of the nonnegative integrable functions u, v. Also, w be a continuous, nonnegative and nondecreasing function defined on Δ and $\Phi \in C^{1}(\Delta, \mathbb{R}_{+})$ be an increasing function with the restriction that $\Phi'(\ell) \neq 0$, for all $\ell \in \Delta$. If*

$$u(\ell) \leq v(\ell) + w(\ell) \int_{a}^{\ell} \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1} u(\rho)\mathrm{d}\rho, \qquad \ell \in \Delta.$$

Then

$$u(\ell) \leq v(\ell) + \int_{a}^{\ell} \sum_{n=0}^{\infty} \frac{(w(\ell)\Gamma(\mu))^{n}}{\Gamma(n\mu)} \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{n\mu-1} v(\rho)\mathrm{d}\rho, \qquad \ell \in \Delta.$$

Corollary 2.1. *[26] Under the conditions of Lemma 2.7, let v be a nondecreasing function on Δ. Then we get*

$$u(\ell) \leq v(\ell)\mathbb{E}_{\mu}\left(\Gamma(\mu)w(\ell)(\Phi(\ell) - \Phi(a))^{\mu}\right), \ \ell \in \Delta. \tag{2.5}$$

Lemma 2.8. *Assume that $\{w_{n}\}$ is a family of continuous functions on Δ, for each $n > 0$ which satisfies*

$$\begin{cases} ^{c}\mathbb{D}_{a^{+}}^{\mu;\Phi}w_{n}(\ell) + \omega^{c}\mathbb{D}_{a^{+}}^{\kappa;\Phi}w_{n}(\ell) = \mathbb{F}(\ell, w_{n}(\ell)), & \ell \in \Delta, \\ w_{n}(a) = w_{a}, \end{cases} \tag{2.6}$$

and $|\mathbb{F}(\ell, w_{n}(\ell))| \leq \mathbb{L}$ ($\mathbb{L} > 0$ independent of n) for each $\ell \in \Delta$. Then, the family $\{w_{n}\}$ is equicontinuous on Δ.

Proof. According to Lemma 2.5, the integral representation of problem (2.6) is given by

$$w_{n}(\ell) = w_{a} + \int_{a}^{\ell} \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}$$
$$\times \mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{F}(\rho, w_{n}(\rho))\mathrm{d}\rho. \tag{2.7}$$

Let now any $\ell_1, \ell_2 \in \Delta$ with $a < \ell_1 < \ell_2 < b$. Then from (2.7) and Lemma 2.2 we have

$$
\begin{aligned}
&|w_n(\ell_2) - w_n(\ell_1)| \\
&\leq \int_a^{\ell_1} \frac{\Phi'(\rho)\left[(\Phi(\ell_1) - \Phi(\rho))^{\mu-1} - (\Phi(\ell_2) - \Phi(\rho))^{\mu-1}\right]}{\Gamma(\mu)} |\mathbb{F}(\rho, w_n(\rho)| d\rho \\
&\quad + \int_{\ell_1}^{\ell_2} \frac{\Phi'(\rho)(\Phi(\ell_2) - \Phi(\rho))^{\mu-1}}{\Gamma(\mu)} |\mathbb{F}(\rho, w_n(\rho)| \, d\rho \\
&\leq \frac{\mathbb{L}}{\Gamma(\mu+1)} [(\Phi(\ell_1) - \Phi(a))^{\mu} + 2(\Phi(\ell_2) - \Phi(\ell_1))^{\mu} - (\Phi(\ell_2) - \Phi(a))^{\mu}] \\
&\leq \frac{2\mathbb{L}}{\Gamma(\mu+1)}(\Phi(\ell_2) - \Phi(\ell_1))^{\mu}.
\end{aligned}
$$

As $\ell_2 \to \ell_1$, the right-hand side of the above inequality tends to zero independently of $\{w_n\}$. Hence, the family $\{w_n\}$ is equicontinuous on Δ. $\qquad\square$

3. Some qualitative properties of solutions for problem (1.1)

In this section, we obtain some qualitative properties of solutions for problem (1.1). To do this, we apply the Φ-fractional Gronwall inequality.

First of all, we present the following theorem that contains the estimates of the solutions of problem (1.1).

Theorem 3.1. *Let $\mathbb{F} : \Delta \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfies the following condition:*

(H_1) *There exists a constant $\mathbb{L} > 0$ such that*

$$|\mathbb{F}(\ell, y) - \mathbb{F}(\ell, x)| \leq \mathbb{L}|y - x|,$$

for all $x, y \in \mathbb{R}$ and $\ell \in \Delta$.

If $\mathfrak{z} \in C(\Delta, \mathbb{R})$ is any solution of the problem (1.1), then

$$|\mathfrak{z}(\ell)| \leq \left(|\mathfrak{z}_a| + \frac{\mathbb{L}\mathbb{F}^* (\Phi(b) - \Phi(a))^{\mu}}{\Gamma(\mu+1)}\right) \mathbb{E}_{\mu}\left(\mathbb{L}(\Phi(b) - \Phi(a))^{\mu}\right), \ \ell \in \Delta,$$

where $\mathbb{F}^* = \sup_{\ell \in \Delta} |\mathbb{F}(\ell, 0)|$.

Proof. Let $\mathfrak{z} \in C(\Delta, \mathbb{R})$ be the solution of the problem (1.1), then by Lemma 2.5, the solution $\mathfrak{z}$ can be represented as follow:

$$\mathfrak{z}(\ell) = \mathfrak{z}_a + \int_a^{\ell} \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1} \mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{F}(\rho, \mathfrak{z}(\rho))d\rho.$$

From Lemma 2.2 and the hypothesis (H_1) we can get

$$|\mathfrak{z}(\ell)| \leq |\mathfrak{z}_a| + \frac{\mathbb{L}\mathbb{F}^* (\Phi(\ell) - \Phi(a))^{\mu}}{\Gamma(\mu+1)} + \frac{\mathbb{L}}{\Gamma(\mu)} \int_a^{\ell} \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}|\mathfrak{z}(\rho)|d\rho.$$

Using Corollary 2.1, we conclude that

$$|\mathfrak{z}(\ell)| \leq \left(|\mathfrak{z}_a| + \frac{\mathbb{L}\mathbb{F}^*\,(\Phi(b) - \Phi(a))^\mu}{\Gamma(\mu + 1)}\right)\mathbb{E}_\mu\left(\mathbb{L}(\Phi(b) - \Phi(a))^\mu\right),\ \ell \in \Delta.$$

In the following theorem, we present the dependence of solution $\mathfrak{z}$ on the initial values.

Theorem 3.2. *Let $\mathbb{F} : \Delta \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the hypothesis (H_1). Suppose $\mathfrak{z}$ and $\bar{\mathfrak{z}}$ are the solutions of the problem*

$$^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}(\ell) + \omega\,^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}(\ell) = \mathbb{F}(\ell, \mathfrak{z}(\ell)),\ \ell \in \Delta, \tag{3.1}$$

corrosponding to $\mathfrak{z}(a) = \mathfrak{z}_a$ and $\bar{\mathfrak{z}}(a) = \bar{\mathfrak{z}}_a$ respectively. Then

$$\|\mathfrak{z} - \bar{\mathfrak{z}}\| \leq \mathbb{E}_\mu\left(\mathbb{L}(\Phi(b) - \Phi(a))^\mu\right)|\mathfrak{z}_a - \bar{\mathfrak{z}}_a|. \tag{3.2}$$

Proof. Let $\mathfrak{z}, \bar{\mathfrak{z}} \in C(\Delta, \mathbb{R})$ be the solutions of the problem (3.1) corresponding to $\mathfrak{z}(a) = \mathfrak{z}_a$ and $\bar{\mathfrak{z}}(a) = \bar{\mathfrak{z}}_a$, respectively. Then by Lemma 2.5, solutions $\mathfrak{z}$ and $\bar{\mathfrak{z}}$ can be represented as follows:

$$\begin{cases} \mathfrak{z}(\ell) = \mathfrak{z}_a + \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{F}(\rho, \mathfrak{z}(\rho))\mathrm{d}\rho, \\ \bar{\mathfrak{z}}(\ell) = \bar{\mathfrak{z}}_a + \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{F}(\rho, \bar{\mathfrak{z}}(\rho))\mathrm{d}\rho. \end{cases}$$

From Lemma 2.2 and the hypothesis (H_1) we get

$$|\mathfrak{z}(\ell) - \bar{\mathfrak{z}}(\ell)| \leq |\mathfrak{z}_a - \bar{\mathfrak{z}}_a| + \frac{\mathbb{L}}{\Gamma(\mu)}\int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}|\mathfrak{z}(\rho) - \bar{\mathfrak{z}}(\rho)|\mathrm{d}\rho.$$

Using Corollary 2.1, we conclude that

$$|\mathfrak{z}(\ell) - \bar{\mathfrak{z}}(\ell)| \leq |\mathfrak{z}_a - \bar{\mathfrak{z}}_a|\mathbb{E}_\mu\left(\mathbb{L}(\Phi(b) - \Phi(a))^\mu\right),\ \ell \in \Delta.$$

Taking supremum over $\ell \in \Delta$, we obtain

$$\|\mathfrak{z} - \bar{\mathfrak{z}}\| \leq |\mathfrak{z}_a - \bar{\mathfrak{z}}_a|\mathbb{E}_\mu\left(\mathbb{L}(\Phi(b) - \Phi(a))^\mu\right).$$

Remark 3.1. *The inequality* (3.2) *exhibits continuous dependence of solutions of the problem* (1.1) *on initial conditions as well as it gives the uniqueness. The uniqueness follows by putting $\mathfrak{z}_a = \bar{\mathfrak{z}}_a$ in* (3.2).

4. Monotone iterative technique for problem (1.1)

The main theme of this section is to discuss the existence and uniqueness of extremal solutions for the problem (1.1). First of all, we give the definitions of lower and upper solutions of the problem (1.1).

Definition 4.1. *A function $\mathfrak{z} \in C(\Delta, \mathbb{R})$ is called a lower solution of* (1.1), *if it satisfies*

$$\begin{cases} ^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z} + \omega\,^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}(\ell) \leq \mathbb{F}(\ell, \mathfrak{z}(\ell)), & \ell \in \Delta, \\ \mathfrak{z}(a) \leq \mathfrak{z}_a. \end{cases} \tag{4.1}$$

If all inequalities of (4.1) *are inverted, we say that $\mathfrak{z}$ is an upper solution of the problem* (1.1).

In order to get the existence and uniqueness of the extremal solutions for the initial value problem (1.1), we give the following assumptions:

(H_2) There exist $\mathfrak{z}_0$, $\tilde{\mathfrak{z}}_0 \in C(\Delta, \mathbb{R})$ such that $\mathfrak{z}_0$ and $\tilde{\mathfrak{z}}_0$ are lower and upper solutions of problem (1.1), respectively, with $\mathfrak{z}_0(\ell) \leq \tilde{\mathfrak{z}}_0(\ell)$ for $\ell \in \Delta$.

(H_3) $\mathbb{F}$ is increasing with respect to the second variable, i.e.

$$\mathbb{F}(\ell, x) \leq \mathbb{F}(\ell, y),$$

for any $\ell \in \Delta$ and

$$\mathfrak{z}_0(\ell) \leq x \leq y \leq \tilde{\mathfrak{z}}_0(\ell).$$

(H_4) There exists a constant $\mathbb{M} \geq 0$ such that

$$0 \leq \mathbb{F}(\ell, y) - \mathbb{F}(\ell, x) \leq \mathbb{M}(y - x),$$

with

$$\mathfrak{z}_0(\ell) \leq x \leq y \leq \tilde{\mathfrak{z}}_0(\ell),$$

for all $\ell \in \Delta$.

Theorem 4.1. *Because of (H_2), (H_3) and that the function $\mathbb{F} : \Delta \times \mathbb{R} \to \mathbb{R}$ is continuous, then there exist monotone iterative sequences $\{\mathfrak{z}_n\}$ and $\{\tilde{\mathfrak{z}}_n\}$ which converge uniformly on Δ to the extremal solutions of the problem (1.1) in the sector $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$, where*

$$[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0] = \Big\{ \mathfrak{z} \in C(\Delta, \mathbb{R}) \,:\, \mathfrak{z}_0(\ell) \leq \mathfrak{z}(\ell) \leq \tilde{\mathfrak{z}}_0(\ell), \ \ell \in \Delta \Big\}.$$

Furthermore, if (H_4) holds, problem (1.1) has a unique solution in $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$.

Proof. For any $\mathfrak{z}_0, \tilde{\mathfrak{z}}_0 \in C(\Delta, \mathbb{R})$, we define

$$
\begin{cases}
{}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}_{n+1}(\ell) + \omega\, {}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}_{n+1}(\ell) = \mathbb{F}(\ell, \mathfrak{z}_n(\ell)), & \ell \in \Delta, \\[2mm]
\mathfrak{z}_{n+1}(a) = \mathfrak{z}_a,
\end{cases}
\tag{4.2}
$$

and

$$
\begin{cases}
{}^c\mathbb{D}_{a^+}^{\mu;\Phi}\tilde{\mathfrak{z}}_{n+1}(\ell) + \omega\, {}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\tilde{\mathfrak{z}}_{n+1}(\ell) = \mathbb{F}(\ell, \tilde{\mathfrak{z}}_n(\ell)), & \ell \in \Delta, \\[2mm]
\tilde{\mathfrak{z}}_{n+1}(a) = \mathfrak{z}_a.
\end{cases}
\tag{4.3}
$$

By Lemma 2.5, we know that problems (4.2) and (4.3) have unique solutions $\mathfrak{z}_n(\ell)$ and $\tilde{\mathfrak{z}}_n(\ell)$, respectively. These solutions are expressed as

$$\mathfrak{z}_{n+1}(\ell) = \mathfrak{z}_a + \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1} \times \mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{F}(\rho, \mathfrak{z}_n(\rho))\mathrm{d}\rho \tag{4.4}$$

and

$$\tilde{\mathfrak{z}}_{n+1}(\ell) = \mathfrak{z}_a + \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1} \times \mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa})\mathbb{F}(\rho, \tilde{\mathfrak{z}}_n(\rho))\mathrm{d}\rho. \tag{4.5}$$

Firstly, let us prove that

$$\mathfrak{z}_0(\ell) \leq \mathfrak{z}_1(\ell) \leq \tilde{\mathfrak{z}}_1(\ell) \leq \tilde{\mathfrak{z}}_0(\ell), \qquad \ell \in \Delta.$$

To this end, set

$$\gamma(\ell) = \mathfrak{z}_1(\ell) - \mathfrak{z}_0(\ell).$$

From (4.2) and Definition 4.1, we obtain

$$
\begin{aligned}
{}^c\mathbb{D}_{a^+}^{\mu;\Phi}\gamma(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\gamma(\ell) &= {}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}_1(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}_1(\ell) - \left({}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}_0(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}_0(\ell)\right) \\
&= \mathbb{F}(\ell, \mathfrak{z}_0(\ell)) - \left({}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}_0(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}_0(\ell)\right) \\
&\geq 0
\end{aligned}
$$

and $\gamma(a) = 0$. Invoking Lemma 2.6, we get $\gamma(\ell) \geq 0$ for any $\ell \in \Delta$. Thus,

$$\mathfrak{z}_0(\ell) \leq \mathfrak{z}_1(\ell),$$

for $\ell \in \Delta$. As the same method, it can be showed that $\tilde{\mathfrak{z}}_1(\ell) \leq \tilde{\mathfrak{z}}_0(\ell)$, for all $\ell \in \Delta$. Now, let

$$\gamma(\ell) = \tilde{\mathfrak{z}}_1(\ell) - \mathfrak{z}_1(\ell).$$

Using (4.2) and (4.3) together with assumptions (H_2) and (H_3), we get

$$
{}^c\mathbb{D}_{a^+}^{\mu;\Phi}\gamma(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\gamma(\ell) = \mathbb{F}(\ell, \tilde{\mathfrak{z}}_0(\ell)) - \mathbb{F}(\ell, \mathfrak{z}_0(\ell)) \geq 0
$$

and $\gamma(a) = 0$. Using Lemma 2.6, we arrive at $\mathfrak{z}_1(\ell) \leq \tilde{\mathfrak{z}}_1(\ell)$, for each $\ell \in \Delta$.

Secondly, we need to show that $\mathfrak{z}_1$ and $\tilde{\mathfrak{z}}_1$ are the lower and upper solutions of problem (1.1), respectively. Taking into account that $\mathbb{F}$ is increasing function with respect to the second variable, we get

$$
\begin{cases}
{}^c\mathbb{D}_{a^+}^{\mu;\Phi}\mathfrak{z}_1(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\mathfrak{z}_1(\ell) = \mathbb{F}(\ell, \mathfrak{z}_0(\ell)) \leq \mathbb{F}(\ell, \mathfrak{z}_1(\ell)), \\
\mathfrak{z}_1(a) = \mathfrak{z}_a
\end{cases}
$$

and

$$
\begin{cases}
{}^c\mathbb{D}_{a^+}^{\mu;\Phi}\tilde{\mathfrak{z}}_1(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\tilde{\mathfrak{z}}_1(\ell) = \mathbb{F}(\ell, \tilde{\mathfrak{z}}_0(\ell)) \geq \mathbb{F}(\ell, \tilde{\mathfrak{z}}_1(\ell)), \\
\tilde{\mathfrak{z}}_1(a) = \mathfrak{z}_a.
\end{cases}
$$

This means that $\mathfrak{z}_1$ and $\tilde{\mathfrak{z}}_1$ are the lower and upper solutions of problem (1.1), respectively. By the above arguments and mathematical induction, we can show that the sequences $\mathfrak{z}_n$ and $\tilde{\mathfrak{z}}_n$, $(n \geq 1)$ are lower and upper solutions of (1.1), respectively, and satisfy the following relation

$$\mathfrak{z}_0(\ell) \leq \mathfrak{z}_1(\ell) \leq \cdots \leq \mathfrak{z}_n(\ell) \leq \cdots \leq \tilde{\mathfrak{z}}_n(\ell) \leq \cdots \leq \tilde{\mathfrak{z}}_1(\ell) \leq \tilde{\mathfrak{z}}_0(\ell), \tag{4.6}$$

for $\ell \in \Delta$.

Thirdly, we show that the sequences $\{\mathfrak{z}_n\}$ and $\{\tilde{\mathfrak{z}}_n\}$ converge uniformly to their limit functions $\mathfrak{z}^*$ and $\tilde{\mathfrak{z}}^*$ respectively. In fact, it follows from (4.6), that the sequences $\{\mathfrak{z}_n\}$ and $\{\tilde{\mathfrak{z}}_n\}$ are uniformly bounded on Δ. Moreover, from Lemma 2.8, the sequences $\{\mathfrak{z}_n\}$ and $\{\tilde{\mathfrak{z}}_n\}$ are equicontinuous on Δ. Hence by

Arzelà-Ascoli's Theorem, there exist subsequences $\{\mathfrak{z}_{n_k}\}$ and $\{\tilde{\mathfrak{z}}_{n_k}\}$ which converge uniformly to $\mathfrak{z}^*$ and $\tilde{\mathfrak{z}}^*$ respectively on Δ. This together with the monotonicity of sequences $\{\mathfrak{z}_n\}$ and $\{\tilde{\mathfrak{z}}_n\}$ implies

$$\lim_{n\to\infty} \mathfrak{z}_n(\ell) = \mathfrak{z}^*(\ell),$$
$$\lim_{n\to\infty} \tilde{\mathfrak{z}}_n(\ell) = \tilde{\mathfrak{z}}^*(\ell),$$

uniformly on $\ell \in \Delta$ and the limit functions $\mathfrak{z}^*$, $\tilde{\mathfrak{z}}^*$ satisfy problem (1.1).

Lastly, we prove the minimal and maximal property of $\mathfrak{z}^*$ and $\tilde{\mathfrak{z}}^*$ on $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$. To do this, let $\mathfrak{z} \in [\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$ be any solution of (1.1). Suppose for some $n \in \mathbb{N}^*$ that

$$\mathfrak{z}_n(\ell) \leq \mathfrak{z}(\ell) \leq \tilde{\mathfrak{z}}_n(\ell), \quad \ell \in \Delta. \tag{4.7}$$

Setting

$$\gamma(\ell) = \mathfrak{z}(\ell) - \mathfrak{z}_{n+1}(\ell).$$

It follows that

$${}^c\mathbb{D}_{a^+}^{\mu;\Phi}\gamma(\ell) + \omega\,{}^c\mathbb{D}_{a^+}^{\kappa;\Phi}\gamma(\ell) = \mathbb{F}(\ell, \mathfrak{z}(\ell)) - \mathbb{F}(\ell, \mathfrak{z}_n(\ell)) \geq 0.$$

Furthermore, $\gamma(a) = 0$. Thus, in light of Lemma 2.6, we have the inequality $\gamma(\ell) \geq 0$, $\ell \in \Delta$, and then $\mathfrak{z}_{n+1}(\ell) \leq \mathfrak{z}(\ell)$, $\ell \in \Delta$. Analogously, it can be obtained that $\mathfrak{z}(\ell) \leq \tilde{\mathfrak{z}}_{n+1}(\ell)$, $\ell \in \Delta$. So, from mathematical induction, it follows that the relation (4.7) holds on Δ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ on both sides of (4.7), we get

$$\mathfrak{z}^*(\ell) \leq \mathfrak{z}(\ell) \leq \tilde{\mathfrak{z}}^*(\ell), \qquad \ell \in \Delta.$$

This means that $\mathfrak{z}^*$, $\tilde{\mathfrak{z}}^*$ are the extremal solutions of (1.1) in $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$. To close the proof it remains to show that the problem (1.1) has a unique solution. In fact, by the foregoing arguments, we know that $\mathfrak{z}^*$, $\tilde{\mathfrak{z}}^*$ are the extremal solutions of the problem (1.1) in $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$ and $\mathfrak{z}^*(\ell) \leq \tilde{\mathfrak{z}}^*(\ell), \ell \in \Delta$. So, it is enough to prove that $\mathfrak{z}^*(\ell) \geq \tilde{\mathfrak{z}}^*(\ell)$, for $\ell \in \Delta$. For this purpose, let

$$u(\ell) = \tilde{\mathfrak{z}}^*(\ell) - \mathfrak{z}^*(\ell),$$

for $\ell \in \Delta$, then by (H_4) and Lemmas 2.2, 2.5, we get

$$0 \leq u(\ell) = \tilde{\mathfrak{z}}^*(\ell) - \mathfrak{z}^*(\ell)$$
$$= \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}(-\omega(\Phi(\ell) - \Phi(\rho))^{\mu-\kappa}) \times (\mathbb{F}(\rho, \tilde{\mathfrak{z}}^*(\rho)) - \mathbb{F}(\rho, \mathfrak{z}^*(\rho)))\mathrm{d}\rho,$$
$$\leq \frac{\mathbb{M}}{\Gamma(\mu)} \int_a^\ell \Phi'(\rho)(\Phi(\ell) - \Phi(\rho))^{\mu-1}u(\rho)\mathrm{d}\rho.$$

By the Gronwall's inequality (Lemma 2.7), we get $u(\ell) \equiv 0$ on Δ. Hence, $\mathfrak{z}^* \equiv \tilde{\mathfrak{z}}^*$ is the unique solution of the problem (1.1). In addition, the unique solution can be obtained by the monotone iterative procedures (4.2) and (4.3) starting from $\mathfrak{z}_0$ or $\tilde{\mathfrak{z}}_0$. Thus, the proof of Theorem 4.1 is finished. $\qquad\square$

5. Numerical results

Here, we present some applications for our analysis.

Example 5.1. *Let us consider problem* (1.1) *with specific data:*

$$\mu = 0.8, \quad \kappa = 0.5, \quad \omega = \frac{2}{\sqrt{\pi}}, \quad a = 0, \quad b = 1, \quad \mathfrak{z}(0) = 1. \tag{5.1}$$

In order to illustrate Theorem 4.1, we take

$$\Phi(\ell) = \sigma(\ell),$$

where $\sigma(\ell)$ is the Sigmoid function [9] which can be expressed as the following form:

$$\sigma(\ell) = \frac{1}{1 + e^{-\ell}}, \tag{5.2}$$

and a convenience of the Sigmoid function is its derivative:

$$\sigma'(\ell) = \sigma(\ell)(1 - \sigma(\ell)).$$

Taking also $\mathbb{F} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ given by

$$\mathbb{F}(\ell, \mathfrak{z}(\ell)) = (\sigma(\ell) - 0.5) \, e^{\mathfrak{z}(\ell) - 3}, \tag{5.3}$$

for $\ell \in [0, 1]$. Clearly, $\mathbb{F}$ is continuous. Moreover, it is easy to verify that $\mathfrak{z}_0(\ell) = 0$, $\tilde{\mathfrak{z}}_0(\ell) = 1 + \ell$ are lower and upper solutions of (1.1), respectively and

$$\mathfrak{z}_0(\ell) \leq \tilde{\mathfrak{z}}_0(\ell),$$

for all $\ell \in [0, 1]$.

On the other hand, from the expression of $\mathbb{F}$, one can see that $\mathbb{F}$ is increasing with respect to the second variable. Thus, by Theorem 4.1 the problem (1.1) with the data (5.1)–(5.3) has extremal solutions in $[\mathfrak{z}_0, \tilde{\mathfrak{z}}_0]$, which can be approximated by the following iterative sequences:

$$\begin{cases} \mathfrak{z}_0(\ell) = 0, \\[2mm] \mathfrak{z}_{n+1}(\ell) = 1 + \displaystyle\int_0^\ell \sigma(\rho)(1 - \sigma(\rho)) \\[2mm] \qquad \dfrac{\mathbb{E}_{0.3,0.8}\left(-\frac{2}{\sqrt{\pi}}(\sigma(\ell) - \sigma(\rho))^{\mu-\kappa}\right)}{(\sigma(\ell) - \sigma(\rho))^{0.2}} \\[2mm] \qquad \times\left((\sigma(\rho) - 0.5)e^{\mathfrak{z}_n(\rho) - 3}\right) d\rho \end{cases} \tag{5.4}$$

and

$$\begin{cases} \tilde{\mathfrak{z}}_0(\ell) = 1 + \ell, \\[2mm] \tilde{\mathfrak{z}}_{n+1}(\ell) = 1 + \displaystyle\int_0^\ell \sigma(\rho)(1 - \sigma(\rho)) \\[2mm] \qquad \dfrac{\mathbb{E}_{0.3,0.8}\left(-\frac{2}{\sqrt{\pi}}(\sigma(\ell) - \sigma(\rho))^{\mu-\kappa}\right)}{(\sigma(\ell) - \sigma(\rho))^{0.2}} \\[2mm] \qquad \times\left((\sigma(\rho) - 0.5)e^{\tilde{\mathfrak{z}}_n(\rho) - 3}\right) d\rho. \end{cases} \tag{5.5}$$

It should be noted at this stage that the exact calculation of the integrals of Eqs (5.4) and (5.5) is far from trivial due to the complicated integrands. Therefore, we implemented the well-known composite trapezoidal rule to these integrals. We first subdivide the interval $I := [0, 1]$ into N subintervals with $h = 1/N$, $\ell_j = \rho_j = jh$ for $j = 0, 1, \cdots, N$. Then, at each node $\ell = l_i$, we applied trapezoidal's quadrature rule to approximate the integrals which is a second-order accurate. Therefore, global error is $O(h^2)$ as the step size h tends to zero. We used $h = 0.05$ in the below examples.

The graphs of $\mathfrak{z}_n$ and $\tilde{\mathfrak{z}}_n$ for $n = 0, 1, 2$ are plotted in Figure 1.
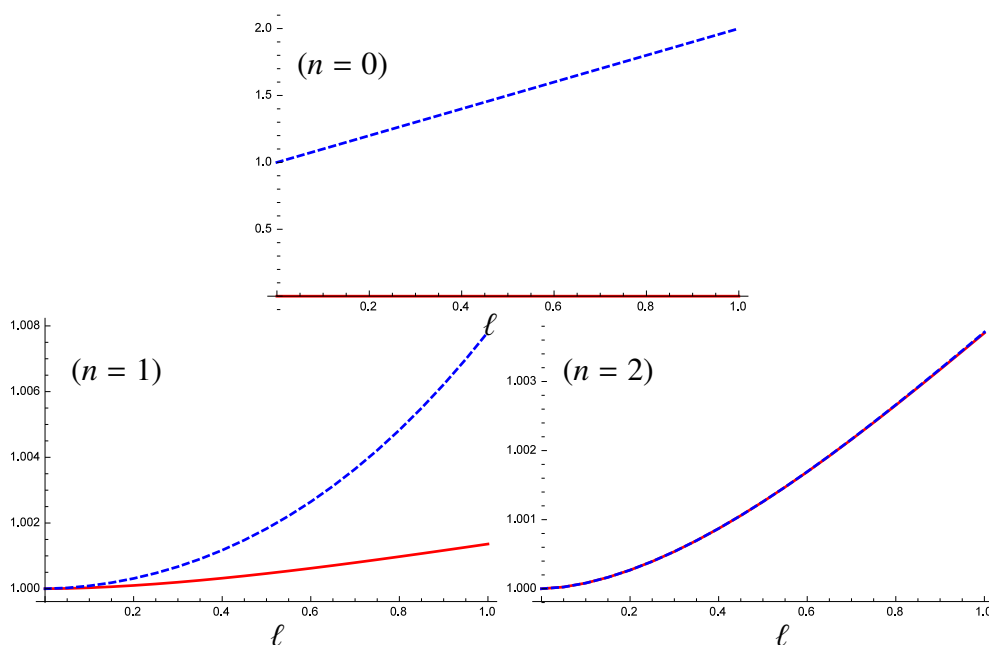

Figure 1. Graphs of lower solutions $\mathfrak{z}_n$ (solid) and upper solutions $\tilde{\mathfrak{z}}_n$ (dashed) for Example 1.

It is clearly observed that the sequences $\mathfrak{z}_n$ and $\tilde{\mathfrak{z}}_n$ converge uniformly and very rapidly. To measure the bound of the error at each iteration n, we use the L_2-norm defined as

$$E_n = \|\tilde{\mathfrak{z}}_n - \mathfrak{z}_n\|^2 = \int_0^1 (\tilde{\mathfrak{z}}_n(\ell) - \mathfrak{z}_n(\ell))^2 d\ell.$$

Table 1 shows the error bounds E_n for $n = 0, 1, 2, 3$. This table clearly states that both lower and upper solutions converges rapidly to the exact solution with almost negligible error after only three iterations.

Table 1. Error bounds E_n ($n = 0, 1, 2, 3$) for Example 1.

n	0	1	2	3
E_n	2.33333	7.46215×10^{-6}	2.0401×10^{-11}	4.01309×10^{-17}

Example 5.2. *Consider the following problem:*

$$\begin{cases} {}^c\mathbb{D}_{0^+}^{0.9}\mathfrak{z}(\ell) + \Gamma(1.6)\,{}^c\mathbb{D}_{0^+}^{0.4}\mathfrak{z}(\ell) = \ell \sin \mathfrak{z}(\ell), \\ \mathfrak{z}(0) = 0.5, \end{cases} \tag{5.6}$$

for $\ell \in [0, 1]$, here,

$$\mu = 0.9, \quad \kappa = 0.4, \quad \omega = \Gamma(1.6), \quad a = 0, \quad b = 1, \quad \Phi(\ell) = \ell,$$

and

$$\mathbb{F}(\ell, \mathfrak{z}(\ell)) = \ell \sin \mathfrak{z}(\ell),$$

for all $\ell \in [0, 1]$.

Obviously, $\mathbb{F}$ is continuous. On the one hand, it is not difficult to verify that the choices $\mathfrak{z}_0(\ell) = 0.5$ and $\tilde{\mathfrak{z}}_0(\ell) = 0.5 + \ell$ are lower and upper solutions of (5.6), respectively, with $\mathfrak{z}_0(\ell) \leq \tilde{\mathfrak{z}}_0(\ell)$. Moreover, for all $\ell \in [0, 1]$ and

$$\mathfrak{z}_0(\ell) \leq x(\ell) \leq y(\ell) \leq \tilde{\mathfrak{z}}_0(\ell),$$

one has

$$0 \leq \mathbb{F}(\ell, y(\ell)) - \mathbb{F}(\ell, x(\ell)) \leq (y(\ell) - x(\ell)).$$

Thus, all the assumptions of Theorem 4.1 hold true. As a result, Theorem 4.1 guarantees that the problem (5.6) has a unique solution, which can be obtained by the following iterative scheme:

$$\mathfrak{z}_{n+1}(\ell) = 0.5 + \int_0^\ell \frac{\mathbb{E}_{0.5, 0.9}\left(-\Gamma(1.6)\sqrt{\ell - \rho}\right)}{(\ell - \rho)^{0.1}} \rho \sin \mathfrak{z}_n(\rho) d\rho,$$

starting from $\mathfrak{z}_0(\ell) = 0.5$ or $\tilde{\mathfrak{z}}_0(\ell) = 0.5 + \ell$.

Applying the same algorithm used in the previous example, we may state the same conclusion that the two sequences $\mathfrak{z}_n$ and $\tilde{\mathfrak{z}}_n$ converge uniformly and very rapidly to the exact solution as shown in Figure 2 and supported by the error analysis in Table 2.
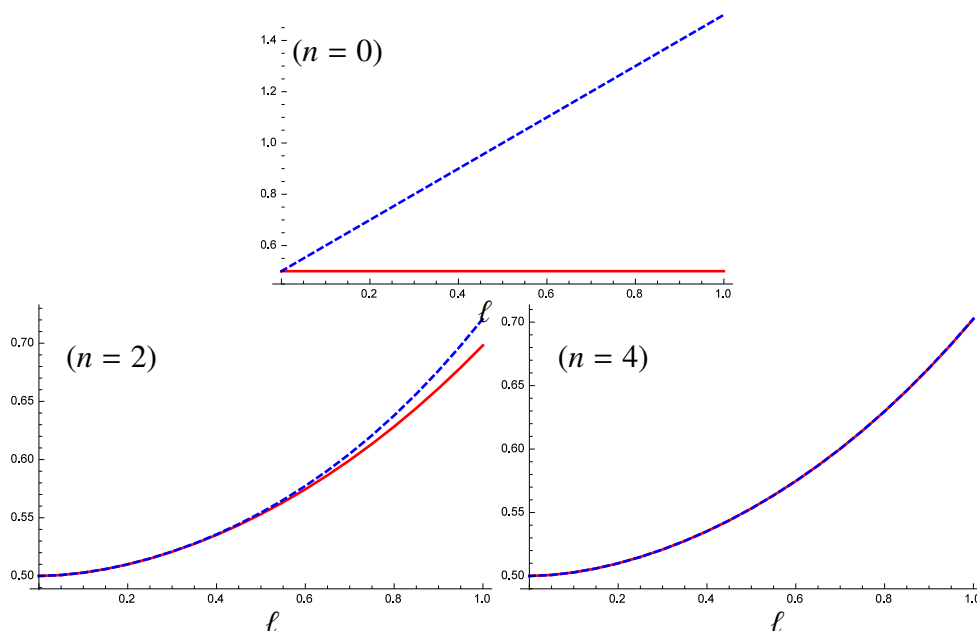

Figure 2. Graphs of lower solutions $\mathfrak{z}_n$ (solid) and upper solutions $\tilde{\mathfrak{z}}_n$ (dashed) for Example 2.

Table 2. Error bounds E_n ($n = 0, 1, 2, 3, 4$) for Example 2.

n	0	1	2	3	4
E_n	0.33333	4.22221×10^{-3}	5.94414×10^{-5}	5.98584×10^{-7}	4.38003×10^{-9}

6. Conclusions

In this paper, we analyzed some qualitative properties of solutions to a new class of fractional differential equations involving two fractional derivatives in the sense of Caputo fractional derivative with respect to a strictly increasing continuous function Φ. As a methodology, we adopted the coupling of Φ-Laplace transform method, Φ-fractional Gronwall inequality and the monotone iterative technique along with the method of upper and lower solutions. Lastly, numerical examples are given to confirm the simplicity and accuracy of the proposed technique. In addition, we gave our findings on a broad platform that covers a wide class of initial value problems that arise in nonlinear analysis and its applications. As a future direction of research, it is desirable to apply this technique to other generalized fractional operators.

Conflict of interest

The authors declare no conflicts of interest.

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